UNIVERSIDADE DE SÃO PAULO

Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2569

Log-New Weibull Extension Regression Models with Censored Data

Giovana Oliveira
Edwin M. M. Ortega
Vicente G. Cancho

Nº 274

RELATÓRIOS TÉCNICOS

São Carlos – SP
Maio/2006

Log-New Weibull Extension Regression Models With Censored Data

Giovana Oliveira*
University of São Paulo

Edwin M. M. Ortega†
University of São Paulo

Vicente G. Cancho. ‡
University of São Paulo

Abstract

In this paper we introduced a regression model considering the new Weibull extension distribution. This distribution can be used to model bathtub-shaped failure rate function. Assuming censored data, we considered a classic analysis and a Bayesian analysis assuming informative priors for the parameters of the model. A Bayesian approach is considered using Markov Chain Monte Carlo Methods, where Gibbs algorithms along with Metropolis steps are used to obtain the posterior summaries of interest. We illustrated methodology with two numeric examples.

Keywords: New Weibull extension distribution; censored data; regression model; survival data.

1 Introduction

Standard lifetime distributions usually very strong restrictions on the data as is well illustrated by their inability to produce bathtub curves, and thus to adequately interprete data with this character. Some distributions were introduced to model this kind data, as the generalized gamma distribution proposed by Stacy (1962), the IDB distribution proposed by Hjorth (1980), the generalized F distribution proposed by Prentice (1974), the exponential-power family proposed by Smith and Bain (1975) , among others. A good review of these models is presented in Rajarshi and Rajarshi (1988). In the last decade were proposed a new class of models to be used for this kind of data based in extension for the Weibull distribution. Mudholkar et al. (1995) introduced, the exponentiated-Weibull distribution, the Xie and Lai (1996) work, in which presents additive Weibull distribution and Xie et al. (2002) proposed the new Weibull extension distribution. In general, the inferencial

*Address: ESALQ, University of São Paulo, Piracicaba, Brasil. E-mail: gosilva@esalq.usp.br
†Address: ESALQ, University of São Paulo, Piracicaba, Brasil. E-mail: edwin@esalq.usp.br
‡Address: ICMC, University of São Paulo, São Carlos, Brasil. E-mail: garibay@icmc.usp.br
[1]Address for correspondence: Departamento de Ciências Exatas, USP Av. Pádua Dias 11 - Caixa Postal 9, 13418-900 Piracicaba - São Paulo - Brazil.
e-mail: edwin@esalq.usp.br

part of these models is carried out using the asymptotic distribution of the maximum likelihood estimators, which in situations when the sample is small, it might present difficult results to be justified.

In this paper, we explored the use of techniques of Chains Markov Monte Carlo (MCMC) to develop an Bayesian inference for the new Weibull extension model proposed by Xei et al.(2002). The approach is ilustrated with real data. We also extended the new Weibull model to regression model. We discussed inference aspects of the regression models following both, a classical and Bayesian approach. Section 2 is considering a briefing study of the new Weibull extension distribution besides the inferential part of this model. In the section 3 we suggest a log-new Weibull extension regression model, in addition with the maximum likelihood estimators. In the section 4 we approach the parameter estimation using the Bayesian inference. In the section 5 the results are applied to real data. Also we have studied residual from a fitted model using the Martingale residual proposed by Therneau el al (1990) and residual through the cumulative hazard function proposed by Lawless (2002).

2 A new Weibull extension distribution

The Weibull family of distributions has been widely used in the analysis of survival data in for medical and engineering application. This family is suitable in situations where the failure rate function is constant or monotone. It is not suitable in situations where the failure rate function is presents a bathtub shape. Recently Xie et al. (2002) presented an extension of the Weibull distribution, called the new Weibull extension family of distributions, which can be adequately fitted to data sets presenting bathtub shaped failure rate functions.

The new Weibull extension distribution considered in Xie et al.(2002) with parameters λ, α and τ considers that the life time T has a density function given by

$$f(t;\lambda,\tau,\alpha)=\lambda\tau\Big(\frac{t}{\alpha}\Big)^{\tau-1}\exp\bigg\{\Big(\frac{t}{\alpha}\Big)^{\tau}+\lambda\alpha\bigg[1-\exp\Big\{\Big(\frac{t}{\alpha}\Big)^{\tau}\Big\}\bigg]\bigg\} \tag{1}$$

where $\lambda>0$ and $\alpha>0$ are scale parameters and $\tau>0$ is shape parameter. The survival function corresponding to the random variable T with new Weibull extension density is given by

$$S(t;\lambda,\tau,\alpha)=P(T\geq t)=\exp\bigg\{\lambda\alpha\bigg[1-\exp\Big\{\Big(\frac{t}{\alpha}\Big)^{\tau}\Big\}\bigg]\bigg\} \tag{2}$$

The corresponding failure rate function has the following form

$$h(t;\lambda,\tau,\alpha)=\lambda\tau\Big(\frac{t}{\alpha}\Big)^{\tau-1}\exp\Big\{\Big(\frac{t}{\alpha}\Big)^{\tau}\Big\} \tag{3}$$

2.1 Characterizing the failure rate function

According to Xie et al. (2002), the failure rate function of the new Weibull extension distribution can be in the bathtub shape, as it will be seen in this section. To study the shape of the failure rate function we have found its derivative that can be written as

$$h^{'}(t;\lambda,\tau,\alpha)=\frac{\lambda\tau}{\alpha}\Big(\frac{t}{\alpha}\Big)^{\tau-2}\exp\Big\{\Big(\frac{t}{\alpha}\Big)^{\tau}\Big\}\Big[\tau\Big(\frac{t}{\alpha}\Big)^{\tau}+(\tau-1)\Big] \tag{4}$$

In order to study better this function one can note that two situations might be considered:

- $\tau \geq 1$
  To any $t > 0$, $h^{'}(t) > 0$ and therefore $h(t)$ is an increasing function.

- $\tau < 1$
  When $h^{'}(t^{*}) = 0$ we have $\tau\Big(\frac{t^{*}}{\alpha}\Big)^{\tau}+\tau-1=0$, so the inflexion point is given by $t^{*}=\alpha\Big(\frac{1}{\tau}-1\Big)^{\frac{1}{\tau}}$; where we observe the failure rate function can have the bathtub shape.

Figure 1 shows the plots of the failure rate function for some different parameter combinations. From figure 1, it can be seen that the failure rate function is an increasing function when $\tau \geq 1$ and $h(t)$ is a bathtub-shaped function when $\tau < 1$.

2.2 Mean and variance for the failure time

The expected value and the variance for the failure time is given by:

$$E(T)=\int_{0}^{+\infty} t dF(t)=\int_{0}^{+\infty}\exp\Big\{\ \lambda\alpha\Big[1-\exp\Big(\Big(\frac{t}{\alpha}\Big)^{\tau}\Big)\Big]\Big\}\ \mathrm{dt}$$

$$\mathrm{Var}(T) \ = \ \int_{0}^{+\infty} t^{2}dF(t)-\mu^{2}=2\int_{0}^{+\infty} t\ \ \exp\Big\{\ \lambda\alpha\Big[1-\exp\Big(\Big(\frac{t}{\alpha}\Big)^{\tau}\Big)\Big]\Big\}\ \mathrm{dt}-\mu^{2}$$

The mentioned integrals are computed numerically.

2.3 Relation the other distributions

The new Weibull extension distribution is mainly related to the model by Chen (2000) with the additional scale parameter. Besides, the new model has Weibull distribution as a especial and asymptotic case.

When $\alpha = 1$, the model is reduced to a proposed model by Chen (2000) where the survival function can be written as $S(t;\lambda,\tau,\alpha)=\exp\Big\{\lambda\Big[1-\exp\Big(\mathrm{t}^{\tau}\Big)\Big]\Big\}$. Also when $\alpha \to \infty$ the survival function is given by $S(t;\lambda,\tau,\alpha)\approx\exp\Big\{-\lambda\alpha^{1-\tau}\mathrm{t}^{\tau}\Big\}$, which is the standard Weibull distribution with two parameters being τ the shape parameter and $\frac{\alpha^{\tau-1}}{\lambda}$ the scale parameter.

On the other hand one can show that when $\tau = 1$, $\alpha \to \infty$ e $\frac{\alpha^{\tau-1}}{\lambda}$ is a constant; the model reduces to a exponential distribution with parameter $\frac{\alpha^{\tau-1}}{\lambda}$.

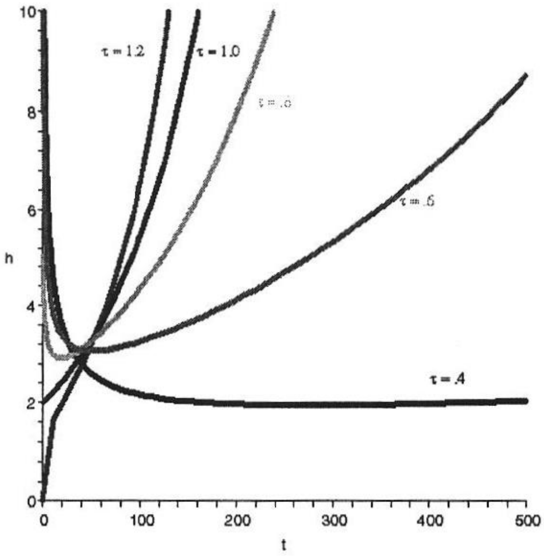

Figure 1: Plots of the failure rate function with $\lambda = 2$, $\alpha = 100$ and τ changing from $0,4$ to $1,2$.

2.4 Model parameter estimation

We assume that the lifetime are independently distributed, and also independent from the censoring mechanism. Considering right-censored lifetime data, we observe $t_i = \min(T_i, C_i)$, where T_i is the lifetime for the *ith* individual and C_i is the censoring for *ith* individual, $i = 1, \dots, n$. Assuming that $T_1, T_2, \dots, T_n$ is a random sample of the random variable T with new Weibull extension distribution (1). The likelihood function of λ, α and τ corresponding to the observed sample is given by

$$\begin{aligned} L(\lambda, \alpha, \tau) &= (\lambda\tau)^r \left[\prod_{i \epsilon F} \left(\frac{t_i}{\alpha}\right)^{\tau - 1} \right] \exp\left\{ \sum_{\mathrm{i}\epsilon \mathrm{F}} \left(\frac{\mathrm{t_i}}{\alpha}\right)^{\tau} + \lambda\alpha \sum_{\mathrm{i}\epsilon \mathrm{F}} \left[1 - \exp\left\{\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\tau}\right\}\right] \right\} \\ &\quad \exp\left\{ \lambda\alpha \sum_{\mathrm{i}\epsilon \mathrm{C}} \left[1 - \exp\left\{\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\tau}\right\}\right] \right\} \end{aligned} \tag{5}$$

where r is observed number of failures, F denotes the set of uncensored observations and C denotes the set of censored observations. The log-likelihood function is given by:

$$\begin{aligned} l(\lambda, \alpha, \tau) &= \mathrm{rlog}(\lambda) + \mathrm{rlog}(\tau) + \sum_{i \epsilon F} \left[\log\left(\frac{t_i}{\alpha}\right)^{\tau - 1} \right] + \sum_{i \epsilon F} \left(\frac{t_i}{\alpha}\right)^{\tau} \\ &\quad + \lambda\alpha \sum_{i=1}^{n} \left[1 - \exp\left\{\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\tau}\right\}\right] \end{aligned}$$

The maximum likelihood estimator $\hat{\lambda}$, $\hat{\alpha}$ and $\hat{\tau}$ of λ, α and τ is obtained by maximizing the log-likelihood, which results in solving the equations

$$\frac{\partial l(\lambda,\alpha,\tau)}{\partial\lambda} = \frac{\mathrm{r}}{\lambda} + \alpha\sum_{i=1}^{n}\Big[1 - \exp\Big\{\Big(\frac{\mathrm{t_i}}{\alpha}\Big)^{\tau}\Big\}\Big] = 0$$

$$\frac{\partial l(\lambda,\alpha,\tau)}{\partial\alpha} = -\frac{\mathrm{r}(\tau-1)}{\alpha} - \frac{\tau}{\alpha}\sum_{i\epsilon F}\Big(\frac{t_i}{\alpha}\Big)^{\tau} + \mathrm{n}\lambda - \lambda\sum_{\mathrm{i}=1}^{\mathrm{n}}\exp\Big\{\Big(\frac{\mathrm{t_i}}{\alpha}\Big)^{\tau}\Big\}\Big[1 - \tau\Big(\frac{\mathrm{t_i}}{\alpha}\Big)^{\tau}\Big] = 0$$

$$\frac{\partial l(\lambda,\alpha,\tau)}{\partial\tau} = \frac{\mathrm{r}}{\tau} + \sum_{i\epsilon F}\log\Big(\frac{\mathrm{t_i}}{\alpha}\Big) + \sum_{\mathrm{i}\epsilon\mathrm{F}}\Big(\frac{\mathrm{t_i}}{\alpha}\Big)^{\tau}\log\Big(\frac{\mathrm{t_i}}{\alpha}\Big) - \lambda\alpha\sum_{\mathrm{i}=1}^{\mathrm{n}}\Big(\frac{\mathrm{t_i}}{\alpha}\Big)^{\tau}\log\Big(\frac{\mathrm{t_i}}{\alpha}\Big)\exp\Big\{\Big(\frac{\mathrm{t_i}}{\alpha}\Big)^{\tau}\Big\} = 0$$

These equations cannot be solved analytically so that statistical software such as Ox or R can be used to solved them. In this paper, software Ox (MAXBFGS subroutine) is used to compute the maximum likelihood estimator (MLE)

3 Log-New Weibull Extension Regression Models

In many practical applications, lifetimes are affected by covariates such as the cholesterol level, blood pressure and many others. The covariate vector is denoted by $\mathbf{x} = (x_1, x_2, \ldots, x_p)^T$ which is related to the responses $Y = \log(T)$ through a regression model. Considering the transformations $\tau = \frac{1}{\delta}$ and $\alpha = \exp(\mu)$. It is also considered that the scale parameter μ of the new Weibull extension model depends on the matrix of explanatory variables X, it follows that the density function of Y can be written as

$$f(y;\lambda,\delta,\mu) = \frac{\lambda}{\delta}\exp\Big(\frac{\mathrm{y}-\mu}{\delta}\Big)\exp\Big\{\mu + \exp\Big(\frac{\mathrm{y}-\mu}{\delta}\Big) + \lambda\exp(\mu)\Big[1 - \exp\Big\{\exp\Big(\frac{\mathrm{y}-\mu}{\delta}\Big)\Big\}\Big]\Big\} \quad (6)$$

where $-\infty < y < \infty$, $\lambda > 0$, $\delta > 0$, and $-\infty < \mu < \infty$. And survival function given by

$$S(y) = \exp\Big\{\lambda\exp\{\mu\}\Big[1 - \exp\Big\{\exp\Big(\frac{\mathrm{y}-\mu}{\delta}\Big)\Big\}\Big]\Big\}. \quad (7)$$

We can write the above model as a log-linear models

$$Y = \mu + \delta Z \quad (8)$$

where the variable Z follows the density

$$f(z) = \lambda \exp\Big\{z + \mu + \exp(z) + \lambda \exp(\mu)[1 - \exp\{\exp(z)\}]\Big\} \quad \forall -\infty < z < \infty \tag{9}$$

We consider now the regression model based on the log-new Weibull extension distribution given in (6) relating the response Y and the covariate vector $\mathbf{x}$, so that the distribution $Y|\mathbf{x}$ can be represents as

$$Y_i = \mathbf{x}_i^T \boldsymbol{\beta} + \delta Z_i, \quad i = 1, \ldots, n, \tag{10}$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^T$, $\delta > 0$ and $\lambda > 0$ are unknown parameters, $\mathbf{x}_i^T = (x_{i1}, x_{i2}, \ldots, x_{ip})$ is the explanatory vector and Z follows the distribution in (9).

In this case, the survival function of $Y|\mathbf{x}$ is given by

$$S(y|\mathbf{x}) = \exp\left\{ \lambda \exp\{\mathbf{x}^{\mathrm{T}}\boldsymbol{\beta}\}\left[1 - \exp\left\{\exp\left(\frac{y - \mathbf{x}^{\mathrm{T}}\boldsymbol{\beta}}{\delta}\right)\right\}\right]\right\}. \tag{11}$$

Moreover, corresponding to the sample $(y_1, x_1), (y_2, x_2), \ldots, (y_n, x_n)$ of n observations, where $y_i = \min(\log(T_i), \log(C_i))$ and $\mathbf{x}_i$ the covariate vector associated with the i-th individual, the logarithm of the likelihood function δ, λ and $\boldsymbol{\beta}$ assuming data censored is given by

$$\begin{aligned} l(\boldsymbol{\theta}) \;=\; & \mathrm{r}\log(\lambda) - \mathrm{r}\log(\delta) + \sum_{\mathrm{i}\epsilon\mathrm{F}} \mathbf{x}_{\mathrm{i}}^{\mathrm{T}}\boldsymbol{\beta} + \sum_{\mathrm{i}\epsilon\mathrm{F}} \mathrm{z}_{\mathrm{i}} + \sum_{\mathrm{i}\epsilon\mathrm{F}} \exp\{\mathrm{z}_{\mathrm{i}}\} \\ & + \sum_{i\epsilon F} \lambda \exp\{\mathbf{x}_{\mathrm{i}}^{\mathrm{T}}\boldsymbol{\beta}\}\Big[1 - \exp\{\exp(\mathrm{z}_{\mathrm{i}})\}\Big] + \sum_{\mathrm{i}\epsilon\mathrm{C}} \lambda \exp\{\mathbf{x}_{\mathrm{i}}^{\mathrm{T}}\boldsymbol{\beta}\}\Big[1 - \exp\{\exp(\mathrm{z}_{\mathrm{i}})\}\Big], \end{aligned} \tag{12}$$

where r is the number of uncensored observation (failures) and $\mathrm{z}_{\mathrm{i}} = \frac{\mathrm{y}_{\mathrm{i}} - \mathbf{x}_{\mathrm{i}}^{\mathrm{T}}\boldsymbol{\beta}}{\delta}$. Maximum likelihood estimates for the parameter vector $\boldsymbol{\theta} = (\lambda, \delta, \boldsymbol{\beta}^T)^T$ can be obtained by maximizing the likelihood function. In this paper, the software Ox (MAXBFGS subroutine) (see Doornik, 1996) was used to compute maximum likelihood estimates (MLE). Covariance matrix estimates for the maximum likelihood estimators $\widehat{\boldsymbol{\theta}}$ can also be obtained using the Hessian matrix. Confidence intervals and hypothesis testing can be conducted using the large sample distribution of the MLE which is a normal distribution with the covariance matrix as the inverse of the Fisher information since regularity conditions are satisfied. More specifically, the asymptotic covariance matrix is given by $\mathbf{I}^{-1}(\boldsymbol{\theta})$ with $\mathbf{I}(\boldsymbol{\theta}) = -E[\ddot{\mathbf{L}}(\boldsymbol{\theta})]$ such that $\ddot{\mathbf{L}}(\boldsymbol{\theta}) = \left\{\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T}\right\}$.

Since it is not possible to compute the Fisher information matrix $\mathbf{I}(\boldsymbol{\theta})$ due to the censored observations (censoring is random and noninformative), but it is possible to use the matrix of second derivatives of the log likelihood, $-\ddot{\mathbf{L}}(\boldsymbol{\theta})$, evaluated at the MLE $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$, which is consistent. The asymptotic normal approximation for $\widehat{\boldsymbol{\theta}}$ may be expressed as $\widehat{\boldsymbol{\theta}}^T \sim N_{(p+2)}\{\boldsymbol{\theta}^T; -\ddot{\mathbf{L}}(\boldsymbol{\theta})^{-1}\}$ where $-\ddot{\mathbf{L}}(\boldsymbol{\theta})$ is the $(p+2)(p+2)$ observed information matrix, obtained from:

$$\ddot{\mathbf{L}}(\boldsymbol{\theta}) = \begin{pmatrix} \mathbf{L}_{\lambda\lambda} & \mathbf{L}_{\lambda\delta} & \mathbf{L}_{\lambda\beta_j} \\ . & \mathbf{L}_{\delta\delta} & \mathbf{L}_{\delta\beta_j} \\ . & . & \mathbf{L}_{\beta_j\beta_s} \end{pmatrix}$$

with the following submatrices

$$\mathbf{L}_{\lambda\lambda} = -\frac{\mathrm{r}}{\lambda^2}$$

$$\mathbf{L}_{\lambda\delta} = \frac{1}{\delta}\sum_{i=1}^{n}\mathrm{z_i h_i}\exp\{\mathbf{x}_\mathrm{i}^\mathrm{T}\boldsymbol{\beta}\}$$

$$\mathbf{L}_{\lambda\beta_j} = \sum_{i=1}^{n}\mathrm{x_{ij}}\exp\{\mathbf{x}_\mathrm{i}^\mathrm{T}\boldsymbol{\beta}\} + \frac{1}{\delta}\sum_{\mathrm{i=1}}^{\mathrm{n}}\mathrm{x_{ij}}\exp\{\mathbf{x}_\mathrm{i}^\mathrm{T}\boldsymbol{\beta}\}\mathrm{h_i}\Big[1-\delta\exp\{-\mathrm{z_i}\}\Big]$$

$$\mathbf{L}_{\delta\delta} = \frac{\mathrm{r}}{\delta^2} + \frac{1}{\delta^2}\sum_{i\epsilon F}\Big\{\mathrm{z_i}\Big[2+\exp\{\mathrm{z_i}\}(\mathrm{z_i}+2)\Big]\Big\} + \frac{\lambda}{\delta^2}\sum_{\mathrm{i=1}}^{\mathrm{n}}\mathrm{z_i}\exp\{\mathbf{x}_\mathrm{i}^\mathrm{T}\boldsymbol{\beta}\}\mathrm{h_i}\Big[-2-\mathrm{z_i}(1+\exp\{\mathrm{z_i}\})\Big]$$

$$\mathbf{L}_{\delta\beta_j} = \frac{1}{\delta^2}\sum_{i\epsilon F}\mathrm{x_{ij}}\Big[1+\exp\{\mathrm{z_i}\}(1+\mathrm{z_i})\Big] + \frac{\lambda}{\delta^2}\sum_{\mathrm{i=1}}^{\mathrm{n}}\mathrm{x_{ij}}\exp\{\mathbf{x}_\mathrm{i}^\mathrm{T}\boldsymbol{\beta}\}\mathrm{h_i}\Big[-1+\mathrm{z_i}(\delta-1-\exp\{\mathrm{z_i}\})\Big]$$

$$\begin{aligned}\mathbf{L}_{\beta_j\beta_s} &= \frac{1}{\delta^2}\sum_{i\epsilon F}\mathrm{x_{ij}x_{is}}\exp\{\mathrm{z_i}\} + \lambda\sum_{\mathrm{i=1}}^{\mathrm{n}}\mathrm{x_{ij}x_{is}}\exp\{\mathbf{x}_\mathrm{i}^\mathrm{T}\boldsymbol{\beta}\}\\ &\quad + \frac{\lambda}{\delta^2}\sum_{i=1}^{n}\mathrm{x_{ij}x_{is}}\exp\{\mathbf{x}_\mathrm{i}^\mathrm{T}\boldsymbol{\beta}\}\mathrm{h_i}\Big[2\delta-1-\delta^2\exp\{-\mathrm{z_i}\}-\exp\{\mathrm{z_i}\}\Big]\end{aligned}$$

in which $j, s = 1, 2, \ldots, p$, $\mathrm{h_i} = \exp\Big\{\mathrm{z_i} + \exp\{\mathrm{z_i}\}\Big\}$ and $\mathrm{z_i} = \frac{\mathrm{y_i} - \mathbf{x}_\mathrm{i}^\mathrm{T}\boldsymbol{\beta}}{\delta}$

4 A Bayesian analysis for the model

The use of the Bayesian method besides being an alternative analysis, it allows the incorporation of previous knowledge of the parameters through informative priori densities. When there is not this information one considers noninformative priori. In the Bayesian approach, the referent information to the model parameters is obtained through posterior marginal distribution. In this way it appears two difficulties. The first it refers to attainment marginal posterior distribution and the second to the calculation of the interest moments. In both cases are necessary integral resolutions that many times do not present analytical solution. In this paper we have used the simulation method of Markov Chain Monte Carlo, such as the Gibbs sampler and Metropolis-Hasting algorithm.

Considere the new Weibull extension distribution (1), censored data and the likelihood function (5) for α, τ and λ. For a Bayesian analysis, we assume the following priori densities for α, τ and λ

- $\alpha \sim \Gamma(\alpha_0, \alpha_1)$, α_0 and α_1 known;
- $\tau \sim \Gamma(\tau_0, \tau_1)$, τ_0 and τ_1 known;
- $\lambda \sim \Gamma(\lambda_0, \lambda_1)$, λ_0 and λ_1 known;

where $\Gamma(a_i, b_i)$ denotes a gamma distribution with mean $\frac{a_i}{b_i}$, variance $\frac{a_i}{b_i^2}$ and density function given by

$$f(\omega; ai, b_i) = \frac{b_i^{a_i} \omega^{a_i - 1} \exp\{-\omega \mathrm{b_i}\}}{\Gamma(a_i)} \tag{13}$$

where $\theta > 0$, $a_i > 0$ and $b_i > 0$.

In the special case where $\alpha_0 = \alpha_1 = \tau_0 = \tau_1 = \lambda_0 = \lambda_1 = 0$, the noninformative case follows, that is

$$\pi(\alpha, \lambda, \tau) \propto \frac{1}{\alpha\lambda\tau} \tag{14}$$

We further assume independence among the parameters α, λ and τ. The joint posteriori distributions for α, λ and τ is given by,

$$\begin{aligned} \pi(\alpha, \lambda, \tau | D) \quad &\propto \quad \lambda^{\lambda_0 - 1} \exp\{-\lambda\lambda_1\} \alpha^{\alpha_0 - 1} \exp\{-\alpha\alpha_1\} \tau^{\tau_0 - 1} \exp\{-\tau\tau_1\} (\lambda\tau)^{\mathrm{r}} \left[\prod_{\mathrm{i} \epsilon \mathrm{F}} \left(\frac{\mathrm{t_i}}{\alpha} \right)^{\tau - 1} \right] \\ & \exp\left[\sum_{\mathrm{i} \epsilon \mathrm{F}} \left(\frac{\mathrm{t_i}}{\alpha} \right)^{\tau - 1} \right] \exp\left\{ \lambda\alpha \sum_{\mathrm{i}=1}^{\mathrm{n}} \left[1 - \exp\left\{ \left(\frac{\mathrm{t_i}}{\alpha} \right)^{\tau} \right\} \right] \right\} \end{aligned} \tag{15}$$

where D denotes the data sets.

It can be shown that the conditional posteriori densities for the Gibbs algorithm are given by

$$\pi(\lambda | \alpha, \tau, D) \propto \lambda^{\lambda_0 r - 1} \exp\left\{ -\lambda\lambda_1 + \lambda\alpha \sum_{\mathrm{i}=1}^{\mathrm{n}} \left[1 - \exp\left\{ \left(\frac{\mathrm{t_i}}{\alpha} \right)^{\tau} \right\} \right] \right\} \tag{16}$$

$$\pi(\alpha | \lambda, \tau, D) \propto \alpha^{\alpha_0 - 1} \left[\prod_{i \epsilon F} \left(\frac{t_i}{\alpha} \right)^{\tau - 1} \right] \exp\left\{ -\alpha\alpha_1 + \left[\sum_{\mathrm{i} \epsilon \mathrm{F}} \left(\frac{\mathrm{t_i}}{\alpha} \right)^{\tau} \right] + \lambda\alpha \sum_{\mathrm{i}=1}^{\mathrm{n}} \left[1 - \exp\left\{ \left(\frac{\mathrm{t_i}}{\alpha} \right)^{\tau} \right\} \right] \right\} \tag{17}$$

$$\pi(\tau | \lambda, \alpha, D) \propto \tau^{\tau_0 + r - 1} \left[\prod_{i \epsilon F} \left(\frac{t_i}{\alpha} \right)^{\tau - 1} \right] \exp\left\{ -\tau\tau_1 + \sum_{\mathrm{i} \epsilon \mathrm{F}} \left(\frac{\mathrm{t_i}}{\alpha} \right)^{\tau} + \lambda\alpha \sum_{\mathrm{i}=1}^{\mathrm{n}} \left[1 - \exp\left\{ \left(\frac{\mathrm{t_i}}{\alpha} \right)^{\tau} \right\} \right] \right\} \tag{18}$$

Observe that we need to use the Metropolis-Hastings algorithm to generate the variables α, λ and τ from the respective conditional posteriori densities since their forms are somewhat complex.

For Bayesian inference, considering model (6) with $\mu(\mathbf{x}_i) = \mathbf{x}_i^T \boldsymbol{\beta}$, assume the following priori densities for δ, λ and $\boldsymbol{\beta}^T$:

- $\lambda \sim \Gamma(c_1, d_1)$, c_1 and d_1 known;
- $\delta \sim \Gamma(c_2, d_2)$, c_2 and d_2 known;
- $\beta_j \sim N(\mu_{0j}, \sigma^2_{0j})$, μ_{0j} and σ^2_{0j} known, j=1,...p.

Noinformative prioris follows by considering $c_1 = c_2 = d_1 = d_2 = 0$ and σ^2_{0j} large.

We assume independence among the parameters. The joint posteriori distribution for δ, λ and $\boldsymbol{\beta}$ is given by:

$$\begin{aligned}\pi(\delta,\lambda,\boldsymbol{\beta}^T|D) \;\propto\;& \lambda^{c_1-1}\exp\{-\lambda d_1\}\delta^{c_2-1}\exp\{-\delta d_2\}\exp\Big\{-\frac{1}{2}\sum_{j=1}^{p}\Big(\frac{\beta_j-\mu_{0j}}{\sigma_{0j}}\Big)^2\Big\}\Big(\frac{\lambda}{\delta}\Big)^r\\ &\exp\Big\{\sum_{i\epsilon F}\mathbf{x}_i^T\boldsymbol{\beta}\Big\}\exp\Big\{\sum_{i\epsilon F} z_i\Big\}\\ &\exp\Big\{\sum_{i\epsilon F}\exp\{z_i\}\Big\}\exp\Big\{\sum_{i=1}^{n}\lambda\exp\{\mathbf{x}_i^T\boldsymbol{\beta}\}\Big[1-\exp\Big\{\exp\{z_i\}\Big\}\Big]\Big\}\end{aligned} \tag{19}$$

where $z_i = \frac{y_i - \mathbf{x}_i^T\boldsymbol{\beta}}{\delta}$.

It can be show that the conditional marginal distributions for the Gibbs algorithm are given by:

$$\pi(\lambda|\delta,\boldsymbol{\beta}^T,D) \propto \lambda^{c_1+r-1}\exp\Big\{-\lambda d_1 + \sum_{i=1}^{n}\Big[\lambda\exp\{\mathbf{x}_i^T\boldsymbol{\beta}\}\Big[1-\exp\Big\{\exp\{z_i\}\Big\}\Big]\Big]\Big\} \tag{20}$$

$$\begin{aligned}\pi(\delta|\lambda,\boldsymbol{\beta}^T,D) \;\propto\;& \delta^{c_2-r-1}\exp\Big\{-\delta d_2 + \Big\{\sum_{i\epsilon F} z_i\Big\} + \Big\{\sum_{i\epsilon F}\exp\{z_i\}\Big\}\\ &+\sum_{i=1}^{n}\Big[\lambda\exp\{\mathbf{x}_i^T\boldsymbol{\beta}\}\Big[1-\exp\Big\{\exp\{z_i\}\Big\}\Big]\Big]\Big\}\end{aligned} \tag{21}$$

$$\begin{aligned}\pi(\beta_j|\lambda,\delta,\boldsymbol{\beta}_{-j},D) \;\propto\;& \exp\Big\{-\frac{1}{2}\Big(\frac{\beta_j-\mu_{0j}}{\sigma_{0j}}\Big)^2\Big\}\exp\Big\{\sum_{i\epsilon F}x_{ij}\beta_j\Big\}\exp\Big\{\sum_{i\epsilon F}\Big(\frac{y_i-x_{ij}\beta_j}{\delta}\Big)\Big\}\\ &\exp\Big\{\sum_{i\epsilon F}\Big[\exp\Big(\frac{y_i-x_{ij}\beta_j}{\delta}\Big)\Big]\Big\}\\ &\exp\Big\{\sum_{i=1}^{n}\lambda\exp^{x_{ij}\beta_j}\Big[1-\exp\Big\{\exp\Big\{\frac{y_i-x_{ij}\beta_j}{\delta}\Big\}\Big\}\Big]\Big\}\end{aligned} \tag{22}$$

Observe that we need to use the Metropolis Hastings algorithm to generate from the distributions of β_j, λ and δ.

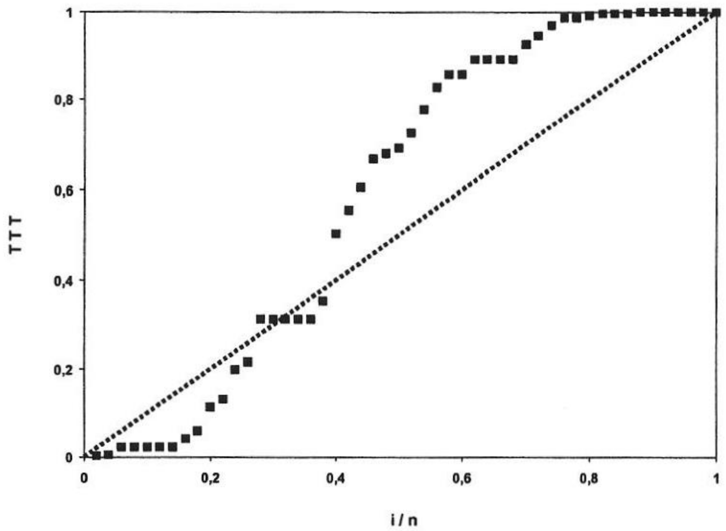

Figure 2: TTT plot for the 50 observations in Aarset (1987)

5 Application

5.1 A reanalysis of the Aarset (1987) data set

To illustrated the approach developed in the previous sections we consider the data set presented in Aarset (1987). The data describe lifetimes for 50 industrial devices put on life test at time zero. The TTT plot, that indicates a bathtub shaped failure rate function, is presented in Figure 2

Table 1: Lifetimes for the 50 devices.

0.1	0.2	1	1	1	1	1	2	3	6	7	11	12	18	18	18	18	18	21
32	36	40	45	46	47	50	55	60	63	63	67	67	67	67	72	75	79	82
82	83	84	84	84	85	85	85	85	85	86	86							

5.1.1 Maximum likelihood results

Considering the data in Aarset, we fit the new Weibull extension model to the data set, using subroutine MaxBFGS in Ox (as the maximization approach), and obtain the following maximum likelihood estimates: $\hat{\alpha} = 13.746645$, $\hat{\tau} = 0.58770374$ and $\hat{\lambda} = 0.008759687$. Since the estimate of τ is smaller than one (with sd=0.06), we have strong indication that the data set presents bathtube shaped failure rate.

These estimates which are presented, however, different from the ones obtained by Xei et al.

(2002) which are given by $\tilde{\alpha} = 110.0909$, $\tilde{\tau} = 0.8408$ and $\tilde{\lambda} = 0.0141$.. For those values, the equations socre are not satisfied (subsection 2.4), that is, the left side of each equation is different from zero, as we show next:

$$\frac{\partial l(\lambda, \alpha, \tau; t)}{\partial \alpha}|_{(\tilde{\alpha},\tilde{\tau},\tilde{\lambda})} = -0.0300265319, \qquad \frac{\partial l(\lambda, \alpha, \tau; t)}{\partial \tau}|_{(\tilde{\alpha},\tilde{\tau},\tilde{\lambda})} = 0.11264210,$$

and

$$\frac{\partial l(\lambda, \alpha, \tau; t)}{\partial \lambda}|_{(\tilde{\alpha},\tilde{\tau},\tilde{\lambda})} = -11.294634.$$

Notice that specially the third equation tremendously deviates from zero. Hence, the authors (Xie et al., 2002) think they are using the correct estimates and indeed they have highly incorrect estimates, so that further inference based on it like the likelihood ratio statistics, for example, may present incorrect values which would put all further inference in tremendous jeopardy, leading to dangerously incorrect inferences.

5.1.2 Bayesian analysis

We consider first the new Weibull extension model (5), considering the prior densities for α, τ and λ given in section 4, with $\alpha_0 = 28$, $\alpha_1 = 2$, $\tau_0 = 0.001$, $\tau_1 = 0.001$, $\lambda_0 = 0.001$ and $\lambda_1 = 0.001$, we generated two parallel independent runs of the Gibbs sampler chain with size 25000, for each parameter discarding the first 5000 iterations, to eliminate the effect of the initial values and to avoid correlation problems, we considered a spacing of size 10, obtaining a sample of size 2000 from each chain, we monitored the convergence of the Gibbs samples using the Gelman and Rubin (1992) method that uses the analysis of variance technique if further iterations are needed. In the Table 2, we report posterior summaries for the parameters, and in Figure 3 we have the approximate marginal posterior densities considering the 4000 Gibbs samples. We also have in the Table 2, the estimated potential scale reduction $\hat{R}$ (see Gelman and Rubin, 1992) which is an index to check the convergence of the algorithm. Since $\hat{R} < 1.1$ for all parameters, it seems that the chains converge.

Table 2: Posterior summaries the modified Weibull model

Parameters	Mean	S.D	95% Credible interval	$\hat{R}$
α	13.77	2.51	(9.24; 19.07)	1.001
τ	0.582	0.0602	(0.4707 ; 0.7028)	1.001
λ	0.00916	0.002608	(0.00488 ; 0.01501)	1.002

5.2 Application Golden shiner data

Survival time for the golden shiner, Notemigonus crysoleucas, were obtained from field experiments conducted in Lake Saint Pierre, Quebec, in 2005 (Laplante et al., unpublished data). Each indi-

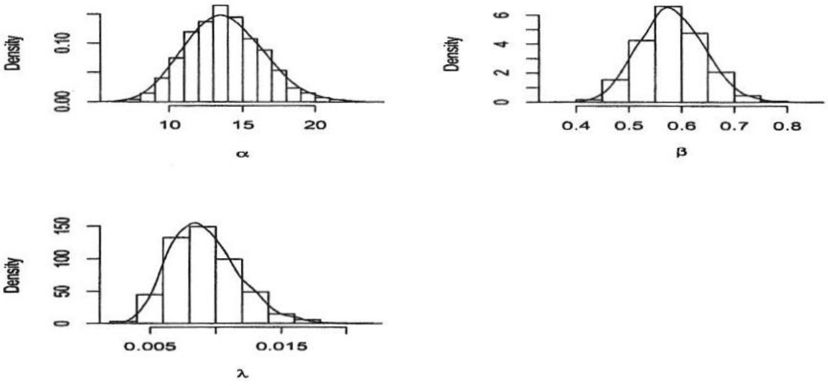

Figure 3: Approximate marginal posterior density for α, τ and λ of modified Weibull model.

vidual fish was attached by means of a monofilament chord to a chronographic tethering device that allowed the fish to swim in midwater. A timer in the device was set off when the tethered fish was captured by a predator. The device was retrieved approximately 24 h after the onset of an experiment, and survival time was then obtained from the difference: time elapsed between onset of experiment and retrieval - time elapsed in device timer since predation event. The variables involved in the study were:

- y_i: survival time observed (in hours);
- $cens_i$: censoring indicator (0=censoring, 1=lifetime observed);
- x_{i1}: north or south bank of the lake (0=north, 1=south);
- x_{i2}: distance over the longitudinal axis of the lake (in km);
- x_{i3}: size of the fish (in cm);
- x_{i4}: depth of the place (in cm);
- x_{i5}: abundance index of macro-thin plants (in percentage);
- x_{i6}: transparency of the water (in cm);
- x_{i7}: initial time.

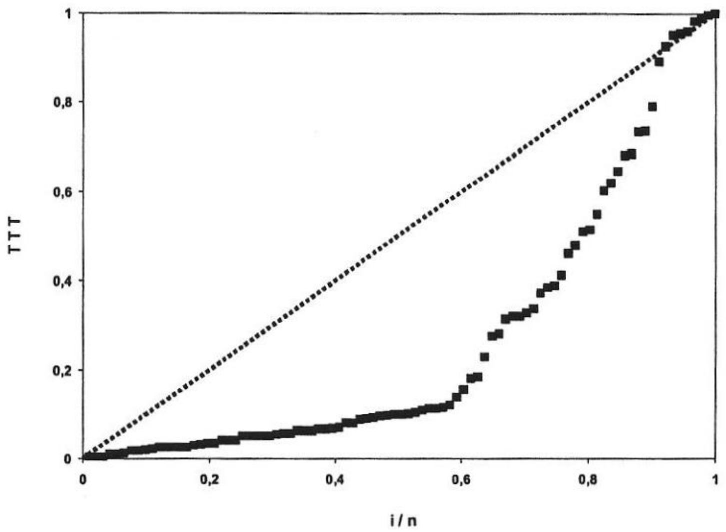

Figure 4: TTT-plot on Golden shiner data

The TTT plot that is in figure 4 indicates a bathtub shaped failure rate function .
We present now results on fitting the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i5} + \beta_6 x_{i6} + \beta_7 x_{i7} + \delta z_i \tag{23}$$

where the variable Y_i follows the log-new Weibull extension distribution given in (6), $i = 1, 2, \ldots, 106$. To obtain the maximum likelihood estimates for the parameters in the model we use the subroutine MAXBFGS in Ox, whose results are given in the Table 3.

We may observe that the variables x_1, x_2, x_3 x_4 and x_5 are significant for the model.

5.2.1 Bayesian analysis

In order to analyze the previous described data, under Bayesian perspective, considering log-new Weibull extension regression model (10) the priori densities given in the section 4 with $\beta_j \sim N(0, 10^2)$, $j = 0, \ldots, 7$, $\lambda \sim \Gamma(1, 1)$ e $\delta \sim \Gamma(0.1, 0.1)$. With the choice generated two parallel chains each one with 110,000 iterations; it was monitored the convergence of Gibbs samples using the Gelman & Rubin (1992) method that uses the variance analysis technique to determine whether more iterations are needed. For each parameter, the 10,000 first iterations were discarded to eliminate the effect to the initial values and to avoid autocorrelation problem were taken samples from 40 to 40 that totalize a final sample of 5,000. In the Table 4, we have reported the posteriori summary of the model parameters together with the results the estimated potencial scale reduction

Table 3: Maximum likelihood estimates for the complete data set

Parameter	Estimate	SE	p-valor
λ	0,0189	0,00004	-
δ	1,5969	0,01913	-
β_0	7,3999	27,9730	0,1617
β_1	-3,3214	0,95106	$< 0,01$
β_2	-0,2371	0,00401	$< 0,01$
β_3	0,2309	0,00364	0,013
β_4	-0,0915	0,00041	$< 0,001$
β_5	-0,1120	0,00201	$< 0,01$
β_6	-0,1840	0,14512	0,629
β_7	-0,0944	0,07118	0,724

(see Gelman & Rubin, 1992), for all parameters. We have observed values $\hat{R}$ very near one that indicate that the chains converged.

Table 4: Posterior summaries for the log-new Weibull extension regression model.

Parameter	Mean	Median	S.D.	2,5%	97.5%	$\hat{R}$
λ	0.01487	0.01504	0.007334	0.00158	0.02926	0.999
δ	1.549	1.298	0.1421	1.298	1.855	0.997
β_0	7.127	7.172	5.413	1.532	12.68	1.006
β_1	-3.435	-3.349	1.058	-5.827	-1.61	1.001
β_2	-0.2316	-0.2241	0.08473	-0.4321	-0.08576	1.001
β_3	0.2248	0.2137	0.07641	0.102	0.4155	1.005
β_4	-0.08425	-0.08199	0.02642	-0.1441	-0.03878	1.003
β_5	-0.09778	-0.09493	0.03773	-0.1832	-0.03516	1.004
β_6	-0.377	-0.3539	0.4117	-1.277	0.3845	1.001
β_7	0.02783	0.02308	0.2129	-0.3821	0.466	1.001

In the Figure 5, we have the approximated marginal posteriori densities for the log-new Weibull extension regression model considering the 5,000 generated samples points.

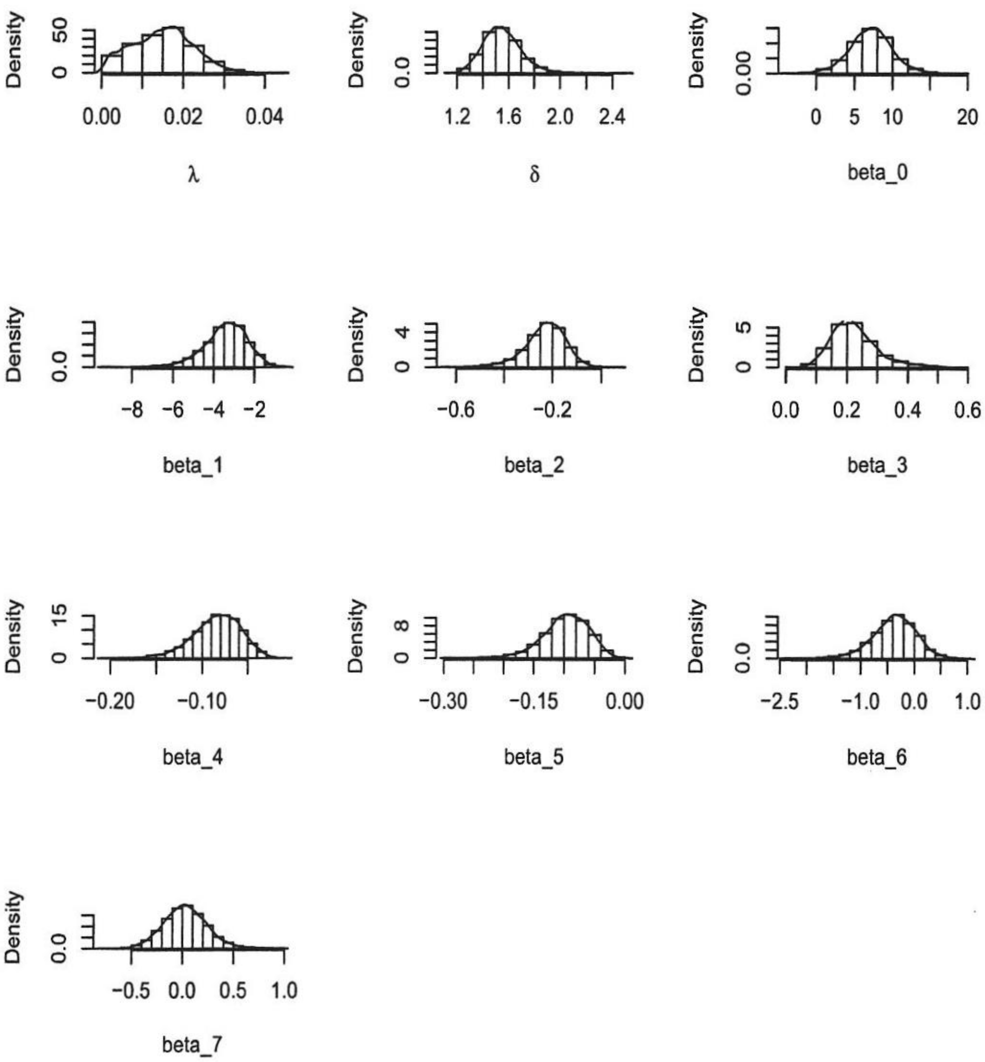

Figure 5: Approximate marginal posterior densities for $\boldsymbol{\beta}$, δ and λ

5.2.2 Residual analysis

In order to study departures from the error assumption as well as presence of outliers we will first consider the deviance residual proposed by Barlow and Prentice (1988) (see also Therneau et al.,

1990) and residual though cumulative hazard function proposed by Lawless (2002).

5.2.3 Deviance Residual

This residual was introduced in counting process and can be written in log-new Weibull extension regression models as

$$r_{M_i} = \delta_i + log[S(y_i, \hat{\boldsymbol{\theta}})]$$

where $\delta_i = 0$ denotes censored observation, $\delta_i = 1$ uncensored and $S(y_i, \hat{\boldsymbol{\theta}})$ is as defined in Section 2. Due to the skewness distributional form of r_{M_i}, it has maximum value $+1$ and minimum value $-\infty$, transformations to achieve a more normal shaped form would be more appropriate for residual analysis. Another possibility is to use the deviance residual (see, for instance, definition in McCullagh and Nelder, 1989, section 2.4) that has been largely applied in generalized linear models (GLMs). Various authors have investigated the use of deviance residuals in GLMs (see, for instance, Williams, 1987; Hinkley et al., 1991; Paula 1995) as well as in other regression models (see, for example, Farhrmeir and Tutz, 1994). In log-new Weibull extension regression models o residual deviance is expressed here as

$$r_{D_i} = \text{sign}(r_{M_i})\Big[-2\Big\{r_{M_i} + \delta_i \log(\delta_i - r_{M_i})\Big\}\Big]^{\frac{1}{2}}$$

where r_{M_i} is the residual martingale corresponding to the log-new Weibull extension regression model.

Calculating the deviance residual for the adjusted model one can observe the Figure 6 a random behavior of the residues, where all observations are on the interval (-3,3) showing no evidence against suppositions of the adjusted model.

5.2.4 Residual though the cumulative hazard function

Lawless (2002) defines the residual for the regression models with the presence of censored data using the cumulative hazard function. In the log-new Weibull extension regression model the residual can be expressed in the following way:

$$e_i = \begin{cases} -\widehat{\lambda}\exp\{\mathbf{x}_i^T\widehat{\boldsymbol{\beta}}\}\Big[1 - \exp\Big\{\exp\Big(\frac{y_i - \mathbf{x}_i^T\widehat{\boldsymbol{\beta}}}{\widehat{\delta}}\Big)\Big\}\Big] & \text{if} \quad i \epsilon F \\ 1 + \exp\{\mathbf{x}_i^T\widehat{\boldsymbol{\beta}}\}\Big[1 - \exp\Big\{\exp\Big(\frac{y_i - \mathbf{x}_i^T\widehat{\boldsymbol{\beta}}}{\widehat{\delta}}\Big)\Big\}\Big] & \text{if} \quad i \epsilon C \end{cases}$$

According to Lawless (2002) plotting on a chart $-\log\Big[S^*(e_i)\Big]$ versus e_i (Figure 7) must be approximately a straight line with an inclination equals to 1, where S^* is an empiric survival function of the e_i.

In the Figure 7 one can observe that the proposed regression model is appropriated to the data.

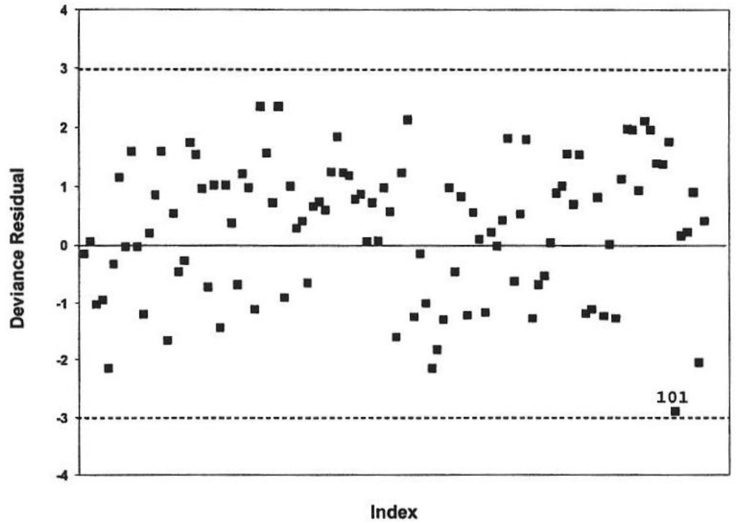

Figure 6: Index plot of the deviance residual r_{D_i}

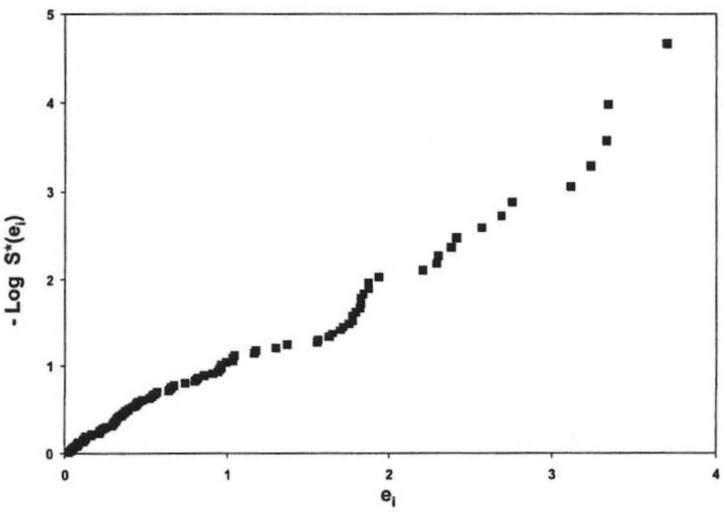

Figure 7: Index plot of the residual though the cumulative hazard function

6 Concluding Remarks

In this paper is proposed a log-new Weibull extension regression model with the presence of censored data as an alternative to model lifetime when the failure rate function presents bathtub shape. We used the algorithm Quasi-Newton to obtain the estimators of maximum likelihood and to realize asymptotics tests for the parameters based on the asymptotic distribution of the maximum likelihood estimators. On the other hand, as an alternative analysis, the paper discusses the use of Markov Chain Monte Carlo methods as a reasonable way to get Bayesian inference for the log-new Weibull extension regression models. In the applications within a real data we observed that both estimators present similar results. Also we showed a good adjustment of the log-new Weibull extension regression model through residual analysis.

References

Aarset, M.V (1987) How to identify bathtub hazard rate, *IEEE Trans. Reliability,* **36** 106-108.

Barlow, W. E., and Prentice, R. L.(1988). Residual for relative risk regression. *Biometrika*, **75**, 65-74.

Chen Z. A. (2000). A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. *Stat Probab Lett*, **49**, 155-61.

Doornik, J. (1996). Ox: An object-oriented matrix programming language. International Thomson Bussiness Press.

Fahrmeir, L. and Tutz, G. (1994). *Multivariate Statistical Modelling Based on Generalized Linear Models.* Springer-Verlag: New York.

Gelman, A. and Rubin, D. B. (1992). Inference from iterative simulation using multiple sequences (with discussion). Statistical Science.

Hinkley, D. V.; Reid, N. and Snell, E. J. (1991). Statistical Theory and Modelling-In honour of Sir David Cox . Chapman & Hall: London.

Hjorth. U. (1980). A realibility distributions with increasing, decreasing, constant and bathtub failure rates. *Technometrics* **22**, 99-107.

Lawless, J. F.(2002). Statistical Models and Methods for Lifetime Data. Wiley: New York.

McCullagh, P. and Nelder, J. A. (1989). *Generalized Linear Models*, 2nd Edition. Chapman and Hall: London.

Mudholkar, G. S; Srivastava, D. K. and Friemer, M. (1995). The exponentiated Weibull family: A reanalysis of the bus-motor-failure data. *Technometrics* **37**, 436-445.

Paula, G. A. (1995). Influence and residuals in restricted generalized linear models. *J. Statist. Comput. Simulation*, **51** 63-79.

Prentice, R. L. (1974). A log-gamma model and its maximum likelihood estimation. *Biometrica*, **61** 539-544.

Rajarchi, S. and Rajarshi, M.B. (1988). Bathtub distributions: a review. *Commun. Statatist. Theory. Meth.*, **17(8)**, 2597-2521.

Stacy, E. W. (1962). A generalization of the gamma distribution. *Ann. Math. Stat.*, **33** 1187-1192.

Smith. R.M. and Bain, L.J. (1975). An exponential power life testing distributions. *Commun. in Statist.*, 469-481.

Therneau, T.M., Grambsch, P.M and Fleming, T.R. (1990). Martingale-based residuals for survival models. *Biometrika* **77**, 147-60.

Williams, D. A. (1987). Generalized linear model diagnostic using the deviance and single case deletion. *Appl. Statist* **36**, 181-191.

Xie, M. and Lai, C.D. (1996). Reliability analysis using an addi tive Weibull model with bathtub-shaped failure rate function. *Reliab Engng Syst Safety*, **52**, 87-93.

Xie, M., Tang, Y. and Goh, T. N. (2002). A modified Weibull extension with bathtub failure rate function. *Reliab Engng Syst Safety*, **76**, 279-285.

Tsai, C. e Wu, X. (1992). Transformation-Model diagnostics. *Technometrics* **34**, 197-202.

Therneau, T.M., Grambsch, P.M and Fleming, T.R. (1990). Martingale-based residuals for survival models. *Biometrika* **77**, 147-60.