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# The Weibull-Negative-Binomial Model Under Latent Failure Causes With Randomized Activation Schemes

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## Abstract

In this paper we propose the Weibull-Negative-Binomial (WNB) distribution under latent failure causes with randomized activation scheme. The proposed model is general by assuming a latent activation structure to explain the occurrence of an event of interest, where the number of competing causes are modeled by negative binomial distribution while the competitive causes may have different activation mechanisms, namely, first, random and last ones. Moreover, we derive some standard properties of the WNB distribution and discuss inference via the maximum likelihood approach. The WNB distribution generalizes various usual lifetime distributions, but particularly the extended Weibull-Geometric distribution introduced by Barreto-Souza *et al.* (2011), when the first activation scheme is considered, and the exponential-Poisson lifetime distribution proposed by Cancho *et al.* (2011), when the last activation scheme is considered. A misspecification simulation study comparing the proposed model in the presence of different activation schemes with the usual Weibull distribution is performed. Application of the proposed model in two sets of real data illustrates its flexibility.

**Keywords:** *Keywords:* Weibull distribution; Negative-Binomial distribution; hazard function;

maximum likelihood estimation; Survival analysis.

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## 1. Introduction

The Weibull distribution is one of the most widely used distributions for modeling the lifetimes data. It is used to model various type of lifetime data with monotone failure rates, however it is not useful to model lifetime data in the presence of bathtub-shaped and unimodal failure rates, which are common in lifetime studies. In order to accommodate such a type of failure rates, various distributions have been proposed in the literature to extend the Weibull distribution. The Weibull-Exponentiated (WE) distribution introduced by Mudholkar & Srivastava (1993), which presents unimodal hazard function, the extended Weibull distribution proposed by Xie *et al.* (2002) and the modified Weibull (MW) distribution introduced by Lai *et al.* (2003) have hazard functions in the form of bathtub. Moreover, Silva *et al.*

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(2010) discussed the modified Beta-Weibull distribution which generalizes MW, WE and Beta-Weibull distribution introduced by Famoye *et al.* (2005). The hazard function of these distributions have monotone decreasing and unimodal forms.

Recently Barreto-Souza *et al.* (2011) proposed Weibull-Geométrica (WG) distribution obtained from composition of Weibull distribution and Geometric distribution, in a competitive causes (or risks) scenario (Basu & Klein (1982)), where only the minimum lifetime was observed, that is only a single cause being activated. This distribution generalizes the Geometric Exponential extended (GEE) distribution proposed by Adamidis *et al.* (2005), Geometric Exponential (GE) distribution introduced by Adamidis & Loukas (1998) and Weibull distribution. The hazard function of WG distribution has forms more general and it is useful to modeling data with unimodal failure rate.

In this paper we introduce the Weibull-Negative-Binomial (WNB) distribution which is general by assuming a latent activation structure to explain the occurrence of an event of interest, where the number of competing causes are modeled by a negative binomial distribution and the competitive causes may have different activation mechanisms. We derive some standard properties of this new distribution, and inferences via the maximum likelihood framework. The proposed distribution includes as particular cases the models proposed by Adamidis & Loukas (1998), Kus (2007), Barreto-Souza *et al.* (2011), Roman *et al.* (2011) e Cancho *et al.* (2011), moreover the WNB distribution is very flexible and can accommodate lifetime data with hazard function which has no monotone forms, such as bathtub and unimodal.

The paper is organized as follows. In Section 2 we introduce the new distribution and plot its probability density function (pdf) and we derive some structural properties of this distribution. In Section 3 we study the inferential procedure based on the likelihood method. The results of a misspecification simulation study comparing the proposed model in the presence of two different activation schemes with the usual Weibull distribution is presented in Section 4. The new distribution is illustrated in Section 5 by considering two sets of real data. Finally we concludes the paper in Section 6.

## 2. The Model

For an individual in a population, let  $M$  denote the unobservable number of causes of the event of the interest (number of latent factors) for such a individual. Assume  $M$  is a discrete random variable taken to have a zero truncated Binomial Negative distribution with parameters  $\eta$  and  $\theta$ , has probability mass function (pmf) given by

$$p_m = \frac{\Gamma(\eta^{-1} + m)}{\Gamma(\eta^{-1})m!} \left( \frac{\eta\theta}{1 + \eta\theta} \right)^m (1 + \eta\theta)^{-1/\eta} \left( 1 - (1 + \eta\theta)^{-1/\eta} \right)^{-1}. \quad (1)$$

$$m = 1, 2, \dots, \text{ for } \theta > 0, \eta \geq -1, \text{ and } \eta > -1/\theta, \text{ so that } E[M] = \frac{\theta}{1 - (1 + \eta\theta)^{-1/\eta}} \text{ and } Var[M] = \frac{\eta\theta^2 + \theta - (\eta\theta^2 + \theta^2 + \theta)(1 + \eta\theta)^{-1/\eta}}{(1 - (1 + \eta\theta)^{-1/\eta})^2}.$$

The time of the cause to produce the event of interest (latent event times) is denoted by  $T_j$ ,  $j = 1, \dots, M$ . We assume that,  $T_1, T_2, \dots$  are independent of  $M$  and  $T_j$  are independent and identically distributed (i.i.d.). The lifetime observed can be define by the random variable  $Y = T_{(R)}$ ,  $R = 1, \dots, M$ , where  $T_{(1)} \leq \dots \leq T_{(M)}$  are the ordered  $T_j$  and  $R$  is dependent of  $M$ . Thus,  $R$  is a threshold variable, which, in many biological process, denotes the resistance factor of the immune system of the individual. For

instance, when the cancer relapses, the random variable  $Y$  has the value given by the  $T_{(R)}$ . It can be fixed as constant, a function of  $N$ , or even be treated as random by specifying a conditional distribution for  $R|N$ .

In this paper we deal with three specified cases for  $R$ . First, we assume that given, the conditional distribution of  $R$  is uniform on  $1, 2, \dots, M$  (random activation scheme). Under this setup, the distribution function for  $Y$  given  $M = m$  and  $R = r$ , is given by

$$F_{Y|m,r}(y) = P[Y \leq y | M = m, R = r] = \sum_{j=r}^m \binom{m}{j} (G(y))^j (1 - G(y))^{m-j}. \quad (2)$$

Moreover, we can demonstrate that the marginal distribution function of  $Y$  is given by

$$F_{\text{random}}(y) = \sum_{m=1}^{\infty} \sum_{r=1}^m P[T_{(R)} \leq y | M = m, R = r] P[R = r | M = m] P[M = m] \quad (3)$$

$$= 1 - \sum_{m=1}^{\infty} \left\{ \sum_{r=0}^m (m-1) B(r, m, G(y)) \right\} \frac{1}{m} P[M = m] \quad (4)$$

$$= 1 - (1 - G(y)) \sum_{m=1}^{\infty} p_m = G(y),$$

where  $B(x, m, G(y))$  is the probability mass function (pmf) of the binomial distribution with parameters  $m$  and  $G(y)$ , and  $P[M = m]$  is given in 1. In this case, the distribution function of  $Y$  is the same the distribution of latent random variable  $T_j$ 's.

If we consider  $R = r$  (fixed), then the marginal distribution of  $Y$  is given by

$$F(y) = \sum_{m=1}^{\infty} IB(G(y); r, m - r + 1) p_m = \sum_{m=1}^{\infty} m \binom{m-1}{r-1} \int_0^{G(y)} u^{r-1} (1-u)^{m-r} du p_m, \quad (5)$$

where  $IB(x; a, b)$  is incomplete beta function and  $p_m = P[M = m]$  is given in (1).

As a second setup, we consider the so-called first activation scheme by supposing that the event of interest happens due to any one of these causes (latent events). Therefore, for  $R = 1$ , conditional upon  $M$ ,  $Y = T_{(1)} = \min\{T_1, \dots, T_M\}$ . In this case, the marginal distribution of  $Y$  in (5) is given by

$$F_{\text{first}}(y) = \frac{1 - (1 + \eta \theta G(y))^{-1/\eta}}{1 - (1 + \eta \theta)^{-1/\eta}}, \quad y > 0. \quad (6)$$

From (6) the density function is

$$f_{\text{first}}(y) = \frac{\theta g(y) (1 + \theta \eta G(y))^{-(1/\eta+1)}}{1 - (1 + \theta \eta)^{-1/\eta}}. \quad (7)$$

Considering (6) and (7), the surviving and hazard function of  $Y$  are given

$$S_{\text{first}}(y) = \frac{(1 + \theta \eta G(y))^{-1/\eta} - (1 + \theta \eta)^{-1/\eta}}{1 - (1 + \theta \eta)^{-1/\eta}}, \quad (8)$$

and

$$h_{\text{first}}(y) = \frac{\theta g(y) (1 + \theta \eta G(y))^{-(1/\eta+1)}}{(1 + \theta \eta G(y))^{-1/\eta} - (1 + \theta \eta)^{-1/\eta}}. \quad (9)$$

As a third setup, we consider the so-called last activation scheme by assuming that the event of interest happens after all the  $M$  causes have been reached. This implies  $R = M$  and  $Y = T_{(1)} = \max\{T_1, \dots, T_M\}$ . The marginal distribution of  $Y$  in (5) is given by

$$F_{\text{last}}(y) = \frac{(1 + \eta\theta - \eta\theta G(y))^{-1/\eta} - (1 + \eta\theta)^{-1/\eta}}{1 - (1 + \eta\theta)^{-1/\eta}}, \quad y > 0. \quad (10)$$

The corresponding density function is given

$$f_{\text{last}}(y) = \frac{\theta f(y)(1 + \theta\eta(1 - G(y)))^{-(1/\eta+1)}}{1 - (1 + \theta\eta)^{-1/\eta}}. \quad (11)$$

Considering (10) e (11), the surviving and hazard function of  $Y$  are given by

$$S_{\text{last}}(y) = \frac{1 - (1 + \theta\eta(1 - G(y)))^{-1/\eta}}{1 - (1 + \theta\eta)^{-1/\eta}}, \quad (12)$$

and

$$h_{\text{last}}(y) = \frac{\theta f(y)(1 + \theta\eta(1 - G(y)))^{-(1/\eta+1)}}{1 - (1 + \theta\eta(1 - G(y)))^{-1/\eta}}. \quad (13)$$

The following proposition shows the relationship between the distribution functions  $F_{\text{random}}(y)$ ,  $F_{\text{first}}(y)$  and  $F_{\text{last}}(y)$ .

**Proposition 2.1.** Under conditions of the models (4), (6) and (10) for any distribution function  $G(y)$ , we have  $F_{\text{last}}(y) \leq F_{\text{random}}(y) \leq F_{\text{first}}(y)$  for all  $y > 0$ .

*Proof.* We know that  $\frac{1 - (1 + \eta\theta G(y))^{-1/\eta}}{G(y)}$  is an increasing function with respect to  $y$ , so  $\liminf \frac{1 - (1 + \eta\theta G(y))^{-1/\eta}}{G(y)} = \lim_{y \rightarrow 0} \frac{1 - (1 + \eta\theta G(y))^{-1/\eta}}{G(y)} = 1 - (1 + \eta\theta)^{-1/\eta}$ , hence  $\frac{1 - (1 + \eta\theta G(y))^{-1/\eta}}{G(y)} \geq 1 - (1 + \theta\eta)^{-1/\eta}$ ,  $\forall y$ . Soon,  $G(y) \leq \frac{1 - (1 + \eta\theta G(y))^{-1/\eta}}{1 - (1 + \eta\theta)^{-1/\eta}}$ , i. e.,  $F_{\text{first}}(y) \geq F_{\text{random}}(y)$ . Similarly to prove  $F_{\text{last}}(y) \leq F_{\text{random}}(y)$ .  $\square$

Note that by considering different choices for the distribution of latent random variables  $T_j$ 's, new families of distributions can be obtained. In this paper, we consider that the random variables  $\{T_i\}_{i=1}^M$  are independent and identically distributed (iid) follows the Weibull distribution  $W(\alpha, \lambda)$  with scale parameter  $\lambda > 0$ , shape parameter  $\alpha > 0$  and density function is given

$$f(t) = \alpha t^{\alpha-1} \exp(\lambda - t^\alpha e^\lambda). \quad (14)$$

Thus, from (4) (6) and (10), we obtain the Weibull distribution (random activation), the WNB distribution under first activation (WNB-FA) and the WNB distribution under last activation (WNB-LA), respectively. The main features of distributions WNB-FA and WNB-LA are presented below.

### 2.1. The first activation scheme of WNB distribution

The non-negative random variable  $Y$  is said to have a WNB-FA distribution with parameter  $\xi = (\alpha, \lambda, \eta, \theta)$  if its pdf is given by

$$f_{\text{first}}(y; \xi) = \frac{\theta \alpha y^{\alpha-1} \exp(\lambda - y^\alpha e^\lambda) [1 + \eta\theta(1 - \exp(-y^\alpha e^\lambda))]^{-(1/\eta+1)}}{1 - (1 + \eta\theta)^{-1/\eta}}, \quad (15)$$

Some special sub-models of the WNB-FA distribution (15) are obtained as follows. For  $\eta = 1$  it leads to the Weibull Geométrica distribution under the first activation (WG-FA) introduced by Barreto-Souza *et al.* (2011). For  $\eta \rightarrow 0$ , we obtain, from (15), the Weibull Poisson distribution under the first activation. For  $\alpha = 1$ , we have the Exponential Negative Binomial distribution (ENB-FA), also when  $\eta = 1$ , we have the Exponential Geometric distribution (EG-FA) (Adamidis & Loukas (1998)), and when  $\eta \rightarrow 0$ , we obtain the Exponential Poisson distribution (EP-FA)(Kus(2007)). Furthermore when  $\theta \rightarrow 0$  the WNB distribution (15) leads to the Weibull distribution (14). The WNB density functions are displayed in Figure 1 with  $\lambda = 0$  for selected values the vector  $\phi = (\alpha, \theta)$  and  $\eta = -1, -0.5, 0.01, 1.0$  e  $10$ .

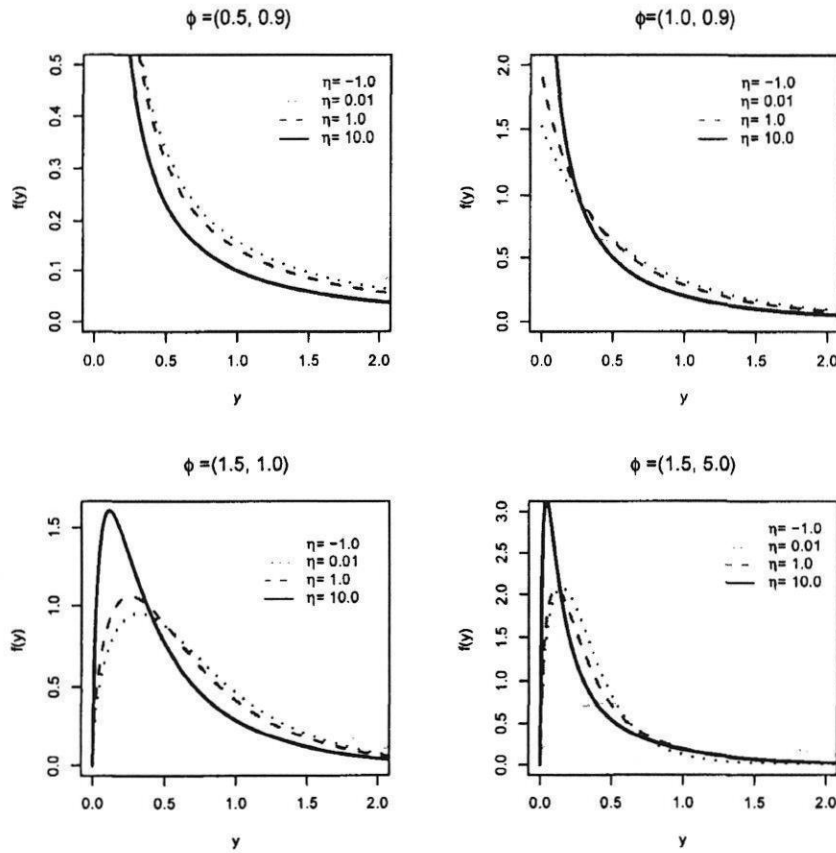


Figure 1: WNB density for selected parameter

Note that the *pdf* of the WNB-FA distribution given in (15) also can be written as

$$f_{\text{first}}(y; \xi) = \sum_{m=1}^{\infty} f(y|m, \alpha, \lambda) p_m, \quad (16)$$

where  $f(y|m, \alpha, \lambda) = m\alpha y^{\alpha-1} \exp(\lambda - me^{\lambda} y^{\alpha})$  has Weibull distribution with scale parameter  $\lambda + \log(m)$ ,

shape parameter  $\alpha$ , and  $p_m$  was given in (1). So the WNB cumulative distribution has the following expression

$$F_{\text{first}}(y; \xi) = \sum_{m=0}^{\infty} F(y|m, \alpha, \lambda) p_m, \quad (17)$$

where  $F = (y|m, \alpha, \lambda)$  denotes the Weibull cumulative function with the scale parameter  $\lambda + \log(m)$ , shape parameter  $\alpha$ . Therefore we can obtain some of its mathematical properties from those of the Weibull distribution.

### 2.1.1. Properties of the WNB distribution under first activation scheme

Let  $Y$  be a random variable having the WNB-FA distribution with parameter vector  $\xi = (\beta, \alpha, \eta, \theta)$ . The survival and hazard functions of  $Y$  are

$$S_{\text{first}}(y; \xi) = \frac{(1 + \eta\theta(1 - \exp(-y^\alpha e^\lambda)))^{-1/\eta} - (1 + \eta\theta)^{-1/\eta}}{1 - (1 + \eta\theta)^{-1/\eta}}, \quad y > 0. \quad (18)$$

and

$$h_{\text{first}}(y; \xi) = \frac{\theta \alpha y^{\alpha-1} \exp(\lambda - y^\alpha e^\lambda) [1 + \eta\theta(1 - \exp(-y^\alpha e^\lambda))]^{-(1/\eta+1)}}{(1 + \eta\theta(1 - \exp(-y^\alpha e^\lambda)))^{-1/\eta} - (1 + \eta\theta)^{-1/\eta}}, \quad y > 0, \quad (19)$$

respectively.

The hazard function (19) is decreasing for  $0 < \alpha \leq 1$  and increasing for  $\alpha > 1$ . However, for other parameter values, it can take different forms. Figura 2 illustrates some of the possible shapes of the hazard function with  $\lambda = 0$  for selected values of the vector  $\phi = (\alpha, \theta)$  and  $\eta = -1.0, -0.5, 0.01, 1.0, 10.0$ . These plots show that the hazard function of the new distribution is much more flexible than the WG-FA distribution.

### 2.1.2. Quantiles and moments

The quantiles  $q$  of the WNB-FA distribution are obtained by inverting the cumulative distribution functions (cdf) and are given by

$$y_q^f = G^{-1}\left(\frac{[1 - q(1 - p_0)]^{-\eta} - 1}{\eta\theta}\right),$$

In particular, the median is  $y_{1/2} = G^{-1}\left(\frac{[1 - \frac{1}{2}(1 - p_0)]^{-\eta} - 1}{\eta\theta}\right)$ , where  $G^{-1}(y) = (-\log(1 - y)/e^\lambda)^{1/\alpha}$ .

The  $k$ th ordinary moment of  $Y$  is given by

$$\mu_k^f = E_{\text{first}}[Y^k] = (e^{-\lambda k/\alpha} \Gamma\left(\frac{k}{\alpha} + 1\right) \sum_{m=1}^{\infty} m^{-k/\alpha} p_m) \quad (20)$$

$$= \frac{e^{-\lambda k/\alpha} \Gamma\left(\frac{k}{\alpha} + 1\right)}{(1 + \eta\theta)^{1/\eta} - 1} \varphi(k, \alpha, \eta, \theta) \quad (21)$$

where  $\varphi(k, \alpha, \eta, \theta) = \sum_{m=1}^{\infty} m^{-k/\alpha} \frac{\Gamma(\eta^{-1} + m)}{\eta^{-1} m!} \left(\frac{\eta\theta}{1 + \eta\theta}\right)^m < \infty$ , indeed, it could be proved by ratio criterion.

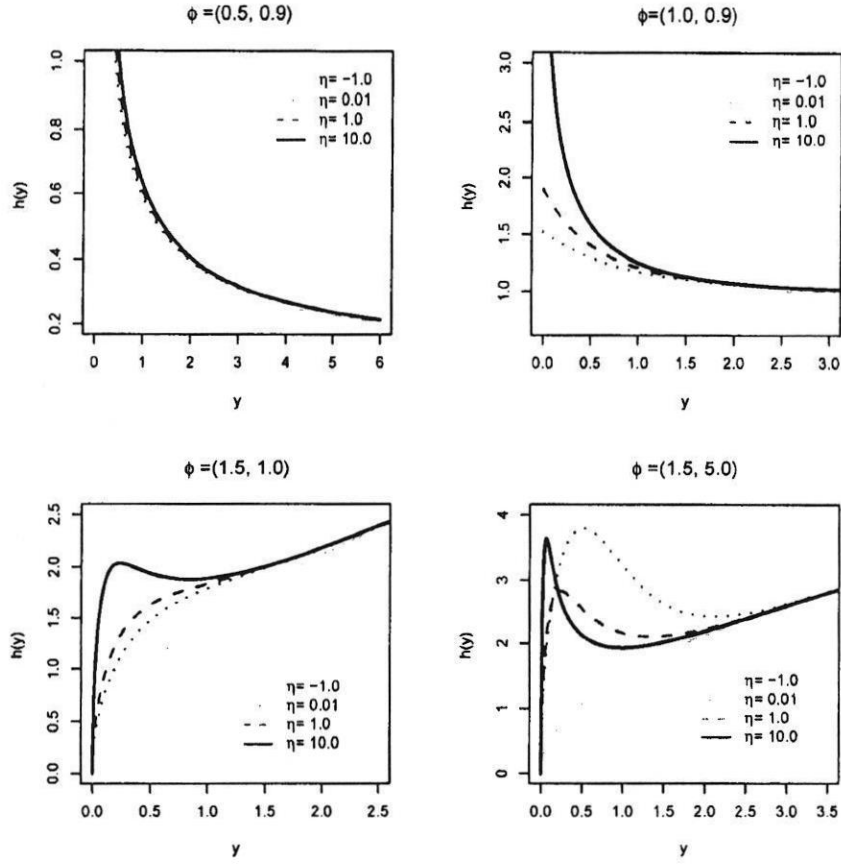


Figure 2: WNB hazard function for selected parameter

As an special case, when  $k/\alpha$  is positive rational, for  $\alpha > 0$  and  $k$  in the positive integers, we can rewriter the  $k$ th ordinary moment of  $Y$  as

$$E_{\text{first}}[Y^k] = \frac{\theta e^{-\lambda k/\alpha} \Gamma_{\alpha}^{\frac{k}{\alpha}} + 1}{(1 + \eta\theta)[(1 + \eta\theta)^{1/\eta} - 1]} F_{\frac{k}{\alpha}+2, \frac{1}{\alpha}+1}([1, \dots, 1, \frac{1+\eta}{\eta}], [2, \dots, 2], \frac{\eta\theta}{1+\eta\theta}).$$

where  $F_{p,q}(\mathbf{n}, \mathbf{d}, \lambda)$  is the generalized hypergeometric function. This function is also know as Barnes's extended hypergeometric function. The definition of  $F_{p,q}(\mathbf{n}, \mathbf{d}, \lambda)$  is

$$F_{p,q}(\mathbf{n}, \mathbf{d}, \lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k \prod_{i=1}^p \Gamma(n_i + k) \Gamma^{-1}(n_i)}{\Gamma(k+1) \prod_{i=1}^q \Gamma(d_i + k) \Gamma^{-1}(d_i)},$$

where  $\mathbf{n} = [n_1, \dots, n_p]$ ,  $p$  is the number of the operands of  $\mathbf{n}$ ,  $\mathbf{d} = [d_1, \dots, d_q]$  and  $q$  is the number of the operands of  $\mathbf{d}$ . Generalized hypergeometric function is quickly and readily available in standard softwares such as Maple. Further the value of series  $\varphi(k, \alpha, \eta, \theta)$  can be obtained by the Maple software too.

Various closed form expressions can be obtained from (20) as particular cases. The central moments  $\mu_p$  and the cumulates ( $\kappa_s$ ) of the  $Y$  are easily obtained from the ordinary moments by

$$\mu_p \sum_{r=0}^p \binom{p}{r} \mu_1'^p \mu_{p-r}' \quad \text{and} \quad \kappa_p = \mu_p' - \sum_{r=1}^{p-1} \binom{p-1}{r-1} \kappa_r \mu_{p-r}',$$

respectively. The skewness and kurtosis measures can be obtained from the classical relationships involving cumulates:  $Skewness(Y) = \kappa_3/\kappa_2^{3/2}$  and  $Kurtosis(Y) = \kappa_4/\kappa_2^2$ . Table 2.1.2 presents the ordinary moments, variance, skewness and kurtosis of the WNB-FA distribution for  $\lambda = 0$ ,  $\alpha = 0.5, 1.0, 1.5$ , and  $\theta = 0.9$ .

Table 1: Ordinary moments, Variance, Skewness and Kurtosis of the WNB-FA distribution for  $\lambda = 0$ ,  $\alpha = 0.5$  and  $\theta = 0.9$

| $\alpha$ | $\eta = 0.01$ | $\eta = 1$ | $\eta = 2$ | $\eta = 5$ | $\eta = 10$ |          |
|----------|---------------|------------|------------|------------|-------------|----------|
| 0.5      | $k = 1$       | 0.823      | 0.575      | 0.447      | 0.274       | 0.170    |
|          | $k = 2$       | 9.001      | 6.180      | 4.767      | 2.892       | 1.778    |
|          | $k = 3$       | 264.057    | 180.891    | 139.402    | 84.461      | 51.896   |
|          | $k = 4$       | 14707.132  | 10071.039  | 7759.992   | 4700.897    | 2888.180 |
|          | Variance      | 8.323      | 5.850      | 4.567      | 2.816       | 1.749    |
|          | Skewness      | 10.117     | 12.059     | 13.645     | 17.376      | 22.053   |
|          | Kurtosis      | 200.249    | 282.506    | 360.321    | 581.168     | 932.944  |
| 1.0      | $k = 1$       | 0.466      | 0.338      | 0.268      | 0.170       | 0.108    |
|          | $k = 2$       | 0.823      | 0.575      | 0.447      | 0.274       | 0.170    |
|          | $k = 3$       | 2.320      | 1.600      | 1.237      | 0.752       | 0.463    |
|          | $k = 4$       | 9.001      | 6.180      | 4.767      | 2.892       | 1.778    |
|          | Variance      | 0.606      | 0.461      | 0.375      | 0.245       | 0.158    |
|          | Skewness      | 2.910      | 3.499      | 3.983      | 5.117       | 6.528    |
|          | Kurtosis      | 15.282     | 20.592     | 25.695     | 40.280      | 63.560   |
| 1.5      | $k = 1$       | 0.448      | 0.333      | 0.268      | 0.175       | 0.113    |
|          | $k = 2$       | 0.527      | 0.376      | 0.296      | 0.185       | 0.116    |
|          | $k = 3$       | 0.823      | 0.575      | 0.447      | 0.274       | 0.170    |
|          | $k = 4$       | 1.576      | 1.091      | 0.844      | 0.514       | 0.317    |
|          | Variance      | 0.326      | 0.265      | 0.224      | 0.154       | 0.103    |
|          | Skewness      | 1.579      | 2.003      | 2.343      | 3.116       | 4.051    |
|          | Kurtosis      | 5.784      | 7.658      | 9.518      | 14.905      | 23.517   |

### 2.1.3. Moment Generation Function

The moment generating function (*mgf*)  $M_{\text{first}}(t) = E_{\text{first}}[\exp(tY)]$  follows from the power series expansion for the exponential function and (16) as

$$M_{\text{first}}(t) = \sum_{m=1}^{\infty} p_m \sum_{n=0}^{\infty} \frac{t^n (e^\lambda m)^{-n/\alpha}}{n!} \Gamma(1 + \frac{n}{2}). \quad (22)$$

If the parameter  $\alpha$  is assumed to be a rational number, expressed as  $k = p/q$  where  $p$  e  $q$  are integer, then this expression (36) can be evaluated analytical with  $t$  replace by  $-t$ , on finds

$$M(t) = \sum_{m=1}^{\infty} \frac{m e^\lambda p_m}{(-t)^\alpha} \frac{p^{\alpha-1/2} q^{1/2}}{(\sqrt{2\pi})^{q+p+2}} G_{p,q}^{m,n} \left( \begin{matrix} \frac{1-\alpha}{p}, & \frac{2-\alpha}{p}, & \dots, & \frac{p-\alpha}{p} \\ \frac{0}{q}, & \frac{1}{q}, & \dots, & \frac{q-1}{q} \end{matrix} \middle| \frac{p^p}{(q m^{-1} e^{-\lambda} (-t)^\alpha)^q} \right),$$

where  $G$  is the Meijer G-function defined by

$$G_{p,q}^{m,n} \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{m=1}^{j-1} \Gamma(b_j + t) \prod_{n=1}^{j-1} \Gamma(1 - a_j - t)}{\prod_{j=n+1}^p \Gamma(a_j + t) \prod_{j=m+1}^q \Gamma(1 - b_j - t)} x^{-t} dt.$$

#### 2.1.4. Order Statistic

The density function  $f_{i:n}^f(y)$  of the  $i$ th order statistic for  $i = 1, \dots, n$  corresponding to the random variable  $Y_1, \dots, Y_n$  following the WNB-FA distribution, can be written as

$$f_{i:n}^f(y) = \frac{1}{B(i, n-i+1)} f(y) \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j [1 - F(y)]^{n+j-1},$$

where  $f(y)$  is given in (15),  $F(y)$  is its cdf and  $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$  is the beta function. Setting  $u = \exp(-y^\alpha e^\lambda)$ , we can write from (16) and (17),

$$f_{i:n}^f(y) = \frac{\alpha e^\lambda}{B(i, n-i+1)} \sum_{m=1}^{\infty} m p_m y^{\alpha-1} u^m \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j \left( \sum_{k=1}^{\infty} u^k p_k \right)^{n+j-1}.$$

Throughout an equation of Gradshtey and Ryzhik for a power series raised to a positive integer  $j$ ,

$$\left( \sum_{i=0}^{\infty} a_i y^i \right)^j = \sum_{i=0}^{\infty} c_{j,i} y^i, \quad (23)$$

whose coefficients  $c_{j,i} = (i a_0)^{-1} \sum_{m=1}^i (j m - i + m) a_m c_{j,i-m}$ , where  $c_{j,0} = a_0^j$ . Hence, consequently, the coefficients  $c_{j,i}$  can be obtained. Using this equation and combining its terms, we obtain

$$f_{i:n}^f(y) = \sum_{k,m=1}^{\infty} \sum_{j=0}^{i-1} \phi(k, m, j) g(y|\alpha, \lambda, m, k), \quad (24)$$

where  $\phi(k, m, j) = \binom{i-1}{j} \frac{(-1)^j m p_m c_{n+j-1,k}}{(m+k) B(i, n-i+1)}$ ,  $g(y|\alpha, \lambda, m, k)$  has Weibull distribution with scale parameter  $\lambda + \log(m+k)$ , shape parameter  $\alpha$  and  $c_{j+n-1,k} = [k p_1]^{-1} \sum_{l=1}^k ((n+j-1)l - k + l) c_{n+j-1,k-l} p_l$  with  $c_{n+j-1,1} = p_1^{n+j-1}$ . From the (37), the moments of the WNB-FA order statistic can be written directly in terms of the Weibull moments as

$$E_{\text{first}}[Y_{i:n}^s] = \sum_{k,m=1}^{\infty} \Gamma\left(\frac{s}{\alpha} + 1\right) \sum_{j=0}^{i-1} \frac{\phi(k, m, j)}{\exp(-\lambda s/\alpha) (m+k)^{-s/\alpha}}. \quad (25)$$

We can compute the moments of the WNB order statistic by the equation (25) or by numerical integration. The moments, variance, skewness and kurtosis of the WNB-FA order statistics listed in Table 2 for  $\lambda = 0$ ,  $\alpha = 1.5$ ,  $\eta = 2$ ,  $\theta = 0.9$ ,  $n = 5$  and  $s = 3$ .

#### 2.1.5. Entropy

The entropy of the random variable  $Y$  with density  $f(y)$  is a measure of variation of the uncertainty. Rényi entropy is defined by

$$I_R(\rho) = \frac{1}{1-\rho} \log \left\{ \int f(y)^\rho dy \right\}, \quad (26)$$

Table 2: Moments, Variance, Skewness and Kurtosis of the WNB-FA statistic for  $\lambda = 0$ ,  $\alpha = 1.5$ ,  $\eta = 2$ ,  $\theta = 0.9$   $n = 5$  and  $s = 3$

| Order statistic | $Y_{1:5}$ | $Y_{2:5}$ | $Y_{3:5}$ | $Y_{4:5}$ | $Y_{5:5}$ |
|-----------------|-----------|-----------|-----------|-----------|-----------|
| $k = 1$         | 0.1931    | 0.2083    | 0.1157    | 0.0344    | 0.0044    |
| $k = 2$         | 0.0573    | 0.0837    | 0.0563    | 0.0192    | 0.0027    |
| $k = 3$         | 0.0228    | 0.0411    | 0.0320    | 0.0122    | 0.0019    |
| $k = 4$         | 0.0113    | 0.0239    | 0.0209    | 0.0087    | 0.0014    |
| Variance        | 0.0201    | 0.0403    | 0.0429    | 0.0180    | 0.0027    |
| Skewness        | 1.3920    | 0.8514    | 1.7533    | 4.2564    | 13.2347   |
| Kurtosis        | 5.8919    | 3.5597    | 5.4621    | 22.0399   | 195.5169  |

where  $f(y)$  is the pdf of the  $Y$ ,  $\rho > 0$  and  $\rho \neq 0$ . For a random variable  $Y$  with a WNB-FA distribution, using the Taylor series for  $[1 + \eta\theta(1 - e^{-u})]^{-\rho(1/\eta+1)}$  and  $(1 - e^{-u})^k$ , we have

$$\int f_{\text{first}}(y)^\rho dy = \left( \frac{\theta}{1 - (1 + \eta\theta)^{-1/\eta}} \right)^\rho \alpha^{\rho-1} \exp\left(\frac{\lambda}{\alpha}(\rho-1)\right) \Gamma(A) \sum_{i,k=0}^{\infty} \frac{(\eta\theta)^k}{(\rho+i)^A} \binom{\rho(1/\eta+1)+k+1}{k} \binom{k+i+1}{i}, \quad (27)$$

where  $A = \frac{\rho(\alpha-1)+1}{\alpha}$ . So from (26) and (27), the entropy of the WNB-FA is given

$$I_R^f(\rho) = \frac{1}{1-\rho} \left\{ \rho \log \left( \frac{\theta}{1 - (1 + \eta\theta)^{-1/\eta}} \right) + (\rho-1) \left( \log \alpha + \frac{\lambda}{\alpha} \right) + \log \left[ \Gamma \left( \frac{\rho(\alpha-1)+1}{\alpha} \right) \right] \right. \\ \left. \log \left[ \sum_{i,k=0}^{\infty} \frac{(\eta\theta)^k}{(\rho+i)^{\frac{\rho(\alpha-1)+1}{\alpha}}} \binom{\rho(1/\eta)+k+1}{k} \binom{k+i+1}{i} \right] \right\}.$$

## 2.2. The last activation scheme of WNB distribution

The non-negative random variable  $Y$  is said to have a WNB-LA distribution with parameter  $\xi = (\alpha, \lambda, \eta, \theta)$  if its pdf is given by

$$f_{\text{last}}(y; \xi) = \frac{\theta \alpha \beta^\alpha y^{\alpha-1} e^{-(\beta y)^\alpha} (1 + \eta \theta e^{-(\beta y)^\alpha})^{-1-1/\eta}}{1 - (1 + \eta \theta)^{-1/\eta}}, \quad y > 0, \quad (28)$$

where  $\xi = (\beta, \alpha, \eta, \theta)$ . The WNB-LA density functions are displayed in Figure 3 for selected values the vector  $\phi = (\beta, \alpha, \theta)$  and  $\eta = -1.0, -0.5, 0.01, 1.0$  and  $10$ .

Note that the pdf of the WNB-LA distribution given in (28) can also be written as

$$f_{\text{last}}(y; \xi) = \sum_{m=1}^{\infty} f(y|m, \alpha, \lambda) p_m, \quad (29)$$

where  $f(y|m, \alpha, \lambda) = \alpha m y^{\alpha-1} \exp(\lambda - y^\alpha e^\lambda) (1 - \exp(-y^\alpha e^\lambda))^{m-1}$ . So the WNB-LA cumulative distribution has expression given by

$$F_{\text{last}}(y; \xi) = \sum_{m=0}^{\infty} (1 - \exp(-y^\alpha e^\lambda))^m p_m. \quad (30)$$

Therefore we can obtain some of its mathematical properties from those of the Weibull distribution.

### 2.2.1. Properties of the WNB distribution under last activation scheme

Let  $Y$  be a random variable having the WNB-LA distribution with parameter vector  $\xi = (\beta, \alpha, \eta, \theta)$ . The cdf of  $Y$  is given by

$$F_{\text{last}}(y; \xi) = \frac{(1 + \eta \theta e^{-(\beta y)^\alpha})^{-1/\eta} - (1 + \eta \theta)^{-1/\eta}}{1 - (1 + \eta \theta)^{-1/\eta}}, \quad y > 0. \quad (31)$$

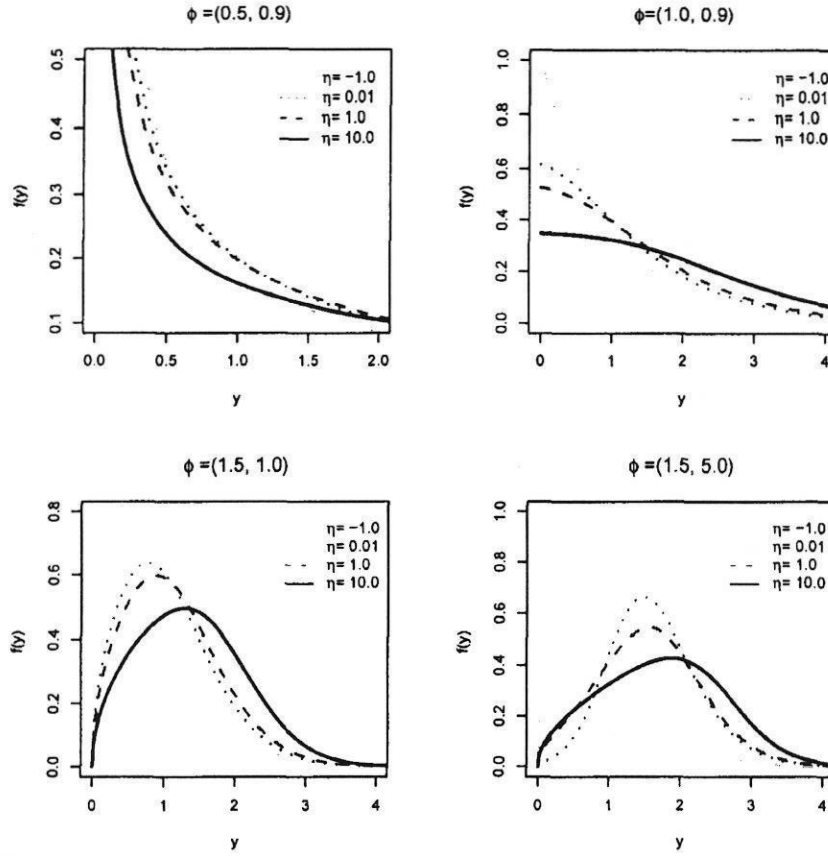


Figure 3: WNB density for selected parameter

The survival and hazard functions of  $X$  are

$$S_{\text{last}}(y; \xi) = \frac{1 - (1 + \eta\theta e^{-(\beta y)^\alpha})^{-1/\eta}}{1 - (1 + \eta\theta)^{-1/\eta}}, \quad y > 0. \quad (32)$$

and

$$h_{\text{last}}(y; \xi) = \frac{\theta\alpha\beta^\alpha y^{\alpha-1} e^{-(\beta y)^\alpha} (1 + \eta\theta e^{-(\beta y)^\alpha})^{-1-1/\eta}}{1 - (1 + \eta\theta e^{-(\beta y)^\alpha})^{-1/\eta}}, \quad y > 0, \quad (33)$$

The hazard function (19) is decreasing for  $0 < \theta \leq 0.5$  and increasing for  $\theta > 0.5$ . However, for other parameter values, it can take different forms. Figure 2 illustrates some of the possible shapes of the hazard function for selected values of the vector  $\phi = (\beta, \alpha, \theta)$  and  $\eta = -1.0, -0.5, 0.01, 1.0, 10.0$ .

### 2.2.2. Quantiles and moments

The quantiles  $q$  of the WNB distribution is obtained by inverting Equation (31) to give

$$y_q^l = \beta^{-1} \left\{ \log \left( [(q + (1 + \eta\theta)^{-1/\eta}(1 - q))^{-\eta} - 1](\eta\theta)^{-1} \right)^{-1} \right\}^{1/\alpha}.$$

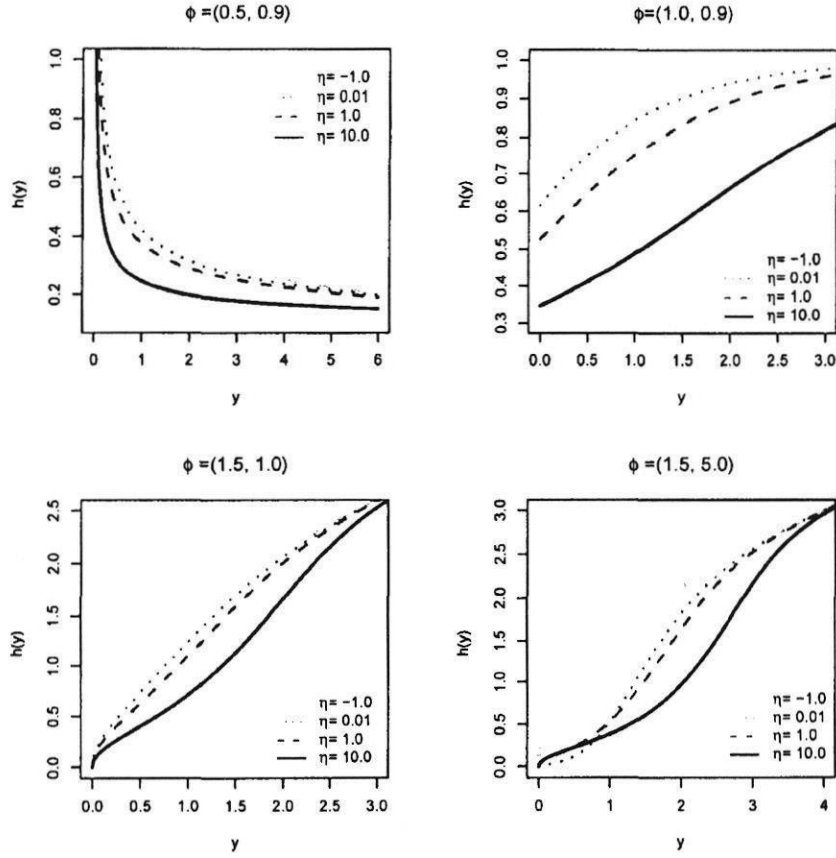


Figure 4: WNB hazard function for selected parameter

In particular, the median is given by

$$y_{0.5}^l = \beta^{-1} \left\{ \log \left( \left[ \left( \frac{1}{2} + \frac{1}{2} (1 + \eta\theta)^{-1/\eta} \right)^{-\eta} - 1 \right] (\eta\theta)^{-1} \right)^{-1} \right\}^{1/\alpha}.$$

The  $k$ th ordinary moment of  $Y$  is given by

$$E_{\text{last}}[Y^k] = \alpha\beta^\alpha \sum_{m=1}^{\infty} mg(k, \alpha, \beta, m)p_m; \quad (34)$$

where  $g(k, \alpha, \eta, \theta) = \int_0^\infty y^{k+\alpha-1} e^{-(\beta y)^\alpha} (1 - e^{-(\beta y)^\alpha})^{m-1} dy$ . From the function (28) and using the expression Taylor series the expression (34) also can be written as

$$E_{\text{last}}[Y^k] = \Gamma(\alpha^{-1} + s - 1) \exp\{\lambda(2 - s - \alpha^{-1})\} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \binom{m-1}{k} (-1)^k m(k+1)^{-(s-1+\alpha^{-1})} p_m. \quad (35)$$

Various expressions such as mean, variance, skewness and kurtosis can be obtained numerically by some mathematical software from (20), such as the Maple. Table 2.2.2 presents the ordinary moments, variance, skewness and kurtosis of the WNB-LA distribution for  $\lambda = 0$ ,  $\alpha = 0.5, 1.0, 1.5$ , and  $\theta = 0.9$ .

Table 3: Ordinary moments, Variance, Skewness and Kurtosis of the WNB-LA distribution for  $\lambda = 0$ ,  $\alpha = 0.5, 1.0, 1.5$ , and  $\theta = 0.9$

| $\alpha$ | $\eta = 0.01$ | $\eta = 1$ | $\eta = 2$ | $\eta = 5$ | $\eta = 10$ |           |
|----------|---------------|------------|------------|------------|-------------|-----------|
| 0.5      | $k = 1$       | 3.240      | 3.009      | 2.832      | 2.470       | 2.112     |
|          | $k = 2$       | 42.042     | 41.111     | 40.301     | 38.357      | 35.993    |
|          | $k = 3$       | 1287.034   | 1279.027   | 1271.536   | 1251.677    | 1223.533  |
|          | $k = 4$       | 72448.653  | 72329.104  | 72212.850  | 71886.022   | 71343.334 |
|          | Variance      | 31.544     | 32.059     | 32.281     | 32.253      | 31.531    |
|          | Skewness      | 5.342      | 5.302      | 5.314      | 5.446       | 5.729     |
|          | Kurtosis      | 58.377     | 57.331     | 57.152     | 58.455      | 62.271    |
| 1.0      | $k = 1$       | 0.731      | 0.642      | 0.580      | 0.468       | 0.370     |
|          | $k = 2$       | 1.620      | 1.504      | 1.416      | 1.235       | 1.056     |
|          | $k = 3$       | 5.119      | 4.912      | 4.742      | 4.362       | 3.940     |
|          | $k = 4$       | 21.021     | 20.555     | 20.151     | 19.178      | 17.996    |
|          | Variance      | 1.086      | 1.092      | 1.079      | 1.016       | 0.919     |
|          | Skewness      | 2.074      | 2.228      | 2.379      | 2.764       | 3.256     |
|          | Kurtosis      | 8.815      | 9.348      | 10.015     | 12.094      | 15.356    |
| 1.5      | $k = 1$       | 0.417      | 0.357      | 0.317      | 0.247       | 0.189     |
|          | $k = 2$       | 0.605      | 0.543      | 0.499      | 0.414       | 0.337     |
|          | $k = 3$       | 1.080      | 1.003      | 0.944      | 0.823       | 0.704     |
|          | $k = 4$       | 2.251      | 2.141      | 2.052      | 1.859       | 1.652     |
|          | Variance      | 0.432      | 0.416      | 0.399      | 0.353       | 0.302     |
|          | Skewness      | 1.651      | 1.909      | 2.120      | 2.603       | 3.177     |
|          | Kurtosis      | 5.320      | 6.216      | 7.089      | 9.501       | 13.063    |

### 2.2.3. Moment Generation Function

The moment generating function (mgf)  $M(t) = E[\exp(tY)]$  follows from the power series expansion for the exponential function and (29),

$$\begin{aligned} M_{\text{last}}(t) &= \int_0^\infty \exp(ty) f_{\text{last}}(y) dy \\ &= \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \binom{m-1}{k} \alpha e^\lambda m (-1)^k p_m g(y, \alpha, \lambda, k, t), \end{aligned} \quad (36)$$

where  $g(y, \alpha, \lambda, k, t) = \int_0^\infty y^{\alpha-1} \exp(-y^\alpha e^\lambda (k+1) + ty) dy$ .

### 2.2.4. Order Statistic

The density function  $f_{i:n}(y)$  of the  $i$ th order statistic for  $i = 1, \dots, n$  corresponding to the random variable  $Y_1, \dots, Y_n$  following the WNB-LA distribution, can be written as

$$f_{i:n}(y) = \frac{1}{B(i, n-i+1)} f(y) \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j [1 - F(y)]^{n+j-1},$$

where  $f(y)$  is given in (15),  $F(y)$  is its cdf and  $B(a, b) = \Gamma(a+b)/(\Gamma(a)\Gamma(b))$  is the beta function. Setting  $u = \exp(-y^\alpha e^\lambda)$ , we can write from (29) and (30),

$$f_{i:n}(y) = \frac{\alpha e^\lambda}{B(i, n-i+1)} \sum_{m=1}^{\infty} m p_m y^{\alpha-1} u^{m-1} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j \left[ 1 - \sum_{k=1}^{\infty} u^k p_k \right]^{n+j-1}.$$

Using the equation of Gradshtey and Ryzhik, we obtain

$$f_{i:n}(y) = \sum_{k,m=1}^{\infty} \sum_{j=0}^{i-1} \sum_{z=0}^{\infty} \phi(k, m, j, z) g(y|\alpha, \lambda, m, k), \quad (37)$$

where  $\phi(k, m, j) = \binom{i-1}{j} \binom{n+j-1}{z} \frac{(-1)^j m p_m c_{z,k}}{(m+k-1)B(i, n-i+1)}$ ,  $g(y|\alpha, \lambda, m, k)$  has Weibull distribution with scale parameter  $\lambda + \log(m+k-1)$ , shape parameter  $\alpha$  and  $c_{z,k} = [kp_1]^{-1} \sum_{m=1}^k (zm-k+m)c_{z,k-m}p_m$  with  $c_{z,1} = p_1^z$ . From the (37), the moments of the WNB-LA order statistic can be written directly in terms of the Weibull moments as

$$E_{\text{last}}[Y_{i:n}^s] = \sum_{k,m=1}^{\infty} \sum_{j=0}^{i-1} \sum_{z=0}^{\infty} \frac{\phi(k, m, j, z)}{\exp(-\lambda s/\alpha)(m+k-1)^{-s/\alpha}}. \quad (38)$$

We can compute the moments of the WNB-LA order statistic by the equation (38) or by numerical integration. The moments, variance, skewness and kurtosis of the WNB-LA order statistics listed in Table 4 for  $\lambda = 0$ ,  $\alpha = 1.5, \eta = 2, \theta = 0.9, n = 5$  and  $s = 3$ .

Table 4: Moments, Variance, Skewness and Kurtosis of the WNB-LA statistic for  $\lambda = 0, \alpha = 1.5, \eta = 2, \theta = 0.9, n = 5$  and  $s = 3$

| Order statistic | $Y_{1:5}$ | $Y_{2:5}$ | $Y_{3:5}$ | $Y_{4:5}$ | $Y_{5:5}$ |
|-----------------|-----------|-----------|-----------|-----------|-----------|
| $k = 1$         | 0.4672    | 0.4749    | 0.2518    | 0.0721    | 0.0088    |
| $k = 2$         | 0.3036    | 0.3993    | 0.2473    | 0.0789    | 0.0105    |
| $k = 3$         | 0.2412    | 0.3799    | 0.2653    | 0.0924    | 0.0131    |
| $k = 4$         | 0.2210    | 0.3987    | 0.3067    | 0.1149    | 0.0173    |
| Variance        | 0.0853    | 0.1738    | 0.1839    | 0.0737    | 0.0104    |
| Skewness        | 0.7868    | 0.3480    | 1.4006    | 3.8049    | 12.1347   |
| Kurtosis        | 3.4247    | 2.1443    | 3.5915    | 16.7051   | 156.1324  |

### 2.2.5. Entropy

From (26) and (27), the entropy of a random variable  $Y$  with a WNB-FA is given by

$$I_R(\rho) = \frac{1}{1-\rho} \left\{ \rho \log \left( \frac{\theta}{1 - (1+\eta\theta)^{-1/\eta\alpha}} \right) + (\rho-1) \left( \log \alpha + \frac{\lambda}{\alpha} \right) + \log \left[ \Gamma \left( \frac{\rho(\alpha-1)+1}{\alpha} \right) \right] \right. \\ \left. \log \left[ \sum_{k=0}^{\infty} (-1)^k \frac{(\eta\theta)^k}{(\rho+k) \frac{\rho(\alpha-1)+1}{\alpha}} \binom{\rho(1/\eta+1)}{k} \right] \right\}.$$

Figure 5 illustrates the surviving functions of the WNB-FA, WNB-LA and Weibull distributions, which shows the flexibility afforded by our proposal.

### 3. Inference

Let  $y = (y_1, y_2, \dots, y_n)$  be a random sample of the WNB-FA distribution with unknown parameter vector  $\xi = (\alpha, \lambda, \eta, \theta)$ . The log-likelihood function  $\ell = \ell(\xi)$  is given by

$$\ell = n[\log \theta + \log \alpha + \lambda - \log(1 - (1+\eta\theta)^{-1/\eta})] + (\alpha-1) \sum_{i=1}^n \log y_i - e^\lambda \sum_{i=1}^n y_i^\alpha - \left( \frac{1}{\eta} + 1 \right) \sum_{i=1}^n \log[1 + \eta\theta(1 - \exp(-y_i^\alpha e^\lambda))] \quad (39)$$

The score function  $U(\xi) = (\partial\ell/\partial\alpha, \partial\ell/\partial\lambda, \partial\ell/\partial\eta, \partial\ell/\partial\theta)^\top$  has components given by

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log y_i - e^\lambda \sum_{i=1}^n y_i^\alpha \log y_i - (\eta+1) \sum_{i=1}^n \theta e^\lambda T_{1,1,1,1}^{(i)}$$

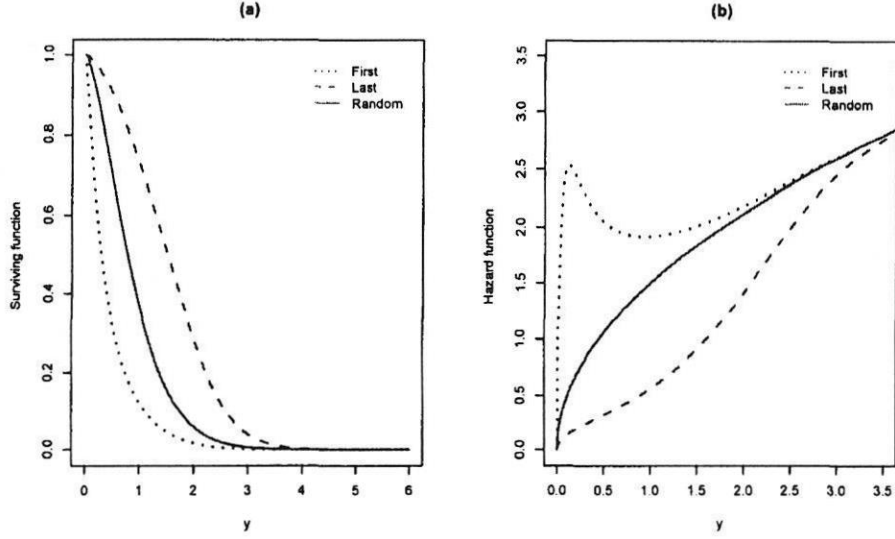


Figure 5: (a) Survival function (b) Hazard function of the WNB-FA, WNB-LA and Weibull distributions with the  $\beta = 0.8$ ,  $\alpha = 1.5$ ,  $\eta = 10$  e  $\theta = 2$ .

$$\begin{aligned}
 \frac{\partial \ell}{\partial \lambda} &= n - e^\lambda \sum_{i=1}^n y_i^\alpha - (1 + \eta) \sum_{i=1}^n \theta e^\lambda T_{1,0,1,1}^{(i)} \\
 \frac{\partial \ell}{\partial \theta} &= \frac{n\theta(1 + \eta\theta)^{-1/\eta} \log(1 + \eta\theta)}{\eta^2(1 - (1 + \eta\theta)^{-1/\eta})} + \sum_{i=1}^n \frac{1}{\eta^2} \log(T_{0,0,-1,0}^{(i)}) - \left(1 + \frac{1}{\eta^2}\right) \theta(T_{0,0,1,0}^{(i)} - T_{0,0,1,1}^{(i)}) \\
 \frac{\partial \ell}{\partial \eta} &= \frac{n}{\theta} - \frac{n(1 + \eta\theta)^{-1/\eta+1}}{1 - (1 + \eta\theta)^{-1/\eta}} - (1 + \eta) \sum_{i=1}^n (T_{0,0,1,0}^{(i)} - T_{0,0,1,1}^{(i)})
 \end{aligned}$$

Where  $T_{j,k,l,m}^{(i)} = y_i^{\alpha j} \{\log y_i\}^k (1 + \eta\theta(1 - \exp(-y_i^\alpha e^\lambda)))^{-l} \exp(-y_i^\alpha e^\lambda)^m$ , for  $j, k, l, m \in \{0, 1, 2\}$  and  $i = 1 \dots n$ .  
 n The maximum likelihood estimate (MLE)  $\hat{\xi}$  of  $\xi$  can be numerically determined from the nonlinear equations  $U(\xi) = 0$ .

For interval estimation and hypothesis tests on the parameter of WNB-FA model, we can obtain the  $4 \times 4$  observed information matrix  $J_n = J_n(\xi)$  given by

$$J_n = \begin{pmatrix} J_{\alpha\alpha} & J_{\alpha\lambda} & J_{\alpha\eta} & J_{\alpha\theta} \\ J_{\lambda\alpha} & J_{\lambda\lambda} & J_{\lambda\eta} & J_{\lambda\theta} \\ J_{\eta\alpha} & J_{\eta\lambda} & J_{\eta\eta} & J_{\eta\theta} \\ J_{\theta\alpha} & J_{\theta\lambda} & J_{\theta\eta} & J_{\theta\theta} \end{pmatrix}$$

where

$$\begin{aligned}
 -J_{\alpha\alpha} &= -\frac{n}{\alpha^2} - e^\lambda \sum_{i=1}^n y_i (\log y_i)^2 + (\eta + 1)\theta e^{2\lambda} \sum_{i=1}^n y_i (T_{1,2,1,1}^{(i)} - \eta\theta T_{1,2,2,2}^{(i)}), \\
 -J_{\alpha\lambda} &= -e^\lambda \sum_{i=1}^n y_i^\alpha \log y_i - (\eta + 1)\theta e^\lambda \sum_{i=1}^n y_i (T_{0,1,1,1}^{(i)} - e^\lambda T_{1,1,1,1}^{(i)} - \eta\theta e^\lambda T_{2,1,2,2}^{(i)}),
 \end{aligned}$$

$$\begin{aligned}
-J_{\alpha\eta} &= -\theta(\eta+1)e^\lambda \sum_{i=1}^n y_i [T_{0,1,1,1}^{(i)} - (\eta+1)\theta(T_{0,1,2,1}^{(i)} - T_{0,1,2,2}^{(i)})], \\
-J_{\alpha\theta} &= -(\eta+1)e^\lambda \sum_{i=1}^n y_i (T_{0,1,1,1}^{(i)} - \eta\theta(T_{0,1,2,1}^{(i)} - T_{0,1,2,2}^{(i)})), \\
-J_{\lambda\lambda} &= -e^\lambda \sum_{i=1}^n y_i^\alpha - (\eta+1)\theta \sum_{i=1}^n e^\lambda (T_{1,0,1,1}^{(i)} - e^\lambda T_{2,0,1,1}^{(i)} - \eta\theta(T_{1,0,2,1}^{(i)} - e^\lambda T_{2,0,2,2}^{(i)})), \\
-J_{\lambda\eta} &= -\theta e^\lambda \sum_{i=1}^n T_{1,0,1,1}^{(i)} - \theta(\eta+1)(T_{1,0,2,1}^{(i)} - T_{1,0,2,2}^{(i)}), \\
-J_{\lambda\theta} &= -(\eta+1)e^\lambda \sum_{i=1}^n T_{1,0,1,1}^{(i)} - \eta\theta(T_{1,0,2,1}^{(i)} - T_{1,0,2,2}^{(i)}), \\
-J_{\eta\eta} &= \frac{n\theta^2}{\eta^2[(1+\eta\theta)^{-1/\eta} - 1]} + \frac{n\theta \log(1+\eta\theta)[(1+\eta\theta)^{1/\eta}(2\eta - \theta \log(1+\eta\theta)) - 2\eta]}{\eta^4[(1+\eta\theta)^{1/\eta} - 1]^2} \\
&\quad + \frac{1}{\eta^3} \sum_{i=1}^n \theta(\eta-1)(T_{0,0,1,0}^{(i)} - T_{0,0,1,1}^{(i)}) - 2 \log T_{0,0,1,0}^{(i)} + (1+1/\eta^2)\theta^2 \sum_{i=1}^n T_{0,0,2,0}^{(i)} - 2T_{0,0,2,1}^{(i)} + T_{0,0,2,2}^{(i)}, \\
-J_{\eta\theta} &= \frac{n\theta(\eta(1+\eta\theta)^{-1/\eta} + 1) \log(1+\eta\theta)}{\eta^2((1+\eta\theta)^{-1/\eta} - 1)^2(1+\eta\theta)} - \left(1 - \frac{1}{\eta} + \frac{1}{\eta^2}\right) \sum_{i=1}^n T_{0,0,1,0}^{(i)} - T_{0,0,1,1}^{(i)} \\
&\quad - \theta \left(\eta + \frac{1}{\eta}\right) \sum_{i=1}^n T_{0,0,2,0}^{(i)} - 2T_{0,0,2,1}^{(i)} + T_{0,0,2,2}^{(i)}, \\
-J_{\theta\theta} &= -\frac{n}{\theta^2} + \frac{n(1+\eta\theta)^{-(2/\eta+2)}[(1+\eta)(1+\eta\theta)^{1/\eta}(1 - (1+\eta\theta)^{-1/\eta}) + 1]}{(1 - (1+\eta\theta)^{-1/\eta})^2} \\
&\quad + \eta(\eta+1) \sum_{i=1}^n T_{0,0,2,0}^{(i)} - 2T_{0,0,2,1}^{(i)} + T_{0,0,2,2}^{(i)}.
\end{aligned}$$

Similarly, let  $\mathbf{y} = (y_1, y_2, \dots, y_n)$  be a random sample of the WNB-LA distribution with unknown parameter vector  $\xi = (\alpha, \lambda, \eta, \theta)$ . The log-likelihood function  $\ell = \ell(\xi)$  is given by

$$\ell = n[\log \theta + \log \alpha + \lambda - \log(1 - (1+\eta\theta)^{-1/\eta})] + (\alpha-1) \sum_{i=1}^n \log y_i - e^\lambda \sum_{i=1}^n y_i^\alpha - \left(\frac{1}{\eta} + 1\right) \sum_{i=1}^n \log[1 + \eta\theta \exp(-y_i^\alpha e^\lambda)] \quad (40)$$

The score function  $U(\xi) = (\partial\ell/\partial\alpha, \partial\ell/\partial\lambda, \partial\ell/\partial\eta, \partial\ell/\partial\theta)^\top$  has components given by

$$\begin{aligned}
\frac{\partial\ell}{\partial\alpha} &= \frac{n}{\alpha} + \sum_{i=1}^n \log y_i - e^\lambda \sum_{i=1}^n y_i^\alpha \log y_i + (\eta+1) \sum_{i=1}^n \theta e^\lambda S_{1,1,1,1}^{(i)} \\
\frac{\partial\ell}{\partial\lambda} &= n - e^\lambda \sum_{i=1}^n y_i^\alpha + (1+\eta) \sum_{i=1}^n \theta e^\lambda S_{1,0,1,1}^{(i)} \\
\frac{\partial\ell}{\partial\theta} &= \frac{n\theta(1+\eta\theta)^{-1/\eta} \log(1+\eta\theta)}{\eta^2(1 - (1+\eta\theta)^{-1/\eta})} + \sum_{i=1}^n \frac{1}{\eta^2} \log(S_{0,0,-1,0}^{(i)}) - \left(1 + \frac{1}{\eta^2}\right) \theta(S_{0,0,1,0}^{(i)} - S_{0,0,1,1}^{(i)}) \\
\frac{\partial\ell}{\partial\eta} &= \frac{n}{\theta} - \frac{n(1+\eta\theta)^{-1/\eta+1}}{1 - (1+\eta\theta)^{-1/\eta}} - (1+\eta) \sum_{i=1}^n (S_{0,0,1,0}^{(i)} - S_{0,0,1,1}^{(i)})
\end{aligned}$$

Where  $S_{j,k,l,m}^{(i)} = y_i^{\alpha j} \{\log y_i\}^k (1 + \eta\theta \exp(-y_i^\alpha e^\lambda))^{-l} \exp(-y_i^\alpha e^\lambda)^m$ , for  $j, k, l, m \in \{0, 1, 2\}$  and  $i = 1 \dots n$ . The maximum likelihood estimate (MLE)  $\hat{\xi}$  of  $\xi$  can be numerically determined from the nonlinear equations  $U(\xi) = 0$ .

For interval estimation and hypothesis tests on the parameter of WNB-FA model, we can obtain the  $4 \times 4$

observed information matrix  $J_n = J_n(\xi)$  given by

$$J_n = \begin{pmatrix} J_{\alpha\alpha} & J_{\alpha\lambda} & J_{\alpha\eta} & J_{\alpha\theta} \\ J_{\lambda\alpha} & J_{\lambda\lambda} & J_{\lambda\eta} & J_{\lambda\theta} \\ J_{\eta\alpha} & J_{\eta\lambda} & J_{\eta\eta} & J_{\eta\theta} \\ J_{\theta\alpha} & J_{\theta\lambda} & J_{\theta\eta} & J_{\theta\theta} \end{pmatrix}$$

where

$$\begin{aligned} -J_{\alpha\alpha} &= -\frac{n}{\alpha^2} - e^\lambda \sum_{i=1}^n y_i (\log y_i)^2 + (\eta + 1)\theta e^{2\lambda} \sum_{i=1}^n y_i (S_{1,2,1,1}^{(i)} + \eta\theta S_{1,2,2,2}^{(i)}), \\ -J_{\alpha\lambda} &= -e^\lambda \sum_{i=1}^n y_i^2 \log y_i - (\eta + 1)\theta e^\lambda \sum_{i=1}^n y_i (S_{0,1,1,1}^{(i)} - e^\lambda S_{1,1,1,1}^{(i)} + \eta\theta e^\lambda S_{2,1,2,2}^{(i)}), \\ -J_{\alpha\eta} &= -\theta(\eta + 1)e^\lambda \sum_{i=1}^n y_i [S_{0,1,1,1}^{(i)} - (\eta + 1)\theta(S_{0,1,2,1}^{(i)} - S_{0,1,2,2}^{(i)})], \\ -J_{\alpha\theta} &= -(\eta + 1)e^\lambda \sum_{i=1}^n y_i (S_{0,1,1,1}^{(i)} - \eta\theta(S_{0,1,2,1}^{(i)} - S_{0,1,2,2}^{(i)})), \\ -J_{\lambda\lambda} &= -e^\lambda \sum_{i=1}^n y_i^2 + (\eta + 1)\theta \sum_{i=1}^n e^\lambda (S_{1,0,1,1}^{(i)} - e^\lambda S_{2,0,1,1}^{(i)} + \eta\theta(S_{1,0,2,1}^{(i)} - e^\lambda S_{2,0,2,2}^{(i)})), \\ -J_{\lambda\eta} &= \theta e^\lambda \sum_{i=1}^n S_{1,0,1,1}^{(i)} - \theta(\eta + 1)(S_{1,0,2,1}^{(i)} - S_{1,0,2,2}^{(i)}), \\ -J_{\lambda\theta} &= (\eta + 1)e^\lambda \sum_{i=1}^n S_{1,0,1,1}^{(i)} - \eta\theta(S_{1,0,2,1}^{(i)} - S_{1,0,2,2}^{(i)}), \\ -J_{\eta\eta} &= \frac{n\theta^2}{\eta^2[(1 + \eta\theta)^{-1/\eta} - 1]} + \frac{n\theta \log(1 + \eta\theta)[(1 + \eta\theta)^{1/\eta}(2\eta - \theta \log(1 + \eta\theta)) - 2\eta]}{\eta^4[(1 + \eta\theta)^{1/\eta} - 1]^2} \\ &\quad + \frac{1}{\eta^3} \sum_{i=1}^n \theta(\eta - 1)(S_{0,0,1,0}^{(i)} - S_{0,0,1,1}^{(i)}) - 2 \log S_{0,0,1,0}^{(i)} + (1 + 1/\eta^2)\theta^2 \sum_{i=1}^n S_{0,0,2,0}^{(i)} - 2S_{0,0,2,1}^{(i)} + S_{0,0,2,2}^{(i)}, \\ -J_{\eta\theta} &= \frac{n\theta(\eta(1 + \eta\theta)^{-1/\eta} + 1) \log(1 + \eta\theta)}{\eta^2((1 + \eta\theta)^{-1/\eta} - 1)^2(1 + \eta\theta)} - \left(1 - \frac{1}{\eta} + \frac{1}{\eta^2}\right) \sum_{i=1}^n S_{0,0,1,0}^{(i)} - S_{0,0,1,1}^{(i)} \\ &\quad - \theta \left(\eta + \frac{1}{\eta}\right) \sum_{i=1}^n S_{0,0,2,0}^{(i)} - 2S_{0,0,2,1}^{(i)} + S_{0,0,2,2}^{(i)}, \\ -J_{\theta\theta} &= -\frac{n}{\theta^2} + \frac{n(1 + \eta\theta)^{-(2/\eta+2)}[(1 + \eta)(1 + \eta\theta)^{1/\eta}(1 - (1 + \eta\theta)^{-1/\eta}) + 1]}{(1 - (1 + \eta\theta)^{-1/\eta})^2} \\ &\quad + \eta(\eta + 1) \sum_{i=1}^n S_{0,0,2,0}^{(i)} - 2S_{0,0,2,1}^{(i)} + S_{0,0,2,2}^{(i)}. \end{aligned}$$

Let us consider the situation when the failure time  $Y$  in Section 2 is not completely observed and it is subject to right censoring. Let  $C$  denote a censoring time. We define the failure time random variable given by  $Z = \min\{Y, C\}$ , such that, in a sample of size  $n$ , we then observe  $z_i = \min\{y_i, c_i\}$  and  $\delta_i = I(y_i \leq c_i)$  is a censor indicator, i.e., when  $\delta_i = 1$  if  $z_i$  is a failure time and  $\delta_i = 0$  if it is right censored, for  $i = 1, \dots, n$ . Then the likelihood function for the vector of parameters  $\xi = (\alpha, \lambda, \eta, \theta)$  in the non-informative censoring is given by

$$L(\xi) \propto \prod_{i=1}^n [f(z_i; \xi)]^{\delta_i} [S(z_i; \xi)]^{1-\delta_i} \quad (41)$$

where  $f(\cdot, \xi)$  and  $S(\cdot, \xi)$  are density and survival distribution presented in Section 2.

Form the likelihood function in (41), the maximum likelihood estimation of the parameter  $\xi$  can be obtained by maximizing the log-likelihood function  $\ell(\xi) = \log L(\xi; \mathbf{D})$  numerically using a common statistical software.

In this paper, the software R (see, R Development Core Team, 2009) was used to compute maximum likelihood estimates. Under suitable regularity conditions, the asymptotic distribution of the maximum likelihood estimator (MLE)  $\hat{\xi}$  is a multivariate normal with the mean vector  $\xi$  and the covariance matrix, which can be estimated by  $\{-\partial^2 \ell(\xi) / \partial \theta \xi \partial \theta \xi^T\}^{-1}$  evaluated at  $\xi = \hat{\xi}$ . The required second derivatives are computed numerically.

The likelihood ratio (LR) statistic is useful for comparing the WNB distribution with some of its special sub-models. We consider the partition  $\xi = (\xi_1^T, \xi_2^T)$  of the WNB distribution, where  $\xi_1$  is a subset of parameters of interest and  $\xi_2$  is a subset of the remaining parameters. The LR statistic for testing the null hypothesis  $H_0 : \xi_1 = \xi_1^{(0)}$  versus the alternative hypothesis  $H_1 : \xi_1 \neq \xi_1^{(0)}$  is given by  $\omega = 2\{\ell(\hat{\xi}) - \ell(\tilde{\xi})\}$ , where  $\hat{\xi}$  and  $\tilde{\xi}$  are the MLEs under the null and alternative hypothesis, respectively. The statistic  $\omega$  is distributed asymptotically (when  $n \rightarrow \infty$ ) has the distribution  $\chi_k^2$ , where  $k$  is the dimension of the subset of interest  $\xi_1$ . Different models can be compared penalizing over-fitting by using the Akaike information criterion given by  $AIC = -2\ell(\hat{\xi}) + 2\#(\xi)$  and the Schwartz-Bayesian criterion given by  $SBC = -2\ell(\hat{\xi}) + \#(\xi) \log(n)$ , where  $\#(\xi)$  is the number of the parameters. The model with the smallest value of anyone of these criteria (among all considered models) is commonly taken as the preferred model for describing the given data set.

#### 4. Simulation

We conducted a misspecification simulation study in order to comparing the proposed WNB model in the presence of the two different activation schemes, namely WNB-FA and WNB-LA, with the usual Weibull distribution. A failure time data was simulated from the quantile function of a distribution (via inversion method) with the parameters  $\alpha = 2$ ,  $\lambda = -3$ ,  $\eta = 15$   $\theta = 50$ . The censoring time were generated from an uniform distribution  $[0, \tau]$ , where  $\tau$  controlled the proportion censored. In the simulation, we consider 0% and 10% of censored observations, and we took the sample size to be  $n = 50, 100$  and  $200$ . For each configuration, we conducted 1,000 replicates and then we fitted the three models (WNB-FA, WNB-LA and Weibull) and calculate the survival function estimative at the median point, its standard error (SE) and the square root of the mean square error (RMSE). Moreover, we calculate the percentage of samples in which the generating distribution was indicated as the best one according to the criteria AIC, evidentiating that, by using a common comparison model measure, it is possible to discriminate between the competitor distributions.

The Table show that percentage of samples in which the generating distribution was indicated as the best model, according to the AIC criteria, increases as the sample size increases. It also happens when the sample presents a percentage of censored observations. Moreover, the RMSE of the best distribution always were the smallest comparing with the other distributions, as it can be observed in Table 4.

Table 5: percentage of samples in which the adjusted model was indicated as the best model according to the criteria AIC

| Model   | N   | 0% of censored |        |         | 10% of censored |        |         |
|---------|-----|----------------|--------|---------|-----------------|--------|---------|
|         |     | WNB-FA         | WNB-LA | Weibull | WNB-FA          | WNB-LA | Weibull |
| WNB-FA  | 50  | 71.1           | 0.2    | 28.7    | 62.1            | 1.5    | 36.4    |
|         | 100 | 92.9           | 0.0    | 7.1     | 88.6            | 0.6    | 10.8    |
|         | 200 | 100.0          | 0.0    | 0.0     | 97.0            | 0.2    | 2.8     |
| WNB-LA  | 50  | 0.1            | 60.9   | 39.0    | 0.4             | 53.0   | 46.6    |
|         | 100 | 0.0            | 84.7   | 15.3    | 0.0             | 81.9   | 18.1    |
|         | 200 | 0.0            | 96.6   | 3.4     | 0.0             | 98.0   | 2.0     |
| Weibull | 50  | 5.5            | 7.2    | 87.3    | 1.0             | 12.0   | 87.0    |
|         | 100 | 1.7            | 9.8    | 88.5    | 3.3             | 8.1    | 88.6    |
|         | 200 | 0.0            | 0.0    | 100.0   | 4.4             | 5.8    | 89.8    |

#### 5. Application

In this section, two sets of real data have been considered. The first set of data refer to the adult numbers of *Tribolium confusum* cultured at 29°C presented by Eugene *et al.* Eugene *et al.* (2002). We fitted data with the

Table 6: Estimative of the survival function at the median point, SE and RMSE

| Model   | N   | Model   | 0% of censored     |        |        | 10% of censored    |        |        |
|---------|-----|---------|--------------------|--------|--------|--------------------|--------|--------|
|         |     |         | $\hat{S}(y_{med})$ | SE     | RMSE   | $\hat{S}(y_{med})$ | SE     | RMSE   |
| WNB-FA  | 50  | WNB-FA  | 0.553              | 0.0426 | 0.0678 | 0.580              | 0.0455 | 0.0918 |
|         |     | WNB-LA  | 0.592              | 0.0531 | 0.4085 | 0.621              | 0.0518 | 0.3799 |
|         |     | Weibull | 0.613              | 0.0362 | 0.3648 | 0.639              | 0.0360 | 0.3386 |
|         | 100 | WNB-FA  | 0.543              | 0.0300 | 0.0527 | 0.572              | 0.0294 | 0.0778 |
|         |     | WNB-LA  | 0.573              | 0.0444 | 0.4264 | 0.605              | 0.0418 | 0.3941 |
|         |     | Weibull | 0.601              | 0.0273 | 0.3764 | 0.629              | 0.0248 | 0.3475 |
|         | 200 | WNB-FA  | 0.515              | 0.0259 | 0.0239 | 0.535              | 0.0272 | 0.0339 |
|         |     | WNB-LA  | 0.519              | 0.0282 | 0.4860 | 0.541              | 0.0381 | 0.4062 |
|         |     | Weibull | 0.568              | 0.0246 | 0.4087 | 0.618              | 0.0246 | 0.3387 |
| WNB-LA  | 50  | WNB-FA  | 0.469              | 0.0595 | 0.4702 | 0.471              | 0.0593 | 0.4720 |
|         |     | WNB-LA  | 0.525              | 0.0534 | 0.0591 | 0.525              | 0.0566 | 0.0617 |
|         |     | Weibull | 0.472              | 0.0569 | 0.4519 | 0.474              | 0.0576 | 0.4533 |
|         | 100 | WNB-FA  | 0.460              | 0.0364 | 0.4583 | 0.462              | 0.0402 | 0.4611 |
|         |     | WNB-LA  | 0.515              | 0.0354 | 0.0385 | 0.518              | 0.0386 | 0.0427 |
|         |     | Weibull | 0.460              | 0.0364 | 0.4372 | 0.463              | 0.0400 | 0.4403 |
|         | 200 | WNB-FA  | 0.455              | 0.0235 | 0.4529 | 0.455              | 0.0267 | 0.4533 |
|         |     | WNB-LA  | 0.512              | 0.0227 | 0.0255 | 0.513              | 0.0256 | 0.0287 |
|         |     | Weibull | 0.455              | 0.0235 | 0.4318 | 0.455              | 0.0267 | 0.4322 |
| Weibull | 50  | WNB-FA  | 0.463              | 0.0618 | 0.3826 | 0.470              | 0.0634 | 0.3901 |
|         |     | WNB-LA  | 0.500              | 0.0631 | 0.4197 | 0.506              | 0.0663 | 0.4142 |
|         |     | Weibull | 0.480              | 0.0558 | 0.0593 | 0.485              | 0.0589 | 0.0609 |
|         | 100 | WNB-FA  | 0.472              | 0.0447 | 0.3890 | 0.471              | 0.0440 | 0.3883 |
|         |     | WNB-LA  | 0.502              | 0.0508 | 0.4160 | 0.500              | 0.0493 | 0.4174 |
|         |     | Weibull | 0.479              | 0.0414 | 0.0464 | 0.480              | 0.0409 | 0.0455 |
|         | 200 | WNB-FA  | 0.446              | 0.0140 | 0.3608 | 0.474              | 0.0314 | 0.3904 |
|         |     | WNB-LA  | 0.496              | 0.0260 | 0.4194 | 0.496              | 0.0417 | 0.4214 |
|         |     | Weibull | 0.446              | 0.0140 | 0.0552 | 0.478              | 0.0301 | 0.0372 |

proposed models (WNB-FA and WNB-LA) and its sub-models (WG-FA, WP-FA, WG-LA, WP-LA and Weibull), the AIC and SBC values of the fitted models were compared among themselves. Moreover these values were compared with the Kw-normal and Beta normal distribution which were fitted by Cordeiro & Castro (2011) too. According to the AIC and SBC criteria, the WNB-FA and WG-FA schemes outperform the remaining adjusted distributions, but the WNB-FA stands out the best one.

The fitted models superimposed on the histogram and the empirical cumulative distribution function of the data in Figure 6 reinforce the result in Table 5 for WNB-FA model. The mean absolute deviation (MAD) between the observed and expected frequencies reaches the minimum value for the WNB-FA model. The MLEs and their standard errors in brackets of the parameters for the WNB-FA model are given by  $\hat{\alpha} = 6.1591$  (0.01915),  $\hat{\lambda} = -33.7471$ , (0.10896),  $\hat{\eta} = 2.7762$  (0.03166) and  $\hat{\theta} = 50.0055$  (1.17701). We note that the standard errors of estimates of the parameters  $\alpha$ ,  $\lambda$ ,  $\eta$  and  $\theta$  of the WNB-FA model are small.

Using the LR statistics to test the null hypothesis  $H_0$ : WG-FA,  $H_0$ : WP-FA,  $H_0$ : Weibull against the alternative hypothesis  $H_1$ : WNB-FA is 9.698 ( $p$ -value = 0.021), 73.390 ( $p$ -value < 0.001) and 78.020 ( $p$  < 0.001) respectively. So choosing the usual significance level (5%), we reject both null hypotheses in favor of the WNB-FA distribution.

The second data set is obtained from Smith and Naylor (1987) represent the breaking strength of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. Unfortunately, the measurement units are not given in the paper. The data set is given by: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.074, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62,

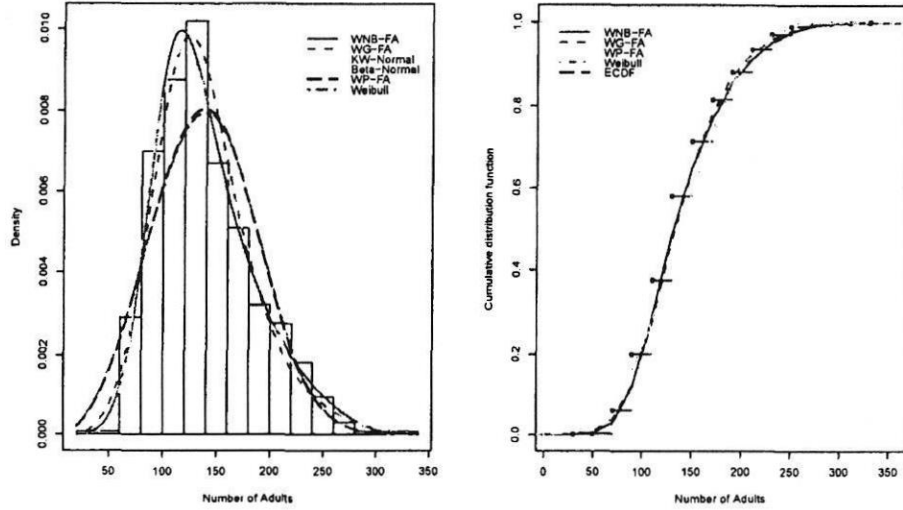


Figure 6: The adjusted density and cumulative distribution functions of the fitted models superimposed on the histogram and the empirical cumulative distribution function of the first dataset.

Table 7: AIC and SBC for adjusted models

| Modelo      | Number of parameters | $\ell(\hat{\xi})$ | AIC    | SBC    |
|-------------|----------------------|-------------------|--------|--------|
| WNB-FA      | 4                    | -3566.1           | 7140.2 | 7158.3 |
| WG-FA       | 3                    | -3570.9           | 7147.9 | 7161.5 |
| Beta normal | 4                    | -3584.5           | 7176.9 | 7195.0 |
| Kw-normal   | 4                    | -3584.7           | 7177.4 | 7195.5 |
| WP-FA       | 3                    | -3602.8           | 7211.6 | 7225.1 |
| Weibull     | 2                    | -3605.1           | 7214.2 | 7223.3 |
| WG-LA       | 3                    | -3605.1           | 7216.1 | 7229.7 |
| WP-LA       | 3                    | -3605.1           | 7216.2 | 7229.8 |
| WNB-LA      | 4                    | -3605.1           | 7218.2 | 7236.3 |

1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89.

Recently, this set of data was adjusted to the Beta Generalized Exponential model introduced by Barreto-Souza *et al.* (2010). Table 9 presents the logarithm of maximum likelihood estimates and the AIC and SBC values of the adjusted models. According to the AIC, the particular case WG under the last activation scheme (WG-LA) stands out as the best one.

Using the LR statistics to test the null hypothesis  $H_0$ : WG-LA,  $H_0$ : WP-LA  $H_0$ : Weibull against the alternative hypothesis  $H_1$ : WNB-LA is 1.305 ( $p$ -value = 0.253), 4.187 ( $p$  = 0.041) and 7.652 ( $p$ -value = 0.022) respectively. So choosing the usual significance level (5%), the distribution and WNB-LA, WG-LA are not significantly different from each other but the Weibull and WP-LA distributions are significantly different from the WNB-LA. Figure 7 show the adjusted density and cumulative distribution functions of the fitted models superimposed on the histogram and the empirical cumulative distribution function, reinforcing the Table 5 results.

Table 8: Observed and expected frequencies of adult for *T.confusum* cultured at 29°C and MAD between the frequencies

| Adult number | Observed | Expected  |             |        |        |
|--------------|----------|-----------|-------------|--------|--------|
|              |          | Kw-normal | Beta normal | WG-FA  | WNB-FA |
| 30           | 1        | 0.21      | 0.22        | 1.40   | 0.47   |
| 50           | 1        | 5.77      | 5.67        | 10.31  | 6.49   |
| 70           | 40       | 37.43     | 37.39       | 36.47  | 33.95  |
| 90           | 96       | 91.85     | 93.70       | 82.18  | 92.02  |
| 110          | 122      | 125.53    | 127.54      | 125.27 | 135.11 |
| 130          | 140      | 123.95    | 123.73      | 133.97 | 127.28 |
| 150          | 92       | 102.41    | 100.77      | 109.17 | 97.37  |
| 170          | 70       | 75.95     | 74.30       | 75.33  | 69.42  |
| 190          | 44       | 52.00     | 50.97       | 47.89  | 48.43  |
| 210          | 38       | 33.18     | 32.80       | 29.42  | 33.22  |
| 230          | 25       | 19.80     | 19.86       | 17.75  | 21.98  |
| 250          | 13       | 11.06     | 11.32       | 10.44  | 13.46  |
| 270          | 4        | 5.79      | 6.08        | 5.80   | 7.09   |
| 290          | 1        | 2.84      | 3.08        | 2.90   | 2.86   |
| 310          | 1        | 1.31      | 1.47        | 1.21   | 0.75   |
| 330          | 2        | 0.56      | 0.66        | 0.38   | 0.10   |
| Total        | 690      | 689.6     | 689.5       | 689.9  | 690    |
| MAD          |          | 4.60      | 4.39        | 5.42   | 4.23   |

Table 9: Maximized log-likelihoods and AIC value for the adjusted distributions.

| Modelo  | Number of parameters | $\ell(\hat{\xi})$ | AIC  | SBC  |
|---------|----------------------|-------------------|------|------|
| WNB-LA  | 4                    | -11.4             | 30.8 | 39.3 |
| WG-LA   | 3                    | -12.0             | 30.1 | 36.5 |
| WP-LA   | 3                    | -13.5             | 32.9 | 39.4 |
| Weibull | 2                    | -15.2             | 34.4 | 38.7 |
| WG-FA   | 3                    | -15.2             | 36.4 | 43.0 |
| WNB-FA  | 4                    | -15.2             | 38.4 | 47.0 |
| WP-FA   | 3                    | -15.2             | 36.4 | 42.9 |

Table 10: Parameters estimates for the models WNB-LA, WG-LA, WP-LA and Weibull

| Modelo  | $\alpha$ | $\lambda$ | $\eta$ | $\theta$ |
|---------|----------|-----------|--------|----------|
| WNB-LA  | 2.00     | 1.02      | 2.25   | 1336.9   |
| WG-LA   | 3.20     | -0.36     | -      | 15.64    |
| WP-LA   | 4.48     | -1.64     | -      | 2.38     |
| Weibull | 5.78     | -2.82     | -      | -        |

## 6. Final comments

In this paper we propose the WNB distribution under a latent activation structure, which generalize several usual lifetime distributions. The flexibility of the WNB distribution is readily seen when we consider its hazard function which can accommodate various shapes. We derive various standard mathematical properties of the proposed model, which the expansions for the moments having a closed form for some special cases. We used the AIC, BSC and LR statistics to compare the WNB model with the its special sub-model. Finally, we fitted the WNB model to two real datasets to show the potentiality of the new distribution.

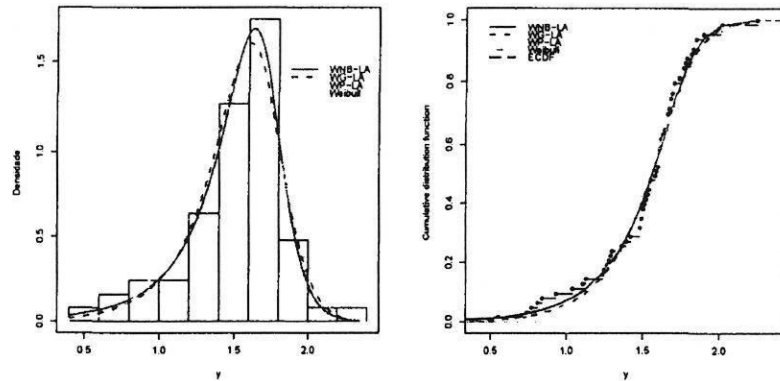


Figure 7: The adjusted density and cumulative distribution functions of the fitted models superimposed on the histogram and the empirical cumulative distribution function of the second dataset.

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