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Infinite Dimensional Dynamics Described by Ordinary Differential Equations

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INTRODUCTION

Due to the possible complexity of infinite dimensional dynamical systems, it is of great interest for mathematicians and applied scientists, to identify those problems that can be regarded as ordinary differential equations.

The purpose of this work is to identify a class of reaction diffusion problems for which we are able to reduce the study of the large time dynamics to the study of the large time dynamics of an ordinary differential equation. An abstract theory is developed to include several reaction diffusion problems with large diffusion and some for which the diffusion coefficient is large in parts of the region and becomes small in some other parts of the region.

The techniques are the invariant manifold theory, a detailed analysis of an eigenvalue problem, some a priori estimates and the converse theory for Liapunov stability as in Yoshizawa [1966].

This problem has been studied by several authors, among them we mention Conway Hoff and Smoller [1978], Hale [1986], Hale and Rocha [1987a,b], Hale and Carvalho [1991], Fusco [1987] and Carvalho and Pereira [1991]. Most of these authors worked in spaces with topology stronger than the uniform topology. By working in energy spaces (the half fractional power space associated with the elliptic operator), we are able to derive conclusions about the global asymptotic dynamics of parabolic problems.

As we will see throughout the text, there are several interesting parameter dependent reaction diffusion problems for which the associated elliptic operator has a finite number of eigenvalues that converge and the remaining diverge to $-\infty$, as the parameter varies. This is an essential ingredient in obtaining invariant manifolds that are graphs over a finite dimensional space and that become flat, as the parameter varies.

The existence stability and smoothness of such invariant manifolds is established in section **I.1**, Theorem 1.2. This result is essential to the proof that, in some cases, the dynamics of the partial differential equation is equivalent to the dynamics of the ordinary differential equation.

We are also concerned with identifying the ordinary differential equation that determines the asymptotic behavior (as seen in the previous works). To that end, we will also need to know what happens to the eigenfunctions associated to those eigenvalues that stay bounded. More specifically, these eigenfunctions must converge in some sense to a step function which is associated to an element of an euclidean space.

To carry out this project we must obtain some a priori estimates which are usually obtained by the use of an energy functional. For this reason we choose to work in energy spaces, where a well known concept of dissipation is available.

The usual method of obtaining a priori estimates for the attractors through energy functionals is not essential in any way in these proofs. They are presented in this way only because it is a method which applies to a very wide class of problems. The results apply whatever method is used to obtain the a priori estimates.

To emphasize how powerful such reductions can be, we remark that the question of finding equilibrium points (or even determining their number) for a parabolic problem with a polynomial nonlinearity may be very hard or even untouchable. However, if this problem can be associated with an ordinary differential equation this question has a much better chance of being answered.

This paper is divided into three main parts, each of them containing several sections.

In Part **I** we introduce the basic results that will be used throughout the text and a few simple immediate applications of these results. The examples considered in this part are reaction diffusion equations in a bounded smooth domain of $\mathbb{R}^n$, $n \leq 3$ for which the associated eigenvalue problem has been studied or it is a simple consequence of a known problem. When considering these examples the only technical difficultness is to set up the problem in the correct space so that the abstract results can be applied.

In Part **II** we study an eigenvalue problem for a strictly elliptic symmetric second order differential operator in a two dimensional domain and use this result to verify the spectral assumptions of the abstract results in **I.1**.

In Part **III** we present some generalizations of the results in Parts **I** and **II** and also make some comments on the interesting features observed for the limiting differential equations.

Recall (see Hale [1988], for example) that, if $T(t) : X \to X$, $t \geq 0$, is a semigroup of transformations on a Banach space X, then a set $\mathcal{A}$ is an attractor if it is a compact invariant set that attracts a neighborhood

$\mathcal{O}$ of itself; that is, $\mathcal{A}$ is compact, $T(t)\mathcal{A} = \mathcal{A}$ for $t \geq 0$ and there is a neighborhood $\mathcal{O}$ of $\mathcal{A}$ such that dist $(T(t)\mathcal{O}, \mathcal{A}) \to 0$ as $t \to \infty$. The set $\mathcal{A}$ is a global attractor if it is an attractor which attracts each bounded set of X. A family of subsets $\{\mathcal{A}_\lambda, \lambda \geq 0\}$ is said to be upper semicontinuous at $\lambda = 0$ if dist $(\mathcal{A}_\lambda, \mathcal{A}_0) \to 0$ as $\lambda \to 0$.

I. THE STATEMENT OF THE RESULTS AND SOME APPLICATIONS

This section is divided into two subsections. In **I.1** we introduce the abstract results that will be used throughout the text and in **I.2** we deal with a few simple examples.

§1. Upper semicontinuity of Attractors for Abstract Parabolic Equations.

In this section we introduce the basic abstract results that will be used throughout this paper. We start by defining invariant manifolds.

Let X and Y be a Banach spaces and $A : D(A) \subset X \to X$ be a sectorial operator. Denote by A^α, the fractional powers of A and by X^α, the associated fractional power spaces (see Henry [1981]).

Definition 1.1. *Let $f : X^\alpha \times Y \to X$, $g : X^\alpha \times Y \to Y$ be locally Lipschitz continuous functions. A set $S \subset X^\alpha \times Y$ is an invariant manifold for a differential equation*

$$\dot{x} = Ax + f(x,y)$$
$$\dot{y} = g(x,y),$$

if there exists $\sigma : Y \to X^\alpha$ such that $S = \{(x,y) \in X^\alpha \times Y : x = \sigma(y)\}$ and, for any $(x_0, y_0) \in S$, there exists a solution $(x(\cdot), y(\cdot))$ of the differential equation on $\mathbb{R}$ such that $(x(t), y(t)) \in S \; \forall \, t \in \mathbb{R}$. An invariant manifold S is said to be exponentially attracting if there are positive constants γ and K such that

$$||x(t) - \sigma(y(t))||_{X^\alpha} \leq K e^{-\gamma t} ||x(0) - \sigma(y(0))||_{X^\alpha},$$

whenever $(x(t), y(t))$ is a solution of the differential equation.

This is the definition that will be used throughout the text. Much more general definitions appear in the literature, see for example Henry [1981].

Let $d > 0$ be a positive parameter, X_d be a Banach space and $A_d : D(A_d) \subset X_d \to X_d$ be a sectorial operator. Denote by A_d^α, the fractional powers of A_d and by X_d^α, the associated fractional power spaces.

Our first result is concerned with existence, stability and smoothness of invariant manifolds. The proof of this result follows the proofs in Henry [1981]. Some extra attention is required since the spaces change with a parameter, but the proof is significantly different and we omit it.

Lemma 1.2. *Let $f_d : X_d^\alpha \times Y \to X_d$ and $g_d : X_d^\alpha \times Y \to Y$ satisfying:*

$$||f_d(x,y) - f_d(z,w)||_{X_d} \leq L(||x - z||_{X_d^\alpha} + ||y - w||_Y),$$
$$||f_d(x,y)||_{X_d} \leq N,$$

$$||g_d(x,y) - g_d(z,w)||_Y \leq L(||x - z||_{X_d^\alpha} + ||y - w||_Y),$$
$$||g_d(x,y)||_Y \leq N,$$

for every $(x,y), (z,w)$ in $X_d^\alpha \times Y$. Assume that

$$||e^{-A_d t} w||_{X_d^\alpha} \leq M e^{-\beta(d)t} ||w||_{X_d^\alpha}, \quad t \geq 0$$
$$||e^{-A_d t} w||_{X_d^\alpha} \leq M t^{-\alpha} e^{-\beta(d)t} ||w||_{X_d}, \quad t > 0,$$

for any $w \in X_d^\alpha$, where $\beta(d) \to \infty$ as $d \to \infty$. Consider the weakly coupled system

$$\left.\begin{aligned} \dot{x} &= A_d x + f_d(x,y), \\ \dot{y} &= g_d(x,y). \end{aligned}\right\} \tag{1.1}$$

Then, for d large enough, there is an exponentially attracting invariant manifold for (1.1)

$$S_d = \{(x,y) : x = \sigma_d(y),\ y \in Y\},$$

where $\sigma_d : Y \to X_d^\alpha$ satisfies

$$s(d) = \sup_{\{y\in Y\}} \|\sigma_d(y)\|_{X_d^\alpha},$$
$$\|\sigma_d(y) - \sigma_d(z)\|_{X_d^\alpha} \leq l(d)\|y - z\|_Y,$$

with $s(d)$, $l(d) \to 0$ when $d \to \infty$. If f_d, g_d are smooth; then, σ_d is smooth and its derivative $D\sigma_d$ satisfy

$$\sup_{y\in Y} \|D\sigma_d(y)\|_{L(Y,X_d^\alpha)} \leq l(d).$$

Assume that $g : Y \to Y$ is Lipschitz continuous in bounded sets of Y and that

$$\dot{y} = g(y) \tag{1.2}$$

has a global attractor $\mathcal{A}$. Assume also that there exists a constant $\mathcal{M} > 0$, independent of d, such that the set

$$\mathcal{B} = \{u \in X_d^\alpha \times Y : \|u\|_{X_d^\alpha \times Y} \leq \mathcal{M}\} \tag{H}$$

attracts bounded sets of $X_d^\alpha \times Y$ under the flow defined by (1.1).

This hypothesis says that if the problem (1.1) has an attractor $\mathcal{A}_d$, then it must be contained in $\mathcal{B}$. The ball $\mathcal{B}$ is not independent of d, since the space and its norm change with the parameter. However, the fact that the radius of $\mathcal{B}$ is fixed will prove essential in the proof of our next result and can be usually be obtained by using energy estimates.

Remark. To simplify the notation, we also denote by $\mathcal{A}$ the set $\{0\} \times \mathcal{A} \subset X_d^\alpha \times Y$.

Suppose that $f_d : X_d^\alpha \times Y \to X$ and $g_d : X_d^\alpha \times Y \to Y$ are Lipschitz continuous in bounded sets of $X_d^\alpha \times Y$ and that for any $r > 0$ there exist positive constants M_f, L_f, depending only on r, such that

$$\|f_d(x,y)\|_{X_d} \leq L_f\|x\|_{X_d^\alpha} + M_f, \quad \|P_d(x,y)\|_Y \leq L_p(d)\|x\|_{X_d^\alpha} + M_p(d), \tag{1.3}$$

for all $(x,y) \in X_d^\alpha \times Y$, $\|(x,y)\|_{X_d^\alpha \times Y} \leq r$, where $P_d(x,y) = g_d(x,y) - g(y)$ and $M_p(d)$, $L_p(d) \to 0$ as $d \to \infty$.

Assume also that one of the following conditions is satisfied:

a) The flow defined by (1.1) is asymptotically smooth,
b) $M_f = 0$ and the flow defined by $\dot{y} = g_d(0,y)$ is asymptotically smooth,
c) $M_f = 0$ and $M_p \equiv 0$.

Theorem 1.3. *Suppose that [H] and (1.3) are satisfied and that either a), b) or c) above is satisfied. Then, there exists a $d_0 > 0$ such that, for $d \geq d_0$, the problem (1.1) has a global attractor $\mathcal{A}_d$ and the family of attractors, $\{\mathcal{A}_d,\ d_0 \leq d \leq \infty\}$, is upper semicontinuous at infinity, where $\mathcal{A}_\infty := \mathcal{A}$. Furthermore, if c) is satisfied, we have that $\mathcal{A}_d = \mathcal{A}\ \forall\, d \geq d_0$.*

Proof. Since (1.2) has a global attractor $\mathcal{A}$ there is a locally Lipschitz continuous function $\Sigma : Y \to \mathbb{R}^+$ such that for any $\Theta \in Y$,

i) $\Sigma(\Theta) = 0$ if $\Theta \in \mathcal{A}$,
ii) $a(d(\Theta, \mathcal{A})) \leq \Sigma(\Theta) \leq b(d(\Theta, \mathcal{A}))$, where a is continuous nondecreasing, $a(r) > 0$ if $r > 0$, $a(r) \to \infty$ as $r \to \infty$ and $b(r)$ is continuous with $b(0) = 0$.
iii) $\dot{\Sigma}_{(1.2)}(\Theta) \leq -\Sigma(\Theta)$, where $\dot{\Sigma}_{(1.2)}$ is the righthand derivative of Σ along the solutions of (1.2).

This result is classical in converse theory of Liapunov sability and we refer the reader to Yoshizawa [1966].

For any $c > 0$, let $\mathcal{B}_c = \{\Theta \in Y : \Sigma(\Theta) < c\}$. Note that, property *ii)* implies that $\mathcal{B}_c$ is bounded.

Using the variation of constants formula (1.5) can be rewritten as

$$\frac{dy}{dt} = g(y) + P_d(x,y), \tag{1.4}$$

$$x(t) = T_d(t)x_0 + \int_0^t T_d(t-s) f_d(x(s), y(s))ds. \tag{1.5}$$

Given $c > 0$ and $\eta > 0$, let $d_0 > 0$ be such that, for all $d \geq d_0$,

$$c - LL_p(d)\eta - LM_p(d) > 0,$$

where L is the Lipschitz constant of Σ on $\mathcal{B}_c$.

Let $(x(t), y(t))$ be the solution of (1.4), (1.5) with initial data (x_0, y_0). If $y(s) \in \mathcal{B}_c$ and $\|x(s)\|_{X_d^\alpha} < \eta$ for $0 \leq s \leq t$, it follows that

$$\begin{aligned}\dot{\Sigma}(y(t)) \leq & -\Sigma(y(t)) + L\|P_d(x(t), y(t))\|_Y \\ \leq & -\Sigma(y(t)) + LL_p(d)\|x(t)\|_{X_d^\alpha} + LM_p(d) < -\Sigma(y(t)) + LL_p(d)\eta + LM_p(d)\end{aligned}$$

and

$$\|x(t)\|_{X_d^\alpha} \leq Me^{-\beta(d)t}\|x_0\|_{X_d^\alpha} + M\int_0^t (t-s)^{-\alpha} e^{-\beta(d)(t-s)} \left[L_f\|x(s)\|_{X_d^\alpha} + M_f\right] ds.$$

Thus, if $z(t) = e^{\frac{\beta(d)}{2}t}\|x(t)\|_{X_d^\alpha}$, we have that

$$\begin{aligned}z(t) \leq & M\|x_0\|_{X_d^\alpha} + ML_f\int_0^t (t-s)^{-\alpha} e^{-\frac{\beta(d)}{2}(t-s)} z(s)ds + MM_f\gamma e^{\frac{\beta(d)}{2}t} \\ \leq & M\|x_0\|_{X_d^\alpha} + MM_f\gamma e^{\frac{\beta(d)}{2}t} + ML_f\gamma v(t)\end{aligned}$$

where $v(t) = \sup_{0\leq s\leq t}\{z(s)\}$ and $\gamma(d) = \int_0^\infty t^{-\alpha}e^{-\frac{\beta(d)}{2}t}dt \to 0$, as $d \to \infty$. Therefore,

$$z(t) \leq v(t) \leq \frac{M}{1 - ML_f\gamma}\left[\|x_0\|_{X_d^\alpha} + M_f\gamma e^{\frac{\beta(d)}{2}t}\right]$$

and

$$\|x(t)\|_{X_d^\alpha} \leq \frac{M}{1 - ML_f\gamma}\left[\|x_0\|_{X_d^\alpha} e^{-\frac{\beta(d)}{2}t} + M_f\gamma\right].$$

Assume that d_0 is chosen such that, for $d \geq d_0$, $1 - ML_f\gamma \geq \frac{1}{2}$ and $M_f\gamma \leq \frac{\eta}{4M}$. Under this condition

$$\|x(t)\|_{X_d^\alpha} \leq 2M\,\|x_0\|_{X_d^\alpha} e^{-\frac{\beta(d)}{2}t} + 2MM_f\gamma. \tag{1.6}$$

For $\|x_0\|_{X_d^\alpha} < \frac{\eta}{4M}$, (1.6) implies $\|x(s)\|_{X_d^\alpha} < \eta$ and $y(s) \in \mathcal{B}_c$, whenever defined.

Therefore, for every $t \geq 0$, $\|x_0\|_{X_d^\alpha} < \frac{\eta}{4M}$ and $y_0 \in \mathcal{B}_c$, (1.6) is satisfied and

$$\dot{\Sigma}(y(t)) \leq -\Sigma(y(t)) + 2LL_p(d)M\left[\|x_0\|_{X_d^\alpha} e^{-\frac{\beta(d)}{2}t} + M_f\gamma\right] + LM_p(d) \tag{1.7}$$

for any $t \geq 0$.

This implies that the ω-limit set of the set $\mathcal{A}_d$ of $\mathcal{B}_{\eta,c} = \{(x,y) \in X_d^\alpha \times Y : \|x\|_{X_d^\alpha} < \frac{\eta}{4M},\ y \in \mathcal{B}_c\}$ is a local attractor for (1.1). Since η and c can be chosen arbitrarily, assume that $\mathcal{B} \subset \mathcal{B}_{\eta,c}$ and $\mathcal{A}_d$ is a global attractor for (1.1). If $M_f = 0$, $M_p \equiv 0$, the above computations also show that $\omega(\mathcal{B}_{\eta,c}) \subset \mathcal{A}$.

It remains to show , in cases a) and b), that the family of attractors $\{\mathcal{A}_d, d_0 \leq d \leq \infty\}$ is upper semicontinuous at infinity.

Consider (1.7) for $\|x_0\|_{X_d^\circ} < \frac{\eta}{4M}$ and $y_0 \in \mathcal{B}_c$. Then,

$$\begin{aligned}\frac{d}{dt}(e^t\Sigma(y(t))) \leq& e^t\dot{\Sigma}(y(t)) + e^t\Sigma(y(t))\\ \leq& 2LL_p(d)M\left[\|x_0\|_{X_d^\circ}e^{-(\frac{\beta(d)}{2}-1)t} + M_f\gamma e^t\right] + LM_p(d)e^t\end{aligned}$$

and

$$\begin{aligned}\Sigma(y(t)) \leq 2LL_p(d)M&\left[\frac{e^{-\frac{\beta(d)}{2}t}-e^{-t}}{1-\frac{\beta(d)}{2}}\|x_0\|_{X_d^\circ} + M_f(1-e^{-t})\right]\\ &+ 2LM_p(d)(1-e^{-t}) + \Sigma(y_0)e^{-t}.\end{aligned} \tag{1.8}$$

From (1.6) and (1.8), for every $(x_0, y_0) \in \mathcal{A}_d$,

$$\|x_0\|_{X_d^\circ} \leq 2MM_f\gamma, \quad \Sigma(y_0) \leq 2LL_p(d)MM_f + 2LM_p(d)$$

and from property *ii)* of Σ

$$\lim_{d\to\infty}\sup_{(x,y)\in\mathcal{A}_d} \operatorname{dist}((x,y),(0,\mathcal{A})) = 0.$$

This concludes the proof.

Corollary 1.4. *Suppose that f_d and g_d are such that Lemma 1.2 and Theorem 1.3 holds. Then, there exists an exponentially attracting invariant manifold S_d given by the graph of $\sigma_d^* : Y \to X_d^\alpha$ such that the attractor $\mathcal{A}_d$ is contained in S_d. Furtheremore, the flow in S_d is given by $(x_d(t), y_d(t)) = ((\sigma_d^*(y_d(t)), y_d(t))$ where $y_d(t)$ is a solution of*

$$\dot{y} = g_d(\sigma_d^*(y), y). \tag{1.9}$$

If f, g are smooth and the flow defined by (1.2) is structurally stable; then, for d large enough, the flow defined by (1.9) is structurally stable and they are topologically equivalent.

The hypothesis $[H]$ is essential to guarantee that $\mathcal{A}_d$ are global attractors. If $[H]$ is not assumed, a similar result holds for local attractors of (1.1).

Corollary 1.5. *Suppose that hypothesis $[H]$ does not hold but all the other hypotheses of Theorem 1.3 are satisfied. Then, there exists $d_0 > 0$ such that, for all $d \leq d_0$, the problem (1.1) has a local attractor $\tilde{\mathcal{A}}_d$. Furthermore, the family of attractors $\{\tilde{\mathcal{A}}_d,\ d_0 \leq d \leq \infty\}$ is upper semicontinuous at infinity, where $\tilde{\mathcal{A}}_\infty := \tilde{\mathcal{A}}$.*

From Corollary 1.5 one can see that the family of local attractors $\tilde{\mathcal{A}}_d$ is such that, for $d_0 \leq d < \infty$ and $u \in \tilde{\mathcal{A}}_d$,

$$\|u\|_{X_d^\circ \times Y} \leq \mathcal{M},$$

for some $\mathcal{M} > 0$. Therefore, one can cut the nonlinearities f_d, g_d to satisfy the hypothesis of Lemma 1.2 and a local version of Corollary 1.4 holds.

Corollary 1.6. *Assume that (1.2) has a global attractor $\mathcal{A}$ and $[H]$, (1.3), b) are satisfied. Assume also that, for some $\epsilon > 0$, $M > 0$ independent of d, the semigroup generated by A_d satisfies*

$$\begin{aligned}\|e^{-A_dt}w\|_{X_d^\circ} &\leq Me^{-\epsilon t}\|w\|_{X_d^\circ}, \quad t \geq 0\\ \|e^{-A_dt}w\|_{X_d^\circ} &\leq Mt^{-\alpha}e^{-\epsilon t}\|w\|_{X_d}, \quad t > 0,\end{aligned}$$

for any $w \in X_d^\alpha$. Then, there exists $d_0 > 0$ such that (1.1) has a global attractor $\mathcal{A}_d$ and $\lim_{d\to\infty}\mathcal{A}_d \subset \mathcal{A}$. Furthermore, if $M_p(d) \equiv 0$, there exists $d_1 > 0$ such that $\mathcal{A}_d \equiv \mathcal{A}$ for $d \geq d_1$.

2. Applications to Parabolic Partial Differential Equations.

Suppose Ω is a bounded smooth domain in $\mathbb{R}^n$, $n \leq 3$. Consider the scalar parabolic partial differential equation

$$u_t = dLu + f(u), \quad \text{in } \Omega \tag{2.1}$$

$$\frac{\partial u}{\partial \nu} = 0, \quad \text{in } \partial\Omega \tag{2.2}$$

where L is defined by

$$Lu = \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k}(a_{kl}\frac{\partial u}{\partial x_l}), \tag{2.3}$$

with $a_{kl} \in C^1(\overline{\Omega}, R)$. Here

$$\frac{\partial u}{\partial \nu} = \sum_{k=1}^{n}(a_{k1}\frac{\partial u}{\partial x_1}, \cdots, a_{kn}\frac{\partial u}{\partial x_n}) \cdot \vec{n} \tag{2.4}$$

and $\vec{n}$ is the outward normal.

We assume that L is strictly elliptic in Ω; that is, there exists a positive number θ such that

$$\sum_{k,l=1}^{n} a_{kl}(x)\xi_k\xi_l \geq \theta|\xi|^2$$

for every $x \in \Omega$, $\xi \in \mathbb{R}^n$.

Due to homogeneous Neumann boundary condition, the above problem can be reduced to the ordinary differential equation

$$\dot{v} = f(v), \tag{2.5}$$

when restricted to the subspace of constant functions. We will use the results of previous section to give information about the global dynamics of (2.1), (2.2); that is, we prove that equation (2.5) dictates the asymptotic behavior of (2.1), (2.2).

Results of this kind have been studied by many authors (see, for example, Conway, Hoff and Smoller [1978], Hale [1986], Hale and Rocha [1987a,b] and Carvalho and Hale [1991]). However, as a consequence of their choice of space, these authors only give information about the local dynamics of the problem. Working in the energy spaces $X_d^{\frac{1}{2}}$ we intend to provide some information about the global dynamics of (2.1), (2.2).

To better describe the results, some additional notation is needed. Let $X = L^2(\Omega)$; then the operator $A_d : D(A_d) \subset X \to X$, defined by $A_d\varphi = -d\,L\varphi$, where

$$D(A_d) = \{\varphi \in H^2(\Omega) : \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega\}.$$

is a sectorial operator. We borrow the notation of the previous section for the fractional powers of A and the associated fractional power spaces. It is assumed throughout that $\alpha = \frac{1}{2}$, unless stated otherwise.

It is well known that,

$$X_d^{\frac{1}{2}} \subset L^q(\Omega) \quad \forall\, q \geq 1, \quad \text{if } n = 2, \tag*{$(2.6)_d$}$$

$$X_d^{\frac{1}{2}} \subset L^6(\Omega), \quad \text{if } n = 3, \tag*{$(2.6)'_d$}$$

with continuous inclusion (see Friedman [1983], Theorem 10.1). It is easy to see that for $d \geq 1$ the embedding constant in $(2.6)_d$ is independent of d, since $\langle A_d^\alpha\phi, \phi\rangle \geq \langle A_1^\alpha\phi, \phi\rangle$ for $d \geq 1$.

Let λ be the first positive eigenvalue of $-L$ with boundary condition (2.2) and assume that $\lambda d \geq 1$. Then, for any $\alpha_1 \geq \alpha_2 > 0$ and $\phi \in X_d^{\alpha_1}$, we have

$$\langle A_d^{\alpha_1}\phi, \phi\rangle \geq \langle A_d^{\alpha_2}\phi, \phi\rangle$$

and if $\phi \perp 1$

$$\langle A_d^{\alpha_1}\phi, \phi\rangle \geq \lambda^{\alpha_1} d^{\alpha_1}\langle \phi, \phi\rangle.$$

To use the results in the previous section the class of nonlinearities in consideration is restricted to those that satisfy some dissipativeness condition and for which the above problem is locally well posed. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function satisfying

$$|f(u) - f(v)|^2 \leq c(1 + e^{\theta|u|^\gamma} + e^{\theta|v|^\gamma})|u - v|^2, \ \gamma < 1, \quad \text{if } n = 2$$

$$|f(u) - f(v)| \leq c(1 + u^2 + v^2)|u - v| \quad \text{if } n = 3 \tag{2.7}$$

and the dissipativeness condition

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} \leq -\delta, \tag{2.8}$$

for some $\delta > 0$.

We will concentrate on the case when $n = 2$, since the case $n = 3$ is much simpler and its proof can be easily carried out from the proof for the case $n = 2$.

Lemma 2.1. *Let $\mathcal{R} = [a, b] \times [a, b]$. For any $p \geq 1$ we have*

$$H_0^1(\mathcal{R}) \subset L^p(\mathcal{R}),$$

with embedding constant proportional to p; that is, there exists a constant $\tilde{K} > 0$ such that

$$\|\phi\|_{L^p(\mathcal{R})} \leq \tilde{K}\, p\, \|\phi\|_{H_0^1(\mathcal{R})},$$

for all $\phi \in H_0^1(\mathcal{R})$.

Proof. Let $\phi \in C_c^1(\mathcal{R})$; then,

$$|\phi(x_1, x_2)| = \left|\int_a^{x_1} \frac{\partial\phi}{\partial x_1}(s, x_2)ds\right| \leq \int_a^b |\frac{\partial\phi}{\partial x_1}(s, x_2)|ds := f_2(x_2)$$

and in the same way

$$|\phi(x_1, x_2)| \leq \int_a^b |\frac{\partial\phi}{\partial x_2}(x_1, s)|ds := f_1(x_1).$$

Therefore,

$$|\phi(x_1, x_2)|^2 \leq f_1(x_1)f_2(x_2)$$

and

$$\|\phi\|_{L^2(\mathcal{R})} \leq \|\frac{\partial\phi}{\partial x_1}\|_{L^1(\mathcal{R})}^{\frac{1}{2}}\|\frac{\partial\phi}{\partial x_2}\|_{L^1(\mathcal{R})}^{\frac{1}{2}}.$$

Let $t \geq 1$. If we apply the above inequality to $\phi = |u|^{t-1}u$ we have that

$$\|u\|_{L^{2t}(\mathcal{R})}^t \leq t\,\|\,|u|^{t-1}\frac{\partial u}{\partial x_1}\|_{L^1(\mathcal{R})}^{\frac{1}{2}}\|\,|u|^{t-1}\frac{\partial u}{\partial x_2}\|_{L^1(\mathcal{R})}^{\frac{1}{2}}$$
$$\leq t\,\|u\|_{L^{q'(t-1)}}^{t-1}\|\frac{\partial u}{\partial x_1}\|_{L^q(\mathcal{R})}^{\frac{1}{2}}\|\frac{\partial u}{\partial x_2}\|_{L^q(\mathcal{R})}^{\frac{1}{2}}.$$

Choose $2t = q'(t-1)$. Then, $t = \frac{q'}{q'-2}$ and

$$\|u\|_{L^{\frac{2q'}{q'-2}}(\mathcal{R})} \leq \frac{q'}{q'-2}\|\frac{\partial u}{\partial x_1}\|_{L^q(\mathcal{R})}^{\frac{1}{2}}\|\frac{\partial u}{\partial x_2}\|_{L^q(\mathcal{R})}^{\frac{1}{2}}.$$

Since $||\phi||_{L^q(\mathcal{R})} \leq |\mathcal{R}|^{\frac{1}{q}-\frac{1}{2}}||\phi||_{L^2(\mathcal{R})}$, for all $\phi \in L^2(\mathcal{R})$, it follows that,

$$
\begin{aligned}
||u||_{L^{\frac{2q'}{q'-2}}(\mathcal{R})} &\leq \frac{q'}{q'-2}|\mathcal{R}|^{\frac{1}{q}-\frac{1}{2}}||\frac{\partial u}{\partial x_1}||^{\frac{1}{2}}_{L^2(\mathcal{R})}||\frac{\partial u}{\partial x_2}||^{\frac{1}{2}}_{L^2(\mathcal{R})} \\
&\leq \frac{q'}{q'-2}\frac{|\mathcal{R}|^{\frac{1}{q}-\frac{1}{2}}}{2}\left[||\frac{\partial u}{\partial x_1}||_{L^2(\mathcal{R})} + ||\frac{\partial u}{\partial x_2}||_{L^2(\mathcal{R})}\right].
\end{aligned}
$$

The result now follows by making $p = \frac{q'}{q'-2}$ and from a density argument.

Corollary 2.2. *Let Ω be a bouded smooth domain in $\mathbb{R}^2$. For any $p \geq 1$ we have*

$$H^1(\Omega) \subset L^p(\Omega),$$

with embedding constant proportional to p; that is, there exists a constant $K > 0$ such that

$$||\phi||_{L^p(\Omega)} \leq K\,p\,||\phi||_{H^1(\Omega)},$$

for all $\phi \in H^1(\Omega)$.

Proof. Let $\mathcal{R}$ be a rectangle such that $\bar{\Omega} \subset \mathcal{R}$. Then, there exists a linear operator $E : H^1(\Omega) \to H^1_0(\mathcal{R})$ such that $E\phi_{|\Omega} = \phi$, and

$$||E\phi||_{H^1_0(\mathcal{R})} \leq e\,||\phi||_{H^1(\Omega)},$$

for all $\phi \in H^1(\Omega)$ and for some $e > 0$. This implies that, for all $\phi \in H^1(\Omega)$,

$$||\phi||_{L^p(\Omega)} \leq ||E\phi||_{L^p(\mathcal{R})} \leq \tilde{K}\,p\,||E\phi||_{H^1_0(\mathcal{R})} \leq \tilde{K}\,e\,p\,||\phi||_{H^1(\Omega)}$$

and the result is proved.

Lemma 2.3. *Under the previous growth assumption on f, the function $f^e : X^{\frac{1}{2}}_d \to X$, given by*

$$f^e(\phi)(x) = f(\phi(x))$$

is Lipschitz continuous in bounded sets of $X^{\frac{1}{2}}_d$; that is, for any $r > 0$, there exists a constant $L_f = L_f(r)$ such that

$$||f^e(\phi) - f^e(\psi)||_X \leq L_f||\phi - \psi||_{X^{\frac{1}{2}}_d},$$

whenever $||\phi||_{X^{\frac{1}{2}}_d} \leq r$, $||\psi||_{X^{\frac{1}{2}}_d} \leq r$.

Proof. The proof is based on the embeddings $(2.6)_d$ and on Corollary 2.2. Let $r > 0$ and $\phi, \psi \in X^{\frac{1}{2}}_d$ be such that $||\phi||_{X^{\frac{1}{2}}_d} \leq r$, $||\psi||_{X^{\frac{1}{2}}_d} \leq r$. Then,

$$
\begin{aligned}
||f^e(\phi) - f^e(\psi)||^2_{L^2(\Omega)} \leq& c\int_\Omega [1 + e^{\theta|\phi(x)|^\gamma} + e^{\theta|\psi(x)|^\gamma}]\,|\phi(x) - \psi(x)|^2 dx \\
\leq& c\left(\int_\Omega [1 + e^{\theta|\phi(x)|^\gamma} + e^{\theta|\psi(x)|^\gamma}]^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |\phi(x) - \psi(x)|^4\,dx\right)^{\frac{1}{2}} \\
\leq& c\,||\phi - \psi||^2_{L^4(\Omega)}\left(\int_\Omega [1 + e^{\theta|\phi(x)|^\gamma} + e^{\theta|\psi(x)|^\gamma}]^2\,dx\right)^{\frac{1}{2}} \\
\leq& 16\,K^2c\,||\phi - \psi||^2_{H^1(\Omega)}\left(\int_\Omega [1 + e^{\theta|\phi(x)|^\gamma} + e^{\theta|\psi(x)|^\gamma}]^2\,dx\right)^{\frac{1}{2}}.
\end{aligned}
$$

Therefore, we only need to prove that

$$\|e^{\theta|\phi(x)|^\gamma}\|_{L^2(\Omega)} \leq C(r),$$

for $\|\phi\|_{X_d^{\frac{1}{2}}} \leq r$, where $C(r)$ depends only upon r. This follows from the following

$$\begin{aligned}
\|e^{\theta|\phi(x)|^\gamma}\|_{L^2(\Omega)} &\leq \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \| |u|^{n\gamma} \|_{L^2(\Omega)} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \|u\|_{L^{2n\gamma}(\Omega)}^{n\gamma} \\
&\leq \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \left(2n\gamma K \|u\|_{H^1(\Omega)}\right)^{n\gamma} = \sum_{n=0}^{\infty} \frac{n^{n\gamma}}{n!} \left(2rp\theta^{\frac{1}{p}} K\right)^{n\gamma}
\end{aligned}$$

and we only need to prove that the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n^{n\gamma}}{n!} x^n$$

is infinity. This follows from the fact that, if $a_k = \frac{k^{k\gamma}}{k!}$, then

$$\frac{a_n}{a_{n+1}} = (n+1)^{1-\gamma} \left(1 + \frac{1}{n}\right)^{-n\gamma} \to \infty$$

as $n \to \infty$. The lemma is now proved.

By standard arguments one can prove that the semigroup associated to (2.1),(2.2) is globally defined and bounded dissipative; that is, there exists a set $\mathcal{O}_d$ in $X_d^{\frac{1}{2}}$ that attracts bounded sets. The aim is to find a constant $M > 0$, independent of d, such that

$$\|u\|_{X_d^{\frac{1}{2}}} \leq M$$

for every $u \in \mathcal{O}_d$, $d \geq d_0 > 0$. Once above constant is found, by Theorem 1.3, the following result follows.

Theorem 2.4. *There exists $d_0 > 0$ such that, for any $d \geq d_0$, the attractor of (2.5) considered as a subset of the space of constant functions in $X_d^{\frac{1}{2}}$ is a global attractor for the problem (2.1),(2.2).*

To prove that the constant M can be chosen independently of $d \geq d_0$ the standard proof that the semigroup associated to (2.1),(2.2) is bounded dissipative is repeated, keeping track of the constants involved. For convinience, we only prove the case $L = \Delta$. The general case can be easily reproduced from it.

Consider the Liapunov functional, $E : X_d^{\frac{1}{2}} \to \mathbb{R}$, defined by

$$E_d(\phi) = \int_\Omega \left[\frac{d}{2} |\nabla\phi(x)|^2 + \frac{b}{2}\phi(x)^2 - F(\phi(x)) \right] dx$$

where $F(u) = \int_0^u f(s)ds$ and $b > 0$.

From the dissipativeness condition it follows that $F(s) \leq -\frac{\delta}{2} s^2 + c_{\frac{\delta}{2}}$ for all $s \in \mathbb{R}$ and from the growth condition that $|F(s)| \leq r(1 + e^{\theta|s|^\gamma})$ for $s \in \mathbb{R}$ and some $r > 0$.

This implies that,

$$\begin{aligned}
E_d(\phi) &\geq \frac{d}{2} \|\nabla\phi\|_{L^2}^2 + \frac{\delta + b}{2} \|\phi\|_{L^2}^2 - c_{\frac{\delta}{2}} |\Omega| \\
&\geq k_1 \|\phi\|_{X_d^{\frac{1}{2}}}^2 - c_{\frac{\delta}{2}} |\Omega|
\end{aligned}$$

and

$$E_d(\phi) \leq C\left(\|\phi\|_{H^1(\Omega)}\right) \leq C\left(\|\phi\|_{X_d^{\frac{1}{2}}}\right).$$

where $C(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and increasing function.

Also, if $u(t)$ is the solution of (2.1), (2.2) satisfying $u(0) = u_0 \in X_d^{\frac{1}{2}}$,

$$\frac{d}{dt}E_d(u(t)) = -\|u_t\|_{L^2}^2 - b\,d\|\nabla u\|_{L^2}^2 + b\int_\Omega uf(u)dx$$
$$\leq -b\,d\|\nabla u(t,\cdot)\|_{L^2}^2 - b\frac{\delta}{2}\|u\|_{L^2}^2 + bc_{\frac{\delta}{2}}|\Omega| \leq -bk_2\|u\|^2_{X_d^{\frac{1}{2}}} + bc_{\frac{\delta}{2}}|\Omega|.$$

The above inequalities imply the following result.

Lemma 2.5. *There exists a constant $M > 0$, independent of d, such that for any $d > 0$ and bounded set B in $X_d^{\frac{1}{2}}$ $\exists\, t_0 = t_0(d, B)$ such that for $t \geq t_0$, $u_0 \in B$*

$$u(t,u_0) \in \{u \in X_d^{\frac{1}{2}} : \|u\|_{X_d^{\frac{1}{2}}} \leq M\},$$

where $u(t,u_0)$ is the solution of (2.1), (2.2) satisfying $u(0,u_0)$.

This proves that (2.1), (2.2) defines a global semigroup which is bounded dissipative. And since A_d is a sectorial operator with compact resolvent we have that the problem (2.1, (2.2) has a compact global attractor $\mathcal{A}_d$ in $X_d^{\frac{1}{2}}$ which from the above estimates must satisfy

$$\mathcal{A}_d \subset \{u \in X_d^{\frac{1}{2}} : \|u\|_{X_d^{\frac{1}{2}}} \leq M\}.$$

Similar computations show the following result.

Corollary 2.6. *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain and $a : \partial\Omega \to \mathbb{R}$ be a continuous function. Consider the scalar parabolic partial differential equation*

$$u_t = Lu + f(u), \tag{2.1'}$$
$$\frac{\partial u}{\partial \nu} = au, \tag{2.2'}$$

where L is as before. Assume that f satisfies the dissipativeness condition (2.8) where δ is strictly bigger then the first eigenvalue of L with boundary condition (2.2'). Then, the problem (2.1'), (2.2') has a global attractor in $H^1(\Omega)$.

The fractional power space X_d^α can be decomposed as

$$X_d^\alpha = Z \oplus Z_\alpha^\perp, \tag{2.9}$$

where Z is the subspace of constant functions in X_d^α, $0 \leq \alpha \leq 1$. If $u \in X_d^{\frac{1}{2}}$ and $u = v + w$, $v \in Z$, $w \in Z_{\frac{1}{2}}^\perp$; then, v is given by

$$v = |\Omega|^{-1}\int_\Omega u(x)dx \in \mathbb{R} \tag{2.10}$$

and

$$w = u - v, \qquad \int_\Omega w(x)dx = 0. \tag{2.11}$$

Let $T_d(t)$ denote the analytic semigroup generated by A_d. Then,

$$\|T_d(t)w\|_{Z_{\frac{1}{2}}^\perp} \leq Ke^{-d\lambda t}\|w\|_{Z_{\frac{1}{2}}^\perp}, \quad t \geq 0$$
$$\|T_d(t)w\|_{Z_{\frac{1}{2}}^\perp} \leq Kt^{-\frac{1}{2}}e^{-d\lambda t}\|w\|_{Z_0^\perp}, \quad t > 0.$$

Suppose that $u(t,\cdot)$ is a solution of (2.1),(2.2) and decompose the equation according to (2.9); that is,

$$u(t,\cdot) = v(t) + w(t,\cdot), \tag{2.12}$$

$v(t) \in Z$, $w(t,\cdot) \in Z_{\frac{1}{2}}^{\perp}$. Then,

$$\begin{aligned}\frac{dv}{dt} &= |\Omega|^{-1}\int_\Omega u_t(t,x)dx = |\Omega|^{-1}\int_\Omega (dLu(t,x) + f(u(t,x)))dx \\ &= |\Omega|^{-1}\int_\Omega f(v(t) + w(t,x))dx,\end{aligned}$$

or

$$\frac{dv}{dt} = P(v,w), \tag{2.13}$$

with

$$P(v,w) = |\Omega|^{-1}\int_\Omega f(v+w)dx. \tag{2.14}$$

The variable w satisfy

$$\left.\begin{aligned} w_t &= dLw + Q(v,w) \\ \frac{\partial w}{\partial \nu} &= 0, \end{aligned}\right\} \tag{2.15}$$

where

$$Q(v,w(x)) = f(v+w(x)) - |\Omega|^{-1}\int_\Omega f(v+w(y))dy$$

Let $Q^e : X_d^{\frac{1}{2}} \to X$ be defined by

$$Q^e(v,w)(x) = Q(v,w(x)). \tag{2.16}$$

The following results can be easily proved using the embeddings $(2.6)_d$ and the growth assumptions on f, as in Lemma 2.1.

Lemma 2.7. *The function $Q^e : Z \oplus Z_{\frac{1}{2}}^{\perp} \to X$ defined by (2.16) is Lipschitz continuous in bounded sets of $Z \oplus Z_{\frac{1}{2}}^{\perp}$; that is, given $r > 0$ $(v,w) \in Z \oplus Z_{\frac{1}{2}}^{\perp}$, there exists a constant $L_Q = L_Q(r)$ such that*

$$\|Q^e(v,w)\|_X \le \frac{L_Q}{d^{\frac{1}{2}}}\|w\|_{Z_{\frac{1}{2}}^{\perp}},$$

for $\|(v,w)\|_{Z \oplus Z_{\frac{1}{2}}^{\perp}} \le r$.

Lemma 2.8. *If P is given by (2.14), the function $\tilde{P} : Z \oplus Z_{\frac{1}{2}}^{\perp} \to \mathbb{R}$ defined by $\tilde{P}(v,w) = P(v,w) - f(v)$ is Lipschitz continuous in bounded sets of $Z \oplus Z_{\frac{1}{2}}^{\perp}$; that is, given $r > 0$ $(v,w) \in Z \oplus Z_{\frac{1}{2}}^{\perp}$, there exists a constant $L_P = L_P(r)$ such that*

$$\|\tilde{P}(v,w)\|_{\mathbb{R}} \le \frac{L_P}{d^{\frac{1}{2}}}\|w\|_{Z_{\frac{1}{2}}^{\perp}},$$

for $\|(v,w)\|_{Z \oplus Z_{\frac{1}{2}}^{\perp}} \le r$.

Remark 1. *It has been shown (see Moser [1971]) that f^e is a bounded operator from H^1 into L^2 for $p < 2$. The proof of this result is very hard, therefore the contribution in this section is the simplicity of its proof.*

Indeed, it has been shown (see Hale and Raugel [1992]) that the operator f^ϵ is a bounded operator from H^1 into $W^{1,s}$ for $s < 2$, but their proof strongly uses the results of Moser [1971].

Remark 2. *One also can consider systems of weakly coupled parabolic partial differential equations. Some additional hypotheses are necessary to guarantee that the system is gradient and bounded dissipative. Assume that $f : \mathbb{R}^N \to \mathbb{R}^N$ is a C^1-function satisfying*

$$\limsup_{|x_i|\to\infty} \frac{f_i(x)}{x_i} \leq -\delta,$$

for $1 \leq i \leq N$ and for some $\delta > 0$. Assume that the Jacobian matrix of f is symmetric; that is,

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i},$$

for $1 \leq i, j \leq N$. Assume also that f satisfies the following growth condition

$$|f(u) - f(v)| \leq c(|u|^2 + |v|^2 + 1)|u - v|,$$

for every $u, v \in \mathbb{R}^N$. Under these hypotheses, Theorem 2.4 holds for systems of weakly coupled parabolic equations.

Examples.

Example 1. Consider the theory of superconductivity of liquids proposed in Abrahams and Tsuneto [1966]. For the case of small smooth domains and Neuman boundary conditions the proposed mathematical model becomes

$$\begin{aligned} u_t &= d_1\Delta u + (1 - u^2 - v^2)u, \\ v_t &= d_2\Delta v + (1 - u^2 - v^2)v. \end{aligned}$$

Using Theorem 2.4, if $d = \max\{d_1, d_2\}$ is large enough, then the above system has a global attractor which is the attractor for

$$\begin{aligned} \dot{u} &= (1 - u^2 - v^2)u \\ \dot{v} &= (1 - u^2 - v^2)v \end{aligned}$$

when considered as a subset of the constant functions in $H^1(\Omega)$. This ODE has the closed unit circle in $\mathbb{R}^2$ as attractor, (0,0) is an unstable equilibrium point and each point in the unit circumference is a stable equilibrium point.

Example 2. The second example comes from mathematical ecology (see for example Conway and Smoller [1977]). In a three dimensional smooth domain Ω consider the following system of parabolic partial differential equations

$$\begin{aligned} u_t &= \alpha\Delta u + f(u, v)u \\ v_t &= \beta\Delta v + g(u, v)v \end{aligned}$$

with homogeneous Newman boundary condition, where it is assumed that f, g satisfy quadratic growth condition, $uf_v(u, v) = vg_u(u, v)$, $\forall (u, v) \in \mathbb{R}^2$ and

$$\limsup_{|u|\to\infty} f(u, v) \leq -\delta, \quad \limsup_{|v|\to\infty} g(u, v) \leq -\delta,$$

for some $\delta > 0$.

By Theorem 2.4, for values of α and β large enough, this system has a global attractor which consists of the attractor for

$$\begin{aligned} \dot{u} &= f(u, v)u \\ \dot{v} &= g(u, v)v \end{aligned}$$

when considered as a subset of the constant functions in $H^1(\Omega)$.

Remark 3. *The results in Carvalho and Hale [1991] can also be considered as an application of the results in Section 1. More specifically, under some conditions on the nonlinearity, one can obtain results similar to the ones in this section, for reaction diffusion equations with dispersion, using Corollary 1.5.*

3. Spectral Assumptions and Upper Semicontinuity.

Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $n \leq 3$ and $d > 0$. Let L_d be a symmetric, strictly elliptic, second order differential operator given by (2.3), where the coefficients $a_{kl}(x,d) = a_{lk}(x,d)$ are real valued continuously differentiable in $\overline{\Omega}$, for each d.

Let $X = L^2(\Omega)$ and consider the self-adjoint operator

$$A_d : D(A_d) \subset X \to X$$

defined by

$$D(A_d) = \{\phi \in H^2(\Omega) : \frac{\partial \phi}{\partial \nu} = e(x,d)\phi\}, \quad A_d\phi = -L_d\phi \quad \forall\, \phi \in D(A_d),$$

where $e(x,d)$ is real valued and continuous in $\partial\Omega$ and $\frac{\partial u}{\partial \nu}$ is defined by (2.4).

Let ℓ be a positive integer and $\Omega_1, \cdots, \Omega_\ell$ be disjoint open subdomains of Ω satisfying $\Omega \subset \cup_{j=1}^{\ell} \overline{\Omega}_j$. Assume that the Lebesgue measure of $\cup_{j=1}^{\ell} \overline{\Omega}_j / \Omega$ is zero. Denote by $\mathcal{X}_{\Omega_j} : \Omega \to \mathbb{R}$ the characteristic function of Ω_j.

Let $\lambda_j(d)$ be the j^{th} eigenvalue of A_d and $\phi_j(d)$ be a corresponding normalized eigenfunction. Assume that there are real numbers $\xi_1 \leq \cdots \leq \xi_\ell$ (without loss of generality, $\xi_1 = 0$) such that

$$\lambda_j(d) \to \xi_j, \quad 1 \leq j \leq \ell,$$

as $d \to \infty$, and that for some $\mu > 0$,

$$\lambda_{\ell+1} \geq \mu d.$$

Assume also that the corresponding eigenfunctions $\phi_1, \cdots, \phi_\ell$ satisfy

$$\|\phi_j(d) - \mathcal{X}_j\|_{L^2(\Omega)} \to 0,$$

as $d \to \infty$, where each $\mathcal{X}_j$ is associated with a vector $k_j \in \mathbb{R}^\ell$ in the following way

$$\mathcal{X}_j = \sum_{i=1}^{\ell} k_j^i \mathcal{X}_{\Omega_i}.$$

From the requirements above and from $\|\phi_j(d)\|_{L^2(\Omega)} \equiv 1$, we clearly have that $\|\mathcal{X}_j\|_{L^2(\Omega)} = 1$.

Consider the following parabolic partial differential equation

$$\left.\begin{aligned} u_t &= L_d u + f(u) \quad x \in \Omega \\ \frac{\partial u}{\partial \nu} &= e(x,d)u, \quad x \in \partial\Omega \end{aligned}\right\} \tag{3.1}$$

Due to technical difficulties that arise in the proofs of Lemmas 3.2 and 3.3, we assume that $f : \mathbb{R} \to \mathbb{R}$ globally Lipschitz continuous. Then, $f^e : X_d^{\frac{1}{2}} \to X$ defined by $f^e(\phi)(x) = f(\phi(x))$ $x \in \Omega$, is globally Lipschitz continuous and takes bounded sets of $X_d^{\frac{1}{2}}$ into bounded sets of X. This implies that for any $u_0 \in X_d^{\frac{1}{2}}$, there exists a unique solution $u(t, u_0)$ of (3.1) in some maximal interval $[0, t_1)$, satisfying $u(0, u_0) = u_0$, and either $t_1 = \infty$ or $\|u(t, u_0)\|_{X_d^{\frac{1}{2}}} \to \infty$ as $t \to t_1^-$.

Assume that the nonlinearity f satisfy the dissipativeness condition (2.8) with $-\delta$ replaced by $\xi_1 - \delta$.

The a priori estimates necessary to guarantee that $[H]$ is satisfied can be obtained, as in the previous section, by considering the Liapunov Functional, $V : X_d^{\frac{1}{2}} \to \mathbb{R}$, defined by

$$V(\phi) = \frac{1}{2}\left[\int_\Omega \sum_{k,l=1}^{n} a_{kl} \frac{\partial \phi}{\partial x_k}\frac{\partial \phi}{\partial x_l} dx + \int_{\partial\Omega} e\,\phi^2 dS + \int_\Omega \phi(x)^2 dx\right] - \int_0^1 F(\phi(x))dx$$

where $F(u) = \int_0^u f(s)ds$. That is, the following results holds.

Lemma 3.1. *There esists a constant $M > 0$, independent of ν, such that for any $\nu > 0$ and bounded set B in $X_d^{\frac{1}{2}}$ there is a $t_0 = t_0(\nu, B)$ such that, for $t \geq t_0$, $u_0 \in B$*

$$u(t, u_0) \in \{u \in X_d^{\frac{1}{2}} : \|u\|_{X_d^{\frac{1}{2}}} \leq M\},$$

where $u(t, u_0)$ is the solution of (3.1) satisfying $u(0, u_0) = u_0$.

Let $v = (v_1, \cdots, v_\ell) \in \mathbb{R}^\ell$, $\Phi_d = (\phi_1(d), \cdots, \phi_\ell(d))$ and $\Phi_d \cdot v = \sum_{j=1}^{\ell} \phi_j(d) v_j$. Consider the following decomposition of $X_d^{\frac{1}{2}}$,

$$X_d^{\frac{1}{2}} = W \oplus W_{d,\frac{1}{2}}^{\perp} \tag{3.2}$$

where

$$W = \text{span}[\phi_1, \cdots, \phi_\ell], \quad W_{d,\frac{1}{2}}^{\perp} = \{\phi \in X_d^{\frac{1}{2}} : \langle \phi, \Phi_d \cdot v \rangle = 0, \ v \in \mathbb{R}^\ell\},$$

$$\langle \phi, \psi \rangle = \int_\Omega \phi(x) \psi(x) dx.$$

If $u(t, x)$ is a solution of (3.1) in $X_d^{\frac{1}{2}}$, it can be written as

$$u(t, x) = \Phi_d(x) \cdot v(t) + w(t, x) \tag{3.3}$$

where $v(t) \in \mathbb{R}^\ell$ and $w(t, \cdot) \in W_{d,\frac{1}{2}}^{\perp}$.

Using the decomposition (3.2) the equation (3.1) can be rewritten as

$$\left.\begin{aligned}
&\frac{dv}{dt} = -\text{diag}(\lambda_1, \cdots, \lambda_\ell) v + \int_\Omega f(\Phi_d(y) \cdot v + w(t, y)) \text{diag}(\phi_1(y), \cdots, \phi_\ell(y)) dy, \\
&w_t = L_d w + f(\Phi_d(x) \cdot v + w(t, x)) - \Phi_d(x) \cdot \int_\Omega \Phi_d(y) f(\Phi_d(y) \cdot v + w(t, y)) dy, \\
&\frac{\partial w}{\partial \nu} = e w.
\end{aligned}\right\} \tag{3.4}$$

From the hypotheses on the eigenvalues of A_d one expects that the component w does not play much role in the asymptotic behavior of (3.1). Therefore,

$$\dot{v}_j \sim -\lambda_j(d) v_j + \int_\Omega f\left(\sum_{i=1}^{\ell} v_i \phi_i(y)\right) \phi_j(y) dy, \quad 1 \leq j \leq \ell.$$

From the assumptions on the eigenfunctions $\phi_j(d)$, one expects that

$$\begin{aligned}
\dot{v}_j &\sim -\lambda_j(d) v_j + \sum_{q=1}^{\ell} \int_{\Omega_q} f\left(\sum_{i=1}^{\ell} v_i \phi_i(y)\right) \phi_j(y) dy \sim -\xi_j v_j + \sum_{q=1}^{\ell} \int_{\Omega_q} f\left(\sum_{i=1}^{\ell} k_i^q v_i\right) k_j^q dy \\
&\sim -\xi_j v_j + \sum_{q=1}^{\ell} |\Omega_q| f\left(\sum_{i=1}^{\ell} k_i^q v_i\right) k_j^q, \quad 1 \leq j \leq \ell.
\end{aligned}$$

Therefore, the following *limiting* ordinary differential equation is associated to (3.1)

$$\dot{v}_j = -\xi_j v_j + f_j(v_1, \cdots, v_\ell), \quad 1 \leq j \leq \ell, \tag{3.5}$$

where

$$f_j(v_1, \cdots, v_\ell) = \sum_{q=1}^{\ell} |\Omega_q| f\left(\sum_{i=1}^{\ell} k_i^q v_i\right) k_j^q, \quad 1 \leq j \leq \ell. \tag{3.6}$$

Under the previous assumptions of f, it is not hard to see that the system of ordinary differential equations (3.8) has a global attractor $\mathcal{A}$. For example one could consider the Liapunov function

$$V(v_1,\cdots,v_n) = \frac{1}{2}\sum_{i=1}^{n}[1+\xi_i]v_i^2 - \sum_{j=1}^{n}|\Omega_j|F\left(\sum_{i=1}^{n} v_i k_i^j\right)$$

to prove the existence of such global attractor.

Next we verify that hypothesis (1.3) of Theorem 1.3 is satisfied.
Let

$$P_j : \mathbb{R}^\ell \oplus W_{d,\frac{1}{2}}^{\perp} \to \mathbb{R}, \quad 1 \le j \le \ell$$

$$P_j(v,\psi) = \int_\Omega f\left(\sum_{i=1}^{\ell} v_i\phi_i(y) + \psi(y)\right)\phi_j(y)dy - f_j(v), \quad 1 \le j \le \ell$$

and

$$Q^e : \mathbb{R}^\ell \oplus W_{d,\frac{1}{2}}^{\perp} \to \mathbb{R}, \quad Q^e(v,\psi)(x) = Q(v,\psi(x))$$

$$Q(v,\psi) = f\left(\sum_{i=1}^{\ell} v_i\phi_i(x) + \psi(x)\right) - \sum_{j=1}^{\ell}\int_\Omega f\left(\sum_{i=1}^{\ell} v_i\phi_i(y) + \psi(y)\right)\phi_j(y)dy\phi_j(x).$$

The following results can now be proved from the hypothesis on the nonlinearity f and on the eigenvalues and eigenfunctions of A_d.

Lemma 3.2. *For $\|(v,w)\|_{X_d^{\frac{1}{2}}} \le r$ there exist $L_{P_j}(r,d)$, $M_{P_j}(r,d)$ such that*

$$\|P_j(v,w)\|_{\mathbb{R}} \le L_{P_j}\|w\|_{X_d^{\frac{1}{2}}} + M_{P_j}(r,d)$$

where $L_{P_j}(r,d)$, $M_{P_j}(r,d) \to 0$ as $d \to \infty$.

Proof. For $\|(v,w)\|_{X_d^{\frac{1}{2}}} \le r$

$$\begin{aligned}
-P_j(v,w) &= \sum_{i=1}^{\ell}|\Omega_i|f\left(\sum_{k=1}^{\ell} v_k k_k^i\right)k_j^i - \int_\Omega f\left(\sum_{k=1}^{\ell} v_k(t)\phi_k(y) + w(t,y)\right)\phi_j(y)dy \\
&= \sum_{i=1}^{\ell}\int_{\Omega_i}\left[f\left(\sum_{k=1}^{\ell} v_k k_k^i\right)k_j^i - f\left(\sum_{k=1}^{\ell} v_k(t)\phi_k(y) + w(t,y)\right)\phi_j(y)\right]dy \\
&= \sum_{i=1}^{\ell}\int_{\Omega_i}\left[f\left(\sum_{k=1}^{\ell} v_k k_k^i\right)k_j^i - f\left(\sum_{k=1}^{\ell} v_k\phi_k(y)\right)k_j^i\right]dy \\
&\quad + \sum_{i=1}^{\ell}\int_{\Omega_i}\left[f\left(\sum_{k=1}^{\ell} v_k\phi_k(y)\right)k_j^i - f\left(\sum_{k=1}^{\ell} v_k\phi_k(y)\right)\phi_j(y)\right]dy \\
&\quad + \sum_{i=1}^{\ell}\int_{\Omega_i}\left[f\left(\sum_{k=1}^{\ell} v_k\phi_k(y)\right)\phi_j(y) - f\left(\sum_{k=1}^{\ell} v_k(t)\phi_k(y) + w(t,y)\right)\phi_j(y)\right]dy.
\end{aligned}$$

Let $M = \left\{\sup_{\{s\in\mathbb{R}\}}|f'(s)|\right\}$. Then,

$$\begin{aligned}
|P_j(v,w)| &\le M\sum_{i=1}^{\ell}\int_{\Omega_i}\left|\sum_{k=1}^{\ell} v_k k_k^i - \sum_{k=1}^{\ell} v_k\phi_k(y)\right||k_j^i|dy \\
&\quad + C(r)\int_\Omega|\mathcal{X}_j(y) - \phi_j(y)|^2dy + \sum_{i=1}^{\ell}\int_{\Omega_i} M|w(y)||\phi_j(y)|dy \\
&\le M\|w(y)\|_{L^2(\Omega)} + M_{P_j}(r,d)
\end{aligned}$$

From the hypotheses on the eigenvalues of A_d, it follows that

$$|P_j(v,w)| \le M\mu^{-\frac{1}{2}}d^{-\frac{1}{2}}\|w(y)\|_{X_d^{\frac{1}{2}}} + M_{P_j}(r,d)$$

and the lemma is proved.

Before we state the next result we need the following facts. Let $K = (k_1,\cdots,k_\ell)$, $k_i = (k_i^1,\cdots,k_i^\ell)^T$, $1 \le i \le \ell$ and $\mathcal{M} = \mathrm{diag}(|\Omega_1|,\cdots,|\Omega_\ell|)$; then, from the orthogonality of the normalized eigenfunctions ϕ_i it follows that

$$\left|\int_\Omega \phi_j(x)\phi_i(x)dx - \sum_{p=1}^{\ell}\int_{\Omega_p} k_j^p k_i^p dx\right| = \left|\int_\Omega [\phi_j(x)[\phi_i(x) - \mathcal{X}_i(x)] + [\phi_j(x) - \mathcal{X}_j(x)]\mathcal{X}_i(x)]dx\right|$$

$$\le \|\phi_i - \mathcal{X}_i\|_{L^2(\Omega)} + \|\phi_j - \mathcal{X}_j\|_{L^2(\Omega)}$$

therefore,

$$\delta_{ij} = \int_\Omega \phi_j(x)\phi_i(x)dx \to \sum_{p=1}^{\ell}|\Omega_p|k_j^p k_i^p$$

and $\sum_{p=1}^{\ell}|\Omega_p|k_j^p k_i^p = \delta_{ij}$. With this information we have $K^T\mathcal{M}K = \mathcal{M}KK^T = I$, and

$$\sum_{j=1}^{\ell}|\Omega_j|\sum_{p=1}^{\ell}k_p^j k_p^i = 1. \tag{3.7}$$

Defining $\mathcal{X}(x) := \sum_{j=1}^{\ell}\int_\Omega \mathcal{X}_j(y)dy\mathcal{X}_j(x)$ we have, from (3.7), that $\mathcal{X}(x) \equiv 1$.

We are now in position to prove the following lemma.

Lemma 3.3. *For* $\|(v,w)\|_{X_d^{\frac{1}{2}}} \le r$ *there exist* $L_Q(r,d)$, $M_Q(r)$ *such that*

$$\|Q^\epsilon(v,w)\|_{L^2(\Omega)} \le L_Q(r,d)\|w\|_{X_d^{\frac{1}{2}}} + M_Q(r)$$

where $L_Q(r,d) \to 0$ *as* $d \to \infty$.

Proof. Let $v = (v_1,\cdots,v_\ell)$, $\phi = (\phi_1,\cdots,\phi_\ell)$ and denote $v\cdot\phi := \sum_{k=1}^{\ell}v_k\phi_k$. Then,

$$\begin{aligned}
Q(v,w) &= f(v\cdot\phi(x) + w(x)) - \sum_{j=1}^{\ell}\int_\Omega f(v\cdot\phi(y) + w(y))\phi_j(y)dy\phi_j(x)\\
&= \sum_{j=1}^{\ell}\int_\Omega f(v\cdot\phi(x) + w(x))\mathcal{X}_j(y)dy\mathcal{X}_j(x)\\
&\quad - \sum_{j=1}^{\ell}\int_\Omega f(v\cdot\phi(y) + w(y))\phi_j(y)dy\phi_j(x)\\
&= \sum_{j=1}^{\ell}\int_\Omega [f(v\cdot\phi(x) + w(x)) - f(v\cdot\phi(y) + w(y))]\phi_j(y)dy\phi_j(x)\\
&\quad - f(v\cdot\phi(x) + w(x))\sum_{j=1}^{\ell}\left[\int_\Omega \phi_j(y)dy\phi_j(x) - \int_\Omega \mathcal{X}_j(y)dy\mathcal{X}_j(x)\right]
\end{aligned}$$

From these computations, it is easy to see that, for $\|(v,w)\|_{X_d^{\frac{1}{2}}} \le r$, there exist constants $L(r)$, $M_Q(r)$ such that

$$\|Q^\epsilon(v,w)\|_{L^2(\Omega)} \le L(r)\|w\|_{L^2(\Omega)} + M_Q(r).$$

By the hypothesis on the eigenvalues of A_d, it follows that

$$\|Q^{\epsilon}(v,w)\|_{L^2(\Omega)} \leq L(r)\mu^{-\frac{1}{2}}d^{-\frac{1}{2}}\|w\|_{X_d^{\frac{1}{2}}} + M_Q(r)$$

and the lemma is proved.

From Theorem 1.4 the following result holds.

Theorem 3.4. *There exists a $d_0 > 0$ such that, for $d \geq d_0$, the problem (3.1), has a global attractor $\mathcal{A}_d$. Furthermore, the family of attractors $\{\mathcal{A}_d,\ d \geq d_0\}$ is upper-semicontinuous at infinity, where $\mathcal{A}_\infty := \mathcal{A}$.*

From Corollary 1.5 and the remarks following it we obtain the result.

Theorem 3.5. *There exists an exponentially attracting local invariant manifold S_d given by the graph of $\sigma_d : \mathbb{R}^{\ell} \to X_d^{\frac{1}{2}}$ such that the attractor $\mathcal{A}_d$ is contained in S_d. The flow in S_d is given by $(v(t),w(t)) = (v(t),\sigma_d(v(t)))$, where $v(t)$ is the solution of*

$$\dot{v} = -diag(\lambda_1,\cdots,\lambda_\ell)v + \begin{pmatrix} f_1(v) \\ \vdots \\ f_\ell(v)\end{pmatrix} + P_d(v,\sigma_d(v)), \tag{3.8}$$

where $P_d(v,w) = (P_1(v,w),\cdots,P_\ell(v,w))^T$. If f is smooth and the flow defined by (3.5) is structurally stable, for d large enough, the flow defined by (3.8) is structurally stable they are topologically equivalent; in addition, the family of attractors $\{\mathcal{A}_d,\ d \geq d_0\}$ is continuous at infinity.

4. Applications to the Results of Carvalho and Pereira [1991]

It happens in some applications that the diffusion coefficient is large in part of the region and becomes small in another part of the region. In this section we consider a class of diffusion coefficients $a \in C^1([0,1],\mathbb{R}^+)$ that are large except in a neighborhood of a finite number of points where they become small. This problem has been studied previously by Fusco [1987] and Carvalho and Pereira [1991]. We obtain that the global asymptotic dynamics is described by an ordinary differential equation that as in **I.3** can be given explicitly.

In what follows we describe, more precisely, the diffusion coefficients a.

Let $\mathcal{N}^n \subset \mathbb{R}^n$ be defined by

$$\mathcal{N}^n = \{\bar{e} = (e_1,\cdots,e_n) \in \mathbb{R}^n : e_i > 0\,,\ 1 \leq i \leq n\},$$

with the partial ordering defined by $\bar{e} \leq \bar{f}$ iff $e_i \leq f_i$, $1 \leq i \leq n$ and let $\mathcal{N}_0^{n+1} \subset \mathbb{R}^{n+1}$ be the set defined by

$$\mathcal{N}_0^{n+1} = \{\bar{\epsilon} = (\epsilon_1,\cdots,\epsilon_{n+1}) \in \mathbb{R}^{n+1} : 0 = \epsilon_1 < \epsilon_2 < \cdots \epsilon_n < \epsilon_{n+1} = 1\}.$$

For $\bar{e} \in \mathcal{N}^n$, $\bar{x} \in \mathcal{N}_0^{n+1}$ and continuous increasing functions $\bar{l}', \bar{a}' : [0,\infty) \to \mathcal{N}^{n+1}$, we say that a function $a : [0,1] \times (0,\infty) \to (0,\infty)$ is a $C_n^1(\bar{e},\bar{l}',\bar{a}',\bar{x},\nu)$ function if, for each ν, $a(\cdot,\nu) \in C^1[0,1]$ and the following is satisfied

$$\left.\begin{array}{lll} a(x,\nu) \geq \dfrac{e_i}{\nu}, & \text{for}\quad x_i + \nu l_i' \leq x \leq x_{i+1} - \nu l_{i+1}' & 1 \leq i \leq n \\ a(x,\nu) \geq \nu a_i, & \text{for}\quad x_i - \nu l_i' \leq x \leq x_i + \nu l_i' & 1 \leq i \leq n+1 \\ a(x,\nu) \leq \nu a_i', & \text{for}\quad x_i - \nu l_i \leq x \leq x_i + \nu l_i & 1 \leq i \leq n+1 \end{array}\right\} \tag{4.1}$$

where $\bar{a} = a'(\bar{0})$, $\bar{l} = l'(\bar{0})$, $\bar{a}' = a'(\bar{\nu})$ and $\bar{l}' = l'(\bar{\nu})$.

Consider the scalar parabolic problem

$$u_t = (au_x)_x + f(u), \quad 0 < x < 1,\ t > 0 \tag{4.2}$$
$$\rho u - (1-\rho)au_x = 0, \quad x = 0,\ t > 0 \tag{4.3}$$
$$\sigma u + (1-\sigma)au_x = 0, \quad x = 1,\ t > 0 \tag{4.4}$$

where $0 \le \rho, \sigma \le 1$.

Next we describe some of the results of Carvalho and Pereira [1991] that will be used in this paper.

First we consider the eigenvalue problem for the operator

$$A_\nu : D(A_\nu) \subset L^2(0,1) \to L^2(0,1)$$

defined by

$$D(A_\nu) = \left\{u \in H^2(0,1) : \rho u(0) - (1-\rho)a(0,\nu)u_x(0) = \sigma u(1) + (1-\sigma)a(1,\nu)u_x(1) = 0\right\},$$
$$A_\nu u = (a(x,\nu)u_x)_x, \quad u \in D(A_\nu),$$

where $0 \le \rho,\ \sigma \le 1$, $a \in C_n^1(\bar{\epsilon}, \bar{l}', \bar{a}', \bar{x}, \nu)$.

Let B be the tridiagonal symmetric matrix

$$B = \begin{bmatrix} m_1 & r_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ r_1 & m_2 & r_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & r_2 & m_3 & r_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & r_3 & m_4 & r_4 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & & & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & r_{n-3} & m_{n-2} & r_{n-2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & r_{n-2} & m_{n-1} & r_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & r_{n-1} & m_n \end{bmatrix}$$

where

$$m_1 = \frac{a_1\rho}{\rho l_1 + a_1(1-\rho)} + \frac{a_2}{2l_2}, \quad m_n = \frac{a_n}{2l_n} + \frac{a_{n+1}\sigma}{\sigma l_{n+1} + a_{n+1}(1-\sigma)},$$
$$m_k = \frac{a_k}{2l_k} + \frac{a_{k+1}}{2l_{k+1}}, \quad k = 2, \cdots, n-1,$$
$$r_k = \frac{-a_{k+1}}{2l_{k+1}}, \quad k = 1, \cdots, n-1.$$

If $\mathcal{M}$ is the matrix $\mathcal{M} = \mathrm{diag}(L_1, \cdots, L_n)$, $L_i = x_{i+1} - x_i$, $1 \le i \le n$, consider the inner product

$$\langle y, z\rangle_{\mathcal{M}} = \langle \mathcal{M}y, z\rangle$$

where $\langle \cdot, \cdot\rangle$ stands for the usual inner product in $\mathbb{R}^n$.

Consider the matrix $\tilde{A} := \mathcal{M}^{-1}B$. Since $\tilde{A}$ is a tridiagonal matrix with non-zero product of the off-diagonal entries, it has n distinct eigenvalues . In addition, its eigenvectors have the Sturm property; that is, the first eigenvector can be chosen with all components positive, the second has one change of sign in the components, the third has two changes of sign and so forth (see Gantmacher [1959]).

With this notation we can state the following result.

Theorem 4.1. *Suppose the above hypotheses are satisfied. Let $\tilde{A}$ be the matrix defined above, $\mu_1 < \mu_2 < \cdots < \mu_n$ the sequence of its eigenvalues, $z_1, z_2, \cdots, z_n$ the corresponding sequence of eigenvectors. Writing*

$z_i := (z_{i1}, \cdots, z_{in})$ we have:

$$\lambda_k = \mu_k + o(1),$$

$$\lambda_{n+1} \geq \tfrac{\tau}{\nu},$$

$$\phi_k(x) = \begin{cases} z_{ki} + o(1), & x \in [x_i + \nu l_i, x_{i+1} - \nu l_{i+1}], \quad i = 1, \cdots, n \\ \\ O(1), & x \in [0,1]. \end{cases}$$

where (λ_k, ϕ_k) is the k^{th} eigenpair of A_ν , $1 \leq k \leq n$.

The next lemma will play an important role in avoiding the technicalities that cause us to assume strong growth hypothesis on the nonlinearity in **I.3**.

Lemma 4.2. *If $(\rho, \sigma) \neq (0,0)$, there exists a constant E independent of ν such that*

$$\langle A\phi, \phi \rangle \geq E\|\phi\|_\infty^2$$

for every $\phi \in X^{\frac{1}{2}}$. If $(\rho, \sigma) = (0,0)$; then, for every $\phi \in X^{\frac{1}{2}}$ such that $\phi \perp 1$ we have

$$\langle A\phi, \phi \rangle \geq E\|\phi\|_\infty^2.$$

Next we use Theorem 4.1 and Theorem 1.3 to obtain the results of Carvalho and Pereira [1991].

Assume that the function f is locally Lipschitz continuous and that it satisfies the dissipativeness condition (2.8).

Lemma 4.3. *The function $f^e : X_\nu^{\frac{1}{2}} \to L^2(\Omega)$ defined by $f^e(\phi)(x) = f(\phi(x))$, is Lipschitz continuous in bounded sets of $X_\nu^{\frac{1}{2}}$.*

Consider the Liapunov Functional $V : X_\nu^{\frac{1}{2}} \to \mathbb{R}$ defined by

$$V(\phi) = \frac{1}{2}\left[\int_0^1 a(s)\phi_x(s)^2 ds + \frac{\sigma}{1-\sigma}\phi(1)^2 + \frac{\rho}{1-\rho}\phi(0)^2 + \int_0^1 \phi(s)^2 ds\right] - \int_0^1 F(\phi(s))ds$$

where $F(u) = \int_0^u f(s)ds$. Proceeding as in Section 3, we can prove the following result:

Lemma 4.4. *There exists a constant $M > 0$, independent of ν, such that for any $\nu > 0$ and bounded set B in $X_\nu^{\frac{1}{2}}$ there is a $t_0 = t_0(\nu, B)$ such that, for $t \geq t_0$, $u_0 \in B$*

$$u(t, u_0) \in \{u \in X_\nu^{\frac{1}{2}} : \|u\|_{X_\nu^{\frac{1}{2}}} \leq M\},$$

where $u(t, u_0)$ is the solution of (4.2), (4.3), (4.4) satisfying $u(0, u_0) = u_0$.

Let A_ν, $\tilde{A}$ and $\mathcal{M}$ be as before, and denote by $\mu_1 < \mu_2 < \cdots < \mu_n$ the sequence of eigenvalues of $\tilde{A}$, by $z_1, z_2, \cdots, z_n$ the corresponding sequence of eigenvectors, normalized with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{M}}$.

Writing $z_i := (z_{i1}, \cdots, z_{in})$ we have, by Theorem 4.1,

$$\lambda_k = \mu_k + o(1)$$

$$\phi_k = \pm \begin{cases} z_{ki} + o(1), & x \in [x_i + \nu l_i, x_{i+1} - \nu l_{i+1}], \quad 1 \leq i \leq n \\ \\ O(1), & x \in [0,1] \end{cases}$$

where (λ_k, ϕ_k) is the k^{th} eigenpair of A_ν, $1 \le k \le n$.

Let A_ν^α denote the fractional powers of A_ν and X_ν^α the associated fractional power spaces. Let

$$W = \text{span}[\phi_1, \cdots, \phi_n], \quad W_\alpha^\perp = \{\psi \in X^\alpha : \langle \phi, \psi \rangle = 0, \forall \phi \in W\}.$$

Then if $u \in X_\nu^{\frac{1}{2}}$ it can be decomposed as

$$u = \sum_{i=1}^{n} v_i \phi_i + w$$

where

$$v_i = \int_0^1 u(x)\phi_i(x)dx, \quad 1 \le i \le n$$
$$w = u - \sum_{i=1}^{n} v_i \phi_i.$$

This decomposition of the space induces a decomposition in the equations (4.2), (4.3), (4.4) as follows. Suppose $u(t,x)$ is a solution of (4.2), (4.3), (4.4); then,

$$u(t,x) = \sum_{i=1}^{n} v_i(t)\phi_i(x) + w(t,x)$$

and

$$\begin{aligned}
&\dot{v}_i(t) = -\lambda_i v_i + P_i(v_1, \cdots, v_n, w), \quad 1 \le i \le n \\
&w_t = (a w_x)_x + Q(v_1, \cdots, v_n, w) \\
&\rho w(0) - (1-\rho) a w_x(0) = 0 \\
&\sigma w(1) + (1-\sigma) a w_x(1) = 0.
\end{aligned}$$

where

$$P_i(v_1, \cdots, v_n, \psi) = \int_0^1 f\left(\sum_{i=1}^{n} v_i \phi_i(y) + \psi(y)\right) \phi_i(y)dy, \quad 1 \le i \le n$$

$$\begin{aligned}
Q(v_1, \cdots, v_n, \psi)(x) = {} & f\left(\sum_{i=1}^{n} v_i \phi_i(x) + \psi(x)\right) \\
& - \sum_{i=1}^{n} \int_0^1 f\left(\sum_{i=1}^{n} v_i \phi_i(y) + \psi(y)\right) \phi_i(y)dy\ \phi_i(x).
\end{aligned}$$

With this information it is possible to guess that the ordinary differential equation that determines the asymptotic behavior is,

$$\left.\begin{aligned}
\dot{v}_1(t) &= -\mu_1 v_1(t) + \sum_{i=1}^{n} [x_{i+1} - x_i] f(v_1(t) z_{1i} + v_2(t)) z_{2i} + \cdots + v_n(t) z_{ni}) z_{1i}, \\
\dot{v}_2(t) &= -\mu_2 v_2(t) + \sum_{i=1}^{n} [x_{i+1} - x_i] f(v_1(t) z_{1i} + v_2(t)) z_{2i} + \cdots + v_n(t) z_{ni}) z_{2i}, \\
&\vdots \\
\dot{v}_n(t) &= -\mu_n v_n(t) + \sum_{i=1}^{n} [x_{i+1} - x_i] f(v_1(t) z_{1i} + v_2(t)) z_{2i} + \cdots + v_n(t) z_{ni}) z_{ni}.
\end{aligned}\right\} \tag{4.5}$$

It is not hard to see (using the dissipativeness condition 2.8) that the system of ordinary differential equations (4.5) has a global attractor $\mathcal{A}$. For example one could consider the Liapunov function

$$V(v_1, \cdots, v_n) = \frac{1}{2} \sum_{i=1}^{n} [1 + \mu_i] v_i^2 - \sum_{j=1}^{n} (x_j - x_{j-1}) F\left(\sum_{i=1}^{n} v_i z_{ij}\right)$$

to prove the existence of such global attractor.

Define $Q^\epsilon : X_\nu^{\frac{1}{2}} \to L^2(\Omega)$ by $Q^\epsilon(v,w)(x) = Q(v,w(x))$ and $\tilde{P}_j : X_\nu^{\frac{1}{2}} \to \mathbb{R}$ by $\tilde{P}_j(v,w) = P_j(v,w) - f_j(v)$. $1 \leq j \leq n$. To apply Theorem 1.4 the following lemmas are needed.

The following results can be proved in the same way as in Section **I.3**

Lemma 4.5. *For $\|(v,w)\|_{X_\nu^{\frac{1}{2}}} \leq r$ there exist $L_{P_j}(r,\nu)$, $M_{P_j}(r,\nu)$ such that*

$$\|\tilde{P}_j(v,w)\|_{\mathbb{R}} \leq L_{P_j}\|w\|_{X_\nu^{\frac{1}{2}}} + M_{P_j}(r,\nu)$$

where $L_{P_j}(r,\nu)$, $M_{P_j}(r,\nu) \to 0$ as $\nu \to 0$.

Lemma 4.6. *For $\|(v,w)\|_{X_\nu^{\frac{1}{2}}} \leq r$ there exist $L_Q(r,\nu)$, $M_Q(r)$ such that*

$$\|\tilde{Q}(v,w)\|_{L^2(\Omega)} \leq L_Q(r,\nu)\|w\|_{X_\nu^{\frac{1}{2}}} + M_Q(r)$$

where $L_Q(r,\nu) \to 0$ as $\nu \to 0$.

From Theorem 1.3 the following result holds.

Theorem 4.7. *There exists a $\nu_0 > 0$ such that the problem (4.2), (4.3), (4.4) has a global attractor $\mathcal{A}_\nu$. Furthermore, the family of attractors $\{\mathcal{A}_\nu,\ 0 \leq \nu \leq \nu_0\}$ is upper-semicontinuous at zero, where $\mathcal{A}_0 := \mathcal{A}$.*

From Corollary 1.4 and following remarks, we obtain the following result.

Theorem 4.8. *There exists an exponentially attracting local invariant manifold S_ν given by the graph of $\sigma_\nu : \mathbb{R}^n \to X_\nu^{\frac{1}{2}}$ such that the attractor $\mathcal{A}_\nu$ is contained in S_ν. The flow in S_ν is given by $(v(t),w(t)) = (v(t),\sigma_\nu(v(t)))$, where $v(t)$ is the solution of*

$$\dot{v} = -diag(\lambda_1,\cdots,\lambda_n)v + P(v,\sigma_\nu(v)), \tag{4.6}$$

where $P(v,w) = (P_1(v,w),\cdots,P_n(v,w))^T$. If f is smooth and the flow defined by (4.8) is structurally stable, for ν small enough, the flow defined by (4.6) is structurally stable they are topologically equivalent; in addition, the family of attractors $\{\mathcal{A}_\nu,\ 0 \leq \nu \leq \nu_0\}$ is continuous at zero.

Next we introduce a first extension of the results of Carvalho and Pereira [1991] to a two dimensional domain. To present the problem, we start with the simplest nontrivial example, leaving the more general cases for **II** and **III** . Let $\Omega = (0,1) \times (0,1)$ and consider the parabolic problem

$$u_t = (a(x)u_x)_x + (b(y)u_y)_y + f(u), \qquad (x,y) \in \Omega,\quad t > 0 \tag{4.7}$$

$$\frac{\partial u}{\partial n} = 0, \qquad (x,y) \in \partial\Omega,\quad t > 0. \tag{4.8}$$

where $a \in C_2^1(\epsilon^{\bar{a}}, l^{\bar{a}'}, \bar{a}', \bar{x}, \nu)$ and $b \in C_2^1(\epsilon^{\bar{b}}, l^{\bar{b}'}, \bar{b}', \bar{y}, \nu)$

We expect that the solutions of this problem converge to a constant on each of the rectangles $R_{ij} = (x_i, x_{i+1}) \times (y_j, y_{j+1})$, $i,j = 1,2$. We prove that this is indeed the case.

Using Theorem 4.1 we can prove the following theorem

Theorem 4.9. *Let A be the operator*

$$A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$$

defined by

$$D(A) = \{u \in H^2(\Omega) : \frac{\partial u}{\partial n} = 0 \quad in \quad \partial\Omega\},$$
$$Au = -(a(x)u_x)_x - (b(y)u_y)_y.$$

Denote by $\lambda_{11} = \lambda_1^a + \lambda_1^b$, $\lambda_{12} = \lambda_2^a + \lambda_1^b$, $\lambda_{21} = \lambda_1^a + \lambda_2^b$, $\lambda_{22} = \lambda_2^a + \lambda_2^b$, λ_5, λ_6, $\cdots$ *the sequence of eigenvalues of* A *and* $\phi_{11} = \phi_1^a\phi_1^b$, $\phi_{12} = \phi_2^a\phi_1^b$, $\phi_{21} = \phi_1^a\phi_2^b$, $\phi_{22} = \phi_2^a\phi_2^b$, ϕ_5, ϕ_6, $\cdots$ *a corresponding sequence of normalized eigenfunctions. Then* $\lambda_{11} = 0$, $\lambda_{12} \to \frac{a_2}{2l_2^a}\frac{1}{x_2(1-x_2)}$, $\lambda_{21} \to \frac{b_2}{2l_2^a}\frac{1}{y_2(1-y_2)}$ *and* $\lambda_{22} \to \frac{a_2}{2l_2^a}\frac{1}{x_2(1-x_2)} + \frac{b_2}{2l_2^a}\frac{1}{y_2(1-y_2)}$ as $\nu \to 0$. *Also, there exists* $\mu > 0$ *such that,* $\lambda_5 \geq \frac{\mu}{\nu}$. *Furthermore, we can choose* $\phi_{11} \equiv 1$ *and* ϕ_{12}, ϕ_{21}, ϕ_{22} *to satisfy*

$$\phi_{12}(x,y) = \begin{cases} \sqrt{\dfrac{1-x_2}{x_2}} + o(1), & (x,y) \in [x_1 + \nu l_1^a, x_2 - \nu l_2^a] \times [0,1] \\ O(1), & (x,y) \in [0,1] \times [0,1] \\ -\sqrt{\dfrac{x_2}{1-x_2}} + o(1), & (x,y) \in [x_2 + \nu l_2^a, x_3 - \nu l_3^a] \times [0,1], \end{cases}$$

$$\phi_{21}(x,y) = \begin{cases} \sqrt{\dfrac{1-y_2}{y_2}} + o(1), & (x,y) \in [0,1] \times [y_1 + \nu l_1^b, y_2 - \nu l_2^b] \\ O(1), & (x,y) \in [0,1] \times [0,1] \\ -\sqrt{\dfrac{y_2}{1-y_2}} + o(1), & (x,y) \in [0,1] \times [y_2 + \nu l_2^b, y_3 - \nu l_3^b] \end{cases}$$

and

$$\phi_{22}(x,y) = \begin{cases} \sqrt{\dfrac{1-x_2}{x_2}}\sqrt{\dfrac{1-y_2}{y_2}} + o(1), & (x,y) \in R_{11} \\ -\sqrt{\dfrac{1-x_2}{x_2}}\sqrt{\dfrac{y_2}{1-y_2}} + o(1), & (x,y) \in R_{12} \\ O(1), & (x,y) \in [0,1]^2 \\ -\sqrt{\dfrac{x_2}{1-x_2}}\sqrt{\dfrac{1-y_2}{y_2}} + o(1), & (x,y) \in R_{21} \\ \sqrt{\dfrac{x_2}{1-x_2}}\sqrt{\dfrac{y_2}{1-y_2}} + o(1), & (x,y) \in R_{22} \end{cases}$$

where $R_{ij}^\nu = [x_i + \nu l_i^a, x_{i+1} - \nu l_{i+1}^a] \times [y_j + \nu l_j^b, y_{j+1} - \nu l_{j+1}^b]$, $i,j = 1,2$.

With this information we proceed to guess the ordinary differential equation that carries the asymptotic behavior.

Let A^α denote the fractional power of A and X^α denote $D(A^\alpha)$ endowed with the graph norm.

Consider the following decomposition of X^α, $\alpha \geq \frac{1}{2}$. Let $W = \text{span}[\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}]$ and $W_\alpha^\perp$ its orthogonal complement in X^α. Then if $u \in X^\alpha$ it can be written as

$$u = v_{11} + v_{12}\phi_{12} + v_{21}\phi_{21} + v_{22}\phi_{22} + w, \tag{4.9}$$

where

$$v_{11} = \int_\Omega u(x,y)dx\,dy,$$
$$v_{12} = \int_\Omega u(x,y)\phi_{12}(x,y)dx\,dy = \int_\Omega u(x,y)\phi_2^a(x)\phi_2^b(y)(x,y)dx\,dy,$$
$$v_{21} = \int_\Omega u(x,y)\phi_{21}(x,y)dx\,dy = \int_\Omega u(x,y)\phi_2^a(x)\phi_2^b(y)(x,y)dx\,dy,$$
$$v_{22} = \int_\Omega u(x,y)\phi_{22}(x,y)dx\,dy = \int_\Omega u(x,y)\phi_2^a(x)\phi_2^b(y)(x,y)dx\,dy,$$
$$w = u - v_{11} - v_{12}\phi_{12} - v_{21}\phi_{21} - v_{22}\phi_{22}.$$

This decomposition of the space induces a decomposition in the equations (4.7), (4.8) as follows. Suppose $u(t, x)$ is a solution of (4.7), (4.8); then,

$$u(t,x,y) = v_{11}(t) + v_{12}(t)\phi_{12}(x,y) + v_{21}(t)\phi_{21}(x,y) + v_{22}(t)\phi_{22}(x,y) + w(t,x,y)$$

and

$$\begin{aligned}
\dot v_{11} &= \int_\Omega u_t(t,\tilde x,\tilde y)d\tilde x\, d\tilde y = \int_\Omega f(u(t,\tilde x,\tilde y))d\tilde x\, d\tilde y,\\
\dot v_{12} &= \int_\Omega u_t(t,\tilde x,\tilde y)\phi_{12}(\tilde x,\tilde y)d\tilde x\, d\tilde y = -\lambda_{12}v_{12} + \int_\Omega f(u(t,\tilde x,\tilde y))\phi_{12}(\tilde x,\tilde y)d\tilde x\, d\tilde y,\\
\dot v_{21} &= \int_\Omega u_t(t,\tilde x,\tilde y)\phi_{21}(\tilde x,\tilde y)d\tilde x\, d\tilde y = -\lambda_{21}v_{21} + \int_\Omega f(u(t,\tilde x,\tilde y))\phi_{21}(\tilde x,\tilde y)d\tilde x\, d\tilde y,\\
\dot v_{22} &= \int_\Omega u_t(t,\tilde x,\tilde y)\phi_{22}(\tilde x,\tilde y)d\tilde x\, d\tilde y = -\lambda_{22}v_{22} + \int_\Omega f(u(t,\tilde x,\tilde y))\phi_{22}(\tilde x,\tilde y)d\tilde x\, d\tilde y,\\
w_t &= (a\,w_x)_x + (b\,w_y)_y + f(u) - \int_\Omega f(u(\tilde x,\tilde y))d\tilde x\, d\tilde y - \int_\Omega f(u(\tilde x,\tilde y))\phi_{12}(\tilde x,\tilde y)d\tilde x\, d\tilde y\ \phi_{12}\\
&\qquad - \int_\Omega f(u(\tilde x,\tilde y))\phi_{21}(\tilde x,\tilde y)d\tilde x\, d\tilde y\ \phi_{21} - \int_\Omega f(u(\tilde x,\tilde y))\phi_{22}(\tilde x,\tilde y)d\tilde x\, d\tilde y\ \phi_{22},\\
\frac{\partial w}{\partial n} &= 0, \qquad (x,y)\in\partial\Omega.
\end{aligned}$$

Proceeding as in the previous section we conclude that the limiting equation should be

$$\begin{aligned}
\dot v_{11} &= f_{11}(v_{11}, v_{12}, v_{21}, v_{22})\\
\dot v_{12} &= -\xi^a\, v_{12} + f_{12}(v_{11}, v_{12}, v_{21}, v_{22})\\
\dot v_{21} &= -\xi^b\, v_{21} + f_{21}(v_{11}, v_{12}, v_{21}, v_{22})\\
\dot v_{22} &= -\xi\, v_{22} + f_{22}(v_{11}, v_{12}, v_{21}, v_{22}).
\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}
f_{11}(v_{11},v_{12},v_{21},v_{22}) :=& x_2y_2f(v_{11} + v_{12}K_1^a + v_{21}K_2^b + v_{22}K_1^aK_1^b)\\
&+ x_2(1-y_2)f(v_{11} + v_{12}K_1^a + v_{21}(-K_2^b) + v_{22}K_1^a(-K_2^b))\\
&+ (1-x_2)y_2f(v_{11} + v_{12}(-K_2^a) + v_{21}K_2^b + v_{22}(-K_2^a)K_1^b)\\
&+ (1-x_2)(1-y_2)f(v_{11} + v_{12}(-K_2^a) + v_{21}(-K_2^b) + v_{22}K_2^aK_2^b)\\
f_{12}(v_{11},v_{12},v_{21},v_{22}) :=& K_1^ax_2y_2f(v_{11} + v_{12}K_1^a + v_{21}K_2^b + v_{22}K_1^aK_1^b)\\
&+ K_1^ax_2(1-y_2)f(v_{11} + v_{12}K_1^a + v_{21}(-K_2^b) + v_{22}K_1^a(-K_2^b))\\
&+ (-K_2^a)(1-x_2)y_2f(v_{11} + v_{12}(-K_2^a) + v_{21}K_2^b + v_{22}(-K_2^a)K_1^b)\\
&+ (-K_2^a)(1-x_2)(1-y_2)f(v_{11} + v_{12}(-K_2^a) + v_{21}(-K_2^b) + v_{22}K_2^aK_2^b),\\
f_{21}(v_{11},v_{12},v_{21},v_{22}) :=& K_1^bx_2y_2f(v_{11} + v_{12}K_1^a + v_{21}K_2^b + v_{22}K_1^aK_1^b)\\
&+ (-K_2^b)x_2(1-y_2)f(v_{11} + v_{12}K_1^a + v_{21}(-K_2^b) + v_{22}K_1^a(-K_2^b))\\
&+ K_1^b(1-x_2)y_2f(v_{11} + v_{12}(-K_2^a) + v_{21}K_2^b + v_{22}(-K_2^a)K_1^b\\
&+ (-K_2^b)(1-x_2)(1-y_2)f(v_{11} + v_{12}(-K_2^a) + v_{21}(-K_2^b) + v_{22}K_2^aK_2^b),\\
f_{22}(v_{11},v_{12},v_{21},v_{22} :=& K_1^aK_1^bx_2y_2f(v_{11} + v_{12}K_1^a + v_{21}K_2^b + v_{22}K_1^aK_1^b)\\
&+ K_1^a(-K_2^b)x_2(1-y_2)f(v_{11} + v_{12}K_1^a + v_{21}(-K_2^b) + v_{22}K_1^a(-K_2^b))\\
&+ (-K_2^a)K_1^b(1-x_2)y_2f(v_{11} + v_{12}(-K_2^a) + v_{21}K_2^b + v_{22}(-K_2^a)K_1^b\\
&+ K_2^aK_2^b(1-x_2)(1-y_2)f(v_{11} + v_{12}(-K_2^a) + v_{21}(-K_2^b) + v_{22}K_2^aK_2^b)),
\end{aligned}$$

with $K_1^a = \sqrt{\frac{1-x_2}{x_2}}$, $K_2^a = \sqrt{\frac{x_2}{1-x_2}}$, $K_2^b = \sqrt{\frac{1-y_2}{y_2}}$, $K_2^b = \sqrt{\frac{y_2}{1-y_2}}$, $\xi^a = \frac{a_2}{2l_2^a}\frac{1}{x_2(1-x_2)}$, $\xi^b = \frac{b_2}{2l_2^a}\frac{1}{y_2(1-y_2)}$ and $\xi = \xi^a + \xi^b$.

In what follows we introduce a change of variables that simplify significantly the above equations and arises naturally.

In the equations derived by Fusco [1987] the variables are the average on each of the intervals, we adopt a similar approach by taking the average on each rectangle; that is,

$$\begin{aligned}
z_{11}(t) &= \frac{1}{x_2 y_2}\int_0^{x_2}\int_0^{y_2} u(t,x,y)dx\,dy,\\
z_{12}(t) &= \frac{1}{x_2(1-y_2)}\int_0^{x_2}\int_{y_2}^{1} u(t,x,y)dx\,dy,\\
z_{21}(t) &= \frac{1}{(1-x_2)y_2}\int_{x_2}^{1}\int_0^{y_2} u(t,x,y)dx\,dy,\\
z_{22}(t) &= \frac{1}{(1-x_2)(1-y_2)}\int_{x_2}^{1}\int_{y_2}^{1} u(t,x,y)dx\,dy,
\end{aligned}$$

whereas the variables in (4.10) are

$$v_{11} = |R_{11}|z_{11} + |R_{12}|z_{12} + |R_{21}|z_{21} + |R_{22}|z_{22},$$

$$v_{12} = K_1^a|R_{11}|z_{11} + K_1^a|R_{12}|z_{12} - K_2^a|R_{21}|z_{21} - K_2^a|R_{22}|z_{22},$$

$$v_{21} = K_1^b|R_{11}|z_{11} - K_2^b|R_{12}|z_{12} + K_1^b|R_{21}|z_{21} - K_2^b|R_{22}|z_{22},$$

$$v_{22} = K_{11}|R_{11}|z_{11} - K_{12}|R_{12}|z_{12} - K_{21}|R_{21}|z_{21} + K_{22}|R_{22}|z_{22}.$$

where $K_{ij} = K_i^a K_j^b$ and $|R_{ij}| = (x_{i+1} - x_i)(y_{j+1} - y_j)$, for $i, j = 1, 2$.

Or in matrix notation

$$\begin{pmatrix} v_{11} \\ v_{12} \\ v_{21} \\ v_{22} \end{pmatrix} = B \begin{pmatrix} z_{11} \\ z_{12} \\ z_{21} \\ z_{22} \end{pmatrix},$$

where $B = A\mathcal{M}$ with

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ K_1^a & K_1^a & -K_2^a & -K_2^a \\ K_1^b & -K_2^b & K_1^b & -K_2^b \\ K_{11} & -K_{12} & -K_{21} & K_{22} \end{bmatrix}$$

and $\mathcal{M} = \text{diag}(|R_{11}|, |R_{12}|, |R_{21}|, |R_{22}|)$.

Note that $A^T A = \mathcal{M}^{-1}$. Therefore, $B^{-1} = A^T$ and

$$\begin{pmatrix} z_{11} \\ z_{12} \\ z_{21} \\ z_{22} \end{pmatrix} = A^T \begin{pmatrix} v_{11} \\ v_{12} \\ v_{21} \\ v_{22} \end{pmatrix}.$$

With these results we obtain

$$\begin{pmatrix} \dot z_{11} \\ \dot z_{12} \\ \dot z_{21} \\ \dot z_{22} \end{pmatrix} = A^T \begin{pmatrix} \dot v_{11} \\ \dot v_{12} \\ \dot v_{21} \\ \dot v_{22} \end{pmatrix} = A^T \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\xi^a & 0 & 0 \\ 0 & 0 & -\xi^b & 0 \\ 0 & 0 & 0 & -\xi \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \\ v_{21} \\ v_{22} \end{pmatrix} + A^T A \mathcal{M} \begin{pmatrix} f(z_{11}) \\ f(z_{12}) \\ f(z_{21}) \\ f(z_{22}) \end{pmatrix}$$

$$= A^T \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\xi^a & 0 & 0 \\ 0 & 0 & -\xi^b & 0 \\ 0 & 0 & 0 & -\xi \end{pmatrix} A \mathcal{M} \begin{pmatrix} z_{11} \\ z_{12} \\ z_{21} \\ z_{22} \end{pmatrix} + A^T A \mathcal{M} \begin{pmatrix} f(z_{11}) \\ f(z_{12}) \\ f(z_{21}) \\ f(z_{22}) \end{pmatrix}$$

$$= -\tilde{A} \begin{pmatrix} z_{11} \\ z_{12} \\ z_{21} \\ z_{22} \end{pmatrix} + \begin{pmatrix} f(z_{11}) \\ f(z_{12}) \\ f(z_{21}) \\ f(z_{22}) \end{pmatrix}$$

where

$$\tilde{A} = \begin{bmatrix} \frac{a_2}{2l_2^a}\frac{1}{x_2} + \frac{b_2}{2l_2^b}\frac{1}{y_2} & -\frac{b_2}{2l_2^b}\frac{1}{y_2} & -\frac{a_2}{2l_2^a}\frac{1}{x_2} & 0 \\ -\frac{b_2}{2l_2^b}\frac{1}{1-y_2} & \frac{a_2}{2l_2^a}\frac{1}{x_2} + \frac{b_2}{2l_2^b}\frac{1}{1-y_2} & 0 & -\frac{a_2}{2l_2^a}\frac{1}{x_2} \\ -\frac{a_2}{2l_2^a}\frac{1}{1-x_2} & 0 & \frac{a_2}{2l_2^a}\frac{1}{1-x_2} + \frac{b_2}{2l_2^b}\frac{1}{y_2} & -\frac{b_2}{2l_2^b}\frac{1}{y_2} \\ 0 & -\frac{a_2}{2l_2^a}\frac{1}{1-x_2} & -\frac{b_2}{2l_2^b}\frac{1}{1-y_2} & \frac{a_2}{2l_2^a}\frac{1}{1-x_2} + \frac{b_2}{2l_2^b}\frac{1}{1-y_2} \end{bmatrix}$$

and the above equations become

$$\left.\begin{aligned} \dot z_{11} &= -\frac{a_2}{2l_2^a}\frac{1}{x_2}(z_{11} - z_{21}) - \frac{b_2}{2l_2^b}\frac{1}{y_2}(z_{11} - z_{12}) + f(z_{11}) \\ \dot z_{12} &= -\frac{a_2}{2l_2^a}\frac{1}{x_2}(z_{12} - z_{22}) - \frac{b_2}{2l_2^b}\frac{1}{1-y_2}(z_{12} - z_{11}) + f(z_{12}) \\ \dot z_{21} &= -\frac{a_2}{2l_2^a}\frac{1}{1-x_2}(z_{21} - z_{11}) - \frac{b_2}{2l_2^b}\frac{1}{y_2}(z_{21} - z_{22}) + f(z_{21}) \\ \dot z_{22} &= -\frac{a_2}{2l_2^a}\frac{1}{1-x_2}(z_{22} - z_{12}) - \frac{b_2}{2l_2^b}\frac{1}{1-y_2}(z_{22} - z_{21}) + f(z_{22}). \end{aligned}\right\} \tag{4.11}$$

From the expression of ξ^a, ξ^b and from the fact that $\lambda_2^a \to \xi^a$, $\lambda_2^b \to \xi^b$ we observe that if x_2 and y_2 are close to zero or close to one, then there is a large gap between λ_{11} and the remaining eigenvalues, hence it is expected that $v_{12}(t)$, $v_{21}(t)$, and $v_{22}(t)$ will not play an important role in the asymptotic behavior and in that case the limiting equation should simply be

$$\dot v_{11}(t) = f(v_{11}(t)).$$

This is in agreement with Hale [1985] where he considers the diffusion coefficient to be large everywhere.

The next step is to prove that the equations derived indeed carry the asymptotic behavior of the parabolic problem (4.7), (4.8).

We start by proving that under suitable conditions on the nonlinearity f the problem (4.7), (4.8) has a compact global attractor $\mathcal{A}_\nu$ that lies in an invariant manifold which is a graph over W, furthermore we prove that this invariant manifold is close to W and W is close in some sense (see Theorem 4.5) to the space of functions

$$E = \{\mathcal{X}_{11}z_{11} + \mathcal{X}_{12}z_{12} + \mathcal{X}_{21}z_{21} + \mathcal{X}_{22}z_{22} : (z_{11}, z_{12}, z_{21}, z_{22}) \in \mathbb{R}^4\},$$

where $\mathcal{X}_{ij} = \mathcal{X}_{[x_i,x_{i+1}]\times[y_j,y_{j+1}]}$, $i,j=1,2$ denotes the characteristic function of the rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$. The space E is isomorphic to $\mathbb{R}^4$.

We are interested in nonlinearities f for which the problem (4.7), (4.8) has a global attractor. Assume that f satisfies the dissipativeness condition (2.8) and, for simplicity, that f is globally Lipschitzian.

Some apriori estimates can be obtained to guarantee that $[H]$ is satisfied by considering the Liapunov Functional, $V : X_\nu^{\frac{1}{2}} \to \mathbb{R}$, defined by

$$V(\phi) = \frac{1}{2}\int_\Omega \left[a(s)\phi_x(s,t)^2 + b(t)\phi_y(s,t)^2 + \phi(s,t)^2\right] ds\,dt - \int_\Omega F(\phi(s,t))ds\,dt$$

where $F(u) = \int_0^u f(s)ds$.

Remark. *We prove, at the end of this section, that the attractor $\mathcal{A}_\nu$ is bounded in $L^\infty(\Omega)$, uniformly in ν. This enable us to cut the nonlinearity without affecting the flow in the attractor, allowing us to assume that the nonlinearity is globally Lipschitzian.*

Also, the equations (4.10) have a global attractor $\mathcal{A}$; that is, the equations (4.11) have a global attractor $\tilde{\mathcal{A}}$. To see that equations (4.10) have a global attractor we recall condition (2.8) and introduce the following Liapunov function

$$V(v_{11}, v_{12}, v_{21}, v_{22}) = \frac{1}{2}\sum_{i,j=1}^{2}[1+\xi_{ij}]v_{ij}^2 - \sum_{k,l=1}^{2}|R_{kl}|F\left(\sum_{i,j=1}^{2} v_{ij}K_{(ij)(kl)}\right),$$

where $K_{(ij)(kl)} = \lim_{\nu\to 0}\phi_{ij}(x,y)$ for $(x,y) \in R_{kl}$, $i,j,k,l = 1,2$ and $\xi_{11} = 0$, $\xi_{12} = \xi^a$, $\xi_{21} = \xi^b$, $\xi_{22} = \xi$.

Consider the decomposition of (4.7),(4.8), given by (4.9), which we rewrite as

$$\left.\begin{aligned}
\dot v_{11} &= P_{11}(v_{11}, v_{12}, v_{21}, v_{22}, w),\\
\dot v_{12} &= -\lambda_{12}v_{12} + P_{12}(v_{11}, v_{12}, v_{21}, v_{22}, w),\\
\dot v_{21} &= -\lambda_{21}v_{21} + P_{21}(v_{11}, v_{12}, v_{21}, v_{22}, w),\\
\dot v_{22} &= -\lambda_{22}v_{22} + P_{22}(v_{11}, v_{12}, v_{21}, v_{22}, w),
\end{aligned}\right\} \tag{4.12}$$

$$\left.\begin{aligned}
w_t &= (a\,w_x)_x + (b\,w_y)_y + Q(v_{11}, v_{12}, v_{21}, v_{22}, w),\\
\frac{\partial w}{\partial n} &= 0. \qquad (x,y) \in \partial\Omega.
\end{aligned}\right\} \tag{4.13}$$

where

$$P_{kl}(v_{11}, v_{12}, v_{21}, v_{22}, w) = \int_\Omega f\left(\sum_{i,j=1}^{2} v_{ij}\phi_{ij}(\tilde x, \tilde y) + w(\tilde x, \tilde y)\right)\phi_{kl}(\tilde x, \tilde y)d\tilde x\,d\tilde y, \quad k,l = 1,2$$

and

$$Q(v_{11}, v_{12}, v_{21}, v_{22}, w) = f\left(\sum_{i,j=1}^{2} v_{ij}\phi_{ij} + w\right) - \sum_{k,l=1}^{2} P_{kl}(v_{11}, v_{12}, v_{21}, v_{22}, w)\,\phi_{kl}.$$

We are now in position to state the theorem that prove our claims.

Theorem 4.10. *Suppose that (4.10) has a compact attractor $\mathcal{A}$. Then, there exists $\nu_0 > 0$ and a Lipschitz continuous function $\sigma_\nu : \mathbb{R}^4 \to W^\perp$, for $\nu \le \nu_0$, such that $\sigma_\nu(v) \to 0$ uniformily for $v := (v_{11}, v_{12}, v_{21}, v_{22}) \in \mathbb{R}^4$, and the set*

$$M_\nu = \{u = \sum_{i,j=1}^{2} v_{ij}\phi_{ij} + \sigma_\nu(v),\ v \in W\}$$

is an exponentially attractive invariant manifold for (4.7), (4.8). Furthermore, the flow in M_ν is given by $u(t,x,y) = \sum_{i,j=1}^{2} v_{ij}(t)\phi_{ij}(x,y) + \sigma_\nu(v(t))(x,y)$ where $v(t)$ is the solution of

$$\left.\begin{aligned}
\dot v_{11} &= P_{11}(v, \sigma_\nu(v)),\\
\dot v_{12} &= -\lambda_{12}v_{12} + P_{12}(v, \sigma_\nu(v)),\\
\dot v_{21} &= -\lambda_{21}v_{21} + P_{21}(v, \sigma_\nu(v)),\\
\dot v_{22} &= -\lambda_{22}v_{22} + P_{22}(v, \sigma_\nu(v)),
\end{aligned}\right\} \tag{4.14}$$

The attractor $\mathcal{A}_\nu$ must lie in M_ν and the family of attractors

$$\{\mathcal{A}_\nu,\ 0 \le \nu \le \nu_0\}$$

is upper-semicontinuous at zero in $\mathbb{R}^4$.

Corollary 4.11. *Suppose f is smooth. Then for ν small the equations (4.14) have a compact attractor $\tilde{\mathcal{A}}_\nu$ and $\lim_{\nu\to 0}\tilde{\mathcal{A}}_\nu \subset \mathcal{A}$. If the flow defined by (4.10) is structurally stable on $\mathcal{A}$; then, $\lim_{\nu\to 0}\tilde{\mathcal{A}}_\nu = \mathcal{A}$, the flow defined by (4.14) is structurally stable on $\mathcal{A}$ in U and they are equivalent.*

We now prove that the attractors $\mathcal{A}_\nu$ are uniformly bounded in $L^\infty(\Omega)$. We start with the following lemma.

Lemma 4.12. *Suppose $\phi \in C^2(\Omega)$ and that $\frac{\partial\phi}{\partial n} = 0$ in $\partial\Omega$. Then, there exists $(\xi,\eta) \in \bar\Omega$ such that*

$$\begin{aligned}
&|\phi(\xi,\eta)| \ge |\phi(x,y)|, \quad (x,y) \in \bar\Omega,\\
&\nabla\phi(\xi,\eta) = 0,\\
&\phi(\xi,\eta)\phi_{xx}(\xi,\eta) \le 0,\\
&\phi(\xi,\eta)\phi_{yy}(\xi,\eta) \le 0.
\end{aligned}$$

Proof. Let $(\xi,\eta) \in \bar\Omega$ be a point where the maximum of $|\phi|$ is attained.

Assume that $(\xi,\eta) \in \partial\Omega$. There are several different possibilities, we assume that $\xi = 0$, $\eta \in [0,1]$, all the others are similar. For this case firstly consider the case $\xi = 0$, $\eta \in (0,1)$, and define

$$\psi(y) = \phi(0,y);$$

then, if $\psi(\eta) \ge 0$ we have that η is a point of maximum for ψ and $\psi''(\eta) \le 0$. Therefore,

$$\phi_y(\xi,\eta) = 0, \quad \phi(\xi,\eta)\phi_{yy}(\xi,\eta) \le 0.$$

Also $\phi_x(\xi,\eta) = 0$ from the hypothesis that $\frac{\partial\phi}{\partial n} = 0$ in $\partial\Omega$ and $\phi_{xx}(\xi,\eta) \le 0$, since for the maximum to be attained the concavity must be non-positive. A similar analysis apply for the case $\phi(\eta) < 0$.

Secondly consider the case $(\xi,\eta) = (0,0)$; then, from the fact that $\frac{\partial\phi}{\partial n} = 0$ we obtain that $\phi_x(0,0) = \phi_y(0,0) = 0$ and for $\phi(0,0) \ge 0$ the concavity of

$$\psi_1(y) = \phi(0,y),\ y \in [0,1]$$

and

$$\psi_2(x) = \phi(x,0),\ x \in [0,1]$$

must be non-positive, since $(0,0)$ is a point of maximum. Similarly for $\phi(0,0) \le 0$.

The case $(\xi,\eta) = (0,1)$ follows in a similar way.

Finally assume that $(\xi,\eta) \in \Omega$. The result follows from the fact that if $\phi(\xi,\eta) \ge 0$ it must be a maximum and if $\phi(\xi,\eta) < 0$ it must be a minimum.

Theorem 4.13. *There is a number $\bar{\xi} > 0$ such that, for $0 < \nu < \nu_0$, $u \in \mathcal{A}_\nu$ satisfies*

$$\|u\|_{C^0(\bar{\Omega})} < \bar{\xi}.$$

Proof. Let $\bar{\xi}$ be such that

$$f(\xi)\xi < 0$$

for $|\xi| \ge \bar{\xi}$. Then, if $u(t,x,y)$ is a solution of (1.4), (1.5) with initial data in X^α we have

$$\begin{aligned}\frac{\partial}{\partial t}u^2(t,x,y) =& 2[a(x)u_{xx}(t,x,y) + b(y)u_{yy}(t,x,y) + f(u(t,x,y))]\,u(t,x,y) + \\ & 2[a'(x)u_x(t,x,y) + b'(y)u_y(t,x,y)]\,u(t,x,y)\,.\end{aligned} \tag{4.15}$$

Thus, if u is an equilibrium, $\max|u(x,y)| \ge \bar{\xi}$; then, $\frac{\partial}{\partial t}u^2 = 0$ and the right hand side of (4.15) is strictly negative, at the point where $\max|u|$ is attained, by Lemma 4.12. This is a contradiction and the lemma is proved for the case when u is an equilibrium point.

If u is not an equilibrium point; then, if for some $t \in \mathbb{R}$, $\max|u(t,x,y)| \ge \bar{\xi}$, from (4.15) we have that

$$u(\tau,\xi,\eta) \ge \bar{\xi} \quad \text{for} \quad \tau \le t;$$

therefore, since $u(\tau,x,y)$ approach an equilibrium point $\bar{u}$ we have that $\|\bar{u}\|_{C^0(\bar{\Omega})} \ge \bar{\xi}$, which is a contradiction and the lemma is proved.

We now make the cut in the nonlinearity in such a way not to affect the flow in the attractors $\mathcal{A}_\nu$, $0 \le \nu \le \nu_0$. Also we choose the cut in such a way as to preserve the dissipativeness and not to affect the attractor for the ODE. Then there exists an attractor $\tilde{\mathcal{A}}_\nu$ for the parabolic problem with the new nonlinearity. Since the flow is not affected in the attractor $\mathcal{A}_\nu$, we have that $\mathcal{A}_\nu \subset \tilde{\mathcal{A}}_\nu$. With this we obtain that there exists a constant M independent of ν such that for any $\nu \in [0,\nu_0]$, such that

$$\|u\|_{X^{\frac{1}{2}}} \le M,$$

for every $u \in \mathcal{A}_\nu$.

We can now cut the nonlinearity f in $X^{\frac{1}{2}}$ and make it uniformly Lipschitz without affecting the flow in a neighborhood of the attractor. Also, the cut can be made smooth and the equivalence between the flows is established.

II. A SELECTED APPLICATION IN $[0,1]\times[0,1]$

§1. An eigenvalue problem.

Let $\Omega = (0,1) \times (0,1)$. We want to study the eigenvalue problem for the operator

$$A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$$

defined by

$$D(A) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial n} = 0 \quad \text{in} \quad \partial\Omega \right\},$$

$$Au = (a(x)u_x)_x + (b(y)u_y)_y, \quad u \in D(A),$$

where $a \in C^1_n(e^{\bar{a}}, l^{\bar{a}'}, \bar{a'}, \bar{x}, \nu)$, $b \in C^1_m(e^{\bar{b}}, l^{\bar{b}'}, \bar{b'}, \bar{y}, \nu)$ (see **I.4** and specially formula 4.1). More specifically we want to study the behavior of the eigenvalues and eigenfunctions of A as the parameter ν tends to zero. We start with a lemma.

Remark. *The eigenvalues λ_{kl} are sum of eigenvalues and the eigenfunctions ϕ_{kl} are product of eignefunctions, for the corresponding scalar problems; that is, $\lambda_{kl} = \lambda_k^a + \lambda_l^b$ and $\phi_{kl}(x,y) = \phi_k^a(x)\phi_l^b(y)$*

Lemma 1.1. *Consider the unbounded linear operator A defined above. Let $\lambda_{11},\cdots,\lambda_{1m},\lambda_{21},\cdots,\lambda_{2m},\cdots,\lambda_{n1},\cdots,\lambda_{nm},\lambda_{nm+1},\lambda_{nm+2},\cdots$ be the sequence of its eigenvalues and $\phi_{11},\cdots,\phi_{1m},\phi_{21},\cdots,\phi_{2m},\cdots,\phi_{n1},\cdots,\phi_{nm},\phi_{nm+1},\phi_{nm+2},\cdots$ a corresponding sequence of normalized eigenfunctions. Then,*

$$\lambda_{11},\cdots,\lambda_{1m},\lambda_{21},\cdots,\lambda_{2m},\cdots,\lambda_{n1},\cdots,\lambda_{nm}$$

remain bounded and there exists $\epsilon > 0$ such that $\lambda_{nm+k} \geq \frac{\epsilon}{\nu}$, for all $k \geq 1$, as $\nu \to 0$. Furthermore the eigenfunctions ϕ_{kl} , $k = 1,\cdots,n$ $l = 1,\cdots,m$ satisfy:

$$\phi_{kl}(x,y) = \begin{cases} \phi_{kl}(x_i + \nu l_i^a, y_j + \nu l_j^b) + o(1), & (x,y) \in R_{ij}^\nu, \quad 1 \leq i < n,\ 1 \leq j < m \\ \\ O(1), & (x,y) \in \bar{\Omega}, \end{cases}$$

where $R_{ij}^\nu = (x_{i-1} + \nu l_{i-1}^a, x_i - \nu l_i^a) \times (y_{j-1} + \nu l_{j-1}^b, y_j - \nu l_j^b)$.

To proceed we need to introduce some notation. Let $R_{kl} = (x_{k-1}, x_k) \times (y_{l-1}, y_l)$ and $R_{kl}^\nu = (x_{k-1} + \nu l_{k-1}^a, x_k - \nu l_k^a) \times (y_{l-1} + \nu l_{l-1}^b, y_l - \nu l_l^b)$. Let ψ_{kl} be a continuous function defined in $\bar{\Omega}$ such that $\psi_{kl}(x,y) = 0$ for $(x,y) \notin R_{kl}$, $\psi_{kl}(x,y) = 1$ for $(x,y) \in R_{kl}^\nu$, linear elsewhere and satisfying the boundary condition. Let $K_{jl}^b = \lim_{\nu\to 0}\phi_j^b(y)$ for $y \in (y_{l-1}, y_l)$, $K_{ik}^a = \lim_{\nu\to 0}\phi_i^a(x)$ for $x \in (x_{k-1}, x_k)$, $L_i^a = (x_i - x_{i-1})$, $L_j^b = (y_j - y_{j-1})$ and $K_{ij}^{kl} = K_{ik}^a K_{jl}^b$.

We define $Q : \mathbb{R}^{mn} \to \mathbb{R}^+$ by

$$Q(\mathcal{K}_{kl}) = \sum_{i=1}^{n} L_i^a \sum_{j=1}^{m-1} \frac{b_j}{2l_j^b}\left|K_{kl}^{ij+1} - K_{kl}^{ij}\right|^2 + \sum_{j=1}^{m} L_j^b \sum_{i=1}^{n-1} \frac{a_i}{2l_i^a}\left|K_{kl}^{i+1j} - K_{kl}^{ij}\right|^2$$

where $(\mathcal{K}_{kl})_{ij} = K_{kl}^{ij}$, $1 \leq i \leq n$, $1 \leq j \leq m$.

We will show that the eigenvalue problem for A is related to the minima of Q in the ellipsoid $\langle y,y\rangle_M = 1$, where $M = \text{diag}(|R_{11}|,\cdots,|R_{1m}|,|R_{21}|,\cdots,|R_{2m}|,\cdots,|R_{n1}|,\cdots,|R_{nm}|)$ and $\langle y,y\rangle_M = \langle My,y\rangle$, with $\langle\cdot,\cdot\rangle$ being the standard inner product in $\mathbb{R}^{nm}$. We want to study this relation in more detail.

It is desirable to find the symmetric matrix associated to the quadratic form Q. Consider the matrix,

$$B = \begin{bmatrix}
r_{(11)(11)} & \cdots & r_{(11)(1m)} & r_{(11)(21)} & \cdots & r_{(11)(2m)} & \cdots & r_{(11)(n1)} & \cdots & r_{(11)(nm)} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
r_{(1m)(11)} & \cdots & r_{(1m)(1m)} & r_{(1m)(21)} & \cdots & r_{(1m)(2m)} & \cdots & r_{(1m)(n1)} & \cdots & r_{(1m)(nm)} \\
r_{(21)(11)} & \cdots & r_{(21)(1m)} & r_{(21)(21)} & \cdots & r_{(21)(2m)} & \cdots & r_{(21)(n1)} & \cdots & r_{(21)(nm)} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
r_{(2m)(11)} & \cdots & r_{(2m)(1m)} & r_{(2m)(21)} & \cdots & r_{(2m)(2m)} & \cdots & r_{(2m)(n1)} & \cdots & r_{(1m)(nm)} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
r_{(n1)(11)} & \cdots & r_{(n1)(1m)} & r_{(n1)(21)} & \cdots & r_{(n1)(2m)} & \cdots & r_{(n1)(n1)} & \cdots & r_{(n1)(nm)} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
r_{(nm)(11)} & \cdots & r_{(nm)(1m)} & r_{(nm)(21)} & \cdots & r_{(nm)(2m)} & \cdots & r_{(nm)(n1)} & \cdots & r_{(nm)(nm)}
\end{bmatrix}$$

where,

$$r_{(ij)(i-1j)} = -L_j^b \frac{a_{i-1}}{2\, l_{i-1}^a}, \qquad 2 \le i \le n\,,\ 1 \le j \le m$$

$$r_{(ij)(ij-1)} = -L_i^a \frac{b_{j-1}}{2\, l_{j-1}^b}, \qquad 1 \le i \le n\,,\ 2 \le j \le m$$

$$r_{(ij)(ij)} = L_i^a \left((1-\delta_{1j}) \frac{b_{j-1}}{2\, l_{j-1}^b} + (1-\delta_{mj}) \frac{b_j}{2\, l_j^b} \right) +$$
$$L_j^b \left((1-\delta_{i1}) \frac{a_{i-1}}{2\, l_{i-1}^a} + (1-\delta_{in}) \frac{a_i}{2\, l_i^a} \right), \qquad 1 \le i \le n\,,\ 1 \le j \le m$$

$$r_{(ij)(ij+1)} = -L_i^a \frac{b_j}{2\, l_j^b}, \qquad 1 \le i \le n\,,\ 1 \le j \le m-1$$

$$r_{(ij)(i+1j)} = -L_j^b \frac{a_i}{2\, l_i^a}, \qquad 1 \le i \le n-1\,,\ 1 \le j \le m$$

$$r_{(ij)(kl)} = 0, \quad \text{otherwise.}$$

One can easily see that B is symmetric with respect to the usual inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^{nm}$.

We now proceed to prove that B is the matrix associated with the quadratic form Q. Consider the following vector in $\mathbb{R}^{nm}$, $\mathcal{K} = (K_{11}, \cdots, K_{1m}, K_{21}, \cdots, K_{2m}, \cdots, K_{n1}, \cdots, K_{nm})^{\mathrm{T}}$. Then,

$$\langle B\mathcal{K}, \mathcal{K} \rangle = \langle \left(\sum_{k=1}^{n} \sum_{l=1}^{m} r_{(ij)(kl)} K_{kl} \right), (K_{ij}) \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{l=1}^{m} r_{(ij)(kl)}\, K_{kl}\, K_{ij}$$

$$= \sum_{i=2}^{n} \sum_{j=1}^{m} (-L_j^b) \frac{a_{i-1}}{2\, l_{i-1}^a} K_{i-1j}\, K_{ij} + \sum_{i=1}^{n} \sum_{j=2}^{m} (-L_i^a) \frac{b_{j-1}}{2\, l_{j-1}^b} K_{ij-1}\, K_{ij} +$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} L_i^a \left((1-\delta_{1j}) \frac{b_{j-1}}{2\, l_{j-1}^b} + (1-\delta_{mj}) \frac{b_j}{2\, l_j^b} \right) Kij^2 +$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} L_j^b \left((1-\delta_{i1}) \frac{a_{i-1}}{2\, l_{i-1}^a} + (1-\delta_{in}) \frac{a_i}{2\, l_i^a} \right) Kij^2 +$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m-1} (-L_i^a) \frac{b_j}{2\, l_j^b} K_{ij+1}\, K_{ij} + \sum_{i=1}^{n-1} \sum_{j=1}^{m} (-L_i^a) \frac{b_j}{2\, l_j^b} K_{ij+1}\, K_{ij}$$

$$= \sum_{i=2}^{n} \sum_{j=1}^{m} L_j^b \frac{a_{i-1}}{2\, l_{i-1}^a} |K_{ij} - K_{i-1j}|^2 + \sum_{i=1}^{n} \sum_{j=2}^{m} L_i^a \frac{b_{j-1}}{2\, l_{j-1}^b} |K_{ij} - K_{ij-1}|^2 +$$

$$\sum_{i=2}^{n} \sum_{j=1}^{m} L_j^b \frac{a_{i-1}}{2\, l_{i-1}^a} K_{i-1j}\, K_{ij} - \sum_{i=1}^{n-1} \sum_{j=1}^{m} L_j^b \frac{a_i}{2\, l_i^a} K_{i+1j}\, K_{ij} +$$

$$\sum_{i=1}^{n} \sum_{j=2}^{m} L_i^a \frac{b_{j-1}}{2\, l_{j-1}^b} K_{ij-1}\, K_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{m-1} L_i^a \frac{b_j}{2\, l_j^b} K_{ij+1}\, K_{ij} +$$

$$\sum_{i=1}^{n-1} \sum_{j=1}^{m} L_j^b \frac{a_i}{2\, l_i^a} K_{ij}^2 - \sum_{i=2}^{n} \sum_{j=1}^{m} L_j^b \frac{a_{i-1}}{2\, l_{i-1}^a} K_{i-1j}^2 +$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m-1} L_i^a \frac{b_j}{2\, l_j^b} K_{ij}^2 - \sum_{i=1}^{n} \sum_{j=2}^{m} L_i^a \frac{b_{j-1}}{2\, l_{j-1}^b} K_{ij-1}^2$$

By changing variables to make all the sums start at one we obtain that the last four lines in the previous expression are zero and the remaining terms are the quadratic form Q at $\mathcal{K}$.

Define $\tilde{A} := M^{-1}B$. Observe that the matrix $\tilde{A}$ is symmetric with respect to the inner product $\langle\cdot,\cdot\rangle_M$ in $\mathbb{R}^{nm}$.

Our main result in this section can now be stated as follows.

Theorem 1.2. *Suppose that the hypotheses of Lemma 1.1 are satisfied. Let $\tilde{A}$ be the matrix defined above, $\mu_{11},\cdots,\mu_{1m},\mu_{21},\cdots,\mu_{2m},\cdots,\mu_{n1},\cdots,\mu_{nm}$ the sequence of its eigenvalues, $z_{11},\cdots,z_{1m},z_{21},\cdots,z_{2m},\cdots,z_{n1},\cdots,z_{nm}$ a corresponding sequence of normalized eigenvectors. Writing $z_{kl} := (z_{kl}^{11},\cdots,z_{kl}^{1m},z_{kl}^{21},\cdots,z_{kl}^{2m},\cdots,z_{kl}^{n1},\cdots,z_{kl}^{nm})$ we have:*

$$\lambda_{kl} = \mu_{kl} + O(\nu^{\frac{1}{2}}),$$

$$\phi_{kl}(x,y) = \begin{cases} z_{kl}^{ij} + o(1), & (x,y) \in R_{ij}^{\nu}, \qquad 1 \le i \le n,\ 1 \le j \le m \\ \\ O(1), & (x,y) \in \tilde{\Omega}. \end{cases}$$

where (λ_{kl},ϕ_{kl}) is the kl^{th} eigenpair of A , $1 \le k \le n \quad 1 \le l \le m$.

Proof: From Theorem 3.1 in **I**, we only need to prove that the eigenvalue μ_{kl} of $\tilde{A}$ can be chosen to satisfy

$$\mu_{ij} = \mu_i^a + \mu_j^b,$$

and the corresponding eigenfunction z_{kl} can be written as

$$z_{ij}^{kl} := z_{ik}^a\, z_{jl}^b.$$

in fact, if z_{ij} is as defined above

$$\begin{aligned}
(\tilde{A}z_{ij})_{pq} &= \left(\sum_{k=1}^{n}\sum_{l=1}^{m} |R_{pq}|^{-1} r_{(pq)(kl)} z_{ij}^{kl}\right) \\
&= |R_{pq}|^{-1}\left[-L_q^b(1-\delta_{p1})\frac{a_{p-1}}{2l_{p-1}^a} z_{ij}^{p-1\,q} - L_p^a(1-\delta_{1q})\frac{b_{q-1}}{2l_{q-1}^b} z_{ij}^{p\,q-1}\right] + \\
&\quad |R_{pq}|^{-1}L_p^a\left((1-\delta_{1q})\frac{b_{q-1}}{2l_{q-1}^b} + (1-\delta_{mq})\frac{b_q}{2l_q^b}\right) z_{ij}^{pq} \quad + \\
&\quad |R_{pq}|^{-1}L_q^b\left((1-\delta_{p1})\frac{a_{p-1}}{2l_{p-1}^a} + (1-\delta_{pn})\frac{a_p}{2l_p^a}\right) z_{ij}^{pq} \quad - \\
&\quad |R_{pq}|^{-1}\left[L_p^a(1-\delta_{mq})\frac{b_q}{2l_q^b} z_{ij}^{p\,q+1} + L_q^b(1-\delta_{pn})\frac{a_p}{2l_p^a} z_{ij}^{p+1\,q}\right] \\
&= z_{jq}^b\left[(L_p^a)^{-1}(1-\delta_{p1})\frac{a_{p-1}}{2l_{p-1}^a}\left(z_{ip}^a - z_{i\,p-1}^a\right) + (L_p^a)^{-1}(1-\delta_{pn})\frac{a_p}{2l_p^a}\left(z_{ip}^a - z_{i\,p+1}^a\right)\right] \\
&\quad + z_{ip}^a\left[(L_q^b)^{-1}(1-\delta_{q1})\frac{b_{q-1}}{2l_{q-1}^b}\left(z_{jq}^b - z_{j\,q-1}^b\right) + (L_q^b)^{-1}(1-\delta_{qm})\frac{b_q}{2l_q^b}\left(z_{jq}^b - z_{j\,q+1}^b\right)\right] \\
&= z_{jq}^b(\tilde{A}^a z_i^a)_p + z_{ip}^a(\tilde{A}^b z_j^b)_q = z_{jq}^b\mu_i^a z_{ip}^a + z_{ip}^a\mu_j^b z_{jq}^b \\
&= (\mu_i^a + \mu_j^b) z_{ij}^{pq}.
\end{aligned}$$

Therefore $\tilde{A}z_{ij} = (\mu_i^a + \mu_j^b)z_{ij} = \mu_{ij}z_{ij}$. Since the family of eigenvectors z_{ij} is an orthonormal basis for $\mathbb{R}^{nm}$ with the inner product $\langle\cdot,\cdot\rangle_M$ we have that these are the only eigenvalues of $\tilde{A}$. The proof is now completed.

Next we consider the question whether or not the results in this section apply only in the case when the diffusion coefficients have the particular form described. We are concerned mainly with the case when the diffusion coefficients satisfy the same kind of estimates, but depend on both variables. It is intuitively expected that exactly the same behavior would be observed. Our primary goal is to justify mathematically these intuitive expectancies.

We start by describing the diffusion coefficients that we are going to consider in this section. Assume that $a, b \in C^1(\overline{\Omega}, R)$, where $\Omega = [0,1] \times [0,1]$. Assume also that $a(\cdot, y) \in C^1_n(e^{\bar{a}}, l^{\bar{a}'}, \bar{a'}, \bar{x}, \nu)$ $\forall$ $y \in [0,1]$ and $b(x, \cdot) \in C^1_m(e^{\bar{b}}, l^{\bar{b}'}, \bar{b'}, \bar{y}, \nu)$ $\forall$ $x \in [0,1]$.

We want to study the eigenvalue problem for the operator

$$A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$$

defined by

$$D(A) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial n} = 0 \quad \text{in} \quad \partial\Omega \right\},$$

$$Au = (au_x)_x + (bu_y)_y, \quad u \in D(A).$$

More specifically we want to study the behavior of the eigenvalues and eigenfunctions of A as the parameter ν tends to zero.

Theorem 1.3. *Suppose the above hypothesis are satisfied. Let $\tilde{A}$ be as before, $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{nm}$ the sequence of its eigenvalues, $z_1, z_2, \cdots, z_{nm}$ a corresponding sequence of normalized eigenvectors. Writing $z_k := (z_k^{11}, \cdots, z_k^{1m}, z_k^{21}, \cdots, z_k^{2m}, \cdots, z_k^{n1}, \cdots, z_k^{nm})$ we have that there is a sequence of eigenfunctions $\phi_1, \phi_2, \phi_3, \cdots$ such that:*

$$\left.\begin{aligned} &\lambda_k = \mu_k + o(1), \\ &\|\phi_k - \textstyle\sum_{i=1}^n \sum_{j=1}^m \mathcal{X}_{R_{ij}} z_k^{ij}\|_{L^2(\Omega)} = o(1) \end{aligned}\right\} \tag{1.1}$$

where (λ_k, ϕ_k) is a k^{th} eigenpair of A, $1 \leq k \leq nm$.

Proof. We start by proving the properties of the eigenvalues. To do that consider the functions a^+, a^- $\in C^1_n(e^{\bar{a}}, l^{\bar{a}'}, \bar{a'}, \bar{x}, \nu)$, b^+, b^- $\in C^1_m(e^{\bar{b}}, l^{\bar{b}'}, \bar{b'}, \bar{y}, \nu)$ and

$$a^-(x) \leq a(x,y) \leq a^+(x),$$
$$b^-(y) \leq b(x,y) \leq b^+(y),$$

for every $(x, y) \in \bar{\Omega}$.

Let λ_k^- be the k^{th} eigenvalue associated to the problem of **I.3** with $a(\cdot)$, $b(\cdot)$ replaced by $a^-(\cdot)$, $b^-(\cdot)$ and A^- be the corresponding sectorial operator. Let λ_k^+ be the k^{th} eigenvalue associated to the problem of **I.3** with $a(\cdot)$, $b(\cdot)$ replaced by $a^+(\cdot)$, $b^+(\cdot)$ and A^+ be the corresponding sectorial operator. Then, by Theorem 3.2 in **I**,

$$\lambda_k^- = \mu_k + o(1),$$
$$\lambda_k^+ = \mu_k + o(1).$$

By the max-min property of eigenvalues, we have that λ_k, λ_k^+, λ_k^- can be written as

$$\lambda_k = \max_{\left\{ \substack{S \subset H^1(\Omega) \\ \dim S = k-1} \right\}} \min_{\{\phi \perp S\}} \langle A\phi, \phi \rangle,$$

$$\lambda_k^+ = \max_{\left\{ \substack{S \subset H^1(\Omega) \\ \dim S = k-1} \right\}} \min_{\{\phi \perp S\}} \langle A^+\phi, \phi \rangle,$$

$$\lambda_k^- = \max_{\left\{ \substack{S \subset H^1(\Omega) \\ \dim S = k-1} \right\}} \min_{\{\phi \perp S\}} \langle A^-\phi, \phi \rangle.$$

respectively.

Therefore, we have that

$$\lambda_k^- \leq \lambda_k \leq \lambda_k^+$$

and the claim for the eigenvalues is proved.

To prove the claim for the eigenfunctions we first assume that the eigenvalues of the matrix $\tilde{A}$ are simple. It follows from the fact that the first mn eigenvalues of A converge to the eigenvalues of $\tilde{A}$ that the first mn eigenvalues of A are simple for ν small enough.

We prove the result by induction. We know that the claim is verified for the first eigenfunction, since 0 is the first eigenvalue for $\tilde{A}$ and for A. Assume that the result is true for the first $k-1$ eigenfunctions. Let ψ_k an the eigenfunction of A^- associated to its eigenvalue λ_k^-. Then we can write,

$$\lambda_k + o(1) = \langle A^- \psi_k, \psi_k \rangle = \sum_{i=1}^{\infty} \sum_{p=1}^{\infty} c_i c_p \lambda_i \langle \phi_i, \phi_p \rangle = \sum_{i=1}^{\infty} c_i^2 \lambda_i$$

From the induction hypothesis we obtain that $c_i = o(1)$ for every $i < k$. Therefore we have

$$\lambda_k + o(1) \geq c_k^2 \lambda_k + (1 - c_k^2)(\lambda_k + \delta)$$

where $\lambda_k + \delta = \lambda_{k+1}$. From the assumption that the eigenvalues of $\tilde{A}$ are all simple we obtain that δ is bounded away from zero. And we have that

$$o(1) = (1 - c_k^2)$$

we choose $c_k = 1 + o(1)$. This also implies that

$$\sum_{i>k} c_{ij}^2 = o(1).$$

This proves that

$$||\phi_k - \psi_k||_{L^2(\Omega)} = o(1).$$

The result now follows from Theorem 1.2, in this section.

Next, we consider the case when the matrix $\tilde{A}$ has multiple eigenvalues. Let μ_k be the first multiple eigenvalue of $\tilde{A}$ and assume that its multiplicity is m. Proceeding as before we have that, for $0 \leq j \leq m-1$

$$\lambda_k + o(1) = \langle A^- \psi_{k+j}, \psi_{k+j} \rangle = \sum_{i=k} (c_i^j)^2 \lambda_i \geq \lambda_k \sum_{i=k}^{k+m-1} (c_i^j)^2 + \left(1 - \sum_{i=k}^{k+m-1} (c_i^j)^2\right)(\lambda_k + \delta),$$

where $\lambda_k + \delta = \lambda_{k+m}$ and δ is bounded away from zero.

This implies that

$$1 - \sum_{i=k}^{k+m-1} (c_i^j)^2 = o(1). \qquad 0 \leq j \leq m-1$$

and that

$$\sum_{i=k+m}^{\infty} (c_i^j)^2 = o(1).$$

Therefore,

$$\psi_{k+j} = \sum_{i=k}^{k+m-1} c_i^j \phi_i + o(1), \qquad 0 \leq j \leq m-1.$$

Let $\vec{c^j} = (c_k^j, \cdots, c_{k+m-1}^j)$ we have that

$$\langle \vec{c^p}, \vec{c^q} \rangle = \delta_{pq} + o(1).$$

If $O := \mathrm{col}(\bar{c^0}, \cdots, \bar{c^{m-1}})$; then, $O^T = O^{-1} + o(1)$. And, if $(\bar{\psi}_k, \cdots, \bar{\psi}_{k+m-1})^T := O^T(\psi_k, \cdots, \psi_{k+m-1})^T$, we have that they are eigenfunctions of A^- and that they satisfy $\|\phi_{k+j} - \bar{\psi}_{k+j}\|_{L^2(\Omega)} = o(1)$.

Now we redefine the sequence of eigenfunctions substituting ψ_{k+j} by $\bar{\psi}_{k+j}$. This together with Theorem 1.2 proves the result.

§2. Applications to dynamics.

Let $\Omega = (0,1) \times (0,1)$. In this section we consider the parabolic equation

$$u_t = (a(u_x)_x + (bu_y)_y + f(u), \quad (x, y) \in \Omega, \ t > 0 \tag{2.1}$$

$$\frac{\partial u}{\partial n} = 0, \quad (x, y) \in \partial\Omega, \ t > 0. \tag{2.2}$$

Let A be the sectorial operator and $\tilde{A}$ and M be the matrices defined in the previous section, and denote by $\mu_{11}, \cdots, \mu_{1m}, \mu_{21}, \cdots, \mu_{2m}, \cdots, \mu_{n1}, \cdots, \mu_{nm}$ the sequence of eigenvalues of $\tilde{A}$, by $z_{11}, \cdots, z_{1m}, z_{21}, \cdots, z_{2m}, \cdots, z_{n1}, \cdots, z_{nm}$ the corresponding sequence of eigenvectors, normalized with respect to the inner product $\langle \cdot, \cdot \rangle_M$.

Writing $z_{kl} := (z_{kl}^{11}, \cdots, z_{kl}^{1m}, z_{kl}^{21}, \cdots, z_{kl}^{2m}, \cdots, z_{kl}^{n1}, \cdots, z_{kl}^{nm})$ we have, by Theorem 1.3 **II.1**,

$$\lambda_{kl} = \mu_{kl} + O(\nu^{1/2})$$

$$\phi_{kl} = \pm \begin{cases} z_{kl}^{ij} + o(1), & x \in R_{ij}, \quad 1 \le i \le n, \ 1 \le j \le m \\ O(1), & x \in \bar{\Omega} \end{cases}$$

where $(\lambda_{kl}, \phi_{kl})$ is the kl^{th} eigenpair of A , $1 \le k \le n$, $1 \le l \le m$.

With this information we can proceed to obtain the ordinary differential equation that determines the asymptotic behavior.

Let A^α be the fractional power of A and X^α be $D(A^\alpha)$ endowed with the graph norm and consider the following decomposition of X^α. Let $W = \mathrm{span}\,[\phi_{kl}, \ 1 \le k \le n, \ 1 \le l \le m]$ and $W_\alpha^\perp$ its orthogonal complement in X^α. Then if $u \in X^{1/2}$ it can be decomposed as

$$u = \sum_{k=1}^{n} \sum_{l=1}^{m} v_{kl}\phi_{kl} + w$$

where

$$v_{kl} = \int_\Omega u(x, y)\phi_{kl}(x, y)dx\, dy, \ 1 \le k \le n, \ 1 \le l \le m,$$

$$w = u - \sum_{k=1}^{n} \sum_{l=1}^{m} v_{kl}\phi_{kl}.$$

This decomposition of the space induces a decomposition in the equations (2.1), (2.2) as follows. Suppose $u(t, x, y)$ is a solution of (2.1), (2.2); then,

$$u(t, x) = \sum_{k=1}^{n} \sum_{l=1}^{m} v_{kl}(t)\varphi_{kl}(x, y) + w(t, x, y)$$

and

$$\dot{v}_{kl}(t) = \int_\Omega u_t(t, \bar{x})\phi_{kl}(\bar{x})d\bar{x} = -\lambda_{kl}v_{kl}(t) + \int_\Omega f(u(t, \bar{x}))\phi_{kl}(\bar{x})d\bar{x}, \tag{2.3}$$

for $1 \le k \le n$, $1 \le l \le m$,

$$\left.\begin{aligned} w_t &= (aw_x)_x + (bw_y)_y + f(u(t, x, y)) - \sum_{k=1}^{n} \sum_{l=1}^{m} \int_\Omega f(u(t, \bar{x}))\phi_{kl}(\bar{x})d\bar{x}\ \phi_{kl}(x, y) \\ \frac{\partial u}{\partial n} &= 0 \qquad (x, y) \in \partial\Omega. \end{aligned}\right\} \tag{2.4}$$

As in **I.3** we guess that the limiting ordinary differential equations are,

$$\dot{v}_{kl}(t) = -\mu_{kl}\, v_{kl}(t) + \sum_{p=1}^{n}\sum_{q=1}^{m} |R_{pq}|\, f\left(\sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij}(t)\, z_{ij}^{pq}\right) z_{kl}^{pq}. \tag{2.5}$$

for $1 \leq k \leq n,\ 1 \leq l \leq m$.

We now introduce a change of variable that we believe simplify the above equations significantly and allow more accurate interpretation of the results. Let

$$v_{kl} = \sum_{i=1}^{n}\sum_{j=1}^{m} |R_{ij}| z_{kl}^{ij} \xi_{ij} \tag{2.6}$$

where $\xi_{ij},\ 1 \leq i \leq n,\ 1 \leq j \leq m$ is the set of variables related to the average on each of the rectangles in the partition.

Introducing the notation,

$$Z^T = \begin{bmatrix} z_{11}^{11} & \cdots & z_{11}^{1m} & z_{11}^{21} & \cdots & z_{11}^{2m} & \cdots & z_{11}^{n1} & \cdots & z_{11}^{nm} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ z_{1m}^{11} & \cdots & z_{1m}^{1m} & z_{1m}^{21} & \cdots & z_{1m}^{2m} & \cdots & z_{1m}^{n1} & \cdots & z_{1m}^{nm} \\ z_{21}^{11} & \cdots & z_{21}^{1m} & z_{21}^{21} & \cdots & z_{21}^{2m} & \cdots & z_{21}^{n1} & \cdots & z_{21}^{nm} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ z_{2m}^{11} & \cdots & z_{2m}^{1m} & z_{2m}^{21} & \cdots & z_{2m}^{2m} & \cdots & z_{2m}^{n1} & \cdots & z_{2m}^{nm} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ z_{n1}^{11} & \cdots & z_{n1}^{1m} & z_{n1}^{21} & \cdots & z_{n1}^{2m} & \cdots & z_{n1}^{n1} & \cdots & z_{n1}^{nm} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ z_{nm}^{11} & \cdots & z_{nm}^{1m} & z_{nm}^{21} & \cdots & z_{nm}^{2m} & \cdots & z_{nm}^{n1} & \cdots & z_{nm}^{nm} \end{bmatrix}$$

$$\mu = \mathrm{diag}(\mu_{11}, \cdots, \mu_{1m}, \mu_{21}, \cdots, \mu_{2m}, \cdots, \mu_{n1}, \cdots, \mu_{nm})$$

$$v = \begin{pmatrix} v_{11} \\ \vdots \\ v_{1m} \\ v_{21} \\ \vdots \\ v_{2m} \\ \vdots \\ v_{n1} \\ \vdots \\ v_{nm} \end{pmatrix}, \qquad \xi = \begin{pmatrix} \xi_{11} \\ \vdots \\ \xi_{1m} \\ \xi_{21} \\ \vdots \\ \xi_{2m} \\ \vdots \\ \xi_{n1} \\ \vdots \\ \xi_{nm} \end{pmatrix}, \qquad F(v) = \begin{pmatrix} f(v_{11}) \\ \vdots \\ f(v_{1m}) \\ f(v_{21}) \\ \vdots \\ f(v_{2m}) \\ \vdots \\ f(v_{n1}) \\ \vdots \\ f(v_{nm}) \end{pmatrix},$$

we can rewrite (2.5) as

$$\dot{v} = -\mu v + Z^T M F(Zv), \tag{2.7}$$

$$v = Z^T M \xi. \tag{2.8}$$

Since the eigenvectors z_{kl} are an orthonormal base for $I\!R^{nm}$ with respect to the inner product $\langle \cdot, \cdot \rangle_M$, we have

$$Z^T M Z = Z^T M^T Z = Id,$$

and thus

$$ZZ^T = M^{-1}.$$

Therefore,

$$Zv = ZZ^T M\xi = \xi$$

and writing the equation (2.7) in the variable ξ, we obtain,

$$\dot{\xi} = Z\dot{v} = -Z\mu Z^T M\xi + ZZ^T MF(\xi).$$

Now, observing that

$$\begin{aligned}\tilde{A}Z &= \left(\tilde{A}z_{11},\cdots,\tilde{A}z_{1m},\tilde{A}z_{21},\cdots,\tilde{A}z_{2m},\cdots,\tilde{A}z_{n1},\cdots,\tilde{A}z_{nm}\right)\\ &= (\mu_{11}z_{11},\cdots,\mu_{1m}z_{1m},\mu_{21}z_{21},\cdots,\mu_{2m}z_{2m},\cdots,\mu_{n1}z_{n1},\cdots,\mu_{nm}z_{nm}) = Z\mu,\end{aligned}$$

we have

$$\dot{\xi} = -\tilde{A}ZZ^T M\xi + F(\xi) = -\tilde{A}\xi + F(\xi).$$

$$\begin{aligned}\dot{\xi}_{kl} = &-|R_{kl}|^{-1}\left[(1-\delta_{k1})L_l^b\frac{a_{k-1}}{2\,l_{k-1}^a}(\xi_{kl}-\xi_{k-1\,l}) + (1-\delta_{1j})L_k^a\frac{b_{l-1}}{2\,l_{l-1}^b}(\xi_{kl}-\xi_{k\,l-1})\right] +\\ &-|R_{kl}|^{-1}\left[(1-\delta_{ml})\,L_k^a\,\frac{b_l}{2\,l_l^b}\,(\xi_{kl}-\xi_{k\,l+1}) + (1-\delta_{kn})L_l^b\,\frac{a_k}{2\,l_k^a}\,(\xi_{kl}-\xi_{k+1\,l})\right] + f(\xi_{kl})\end{aligned}$$

for $1 \leq k \leq n$, $1 \leq l \leq m$.

Assume that the function f satisfies the dissipativeness condition (2.8), where $-\delta$ is replaced by $\mu_{11}-\delta$.

This implies that the parabolic problem (2.1), (2.2) has a global attractor $\mathcal{A}_\nu$ and also, one can verify that the hypothesis $[\,H\,]$ is satisfied.

By considering a Liapunov function (which can be obtained as in **I.3**) one can prove that the equations (2.5) have a global attractor $\mathcal{A}$.

Consider the decomposition (2.3), (2.4) of the problem (2.1), (2.2), which we rewrite as

$$\dot{v}_{kl}(t) = -\lambda_{kl}v_{kl}(t) + P_{kl}(v,w), \tag{2.9}$$

for $1 \leq k \leq n$, $1 \leq j \leq m$ and

$$\left.\begin{aligned} &w_t = (aw_x)_x + (bw_y)_y + Q(v,w) \qquad (x,y)\in\Omega\\ &\frac{\partial u}{\partial n} = 0 \qquad (x,y)\in\partial\Omega.\end{aligned}\right\} \tag{2.10}$$

where

$$P_{kl}(v,w) = \int_\Omega f\left(\sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij}(t)\phi_{ij}(\tilde{x},\tilde{y}) + w(\tilde{x},\tilde{y})\right)\phi_{kl}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y},$$

$$Q(v,w)(x,y) = f\left(\sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij}(t)\phi_{ij}(x,y) + w(x,y)\right) - \sum_{k=1}^{n}\sum_{l=1}^{m} P_{kl}(v,w)\phi_{kl}(x,y).$$

Using the variation of constants formula we rewrite (2.9) as

$$w(t) = e^{-At}w_0 + \int_0^t e^{-A(t-s)}Q(v(s),w(s))ds. \tag{2.11}$$

The semigroup $T(t) := e^{-At}$ when restricted to $W^\perp$ satisfy the following properties:

$$\left.\begin{aligned}\|T(t)w\|_{W_{1/2}^\perp} &\leq K\,t^{-1/2}e^{-(\mu/\nu)t}\|w\|_{W_0^\perp},\\ \|T(t)w\|_{W_{1/2}^\perp} &\leq K\,e^{-(\mu/\nu)t}\|w\|_{W_{1/2}^\perp},\end{aligned}\right\} \tag{2.12}$$

where K is independent of ν. This can be done easily since the generator A is selfadjoint.

We are now in position to state our main theorem in this section.

Theorem 2.2. *Suppose that (2.5) has a compact attractor $\mathcal{A}$ and $V \subset \mathbb{R}^{nm}$ is a neighborhood of $\mathcal{A}$ such that $\omega(V) \subset \mathcal{A}$. Then, for any neighborhood U of $\mathcal{A}$ such that $\bar{U} \subset V$ there exists a $\nu_0 > 0$ and a Lipschitz continuous function $\sigma_\nu : \bar{U} \to W^\perp$, for $\nu \le \nu_0$, such that $\sigma_\nu(v) \to 0$ uniformily for $v \in \bar{U}$, and the set*

$$M_\nu = \{u = \sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij}\phi_{ij} + \sigma_\nu(v),\ v \in U\}$$

is an exponentially attractive invariant manifold for (2.1), (2.2). Furthermore, the flow in M_ν is given by $u(t,x,y) = \sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij}(t)\phi_{ij}(x,y) + \sigma_\nu(v(t))$, where $v(t)$ is the solution of

$$\dot{v}_{kl} = -\mu_{kl}\ v_{kl}\ +\ P_{kl}(v,\sigma_\nu(v)). \tag{2.13}$$

for $1 \le k \le n$, $1 \le j \le m$. The attractor $\mathcal{A}_\nu$ must lie in M_ν and the family of attractors

$$\{\mathcal{A}_\nu,\ 0 \le \nu \le \nu_0\}$$

is upper-semicontinuous at zero in $\mathbb{R}^{nm}$.

The proof of this result is essentially the same as the proof of Theorem 1.3 and we omit it.

Corollary 2.3. *Suppose f is smooth. Then for ν small the equations (2.13) have a compact attractor $\tilde{\mathcal{A}}_\nu$ in $\bar{U}$ and $\lim_{\nu\to 0}\tilde{\mathcal{A}}_\nu \subset \mathcal{A}$. If the flow defined by (2.5) is structurally stable on $\mathcal{A}$ in U then $\lim_{\nu\to 0}\tilde{\mathcal{A}}_\nu = \mathcal{A}$ and the flow defined by (2.13) is structurally stable on $\mathcal{A}$ in U and they are equivalent.*

III. SOME COMMENTS AND GENERALIZATIONS

1. An Eigenvalue Problem in a Two Dimensional Domain.

In this section we consider the case of a domain $\Omega \subset \mathbb{R}^2$ which is not a cartesian product. Let Ω be an open bounded smooth domain in $\mathbb{R}^2$ and assume that the intersection of Ω with $(x_0, x_2) \times \mathbb{R}$ is the rectangle $(x_0, x_2) \times (y_1, y_2)$, where $x_0 < x_2$ are constants.

Let $a_1, e_1, l_1,$, be positive constants, $x_1 \in (x_0, x_2)$ and ν be a positive parameter. Let $a_1'(\cdot), l_1'(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous and increasing functions of the parameter ν satisfying $a_1'(0) = a_1,\ l_1'(0) = l_1$. Assume that $\mathbf{a}_\nu, \mathbf{b}_\nu : \overline{\Omega} \to \mathbb{R}^+$ are functions in $C^1(\overline{\Omega})$ satisfying

$$\begin{aligned}
\mathbf{a}_\nu(x,y) &\ge \frac{e_1}{\nu}, \quad \forall (x,y) \in \Omega_i^{\nu'} \\
\mathbf{a}_\nu(x,y) &\ge a_1\nu, \quad \forall (x,y) \in \Omega \\
\mathbf{a}_\nu(x,y) &\le a_1'\nu, \quad \forall (x,y) \in \Omega \setminus \cup_{i=1}^{2} \Omega_i^\nu \\
\mathbf{b}_\nu(x,y) &\ge \frac{e_1}{\nu}, \quad \forall (x,y) \in \Omega,
\end{aligned}$$

where $\Omega_1^{\nu'} = \{(x,y) \in \Omega : x < x_1 - \nu l_1'\}$, $\Omega_1^\nu = \{(x,y) \in \Omega : x < x_1 - \nu l_1\}$, $\Omega_2^{\nu'} = \{(x,y) \in \Omega : x > x_1 + \nu l_1'\}$ and $\Omega_2^\nu = \{(x,y) \in \Omega : x > x_1 + \nu l_1\}$.

Consider the symmetric, second order elliptic differential operator

$$(L_\nu\phi)(x,y) = \frac{\partial}{\partial x}\left(\mathbf{a}_\nu(x,y)\frac{\partial}{\partial x}\phi(x,y)\right) + \frac{\partial}{\partial y}\left(\mathbf{b}_\nu(x,y)\frac{\partial}{\partial y}\phi(x,y)\right).$$

Let $X = L^2(\Omega)$ and consider the operator A_ν defined by

$$\begin{aligned}
D(A_\nu) &= \{\phi \in H^2(\Omega) : \frac{\partial\phi}{\partial n} = 0 \text{ in } \partial\Omega\} \\
(A_\nu\phi)(x,y) &= -(L_\nu\phi)(x,y),\ \forall\phi \in D(A_\nu).
\end{aligned}$$

It is known that A_ν is an unbounded, self adjoint operator with compact resolvent. Let $\{\lambda_n(\nu)\}$ be the ordered sequence of eigenvalues of A_ν and $\{\phi_n(\nu)\}$ be a corresponding sequence of normalized eigenfunctions.

Our goal is to study the behavior of the eigenvalues and eigenfunctions of A_ν as $\nu \to 0$. Our first result says that the first two eigenvalues of A_ν stay bounded whereas the others blow up as $\nu \to 0$; that is,

Lemma 1.1. *Let $\{\lambda_n(\nu)\}$ and $\{\phi_n(\nu)\}$ be as above. Then, $\lambda_1(\nu) \equiv 0$,*

$$\lim_{\nu \to 0} \lambda_2(\nu) = \frac{a_1}{2l_1} \frac{y_2 - y_1}{|\Omega_1| \, |\Omega_2|} |\Omega|,$$

and $\lambda_3(\nu) \to \infty$ as $\nu \to 0$. Furthermore, for any sequence $\{\nu_j\}$, $\nu_j \to 0$ as $j \to \infty$, there exists a subsequence $\{\nu_{j_k}\}$ of $\{\nu_j\}$, such that $\|\phi_2(\nu_{j_k}) - \mathcal{X}\|_{L^2(\Omega)} \to 0$ as $j_k \to \infty$, where

$$\mathcal{X} = \sum_{i=1}^{2} k_i \mathcal{X}_{\Omega_i},$$

with k_i, $i = 1,2$ satisfying $|\Omega_1| {k_1}^2 + |\Omega_2| {k_2}^2 = 1$, $\quad |\Omega_1| k_1 + |\Omega_2| k_2 = 0$.

Proof: Consider the functional $J_\nu : H^1(\Omega) \to \mathbb{R}$ defined by

$$J_\nu \phi = \int_\Omega \left[\mathbf{a}_\nu(x,y) \left(\frac{\partial}{\partial x} \phi(x,y) \right)^2 + \mathbf{b}_\nu(x,y) \left(\frac{\partial}{\partial y} \phi(x,y) \right)^2 \right] dx \, dy.$$

It is easy to see that $\lambda_1(\nu) \equiv 0$ and $\phi_1(\nu) \equiv |\Omega|^{-\frac{1}{2}}$. Also, we know that the second eigenvalue $\lambda_2(\nu)$ must satisfy

$$\lambda_2(\nu) = \min \left\{ \frac{J_\nu \phi}{\|\phi\|_{L^2}^2} \; : \phi \in H^1(\Omega), \; \phi \perp \phi_1(\nu) \right\}.$$

Therefore, to prove that $\lambda_2(\nu)$ remains bounded as $\nu \to 0$ we must exhibt a family of functions $\psi_\nu \in H^1(\Omega)$, $\|\psi_\nu\|_{L^2(\Omega)} = 1$ such that $J_\nu \psi_\nu$ remains bounded as $\nu \to 0$. Let $\psi_\nu : \Omega \to \mathbb{R}$, $\psi_\nu \perp \phi_1(\nu)$ be continuous and such that

$$\psi_\nu(x,y) = \begin{cases} K_1 + o(1), & (x,y) \in \Omega_1^\nu \\ K_2 + o(1), & (x,y) \in \Omega_2^\nu \\ \dfrac{\partial}{\partial x} \psi_\nu(x,y) = \dfrac{K_2 - K_1}{2\nu l_1} + o(1), \quad \dfrac{\partial}{\partial y} \psi_\nu(x,y) = 0, & (x,y) \in \Omega \setminus \cup_{i=1}^2 \overline{\Omega_i^\nu} \end{cases}$$

where $K_1 = -|\Omega|^{-\frac{1}{2}} \left(\frac{|\Omega_2|}{|\Omega_1|} \right)^{\frac{1}{2}}$ and $K_2 = \; |\Omega|^{-\frac{1}{2}} \left(\frac{|\Omega_1|}{|\Omega_2|} \right)^{\frac{1}{2}}$. Then,

$$\begin{aligned} L_\nu \psi_\nu &= \int_{\Omega \setminus \cup_{i=1}^2 \overline{\Omega_i^\nu}} \left[\mathbf{a}_\nu(x,y) \left(\frac{\partial}{\partial x} \psi_\nu(x,y) \right)^2 + \mathbf{b}_\nu(x,y) \left(\frac{\partial}{\partial y} \psi_\nu(x,y) \right)^2 \right] dx \, dy \\ &= \int_{\Omega \setminus \cup_{i=1}^2 \overline{\Omega_i^\nu}} \mathbf{a}_\nu(x,y) \left(\frac{\partial}{\partial x} \psi_\nu(x,y) \right)^2 dx \, dy \\ &= \int_{\Omega \setminus \cup_{i=1}^2 \overline{\Omega_i^\nu}} \mathbf{a}_\nu(x,y) dx \, dy \left(\frac{K_2 - K_1}{2\nu l_1} \right)^2 dx \, dy \\ &\le a_1' \nu \left| \Omega \setminus \cup_{i=1}^2 \overline{\Omega_i^\nu} \right| \left(\frac{K_2 - K_1}{2\nu l_1} \right)^2 + o(1) = a_1' \nu \, 2\nu l_1 \, (y_2 - y_1) \left(\frac{K_2 - K_1}{2\nu l_1} \right)^2 + o(1) \\ &= \frac{a_1}{2l_1} \frac{y_2 - y_1}{|\Omega_1| \, |\Omega_2|} |\Omega| + o(1), \end{aligned}$$

and $\|\psi_\nu\|_{L^2(\Omega)} = 1 + o(1)$. This implies that

$$\lambda_2(\nu) \le \frac{L_\nu \psi_\nu}{\|\psi_\nu\|_{L^2}^2} \le \frac{a_1}{2l_1} \frac{y_2 - y_1}{|\Omega_1| \, |\Omega_2|} |\Omega| + o(1)$$

and $\limsup_{\nu\to 0}\lambda_2(\nu) \leq \frac{a_1}{2l_1}\frac{y_2-y_1}{|\Omega_1||\Omega_2|}|\Omega|$, proving that there are at least two eigenvalues that stay bounded as $\nu \to 0$.

To prove that there are only two eigenvalues that stay bounded we must study the behavior of the eigenfunctions associated to those eigenvalues that stay bounded. Let $\lambda(\nu)$, $\phi(\nu)$ be an eigenpair such that $\sup_{0\leq\nu\leq 1}\lambda(\nu) < \infty$. Then, there are constants k_1 and k_2 such that

$$||\phi(\nu) - \mathcal{X}||_{L^2(\Omega)} \to 0,$$

as $\nu \to 0$, where $\mathcal{X} = \sum_{i=1}^{2} k_i \mathcal{X}_{\Omega_i}$. To prove that notice that there is a constant $m \geq 0$ such that,

$$\int_\Omega \left[\mathrm{a}_\nu(x,y)\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2 + \mathrm{b}_\nu(x,y)\left(\frac{\partial}{\partial y}\phi(x,y)\right)^2 \right] dx\,dy \leq m.$$

Therefore,

$$\int_{\Omega_i^{\nu'}} \left[\left(\frac{\partial}{\partial x}\phi(x,y)\right)^2 + \left(\frac{\partial}{\partial y}\phi(x,y)\right)^2 \right] dx\,dy \leq \frac{m}{e_1}\nu.$$

This implies that for any $0 < \nu \leq \nu_0$ we have

$$||\phi(\nu)||_{H^1(\Omega_i^{\nu_0'})} \leq 1 + \frac{m}{e_1}\nu_0,$$

which implies that there is a sequence ν_j and a constant k_i such that

$$||\phi(\nu_j) - k_i||_{L^2(\Omega_i^{\nu_0'})} \to 0 \quad \text{as} \quad \nu_j \to 0.$$

Next we want to prove that given $\nu_1 < \nu_0$ we have that $||\phi(\nu_j) - k_i||_{L^2(\Omega_i^{\nu_1'})} \to 0$. It is easy to see that for j large $||\phi(\nu_j)||_{H^1(\Omega_i^{\nu_0'})} \leq 1 + \frac{m}{e_1}\nu_1$. Therefore, there exist a subsequence $\nu_{\nu_{j_k}} \to 0$ and constant $\tilde{k}_i$ such that

$$||\phi(\nu_{j_k}) - \tilde{k}_i||_{L^2(\Omega_i^{\nu_1'})} \to 0 \quad \text{as} \quad \nu_{j_k} \to 0.$$

Since

$$||\phi(\nu_{j_k}) - k_i||_{L^2(\Omega_i^{\nu_0'})} \to 0 \quad \text{as} \quad \nu_{j_k} \to 0$$

we have that $k_i = \tilde{k}_i$. Hence, the limit does not depend on the subsequence and we have that

$$||\phi(\nu_j) - k_i||_{L^2(\Omega_i^{\nu_1'})} \to 0 \quad \text{as} \quad \nu_j \to 0.$$

Remark: So far, we have not proved that the constants k_i, $i = 1, 2$ do not depend upon the sequence ν_j. We have only proved that the constants do not depend upon ν_0.

Assume that,

$$||\phi_\nu||_{L^2(\Omega\setminus\cup_{i=1}^2\Omega_i^{\nu'})} \to 0 \tag{1.1}$$

as $\nu \to 0$. Then k_i must satisfy

$$|\Omega_1|{k_1}^2 + |\Omega_2|{k_2}^2 = 1, \quad |\Omega_1|k_1 + |\Omega_2|k_2 = 0. \tag{1.2}$$

Therefore if $\lambda_3(\nu)$ stays bounded we have that there exist a sequence $\nu_j \to 0$ and constants $k_i, \tilde{k}_i$, $i = 1, 2$ such that for any $\nu_1 > 0$

$$||\phi_2(\nu_j) - k_i||_{L^2(\Omega_i^{\nu_1'})} \to 0 \quad \text{as} \quad \nu_j \to 0$$

and

$$||\phi_3(\nu_j) - \tilde{k}_i||_{L^2(\Omega_i^{\nu_1'})} \to 0 \quad \text{as} \quad \nu_j \to 0.$$

Since $\phi_2 \perp \phi_3$, we have a contradiction since (1.2) implies

$$|\Omega_1|k_1\bar{k}_1 + |\Omega_2|k_2\bar{k}_2 \neq 0.$$

Therefore it only remains to prove (1.1). To do this we consider two points (x,y), (t,s) in the set $\Omega \backslash \cup_{i=1}^{1} \Omega_i^{\nu'}$. Then the following is satisfied

$$\begin{aligned}
|\phi(x,y) - \phi(s,t)|^2 &\leq 2|\phi(x,y) - \phi(x,t)|^2 + |\phi(x,t) - \phi(s,t)|^2 \\
&= 2\left|\int_t^y \phi_y(x,\tau)d\tau\right|^2 + 2\left|\int_s^x \phi_x(\theta,t)d\theta\right|^2 \\
&\leq 2(y_2 - y_1)\int_{y_1}^{y_2} (\phi_y(x,\tau))^2 d\tau + 4\nu l_1' \int_{x_1-\nu l_1'}^{x_1+\nu l_1'} (\phi_x(\theta,t))^2 d\theta
\end{aligned}$$

and

$$\begin{aligned}
&\int_{y_1}^{y_2}\int_{y_1}^{y_2}\int_{x_1-\nu l_1'}^{x_1+\nu l_1'} |\phi(x,y) - \phi(s,t)|^2 dx\, dy\, dt \\
&\qquad \leq 2(y_2 - y_1)\int_{y_1}^{y_2}\int_{y_1}^{y_2}\int_{x_1-\nu l_1'}^{x_1+\nu l_1'}\int_{y_1}^{y_2} (\phi_y(x,\tau))^2 d\tau dx\, dy\, dt \\
&\qquad + 4\nu l_1' \int_{y_1}^{y_2}\int_{y_1}^{y_2}\int_{x_1-\nu l_1'}^{x_1+\nu l_1'}\int_{x_1-\nu l_1'}^{x_1+\nu l_1'} (\phi_x(\theta,t))^2 d\theta dx\, dy\, dt \\
&\qquad \leq 2(y_2 - y_1)^3 \int_{x_1-\nu l_1'}^{x_1+\nu l_1'}\int_{y_1}^{y_2} (\phi_y(x,\tau))^2 d\tau dx \\
&\qquad + 4\nu l_1'(y_2 - y_1)2\nu l_1' \int_{y_1}^{y_2}\int_{x_1-\nu l_1'}^{x_1+\nu l_1'} (\phi_x(\theta,t))^2 d\theta\, dt \\
&\qquad \leq \left[2(y_2 - y_1)^3 \frac{m}{\epsilon_1} + 8{l_1'}^2 (y_2 - y_1)\frac{m}{a_1}\right]\nu.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left(\int_{y_1}^{y_2}\int_{x_1-\nu l_1'}^{x_1+\nu l_1'} |\phi(x,y)|^2 dx\, dy\right)^{\frac{1}{2}} - \left(2\nu l_1' \int_{y_1}^{y_2} |\phi(s,t)|^2 dt\right)^{\frac{1}{2}} \\
&\qquad\qquad \leq \left[m\left[\frac{2(y_2 - y_1)^2}{\epsilon_1} + \frac{8{l_1'}^2}{a_1}\right]\nu\right]^{\frac{1}{2}}.
\end{aligned} \tag{1.3}$$

Next we consider the case when $s = x_1 - \nu l_1'$ which, for convenience, we continue to denote by s and the following holds

$$\begin{aligned}
\int_{y_1}^{y_2}\int_{y_1}^{y_2}\int_{x_0}^{s} |\phi(x,y) - \phi(s,t)|^2 dx\, dy\, dt &\leq 2(y_2 - y_1)^3 \int_{x_0}^{s}\int_{y_1}^{y_2} (\phi_y(x,\tau))^2 d\tau dx \\
&+ 2(x_0 - s)^2(y_2 - y_1)\int_{y_1}^{y_2}\int_{x_0}^{s} (\phi_x(\theta,t))^2 d\theta\, dt \\
&\leq \left[2(y_2 - y_1)^3 \frac{m}{\epsilon_1} + 2(x_0 - s)^2(y_2 - y_1)\frac{m}{\epsilon_1}\right]\nu.
\end{aligned}$$

This implies that

$$\left(\frac{1}{y_2 - y_1}\int_{y_1}^{y_2} |\phi(s,t)|^2 dt\right)^{\frac{1}{2}} - \left(\frac{1}{(s - x_0)(y_2 - y_1)}\int_{y_1}^{y_2}\int_{x_0}^{s} |\phi(x,y)|^2 dx\, dy\right)^{\frac{1}{2}} \leq O(\nu^{\frac{1}{2}}) \tag{1.4}$$

where $O(\nu^{\frac{1}{2}}) = \left[\frac{\left[2(y_2-y_1)^2\frac{m}{\epsilon_1}+2(x_0-s)^2\frac{m}{\epsilon_1}\right]\nu}{(s-x_0)(y_2-y_1)}\right]^{\frac{1}{2}}$ and

$$\left(\frac{1}{y_2 - y_1}\int_{y_1}^{y_2} |\phi(s,t)|^2 dt\right)^{\frac{1}{2}} \leq O(\nu^{\frac{1}{2}}) + \left[\frac{1}{(s - x_0)(y_2 - y_1)}\right]^{\frac{1}{2}}. \tag{1.5}$$

The inequality (1.5) combined with (1.3) implies that there is only two eigenvalues that stays bounded.

The next step is to prove that

$$\liminf_{\nu \to 0} \lambda_2(\nu) \geq \frac{a_1}{2l_1}\frac{y_2 - y_1}{|\Omega_1|\,|\Omega_2|}|\Omega|.$$

We start by proving a few estimates for the second eigenfunction. which for convenience we denote $\phi = \phi(\nu)$,

$$\begin{aligned}
\int_\Omega \left[\mathbf{a}_\nu(\phi_x)^2 + \mathbf{b}_\nu(\phi_y)^2\right] dx\,dy &\geq \int_{x_1-\nu l_1}^{x_1+\nu l_1}\int_{y_1}^{y_2} \mathbf{a}_\nu\,(\phi_x)^2\,dx\,dy \\
&\geq a_1\nu\int_{x_1-\nu l_1}^{x_1+\nu l_1}\int_{y_1}^{y_2} (\phi_x)^2\,dx\,dy \geq \frac{a_1}{2l_1}\int_{y_1}^{y_2}\left(\int_{x_1-\nu l_1}^{x_1+\nu l_1}\phi_x\,dx\right)^2 dy \\
&= \frac{a_1}{2l_1}\int_{y_1}^{y_2} |\phi(x_1+\nu l_1, y) - \phi(x_1 - \nu l_1, y)|^2\,dy \\
&\geq \frac{a_1}{2l_1}\left[\left(\int_{y_1}^{y_2}(\phi(x_1+\nu l_1,y))^2\,dy\right)^{\frac{1}{2}} - \left(\int_{y_1}^{y_2}(\phi(x_1-\nu l_1,y))^2\,dy\right)^{\frac{1}{2}}\right]^2
\end{aligned}$$

and, if $\nu_j \to 0$ is such that

$$\phi(\nu_j) \to \mathcal{X} \text{ in } L^2(\Omega), \quad \text{where} \quad \mathcal{X} = \begin{cases} k_1, & x \in \Omega_1 \\ k_2, & x \in \Omega_2 \end{cases} \tag{1.6}$$

with k_1, k_2 satisfying (1.2). Then, from (1.6), we obtain

$$\frac{1}{(y_2 - y_1)(s - x_0)}\int_{y_1}^{y_2}\int_{x_0}^{s} \phi(x,y)^2\,dx\,dy \to {k_1}^2$$

which by (1.4) implies that

$$\frac{1}{y_2 - y_1}\int_{y_1}^{y_2} \phi(x_1 - \nu l_1, y)^2\,dy \to {k_1}^2.$$

In the same way

$$\frac{1}{y_2 - y_1}\int_{y_1}^{y_2} \phi(x_1 + \nu l_1, y)^2\,dy \to {k_2}^2.$$

Therefore,

$$\begin{aligned}
\liminf_{\nu \to 0}\int_\Omega \left[\mathbf{a}_\nu(x,y)(\phi_x(x,y))^2 + \mathbf{b}_\nu(x,y)(\phi_y(x,y))^2\right] dx\,dy &\geq \frac{a_1}{2l_1}(y_2 - y_1)(k_2 - k_1)^2 \\
&= \frac{a_1}{2l_1}\frac{y_2 - y_1}{|\Omega_1|\,|\Omega_2|}|\Omega|
\end{aligned}$$

and the result is proved.

If $\mathbf{b}_\nu$ is assumed to be large everywhere in Ω, one cannot expect to have a curve Γ different from a straight vertical line separating the regions Ω_1 and Ω_2. One solution to that problem would be to make $\mathbf{b}_\nu$

small along Γ. However, the proof given in Lemma 1.1 does not apply since, in this case, we were not able to prove the convergence (1.1).

This observation seems to indicate that, by fixing the elliptic operator, we somehow fix the type of partition of the domain that we can allow.

§2. Some Comments on Bifurcation.

Consider the scalar parabolic problem

$$u_t = (au_x)_x + f(u), \quad 0 < x < 1, \quad t > 0 \tag{2.1}$$

$$u_x(0) = u_x(1) = 0, \quad t > 0. \tag{2.2}$$

Assume that $a \in C^2([0,1], \mathbb{R})$, $a(x) > 0$, $x \in [0,1]$. We also assume that a is large except in a neighborhood of a point where it becomes small. To describe the coefficient a, let $0 = x_0 < x_1 < x_2 = 1$ be a partition of the interval [0,1] and let l_1 be a positive constant and l_1' be a function of a positive parameter ν that approach l_1 from above as $\nu \to 0$. Then if $e_1(\nu), e_2(\nu)$ are positive functions of the ν that go to infinity as the parameter goes to zero, if $a_1(\nu) \le a_1'(\nu)$ are positive functions of ν that go to zero as the parameter goes to zero (and ν is sufficiently small), let a be such that

$$\left.\begin{array}{lll} a(x) \ge e_1(\nu), & \text{for} & x_0 \le x \le x_1 - \nu l_1', \\ a(x) \ge e_2(\nu), & \text{for} & x_1 + \nu l_1' \le x \le x_2, \\ a(x) \ge a_1(\nu), & \text{for} & x_1 - \nu l_1' \le x \le x_1 + \nu l_1', \\ a(x) \le a_1'(\nu), & \text{for} & x_1 - \nu l_1 \le x \le x_1 + \nu l_1. \end{array}\right\} \tag{2.3}$$

We also assume that

$$\left.\begin{array}{l} l_i' - l_i = o(1), \\ a_i' - a_i = o(1). \end{array}\right\} \tag{2.4}$$

We borrow the notation of **I.4** and concentrate in the generalization of the results for the case when the diffusion coefficient has two bumps and one well. However similar generalizations follow, in the same way, for the more general situations.

Lemma 2.1. *Assume that* $\frac{a_1'(\nu)}{\nu \min\{e_i(\nu), i=1,2\}} = o(1)$ *and that* $\frac{\nu^2 \min\{e_i(\nu), i=1,2\}}{a(\nu)} = O(1)$. *Consider the operator*

$$A : D(A) \subset L^2(0,1) \to L^2(0,1)$$

defined by

$$D(A) = \{u \in H^2(0,1) : u_x(0) = u_x(1) = 0\},$$
$$Au = -(au_x)_x.$$

Let $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ *be the sequence of its eigenvalues and* $\phi_1, \phi_2, \phi_3, \cdots$ *a corresponding sequence of normalized eigenfunctions. Then* $\lambda_1 = 0$,

$$\liminf_{\nu \to 0} \frac{a_1(\nu)}{2\nu l_1} \frac{1}{x_1(1-x_1)} \le \liminf_{\nu \to 0} \lambda_2(\nu) \le \limsup_{\nu \to 0} \lambda_2(\nu) \le \limsup_{\nu \to 0} \frac{a_1'(\nu)}{2\nu l_1} \frac{1}{x_1(1-x_1)}$$

and $\lambda_3 \to \infty$ *as* $\nu \to 0$. *Furthermore, we can choose* $\phi_1 = 1$ *and* ϕ_2 *satisfying*

$$\phi_2(x) = \begin{cases} -\sqrt{\dfrac{1-x_1}{x_1}} + o(1), & x \in [0, x_1 - \nu l_1'] \\ o(\nu^{-\frac{1}{2}}), & x \in [x_1 - \nu l_1', x_1 + \nu l_1'] \\ \sqrt{\dfrac{x_1}{1-x_1}} + o(1), & x \in [x_1 + \nu l_1', 1] \end{cases}$$

The proof of this results is similar to the proof of Lemma 1.1 in Carvalho and Pereira [1991] and we omit it

Assume that $\liminf_{\nu \to 0} \frac{a_1(\nu)}{2\nu l_1} = \limsup_{\nu \to 0} \frac{a_1'(\nu)}{2\nu l_1} = a_1$ and consider the following system of ordinary differential equations

$$\left.\begin{aligned} \dot{z}_1 &= \frac{a_1}{2l_1 x_1}(z_2 - z_1) + f(z_1), \\ \dot{z}_2 &= \frac{-a_1}{2l_1(1-x_1)}(z_2 - z_1) + f(z_2). \end{aligned}\right\} \tag{2.5}$$

We proved that it determines the asymptotic behavior of (2.1), (2.2) for small values of the parameter ν. Consider now the the following cases:

1) $\frac{a_1(\nu)}{\nu} \to \infty$ as $\nu \to 0$.

By Lemma 2.1, $\lambda_2 \to \infty$ as $\nu \to 0$ and the limiting equation must be

$$\dot{v} = f(v), \tag{2.6}$$

where v is identified to the average of u in the interval $[0,1]$.

2) $\frac{a_1'(\nu)}{\nu} \to 0$ as $\nu \to 0$.

By Lemma 2.1 we have that $\lambda_2 \to 0$ as $\nu \to 0$. In this case, the limiting equations must be

$$\begin{aligned} \dot{z}_1 &= f(z_1), \\ \dot{z}_2 &= f(z_2), \end{aligned} \tag{2.7}$$

where z_i is identified to the average of u in the interval $[x_{i-1}, x_i]$, $i = 1, 2$.

An interesting phenomenon is observed in this two cases. Suppose $f(u) = u - u^3$. According to the rates of growth (decay) of the coefficients defining a, we can obtain a single equation with three equilibrium points or a system of two decoupled equations with nine equilibrium points as limiting equation. If we introduce the case when the second eigenvalue converges, we obtain the system (2.5) as limiting equation. By varying a_1 we can go from case 1) to case 2); that is, from three equilibrium points to nine equilibrium points.

To illustrate this, consider $x_1 = 1/2$ and rewrite the equations (2.5) as

$$\left.\begin{aligned} \dot{z}_1 &= r(z_2 - z_1) + f(z_1), \\ \dot{z}_2 &= -r(z_2 - z_1) + f(z_2), \end{aligned}\right\} \tag{2.8}$$

where $r = \frac{a_1}{l_1}$. In this case we can compute the equilibrium points explicitly and they have simple expressions, namely:

$$(z_1, z_2) = (\ 0,0\ ), \qquad (z_1, z_2) = (\ 1,1\ ), \qquad (z_1, z_2) = (\ -1,-1\ ),$$

$$(z_1, z_2) = (\ -\sqrt{1-2r}, \sqrt{1-2r}\ ), \qquad (z_1, z_2) = (\ \sqrt{1-2r}, -\sqrt{1-2r}\ ),$$

$$(z_1, z_2) = \left(-\frac{\sqrt{1 - r - \sqrt{1-2r-3r^2}}}{\sqrt{2}}, \frac{\sqrt{1 - r + \sqrt{1-2r-3r^2}}}{\sqrt{2}} \right),$$

$$(z_1, z_2) = \left(\frac{\sqrt{1 - r - \sqrt{1-2r-3r^2}}}{\sqrt{2}}, -\frac{\sqrt{1 - r + \sqrt{1-2r-3r^2}}}{\sqrt{2}} \right),$$

$$(z_1, z_2) = \left(-\frac{\sqrt{1 - r + \sqrt{1-2r-3r^2}}}{\sqrt{2}}, \frac{\sqrt{1 - r - \sqrt{1-2r-3r^2}}}{\sqrt{2}} \right),$$

$$(z_1, z_2) = \left(\frac{\sqrt{1 - r + \sqrt{1-2r-3r^2}}}{\sqrt{2}}, -\frac{\sqrt{1 - r - \sqrt{1-2r-3r^2}}}{\sqrt{2}} \right)$$

We observe that for $r \geq 1/2$ the only possible equilibrium points must satisfy $z_1 = z_2$ which implies that the only equilibrium points are (0,0), (1,1) and (-1,-1). Note that as the parameter r varies from zero to 1/2 we observe a bifurcation at $r = 1/3$ where four roots cease to exist and a bifurcation as $r = 1/2$ where two roots cease to exist and there is no bifurcation for $r > 1/2$. All the bifurcations are pitchfork bifurcations.

Remark. *Lemma 2.1 can be generalized to include any number of wells. It implies that the rate at which the deep well goes to zero is critical in the case considered by Fusco [1987] and Carvalho and Pereira [1991].*

§3. Systems.

The results in **I.3** can also be carried on for systems of weakly coupled parabolic equations. To ilustrate that, consider the following problem

$$\left.\begin{aligned} &u_t = (au_x)_x + f(u,v), \quad 0 < x < 1, \quad t > 0 \\ &u_x(0) = u_x(1) = 0, \quad t > 0 \\ &v_t = (bv_x)_x + g(u,v), \quad 0 < x < 1, \quad t > 0 \\ &v_x(0) = v_x(1) = 0, \quad t > 0. \end{aligned}\right\} \tag{3.1}$$

where $a \in C_2^1(e^{\bar{a}}, l^{\bar{a}'}, \bar{a}', x_1, \nu)$ and $b \in C_2^1(e^{\bar{b}}, l^{\bar{b}'}, \bar{b}', y_1, \nu)$. We assume that $x_1 \leq y_1$. Let $\lambda_1^a, \lambda_1^a, \cdots \ \phi_1^a, \phi_2^a, \cdots$ be the eigenvalues and a corresponding sequence of eigenfunctions of $A^a u = -(au_x)x$ with homogeneous Neumann boundary conditions and $\lambda_1^b, \lambda_1^b, \cdots \ \phi_1^b, \phi_2^b, \cdots$ be the eigenvalues and a corresponding sequence of eigenfunctions of $A^b u = -(bu_x)x$ with homogeneous Neumann boundary conditions. With the operators A^a and A^b, we associate the corresponding fractional power spaces X_α^a and X_β^a. By Theorem 3.1 in **I** we have that

$$\lambda_1^a \equiv \lambda_1^b \equiv 0, \quad \lambda_2^a \to \frac{a_1}{2l_1^a}\frac{1}{\sqrt{x_1(1-x_1)}} \quad \lambda_2^b \to \frac{b_1}{2l_1^b}\frac{1}{\sqrt{y_1(1-y_1)}} \quad \lambda_3^a \geq \frac{\mu^a}{\nu} \quad \lambda_3^b \geq \frac{\mu^b}{\nu}$$

$$\phi_1^a \equiv \phi_1^b \equiv 1, \quad \phi_2^a \to \begin{cases} -\sqrt{\dfrac{1-x_1}{x_1}}, & x \in (0, x_1) \\ \sqrt{\dfrac{x_1}{1-x_1}}, & x \in (x_1, 1) \end{cases} \qquad \phi_2^b \to \begin{cases} -\sqrt{\dfrac{1-y_1}{y_1}}, & y \in (0, y_1) \\ \sqrt{\dfrac{y_1}{1-y_1}}, & y \in (y_1, 1). \end{cases}$$

By projecting the equations on the eigenspaces associated to the eigenvalues that converge we obtain the following set of limiting ordinary differential equations.

$$\left.\begin{aligned} \dot{z}_1 &= x_1 f(z_1 + k_1^a z_2, \xi_1 + k_1^b \xi_2) + (y_1 - x_1) f(z_1 - k_2^a z_2, \xi_1 + k_1^b \xi_2) \\ &\qquad + (1 - y_1) f(z_1 - k_2^a z_2, \xi_1 - k_2^b \xi_2), \\ \dot{z}_2 &= -\frac{a_1}{2l_1^a}\frac{1}{x_1(1-x_1)} z_2 + k_1^a x_1 f(z_1 + k_1^a z_2, \xi_1 + k_1^b \xi_2) \\ &\quad - k_2^a (y_1 - x_1) f(z_1 - k_2^a z_2, \xi_1 + k_1^b \xi_2) - k_2^a (1 - y_1) f(z_1 - k_2^a z_2, \xi_1 - k_2^b \xi_2), \\ \dot{\xi}_1 &= x_1 g(z_1 + k_1^a z_2, \xi_1 + k_1^b \xi_2) + (y_1 - x_1) g(z_1 - k_2^a z_2, \xi_1 + k_1^b \xi_2) \\ &\qquad + (1 - y_1) g(z_1 - k_2^a z_2, \xi_1 - k_2^b \xi_2), \\ \dot{\xi}_2 &= -\frac{b_1}{2l_1^b}\frac{1}{y_1(1-y_1)} z_2 + k_1^b x_1 g(z_1 + k_1^a z_2, \xi_1 + k_1^b \xi_2) \\ &\quad + k_1^b (y_1 - x_1) g(z_1 - k_2^a z_2, \xi_1 + k_1^b \xi_2) - k_2^b (1 - y_1) g(z_1 - k_2^a z_2, \xi_1 - k_2^b \xi_2). \end{aligned}\right\} \tag{3.2}$$

By introducing the change of variables

$$\begin{aligned} \eta_1 &= z_1 + k_1^a z_2 \qquad & \theta_1 &= \xi_1 + k_1^b \xi_2 \\ \eta_2 &= z_1 - k_2^a z_2 \qquad & \theta_2 &= \xi_1 - k_2^b \xi_2 \end{aligned}$$

we can transform the above equations into

$$
\left.
\begin{aligned}
\dot{\eta}_1 &= \frac{a_1}{2l_1^a x_1}(\eta_2-\eta_1)+f(\eta_1,\theta_1),\\
\dot{\eta}_2 &= \frac{-a_1}{2l_1^a(1-x_1)}(\eta_2-\eta_1)+\frac{y_1-x_1}{1-x_1}f(\eta_2,\theta_1)+\frac{1-y_1}{1-x_1}f(\eta_2,\theta_2),\\
\dot{\theta}_1 &= \frac{b_1}{2l_1^b y_1}(\theta_2-\theta_1)+\frac{x_1}{y_1}g(\eta_1,\theta_1)+\frac{y_1-x_1}{y_1}g(\eta_2,\theta_1),\\
\dot{\theta}_2 &= \frac{-b_1}{2l_1^b(1-y_1)}(\theta_2-\theta_1)+g(\eta_2,\theta_2),
\end{aligned}
\right\}
\tag{3.3}
$$

These results can be proved for an arbitrary number of bumps and wells and for an arbitrary number of weakly coupled equations.

We remark that if the system of parabolic equations is gradient, then the system of ordinary differential equations that we obtain will also be gradient. Suppose we start with the system (3.1) and that we impose the following additional hypotheses

$$
\limsup_{|u|\to\infty}\frac{f(u,v)}{u}\le-\delta \qquad \limsup_{|v|\to\infty}\frac{g(u,v)}{v}\le-\delta
$$

for some $\delta>0$, and

$$
\frac{\partial f}{\partial v}(s,t)=\frac{\partial g}{\partial u}(s,t), \quad \forall (s,t)\in\mathbb{R}^2.
$$

Then, the system (3.1) is a gradient system and the functional $V: X_a^{\frac{1}{2}}\times X_b^{\frac{1}{2}}\to\mathbb{R}^+$ defined by

$$
V(\phi,\psi)=\int_0^1\frac{1}{2}\left[a(x)\phi'(x)^2+\phi(x)^2+b(x)\psi'(x)^2+\psi(x)^2-F(\phi(x),\psi(x))\right]dx,
$$

where $F(u,v)=\int_0^u f(s,v)ds+\int_0^v g(u,s)ds$, is a Liapunov Functional for (3.1).

By decomposing u and v in the usual way, by taking the limit and then changing variables we obtain the following functional for the equations (3.3)

$$
\begin{aligned}
V(\bar{\eta},\bar{\theta}) =&\frac{1}{2}\left(\frac{1}{x_1}\eta_1+\frac{1}{1-x_1}\eta_2\right)^2+\frac{1}{2}\frac{1}{x_1(1-x_1)}(\eta_1-\eta_2)^2+\\
&\frac{1}{2}\left(\frac{1}{y_1}\theta_1+\frac{1}{1-y_1}\theta_2\right)^2+\frac{1}{2}\frac{1}{y_1(1-y_1)}(\theta_1-\theta_2)^2+\\
&\frac{1}{2}\frac{a_1}{2l_1^a}\frac{1}{[x_1(1-x_1)]^{\frac{3}{2}}}(\eta_1-\eta_2)^2+\frac{1}{2}\frac{b_1}{2l_1^b}\frac{1}{[y_1(1-y_1)]^{\frac{3}{2}}}(\theta_1-\theta_2)^2-G(\bar{\eta},\bar{\theta}),
\end{aligned}
$$

where $(\bar{\eta},\bar{\theta})=(\eta_1,\eta_2,\theta_1,\theta_2)$ and

$$
\begin{aligned}
G(\bar{\eta},\bar{\theta})=&\frac{1}{2}\left[x_1\int_0^{\eta_1}f(s,\theta_1)ds+(y_1-x_1)\int_0^{\eta_2}f(s,\theta_1)ds+(1-y_1)\int_0^{\eta_2}f(s,\theta_2)ds\right]\\
&+\frac{1}{2}\left[x_1\int_0^{\theta_1}g(\eta_1,s)ds+(y_1-x_1)\int_0^{\theta_1}g(\eta_2,s)ds+(1-y_1)\int_0^{\theta_2}g(\eta_2,s)ds\right].
\end{aligned}
$$

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