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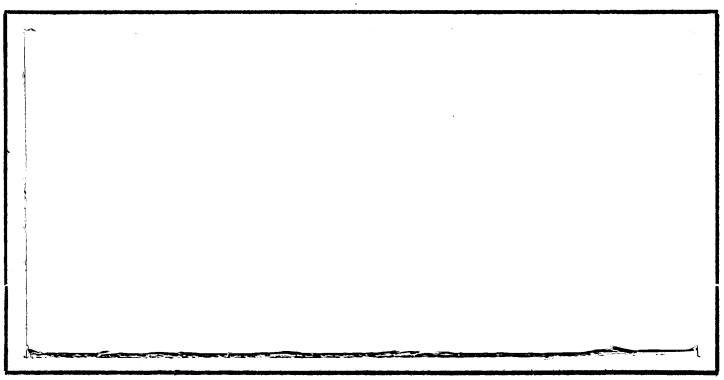

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USE OF BAYESIAN METHODS IN ACCELERATED LIFE TESTS WITH AN WEIBULL DISTRIBUTION AND THE POWER RULE MODEL

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SUMMARY

"We present a Bayesian analysis for accelerated life tests considering a Weibull distribution for the life times and the power rule model for the stress-response relation, considering a noninformative Jeffreys prior density for the parameters. We illustrate the proposed Bayesian method considering a data set introduced by Nelson (1972)".

KEY WORDS: Accelerated Life Tests, Weibull Distribution, Power Rule Model, Bayesian Analysis.

1. INTRODUCTION

Accelerated life tests have been used for a long time by industrial researchers when product life under normal operating conditions is very long, but estimates of length of life are needed in a relatively short time (see for exemple, Mann, Schafer and Singpurwalla, 1974; or Nelson, 1990). Let us consider T a random variable denoting the life time of a device with a Weibull density,

$$f(t;\lambda,p) = \lambda p(\lambda t)^{p-1}\exp\{-(\lambda t)^p\} \qquad (1)$$

where $t > 0, \lambda, p > 0$ and assume a stress variable V changing the scale parameter λ but with common shape parameter p for all stress levels.

Considering k levels of the stress variable V, assume the power rule model given by,

$$\lambda_i^{-1} = \frac{\alpha}{V_i^{\beta}} \qquad (2)$$

where $i=1,2,\ldots,k$, $\alpha>0$ and $-\infty<\beta<\infty$ are unknown parameters. With k stress levels, we assume a type II censoring mechanism, that is, the experiment terminates when we observe r_i failures for each level i of stress. Thus, with n_i units at the beginning of each test with stress V_i, we have the ordered uncensored observations given by $t_{1i},t_{2i},\ldots,t_{r_i i}$ and $n_i - r_i$ censored observations equal to $t_{r_i i}$, $i=1,2,\ldots,k$.

Considering the data of k stress levels $V_1,V_2,\ldots,V_k$ taken at random, the likelihood function for α, β and p is given by

$$L(\alpha,\beta,p) = \frac{r!}{\alpha^{pr}}\Big(\prod_{i=1}^{k} V_i^{\beta p r_i}\Big)\Big(\prod_{i=1}^{k}\prod_{j=1}^{r_i} t_{ij}^{p-1}\Big) \times$$

$$\times \exp\Big\{-\frac{1}{\alpha^p}\sum_{i=1}^{k} V_i^{\beta p} A_i(p)\Big\} \qquad (3)$$

where $A_i(p) = \sum_{j=1}^{r_i} t_{ij}^p + (n_i-r_i)t_{ir_i}^p$ and $r = \sum_{i=1}^{k} r_i$.

2. A BAYESIAN ANALYSIS ASSUMING α, β AND p UNKNOWN

The prior density for α, β and p can be written in the form $\pi(\alpha,\beta,p) = \pi(\alpha,\beta|p)\pi_0(p)$. Using the Jeffreys multiparameter rule (see for example, Box and Tiao, 1973), that is, $\pi(\alpha,\beta|p) \propto \{\det I_p(\alpha,\beta)\}^{1/2}$, where $I_p(\alpha,\beta)$ is the Fisher information matrix given p, and a locally uniform prior for $\ln p$ (that is, $\pi_0(p) \propto p^{-1}$, $p>0$), we consider the noninformative prior for α, β and p given by

$$\pi(\alpha,\beta,p) \propto p/\alpha, \quad -\infty<\beta<\infty, \ \alpha,p>0. \qquad (4)$$

The joint posterior density for α, β and p is given by,

$$\pi(\alpha,\beta,p|\text{data}) \propto \frac{\Gamma^{r+1}}{\alpha^{pr+1}}\Big(\prod_{i=1}^{k} V_i^{\beta p r_i}\Big)\Big(\prod_{i=1}^{k}\prod_{j=1}^{r_i} t_{ij}^{p-1}\Big) \times$$

$$\times \exp\{-\frac{1}{\alpha^p}\sum_{i=1}^{k} V_i^{\beta p}A_i(p)\} \qquad (5)$$

where $-\infty < \beta < \infty$ and $\alpha, p > 0$.

An approximate marginal posterior density for p, using the Laplace's method for approximation of integrals (see for example, Tierney and Kadane, 1986; or Kass, Tierney and Kadane, 1990), is given by

$$\pi(p|data) \propto \frac{(pr+1)^{r-1}e^{-\frac{(pr+1)}{p}}\hat{\beta}^{pr}(\prod_{i=1}^{k} V_i)(\prod_{j=1}^{k}\prod_{j=1}^{r_i} t_{ij}^{p-1})}{\{\sum_{i=1}^{k} V_i^{\hat{\beta}p}A_i(p)\}^r} \times$$

$$\times \{\frac{\sum_{i=1}^{k} V_i^{\hat{\beta}p}(\ell n V_i)^2 A_i(p)}{\sum_{i=1}^{k} V_i^{\hat{\beta}p}A_i(p)} - (\frac{\sum_{i=1}^{k} V_i^{\hat{\beta}p}(\ell n V_i)A_i(p)}{\sum_{i=1}^{k} V_i^{\hat{\beta}p}A_i(p)})^2\}^{-1/2} \qquad (6)$$

where $p > 0$ and $\hat{\beta}$ is given by

$$\sum_{i=1}^{k} r_i \ell n V_i = \frac{(pr+1)\sum_{i=1}^{k}(V_i^{\hat{\beta}})^p(\ell n V_i)A_i(p)}{p\sum_{i=1}^{k}(V_i^{\hat{\beta}})^p A_i(p)}.$$

We also can use de Laplace's method to find the marginal posterior densities for α and β.

Usually, industrial researchers have interest to obtain inferences on the mean life time under the normal use stress level V_1 given by $\theta_1 = \Gamma(1+1/p)/\lambda_1$, where $\lambda_1^{-1} = \alpha/V_1^{\beta}$. Considering the transformation of variables $\theta_1 = \alpha\Gamma(1+1/p)/V_1^{\beta}$, $\beta = \beta$ and $p = p$, we obtain from (5); the joint posterior density for θ_1, β and p. Using the Laplace's method, we find an approximate joint marginal posterior density for θ_1 and p given by

$$\pi(\theta_1, p|data) \propto \frac{p^r\{\Gamma(1+1/p)\}^{p(r-1/2)}}{\theta_1^{p(r-1/2)+1}}(\prod_{i=1}^{k}\prod_{j=1}^{r_i} t_{ij}^{p-1}) \times$$

$$\times \frac{\exp\{\hat{\beta}p \sum_{i=1}^{k} r_i \ell n V_i - \hat{\beta}p r \ell n V_1 - [\frac{\Gamma(1+1/p)}{\theta_1}]^p \sum_{i=1}^{k} A_i(p)(\frac{V_i}{V_1})^{\hat{\beta}p}\}}{\{\sum_{i=1}^{k} A_i(p)(\frac{V_i}{V_1})^{\hat{\beta}p}(\ell n(\frac{V_i}{V_1}))^2\}^{1/2}} \qquad (7)$$

where $\hat{\beta}$ is given by

$$(\frac{\theta_1}{\Gamma(1+1/p)})(\sum_{i=1}^{k} r_i \ell n V_i - r \ell n V_1) = \sum_{i=1}^{k} A_i(p)(\frac{V_i}{V_1})^{\hat{\beta}p} \ell n(\frac{V_i}{V_1}),$$

and θ_1, $p > 0$.

3. AN EXAMPLE

Consider the data of Table 1 representing the times of breakdown of an insulating fluid subject to various constant elevated test voltages (data given in Nelson, 1972). Elevated test voltages were employed to save time in collecting the breakdown data and the work was carried out in a laboratory with a pair of parallel plate electrodes of a certain area and separation. The electrical stress is reported as a voltage.

Assuming an Weibull distribution with density (1) for the life data of Table 1, and the power rule model (2), we have in Figure 1 the graph of the approximate marginal posterior density for p given in (6). The mode of the marginal posterior density for p is given by $\tilde{p} = 0.86$. A 95% HPD interval for p is given by (0.74; 0.99). In Figure 2, we have contour plots for the joint marginal posterior density for $\ell n p$ and $\ell n \theta_1$ obtained from (7).

i	VOLTAGE (V_i)	n_i	BREAKDOWN TIMES (t_{ji})
1	26	3	5.79, 1579.52, 2323.70
2	28	5	68.85, 108.29, 110.29, 426.07, 1067.60
3	30	11	7.74, 17.05, 20.46, 21.02, 22.66, 43.40, 47.30, 139.07, 144.12, 175.88, 194.90
4	32	15	0.27, 0.40, 0.69, 0.79, 2.75, 3.91, 9.88, 13.95, 15.93, 27.80, 53.24, 82.85, 89.29, 100.58, 215.10
5	34	19	0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.00, 31.75, 32.52, 33.91, 36.71, 72.89
6	36	15	0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.90, 3.67, 3.99, 5.35, 13.77, 25.50
7	38	8	0.09, 0.39, 0.47, 0.73, 0.74, 1.13, 1.40, 2.38

TABLE 1 – Breakdown Times in Minutes for Insulating Fluids.

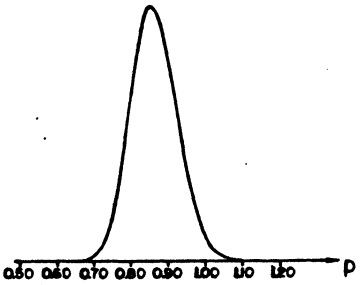

Figure 1 : Marginal Posterior Density for p

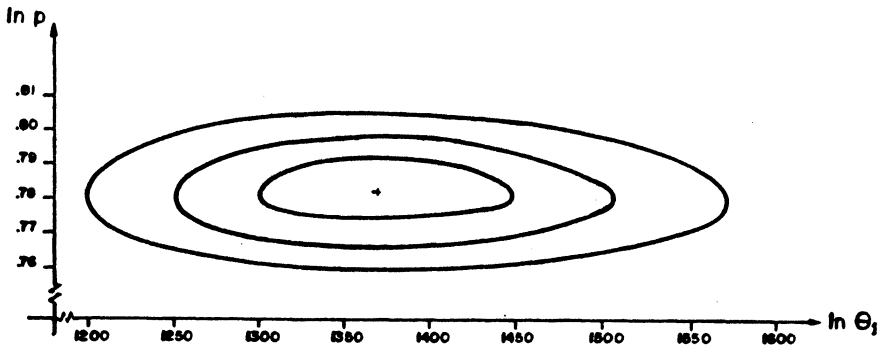

Figure 2 : Marginal Posterior Density for ln θ_1 and ln p.

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