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ON BIFURCATION AND SYMMETRY OF SOLUTIONS OF NONLINEAR D_m-EQUIVARIANT EQUATIONS *

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ABSTRACT. The object of this work is to study existence, bifurcation and symmetries of small solutions of the nonlinear equation: $Lx = N(x,p,\varepsilon)+\mu f$, which is supposed to be equivariant under the action of the group D_m. We assume that L is a linear operator, $N(\cdot,p,\varepsilon)$ is a nonlinear operator, both defined in a Banach space X, with values in a Banach space Z, p, ε and μ are small real parameters and f is a fixed element of Z. We show that the symmetries of the equation imply the existence of symmetric solutions. We also prove that under a generic condition the symmetric solutions are the only feasible ones. Some examples of nonlinear ordinary and partial differential equations are analized.

AMS (MOS) subject classifications. primary 34A34, 34C15, 34C25

1. INTRODUCTION

Let X, Z be Banach spaces. Consider the equation,

$$Lx = N(x,p,\varepsilon) + \mu f \tag{1.1}$$

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where $L : X \to Z$ is a continuous linear operator, $N(\cdot, p, \varepsilon) : X \to Z$ is a nonlinear operator, f is a fixed element in Z and p, ε, μ are small real parameters.

If $\Gamma, \tilde{\Gamma}$ are representations of D_m, $m \geq 2$, over X and Z, respectively, and if equation (1.1) is $(D_m, \Gamma, \tilde{\Gamma})$-equivariant, our main results state that, under certain conditions, there exist small symmetric solutions. We also analyze the variation of the number of such solutions as (p, μ, ε) varies in a small neighborhood of the origin. Under certain generic conditions we show that the symmetric solutions are the only feasible ones.

Previous works have been developed by many authors to Duffing's Equation and more generally to equations of the form

$$\ddot{x} + x = g(x, p) + \mu f \tag{1.2}$$

where f is $2\pi/m$-periodic, $m \geq 1$ is an integer, g is nonlinear, p and μ are small parameters.

Hale and Rodrigues [4] studied Duffing's Equation

$$\ddot{x} + x = px + x^3 + \mu f \ , \tag{1.3}$$

where f is an even, 2π-periodic real function, such that $\int_0^{2\pi} f(s) \cos s \, ds \neq 0$. Besides describing the bifurcations they proved that the only small 2π-periodic solutions of (1.3) are even functions of t. Using these results they studied in [5], the damped Duffing's Equation:

$$\ddot{x} + x = px + x^3 + \delta \dot{x} + \mu \cos t \ .$$

Rodrigues and Vanderbauwhede [6] generalized these results to abstract nonlinear equations in a Banach space, and more applications were given, including to nonlinear partial differential equations.

Furkotter and Rodrigues [1],[2] considered the case where f is $2\pi/m$-periodic, with $m \geq 2$. Under a certain generic conditions, they proved that the small 2π-periodic solutions of (1.2) maintain some symmetry properties of the forcing term f, when $\mu \neq 0$. Vanderbauwhede [8] also presents some related results, for systems of ordinary differential equations.

In §2, using the equivariant Liapunov-Schmidt method, the implicit function theorem, we reduce equation (1.1) to an equivalent symmetric complex equation.

In §3 we present the normal form of complex D_m-equivariant functions and using these results we find a set of solutions of (1.1) which are either "harmonic" or "subharmonic". Under our assumptions, if f is $(D_m, \tilde{\Gamma})$-invariant the equation $Lx = f$ has a unique solution, say Kf, that is (D_m, Γ)-invariant.

We show that each solution of (1.1), determined above, has at least one symmetry of Kf. We also find the bifurcation surfaces in the (p, μ, ε)-parameter space.

In §4 we discuss the special case of equation (1.1), where the operators L and $N(\cdot, p, \varepsilon)$ are $(O(2), \Gamma, \tilde{\Gamma})$-equivariant and f is $(D_m, \tilde{\Gamma})$-invariant. We show that, under a specific condition, the small solutions of (1.1) must necessarily inherit some symmetry of Kf. We also study the genericity of such condition.

In §5 we present some applications of our results to nonlinear ordinary and partial differential equations. We point out that in some respects our examples are more general than the ones presented in the cited earlier papers.

The results of this paper indicate that when one truncates the Taylor series of the nonlinearity, it is important to consider terms up to a sufficient large order. Otherwise the informations of our theorems may be lost. For example if one intends to apply our results to the pendulum equation $\ddot{v} + \frac{g}{L}\sin v = \mu(1 + \cos m w t)$ for w close to $w_0 = \sqrt{g/L}$, one should use all the terms of the Taylor expansion of $\sin v$, around $v = 0$, up to the order v^m if m is odd and up to the order v^{m+1} if m is even.

2. HYPOTHESES AND LIAPUNOV-SCHMIDT REDUCTION

In this work we will be concerned with the abstract equation:

$$Lx = N(x, p, \varepsilon) + \mu f \tag{2.1}$$

where f belongs to the Banach space Z. Let X be a Banach space and $L{:}X \mapsto Z$ be a continuous linear operator. We assume that $\lambda = (p, \mu, \varepsilon)$ varies in an open neighborhood Λ of origin in $\mathbb{R}^3$ and $x \in B_r = \{x \in X : |x| < r\}$. We indicate by $BC^k(B_r \times \Omega, Z)$ the space of all functions defined on $B_r \times \Omega$ with values in Z, which have bounded and continuous derivatives up to the order k in $B_r \times \Omega$, where Ω is an open neighborhood of the origin in $\mathbb{R}^2$.

Throughout this paper we suppose that $m \geq 2$ and $s \in \{0, 1\}$ are fixed integers.

As a basic reference for notations and for some general results that will be mentioned in this chapter, we indicate Vanderbauwhede [7] and also Golubitsky [3].

Consider the following hypotheses:

(H_1) $L : X \mapsto Z$ is a Fredholm operator with zero index and $\ker L = \operatorname{span}\{u_1, u_2\}$.

(H_2) There exist representations $\Gamma, \tilde{\Gamma}$ of the group $O(2)$ over X and Z, respectively, and continuous projections P, Q over X and Z, respectively, such that the followings conditions are satisfied:

$$PX = \ker L,\ QZ = \text{span } \{w_1, w_2\},$$

$$(I - Q)Z = R(L),$$

$$\Gamma(\phi)u_1 = \cos\phi u_1 + \sin\phi u_2 ,$$

$$\Gamma(\phi)u_2 = -\sin\phi u_1 + \cos\phi u_2 ,$$

$$\Gamma_\tau u_1 = u_1,\quad \Gamma_\tau u_2 = -u_2 ,$$

$$\tilde{\Gamma}(\phi)w_1 = \cos\phi w_1 + \sin\phi w_2 ,$$

$$\tilde{\Gamma}(\phi)w_2 = -\sin\phi w_1 + \cos\phi w_2 ,$$

$$\tilde{\Gamma}_\tau w_1 = w_1 \quad \text{and} \quad \tilde{\Gamma}_\tau w_2 = -w_2 .$$

(H_3) i) $\tilde{\Gamma}\left(\frac{2\pi}{m}\right) Lx = L(\Gamma\left(\frac{2\pi}{m}\right) x)$, $(-1)^s\tilde{\Gamma}_\tau Lx = L\Gamma_\tau x$;

ii) $\tilde{\Gamma}(2\pi/m)f = f$ and $(-1)^s\tilde{\Gamma}_\tau f = f$;

iii) $N \in BC^{m+3}(B_r \times \Omega, Z)$, is an odd function of x for each fixed $\lambda = (p, \varepsilon)$ in Ω, $N(x, p, \varepsilon) = [pA + \varepsilon B]x + Q_3x^3 + R(x, p, \varepsilon)$, where $A, B \in \mathcal{L}[X, Z)$, $Q_3 \in \mathcal{L}^3(X, Z)$, $R(x, p, \varepsilon) = o(|px| + |\varepsilon x| + |x|^3)$ as $(x, p, \varepsilon) \to (0, 0, 0)$, and $\tilde{\Gamma}\left(\frac{2\pi}{m}\right) N(x, p, \varepsilon) = N(\Gamma\left(\frac{2\pi}{m}\right) x, p, \varepsilon)$, $(-1)^s\tilde{\Gamma}_\tau N(x, p, \varepsilon) = N(\Gamma_\tau x, p, \varepsilon)$.

iv) $\tilde{\Gamma}(\phi)A = A\Gamma(\phi)\ \forall\, \phi \in \mathbb{R}$ and $QAu_1 = w_{1+s}$.

It follows from (H_2) that P and Q can be chosen in such a way that they are respectively $(O(2), \Gamma)$ and $(O(2), \tilde{\Gamma})$ equivariant. Also from (ii), one obtains $Qf = 0$. Hypotheses (H_2) implies that L restricted to $(I - P)X$ has a continuous inverse $K : R(L) \mapsto (I - P)X$, satisfying $KLx = (I - P)x$ and $LK(I - Q)z = (I - Q)z$, $\forall x \in X$, $\forall z \in Z$.

Let D_m be the subgroup of $O(2)$ generated by the matrices $R_{\frac{2\pi}{m}}$ (rotation) and $\tau = \text{diag}(1, -1)$ (reflection). We will indicate by $\tilde{\Gamma}_j$ the representation of D_m over Z defined by: $\tilde{\Gamma}_j(R_{\frac{2\pi}{m}}) = \tilde{\Gamma}(\frac{2\pi}{m})$ and $\tilde{\Gamma}_j(\tau) = (-1)^j\tilde{\Gamma}_\tau$, $j = 0, 1$.

It follows from (H_3) that equation (1.1) is $(D_m, \Gamma, \tilde{\Gamma}_s)$-equivariant. By application of the equivariant Liapunov-Schmidt reduction method, writing $x \stackrel{\text{def}}{=} u + v$, $u \in \ker L$, $v \in (I - P)X$, the equation (1.1) is reduced to the systems:

$$v = K(I - Q)[N(u + v, p, \varepsilon) + \mu f] \tag{2.2.a}$$

$$0 = QN(u+v,p,\varepsilon)\ . \tag{2.2.b}$$

From the implicit function theorem it follows that (2.2.a) has a unique solution $v^*(u,\lambda)$, in a neighborhood $V \subset (I-P)X$, for (u,λ) in a small neighborhood $U \times \Lambda_1 \subset \ker L \times \Lambda$ of (0,0), such that $v^*(0,0)=0$. Moreover $v^*(u,\lambda)$ is a smooth function of (u,λ) in $U \times \Lambda_1$.

The neighborhoods U and V can be chosen in such a way that $\Gamma(g)U = U$ and $\Gamma(g)V = V$, $\forall g \in D_m$. The function $v^*(u,\lambda)$ is (D_m,Γ)-equivariant with respect to u.

If we substitute v^* in (2.2.b) we obtain the following $(D_m,\Gamma,\tilde{\Gamma}_*)$-equivariant equation,

$$\tilde{M}(u,\lambda) \stackrel{\text{def}}{=} QN(u+v^*(u,\lambda),p,\varepsilon) = 0\ . \tag{2.3}$$

Identifying the subspaces $\ker L$ and QZ with the set of complex numbers $\mathbb{C}$, by the isomorphisms:

$$\mathcal{X}(au_1+bu_2) \stackrel{\text{def}}{=} a+bi$$

$$\sigma(aw_1+bw_2) \stackrel{\text{def}}{=} a+bi$$

where $a,b \in \mathbb{R}$, we obtain the equivalent complex equation:

$$M(z,\lambda) \stackrel{\text{def}}{=} \sigma\tilde{M}(\mathcal{X}^{-1}z,\lambda) = 0\,, \tag{2.4}$$

where λ e Λ_1 and $z \in \mathcal{X}(U)$.

In the following lemma we show that equation (2.4) inherits the symmetries of the original equation.

LEMMA 2.1 *If hypotheses (H_1), (H_2), and (H_3), are satisfied, then for $(z,\lambda) \in \mathcal{X}(U) \times \Lambda_1$ we have:*

$$e^{i\frac{2\pi}{m}}M(z,\lambda) = M(e^{i\frac{2\pi}{m}}z,\lambda)\ , \tag{2.5.a}$$

$$(-1)^s\overline{M(z,\lambda)} = M(\bar{z},\lambda)\ . \tag{2.5.b}$$

The proof of the above lemma follows in a natural way from the relations: $[\mathcal{X}\Gamma(\phi)\mathcal{X}^{-1}]z = [\sigma\tilde{\Gamma}(\phi)\sigma^{-1}]z = e^{i\phi}z$, and $[\mathcal{X}\Gamma_\tau\mathcal{X}^{-1}]z = [\sigma\tilde{\Gamma}_\tau\sigma^{-1}]z = \bar{z}$, $\forall z \in \mathbb{C}$.

3. THE BIFURCATION EQUATIONS

In this chapter, we show that the symmetry assumptions of the original equation, imply that the complex form of the bifurcation equation can be written in a convenient normal form, that will be very helpful to obtain our main results.

In the sequel we will denote by B_r^k the open ball of radius r, centered in the origin in $\mathbb{R}^k$.

LEMMA 3.1 *Let $0 < \varepsilon < r$, $H \in BC^2(B_r^{m+n}, \mathbb{R})$ and $f \in C^1(B_\varepsilon^n, \mathbb{R}^m)$ with $f(0) = 0$ and such that:*

$$Gr(f) = \{(x, f(x))/x \in B_\varepsilon^n\} \subset H^{-1}(\{0\}) .$$

Then, there exists $\tilde{H} : B_\varepsilon^{m+n} \to \mathcal{L}(\mathbb{R}^m, \mathbb{R})$ of class C^1, satisfying:

$$H(x,y) = \tilde{H}(x,y)(y - f(x)), \ \forall (x,y) \in B_\varepsilon^{m+n} .$$

PROOF. The statements of the lemma follow essentially from the following equalities:

$$\begin{aligned} H(x,y) &= \int_0^1 \frac{d}{d\theta} H\left(x, f(x) + \theta(y - f(x))\right) d\theta \\ &= \int_0^1 \frac{\partial H}{\partial y}\left(x, f(x) + \theta(y - f(x))\right) d\theta \cdot [y - f(x)] . \end{aligned}$$

COROLLARY 3.2 *If $H \in BC^2(B_r^2, \mathbb{R})$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a nontrivial linear functional such that $\ker \varphi \cap B_r^2 \subset H^{-1}(\{0\})$, then, there exist $0 < \varepsilon < r$ and $\tilde{H} \in BC^1(B_\varepsilon^2, \mathbb{R})$, such that:*

$$H(x,y) = \tilde{H}(x,y) \cdot \varphi(x,y), \ \forall (x,y) \in B_\varepsilon^2 .$$

THEOREM 3.3 *Let $m \geq 2$, $s \in \{0,1\}$ and $q \geq 1$ be fixed integers and $B_r^2 \subset \mathbb{C}$. If M belongs to space $BC^{m+q}(B_r^2, \mathbb{C})$ and satisfies the relations (2.5.a) and (2.5.b), then, there exist a real number ε, $0 < \varepsilon < r$, and real functions $f = f_{m,s}$, $g = g_{m,s}$, defined in $B_\varepsilon^2 \subset \mathbb{C}$, of class C^q and C^{q-1} respectively, such that:*

$$(-i)^s M(z) = f(z) z + g(z)(\bar{z})^{m-1} , \tag{3.1}$$

where

$$f(z \cdot e^{i\frac{2\pi}{m}}) = f(\bar{z}) = f(z) \quad \text{and} \tag{3.2.a}$$

$$g(z \cdot e^{i\frac{2\pi}{m}}) = g(\bar{z}) = g(z) , \quad \forall z \in B_\varepsilon^2 . \tag{3.2.b}$$

PROOF. Let $H(z) \stackrel{\text{def}}{=} Im\,[(-i)^s M(z)\bar{z}] : B_r^2 \mapsto \mathbb{R}$. Following the ideas of [8] our assumptions imply that $H(ze^{i\frac{2\pi}{m}}) = H(z)$ and $H(\bar{z}) = -H(z)$. Thus $H(z)$ vanishes on the zeros of $\varphi_j(x,y) \stackrel{\text{def}}{=} \sin(j\frac{\pi}{m})x - \cos(j\frac{\pi}{m})y$.

A successive application of Corollary 3.1, gives for z in B_ε^2 :

$$H(z) = \tilde{H}(z) \prod_{j=0}^{m-1} \varphi_j(z) .$$

where $\tilde{H}$ is a C^q function.

The above remarks, applied to $w(z) \stackrel{\text{def}}{=} Im(\bar{z})^m$, instead to $H(z)$, imply that there exists a function $\tilde{w}(z)$ such that

$$w(z) = \tilde{w}(z) \prod_{j=0}^{m-1} \varphi_j(z) .$$

Since the functions $w(z)$ and $\prod_{j=0}^{m-1} \varphi_j(z)$ are homogeneous polynomials of the same degree, it follows that there exists a nonzero constant c such that

$$\prod_{j=0}^{m-1} \varphi_j(z) = c^{-1} w(z) \stackrel{\text{def}}{=} g(z) ,$$

and so

$$H(z) = g(z) Im(\bar{z})^m ,$$

for $z \in B_\varepsilon^2$, where $\varepsilon > 0$ is sufficiently small.

Let $W(z) \stackrel{\text{def}}{=} (-i)^s M(z) - g(z)(\bar{z})^{m-1} \stackrel{\text{def}}{=} W_1(z) + iW_2(z)$, where W_1, W_2 are real functions. From the definition of H and from $H(z) = g(z)Im(\bar{z})^m$ it follows that

$$0 = Im\,[w(z) \cdot \bar{z}] = W_2(z)x - W_1(z)y ,$$

where $z = x + iy$. Therefore, $W_2(z) = 0$ if $y = 0$. Using Corollary 3.1 we see that there exists a C^{q-1} function f such that $W_2(z) = f(z)y$. The relationship between W_1 and W_2 implies that $W_1(z) = f(z)x$ and so $W(z) = f(z)z$. Therefore,

$$(-i)^s M(z) = f(z)z + g(z)(\bar{z})^{m-1} ,$$

for z in a sufficiently small neighborhood of origin and this completes the proof of (3.1).

Since the function $w(z) = Im(\bar{z})^m$ has the same symmetry properties of the function $H(z)$ it follows that (3.2.b) holds.

From (3.1) and (2.5.b), respectively, we obtain:

$$f(z) \cdot |z|^2 = (-i)^s M(z) \cdot \bar{z} - g(z)(\bar{z})^m$$

and

$$(\overline{(-i)^s M(z)\bar{z}}) = (-i)^s M(\bar{z}) \cdot z ,$$

which imply (3.2.a).

COROLLARY 3.4 *Suppose hypotheses (H_1), (H_2) and (H_3) are satisfied. Then, there exist C^1 real functions $f \stackrel{\text{def}}{=} f_{m,s}$ and $g \stackrel{\text{def}}{=} g_{m,s}$ defined in $B_\varepsilon^2 \times \Lambda_1$, such that the bifurcation equation (2.4) is reduced to:*

$$(-i)^s M(z, \lambda) = f(z, \lambda)z + g(z, \lambda)(\bar{z})^{m-1} \tag{3.3}$$

where

$$f(ze^{i\frac{2\pi}{m}},\lambda) = f(\bar{z},\lambda) = f(z,\lambda)$$

and

$$g(ze^{i\frac{2\pi}{m}},\lambda) = g(\bar{z},\lambda) = g(z,\lambda)\ .$$

If we let $z \stackrel{\text{def}}{=} re^{i\phi}$, $r \in \mathbb{R}$, $\phi \in [0,\pi)$ in equation (3.3) we obtain the real equations:

$$F(r,\phi,\lambda) \stackrel{\text{def}}{=} r\left[f(z,\lambda) + g(z,\lambda)r^{m-2}\cos m\phi\right] = 0 \tag{3.4.a}$$

$$G(r,\phi,\lambda) \stackrel{\text{def}}{=} r^{m-1}\sin(m\phi)g(z,\lambda) = 0 \tag{3.4.b}$$

For each fixed λ , to each solution (r,ϕ) of the above system there corresponds the solution:

$$\begin{aligned} x(r,\phi,\lambda) &= \mathcal{X}^{-1}(re^{i\phi}) + v^*(\mathcal{X}^{-1}re^{i\phi},\lambda) \\ &= r\Gamma(\phi)u_1 + v^*(r,\phi,\lambda) \end{aligned}$$

of the original equation (1.1), where $v^*(r,\phi,\lambda) \stackrel{\text{def}}{=} v^*(r\Gamma(\phi)u_1,\lambda)$.

Let $J(r,\phi,\lambda) \stackrel{\text{def}}{=} f(z,\lambda) + r^{m-2}\cos m\phi\, g(z,\lambda)$.

LEMMA 3.5 *Suppose hypotheses* (H_1), (H_2) *and* (H_3) *are satisfied. Then, for each fixed* ϕ *in* $[0,\pi)$, *the function* $J(r,\phi,\lambda)$ *has the following representation in a neighborhood of* $(r,\lambda) = (0,0)$:

$$\begin{aligned} J(r,\phi,\lambda) &= p + \gamma_s(\phi)\varepsilon + \lambda_s(\phi)\mu^2 + \alpha_s(\phi)r^2 \\ &\quad + \xi_s(\phi)r\mu + o(|p| + |\varepsilon| + \mu^2 + r^2)\ , \end{aligned} \tag{3.5}$$

where

$$\gamma_s(\phi) \stackrel{\text{def}}{=} Re\left[(-i)^s e^{-i\phi}\sigma QB\Gamma(\phi)u_1\right]\ ,$$

$$\alpha_s(\phi) \stackrel{\text{def}}{=} Re\left[(-i)^s e^{-i\phi}\sigma QQ_3((\Gamma(\phi)u_1)^3)\right]\ ,$$

$$\lambda_s(\phi) \stackrel{\text{def}}{=} 3Re\left[(-i)^s e^{-i\phi}\sigma QQ_3(\Gamma(\phi)u_1,(Kf)^2)\right]\ ,$$

and

$$\xi_s(\phi) \stackrel{\text{def}}{=} \begin{cases} 0 \quad \text{if} \quad m \neq 3\ , \\ 3Re\left[(-i)^s\sigma QQ_3((\Gamma(\phi)u_1)^2,Kf)\right] \quad \text{if} \quad m = 3\ . \end{cases}$$

The proof follows by using Taylor's expansion.

REMARK 3.6 For $m > 2$ the coefficients $\gamma_s(\phi)$ and $\lambda_s(\phi)$ are independent of ϕ and they are given by:

$$\gamma_s \stackrel{\text{def}}{=} Re\left[(-i)^s \sigma QBu_1\right] ,$$

$$\lambda_s \stackrel{\text{def}}{=} 3Re\left[(-i)^s \sigma QQ_3(u_1,(Kf)^2)\right] .$$

For $m = 3$, the coefficients $\alpha_s(j\frac{\pi}{3})$ $j = 0,1,2$ are independent of j and they are given by:

$$\alpha_s \stackrel{\text{def}}{=} Re\left[(-i)^s \sigma QQ_3({u_1}^3)\right] .$$

If the homogeneous polynomial $Q_3(x^3)$ given in (H_3-iii) satisfies $\bar{\Gamma}(\phi)Q_3(x^3)= Q_3((\Gamma(\phi)x)^3)$ for any ϕ in $\mathbb{R}$ then $\alpha_s(\phi)$ is independent of ϕ and it is given as above.

Now, let us analyze the bifurcation equations (3.4.a) and (3.4.b).

We remark that $r = 0$ solves both equations. The corresponding solution $x(0,\phi,\lambda) = v^*(0,\phi,\lambda)$ does not depend on ϕ and it is $\Gamma(\frac{2\pi}{m})$-invariant. This solutions will be called "harmonic".

Since $G(r,\phi_j,\lambda) = 0$ if $\phi_j = j\frac{\pi}{m}$ $j = 0,\ldots,m-1$, in this case we only have to solve $J(r,\phi_j,\lambda) = 0$. If $r \neq 0$ solves this equation then the corresponding solution $x(r,\phi_j,\lambda)$ of (2.1) is $\Gamma(2\pi)$-invariant and it will be called a "subharmonic" solution with phase ϕ_j.

To find the points where bifurcation occurs we have to solve the system:

$$\begin{aligned} J(r,\phi_j,\lambda) &= p + \gamma_s(\phi_j)\varepsilon + \lambda_s(\phi_j)\mu^2 + \alpha_s(\phi_j)r^2 \\ &\quad + \xi_s(\phi_j)r\mu + o(|\varepsilon| + |p| + \mu^2 + r^2) = 0 , \end{aligned} \tag{3.6.a}$$

$$J_r(r,\phi_j,\lambda) = 2\alpha_s(\phi_j)r + \xi_s(\phi_j)\mu + o(|\varepsilon| + |p| + |r| + \mu^2) = 0 . \tag{3.6.b}$$

A simple calculation shows that $\det\left\{\frac{\partial(J,J_r)}{\partial(p,r)}\right\}_{r=\lambda=0} = 2\alpha_s(\phi_j)$. If $\alpha_s(\phi_j) \neq 0$, it follows from the implicit function theorem that the above system has a unique solution $r = r(\mu,\varepsilon,\phi_j)$ and $p = p(\mu,\varepsilon,\phi_j)$, in a small neighborhood of $r = 0$, $\lambda = 0$.

From definition of $J(r,\phi,\lambda)$ and (3.3) one obtains:

$$J(r,\phi_j,\lambda) = f(re^{i\phi_j},\lambda) + r^{m-2}g(re^{i\phi_j},\lambda)$$

and

$$f(re^{i(\phi_j+\frac{2\pi}{m})},\lambda) = f(re^{i\phi_j}) .$$

These equalities imply that $J_r(0, \phi_j, \lambda) = 0$, $j = 0, \ldots, m-1$ and $m \neq 3$. The substitution of $r = 0$ in the first equation gives

$$p = -\gamma_s(\phi_j)\varepsilon - \lambda_s(\phi_j)\mu^2 + o(|\varepsilon| + \mu^2)\ .$$

Then the unique solution of the system is given by $r = r(\mu, \phi_j, \varepsilon) = 0$ and $p = p(\mu, \varepsilon, \phi_j) = -\gamma_s(\phi_j)\varepsilon - \lambda_s(\phi_j)\mu^2 + o(|\varepsilon| + \mu^2)$.

For $m = 3$ the functions $r = r(\mu, \varepsilon, \phi_j)$ and $p = p(\mu, \varepsilon, \phi_j)$ satisfy $p = p(\mu, \varepsilon, 0) = p(\mu, \varepsilon, \frac{\pi}{3})$ and $r(\mu, \varepsilon, 0) = -r(\mu, \varepsilon, \frac{\pi}{3})$.

¿From (3.6.b) one obtains $r = \frac{\eta}{2\alpha_s}\mu + o(|\varepsilon| + |p| + \mu^2)$, where $\eta \stackrel{\text{def}}{=} -\xi_s(0)$, and the substitution in (3.6.a) gives

$$p = -\gamma_s\varepsilon + \left(\frac{\eta^2}{4\alpha_s} - \lambda_s\right)\mu^2 + o(|\varepsilon| + \mu^2)\ .$$

Then we have proved the following:

THEOREM 3.7 *If hypotheses (H_1), (H_2), and (H_3), are satisfied, $\phi_j = j\frac{\pi}{m}$, $j = 0, \ldots, m-1$ and $\alpha_s(\phi_j) \neq 0$, then in a small neighborhood of $\lambda = 0$, the bifurcation surfaces (of subharmonic solutions with phases $\phi_j = j\frac{\pi}{m}$) are given by:*

$$p = -\gamma_s(j\frac{\pi}{2})\varepsilon - \lambda_s(j\frac{\pi}{2})\mu^2 + o(|\varepsilon| + \mu^2), \quad j = 0, 1 \quad \text{if} \quad m = 2\ ,$$

$$p = -\gamma_s\varepsilon - \lambda_s\mu^2 + o(|\varepsilon| + \mu^2), \quad \text{if} \quad m > 3$$

and

$$p = -\gamma_s\varepsilon + \left(\frac{\eta}{4\alpha_s} - \lambda_s\right)\mu^2 + o(|\varepsilon| + \mu^2), \quad \text{if} \quad m = 3\ .$$

The total number of harmonic and subharmonic solutions with phase ϕ_j, varies with λ , according to Fig. 3.1, Fig. 3.2 and Fig. 3.3 for $m = 2$, $m = 3$ and $m > 3$ respectively, for $\gamma_s(\frac{\pi}{2}) < 0 < \gamma_s(0)$, $0 < \lambda_s(\frac{\pi}{2}) < \lambda_s(0)$, $\left(\frac{\eta^2}{4\alpha_s} - \lambda_s\right) < 0$ and $\gamma_s < 0 < \lambda_s$. Furthermore the solutions $x(r, \phi_j, \lambda)$ of the original equation (2.1) inherit some symmetry of Kf, that is, $\Gamma(-\phi_j)x(r, \phi_j, \lambda)$ and $\Gamma(-\phi_j)Kf$ are Γ_τ-invariant for $j = 0, \ldots, m-1$.

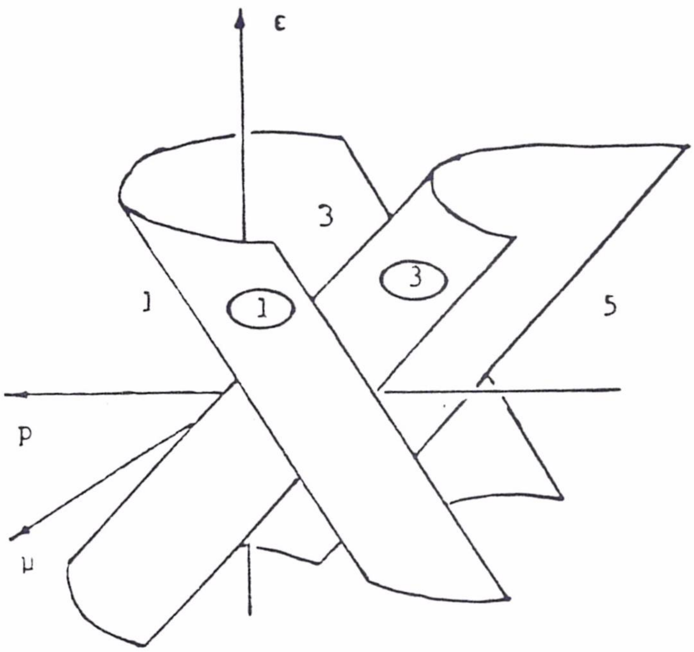

FIGURE 3.1

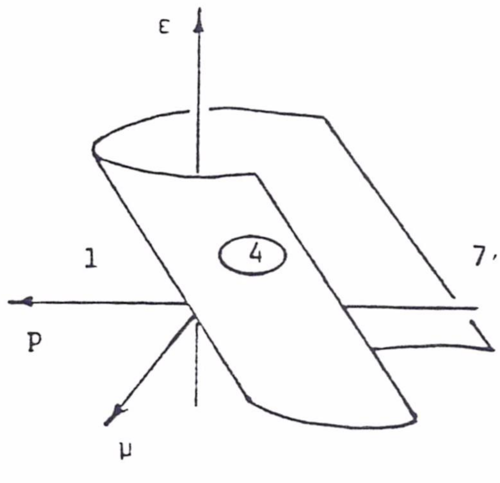

FIGURE 3.2

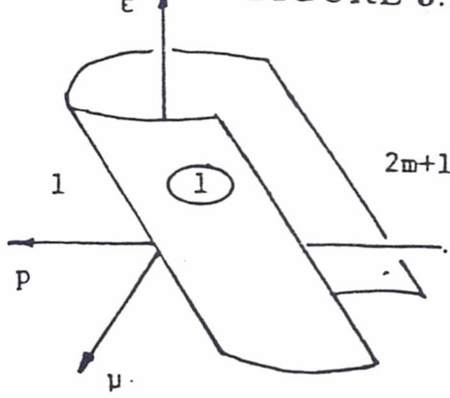

FIGURE 3.3

PROOF. It remains to prove the last statament of the theorem.

From our symmetry assumptions it follows that the function $K(I-Q)N: B_r \times \Omega \subset X \times \mathbb{R}^2 \mapsto X$ is $\Gamma(\frac{2\pi}{m})$-equivariant.

Therefore,

$$\Gamma(\phi_j)K(I-Q)N(\Gamma(-\phi_j)x,p,\varepsilon) = \Gamma(-\phi_j)K(I-Q)N(\Gamma(\phi_j)x,p,\varepsilon). \quad (3.7)$$

for $\phi_j = j\frac{\pi}{m}$, $j = 0,\dots,m-1$.

¿From $\tilde{\Gamma}(\frac{2\pi}{m})f = f$ one obtains

$$\Gamma(-j\frac{\pi}{m})Kf = \Gamma(j\frac{\pi}{m})Kf \quad (3.8)$$

for $j = 0,\dots,m-1$.

It follows from the implicit functions theorem and from the identities (3.7) and (3.8) that $\Gamma(-\phi_j)v^*(r,\phi_j,\lambda)$ is Γ_τ-invariant, where $\phi_j = j\frac{\pi}{m}$, $j = 0,\dots,m-1$. Therefore the solution $x(r,\phi_j,\lambda) = r\Gamma(\phi_j)\mu_1 + v^*(r,\phi_j,\lambda)$, with phase ϕ_j of (1.1), is such that $\Gamma(-\phi_j)x(r,\phi_j,\lambda)$ is Γ_τ-invariant, like $\Gamma(-\phi_j)Kf$.

4. GENERICITY FOR "AUTONOMOUS" NONLINEARITIES

In the sequel we consider the special case of equation (1.1) by taking $\varepsilon = 0$. The operators L and N should present an $(O(2),\Gamma,\tilde{\Gamma}_s)$-equivariance and f is $(D_m,\tilde{\Gamma}_s)$-invariant. Here $\tilde{\Gamma}_s$, $s = 0,1$ indicate representations of the group $O(2)$ over Z which are extensions of the ones defined in §2 for the subgroup D_m. We still assume that $m \geq 2$ and $s \in \{0,1\}$ are fixed integers. We will be concerned with the equation:

$$Lx + N(x,p) + \mu f \ , \quad (4.1)$$

with the assumptions (H$_1$), (H$_2$), and with the following additional hypotheses:

(H$_4$) (i) $N(x,p) = pAx + \tilde{N}(x) + o(|px|) : B_r \times \Lambda \subset X \times \mathbb{R} \to Z$, where $A \in \mathcal{L}(X,Z)$, $\tilde{N} \in BC^{m+3}(B_r,Z)$ is odd function in x with $\tilde{N}_x(0) = 0$;

(ii) $\tilde{\Gamma}(\phi)Lx = L(\Gamma(\phi)x)$, $(-1)^s\tilde{\Gamma}_\tau Lx = L\Gamma_\tau x$, $\forall\phi \in \mathbb{R}$;

(iii) $\tilde{\Gamma}(\phi)N(x,p) = N(\Gamma(\phi)x,p)$,
$(-1)^s\tilde{\Gamma}_\tau N(x,p) = N(\Gamma_\tau x,p)$, $\forall\phi \in \mathbb{R}$;

(iv) $\tilde{\Gamma}(\frac{2\pi}{m})f = f$ and $(-1)^s\tilde{\Gamma}_\tau f = f$;

v) $QAu_1 = w_{1+s}$.

Since (H_4) implies (H_3) it follows from §3 that equations (4.1) has a set of small solutions which are either harmonic or subharmonic solutions with phases $\phi_j = j\frac{\pi}{m}$, $j = 0, \ldots, m-1$.

Under certain conditions, we will prove that those are the only feasible solutions of equation (4.1). We also analyze the genericity of such condition.

LEMMA 4.1 *Under hypotheses (H_1), (H_2) and (H_4), the function $G(r, \phi, \lambda)$ defined by* (3.4.b) *with* $\lambda = (p, \mu)$ *satisfies:*

$$G(r, \phi, p, 0) = 0 \; . \tag{4.2}$$

If, in addition m is even then

$$G(r, \phi, p, -\mu) = G(r, \phi, p, \mu) \; . \tag{4.3}$$

PROOF. The first statement follows from the fact that for $\mu = 0$ the function $(-i)^s e^{-is\phi} M(z, \lambda)$ is real, where $z = re^{i\phi}$. The second is consequence of the identity:

$$v^*(u, p, \mu) = -v^*(-u, p, -\mu) \; ,$$

which follows from the implicit function theorem and (2.2.a).

COROLLARY 4.2 *Under the hypotheses (H_1), (H_2) and (H_4), the function $G(r,\phi,p,\mu)$ has the form:*

$$G(r,\phi,p,\mu) = \begin{cases} r^{m-1}\mu \sin m\phi(\rho + \cdots) & \text{if } m \text{ is odd,} \\ r^{m-1}\mu^2 \sin m\phi(\rho + \cdots) & \text{if } m \text{ is even;} \end{cases} \tag{4.4}$$

where $(\cdots)$ are terms of order $O(|r| + |p| + |\mu|)$, as $(r,p,\mu) \to (0,0,0)$ and $\rho = \rho(N, f)$ is given by:

$$\rho(\tilde{N}, f) = \begin{cases} \frac{1}{2} g_{\mu\mu}(0,0) & \text{for } m \text{ even ,} \\ g_\mu(0) & \text{for } m \text{ odd ;} \end{cases} \tag{4.5}$$

where g is as in (3.2.b).

THEOREM 4.3 *Suppose the hypotheses (H_1), (H_2), (H_4) are satisfied and $\alpha_* > 0$. If $\rho = \rho(\tilde{N}, f) \neq 0$ then the only small solutions of equation (4.1) are either harmonic or subharmonic with phases $\phi_j = j\frac{\pi}{m}$, $j = 0, \ldots, m-1$, where $\lambda = (p,\mu)$ is small and $\mu \neq 0$.*

PROOF. From (4.4), if $\rho \neq 0$ and $\mu \neq 0$, $G(r,\phi,p,\mu) = 0$ implies $r = 0$ or $\phi = \phi_j = j\frac{\pi}{m}$, $j = 0, \ldots, m-1$.

For $r = 0$ the solution $x(0,\phi,\lambda) = v^*(0,\phi,\lambda)$ is harmonic and $\Gamma\left(\frac{2\pi}{m}\right)$-invariant. If $r \neq 0$ and $\phi = \phi_j$ the corresponding solution $x(r,\phi_j,\lambda) = \Gamma(\phi_j)u_1 + v^*(r,\phi_j,\lambda)$ is subharmonic with phases ϕ_j , as we pointed out in §3.

LEMMA 4.4 *If m is even, under the assumptions (H_1), (H_2) and (H_4) the solution of the auxiliary equation: $v^* \stackrel{\text{def}}{=} v^*(r,\phi,o,\mu)$, has the following form:*

$$v^* = \mu K f + W_1 + W_2 + O(|r| + |\mu|)^{m+3}$$

where,

$$W_1 \stackrel{\text{def}}{=} W_1(Q_3, \ldots, Q_{m-1}, f, r, \phi, \mu) = O(|r| + |\mu|)^3 \text{ ,}$$

$$W_2 \stackrel{\text{def}}{=} W_2(Q_{m+1}, f, r, \phi, \mu) = O(|r| + |\mu|)^{m+1}$$

and $Q_k = \frac{1}{k!}\frac{\partial^k \tilde{N}}{\partial x^k}(0)$, $k = 0, \ldots, m+1$.

PROOF. The result follows just by using induction on k.

LEMMA 4.5 *Under the hypotheses* (H_1), (H_2) *and* (H_4) *with* m *even, the coefficient* $\rho = \rho(\tilde{N}, f)$ *defined by* (4.5), *is continuous in* $(\tilde{N}, f)$ *and has the representation:*

$$\rho(\tilde{N}, f) = \delta(Q_{m+1}, (Kf)^2) + \bar{\delta}(Q_3, \ldots, Q_{m-1}, f) \tag{4.6}$$

where $\delta(\cdot,\cdot,\cdot)$ *is a continuous trilinear real function defined in* $\mathcal{L}^{m+1}(X, Z) \times X^2$ *by*

$$\delta(Q_{m+1}, f_1, f_2) \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=} -\binom{m+1}{2} \frac{(-i)^s}{2^{m-1}} \left[\sum_{j=0}^{(m-2)/2} \binom{m-1}{2j} (-1)^j \sigma Q Q_{m+1}\left(u_1^{m-1-2j}, u_2^{2j}, f_1, f_2\right)\right.$$

$$\left. + \, i \sum_{j=0}^{(m-2)/2} \binom{m-1}{2j+1} (-1)^j \sigma Q Q_{m+1}\left(u_1^{m-1-(2j+1)}, u_2^{2j+1}, f_1, f_2\right)\right]$$

and $\bar{\delta}(Q_3, \ldots, Q_{m-1}, f)$ *is a function of* f *and* Q_{2j+1} *with* $1 \leq j \leq \frac{m}{2} - 1$ *(* $\bar{\delta}$ *does not depend on* Q_k*, for* $k > m-1$ *)*.

PROOF. By (4.4) it follows that:

$$\rho(\tilde{N}, f) = \frac{1}{\pi} \int_0^{2\pi} \frac{\partial^{m+1}}{\partial r^{m-1} \partial \mu^2} G(0, \phi, 0, 0) \sin m\phi \, d\phi \,. \tag{4.7}$$

If we let $I(s, \phi) \stackrel{\text{def}}{=} Im\left[(-i)^s e^{-i\phi} \sigma Q\right]$ we obtain from (2.3), (2.4) and Lemma 4.2 that:

$$\begin{aligned}
G(r, \phi, 0, \mu) &= Im\left[(-i)^s e^{-i\phi} M(z, \lambda)\right] \\
&= I(s, \phi) N(r\Gamma(\phi)u_1 + v^*(r, \phi, 0, \mu), 0) \\
&= I(s, \phi) N(r\Gamma(\phi)u_1 + \mu K f + W_1 + W_2 + W, 0) \\
&= I(s, \phi) \sum_{j=1}^{(m-2)/2} Q_{2j+1}(r\Gamma(\phi)u_1 + \mu K f + W_1)^{2j+1} \\
&\qquad + I(s, \phi) Q_{m+1}(r\Gamma(\phi)u_1 + \mu K f)^{m+1} + O\left((|r| + |\mu|)^{m+3}\right) ,
\end{aligned}$$

where $W = O(|r| + |\mu|)^{m+3}$.

The coefficient of the term of order $r^{m-1}\mu^2$ in the Taylor expansion of $I(s, \phi) Q_{m+1}(r\Gamma(\phi)u_1 + \mu K f)$, in r, μ is given by:

$$\delta(Q_{m+1}, (Kf)^2) \left[\sin m\phi + (-1)^j 2^{m-1} \sum_{k=1}^{m-1} a_k \sin k\phi\right] .$$

If we let

$$\tilde{\delta}(Q_3, \ldots, Q_{m-1}, f) \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=} \tfrac{1}{\pi} \int_0^{2\pi} \sum_{j=1}^{(m-2)/2} I(s,\phi) \tfrac{\partial^{m+1}}{\partial r^{m-1} \partial \mu^2} \{Q_{2j+1}(r\Gamma(\phi)u_1$$

$$+ \quad \mu K f + W_1)^{2j+1}\}_{r=\mu=0} \sin m\phi \, d\phi$$

we obtain from (4.7) the identity (4.6).

The continuity of $\rho(\tilde{N}, f)$ follows from (4.7).

REMARK 4.6 If m is odd the coefficient $\rho(\tilde{N}, f)$, defined by (4.5) is continuous in $(\tilde{N}, f)$ and has representation:

$$\rho(\tilde{N}, f) = \nu(Q_m, Kf) + \tilde{\nu}(Q_3, \ldots, Q_{m-2}, f)$$

where $\nu(\cdot,\cdot)$ is a continuous bilinear real function defined in $\mathcal{L}^m(X, Z) \times X$ by:

$$\nu(Q_m, f) \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=} - \tfrac{m(-i)^s}{2^{m-1}} \left[\sum_{j=0}^{(m-1)/2} \binom{m-1}{2j} (-1)^j \sigma Q Q_m \left(u_1^{m-1-2j}, u_2^{2j}, f\right) \right.$$

$$\left. + \, i \sum_{j=0}^{(m-1)/2} \binom{m-1}{2j+1} (-1)^j \sigma Q Q_m \left(u_1^{m-2(j+1)}, u_2^{2j+1}, f\right) \right]$$

and $\tilde{\nu}(Q_3, \ldots, Q_{m-2}, f)$ is a continuous function which does not depend on Q_k for $k > m-2$.

For each pair (m, s) of integer numbers, $m \geq 2$, $s \in \{0, 1\}$, we consider the Banach spaces:

$$\mathcal{H}^s_m \stackrel{\text{def}}{=} \Big\{ \tilde{N} \in B^{m+3}(B,Z) \Big/ \tilde{N} \quad \text{is odd and}$$
$$(O(2),\Gamma,\tilde{\Gamma}_s)\text{ - equivariant with } \tilde{N}_x(0)=0 \Big\} ,$$

$$\mathcal{D}^s_m \stackrel{\text{def}}{=} \Big\{ f \in Z \Big/ f \text{ is } (D_m,\tilde{\Gamma}_s)\text{ - invariant} \Big\} .$$

The next theorem provides a sufficient condition in such a way that $\rho(\tilde{N},f) \neq 0$ holds generically in $\mathcal{H}^s_m \times \mathcal{D}^s_m$.

THEOREM 4.7 *Suppose the hypotheses* (H_1), (H_2), (H_4) *are satisfied where* $m \geq 2$ *is even. If there exist* $f_0 \in \mathcal{D}^s_m$ *and* $Q^0_{m+1} \in \mathcal{L}^{m+1}(X,Z)$, *such that the homogeneous polynomial* $x \in X \mapsto Q^0_{m+1}(x^{m+1}) \in Z$ *belongs to* $\mathcal{H}^s_m$ *and* $\delta(Q^0_{m+1},(Kf_0)^2) \neq 0$, *then*

$$\Sigma_\rho \stackrel{\text{def}}{=} \Big\{ (\tilde{N},f) \in \mathcal{H}^s_m \times \mathcal{D}^s_m \Big/ \rho(\tilde{N},f) \neq 0 \Big\}$$

is open and dense in $\mathcal{H}^s_m \times \mathcal{D}^s_m$.

PROOF. For a fixed element $(\tilde{N},f)$ in $\mathcal{H}^s_m \times \mathcal{D}^s_m - \Sigma_\rho$ consider the sequence (N_k,f_k) in $\mathcal{H}^s_m \times \mathcal{D}^s_m$ defined by:

$$(N_k,f_k) = \begin{cases} (\tilde{N}, f+\frac{1}{k}f_0) & \text{if } (\tilde{N}, f+\frac{1}{k}f_0) \in \Sigma_\rho , \\ (\tilde{N}+\frac{1}{k}Q^0_{m+1}, f+\frac{1}{k}f_0) & \text{if } (\tilde{N}, f+\frac{1}{k}f_0) \notin \Sigma_\rho . \end{cases}$$

From the hypotheses $\delta(Q^0_{m+1},K^2f_0) \neq 0$, it follows that the polynomial. $q(\xi) \stackrel{\text{def}}{=} \delta\left(Q^0_{m+1},(Kf+\xi Kf_0)^2\right)$ does not vanish in $I=(-\varepsilon,\varepsilon)-\{0\}$ if $\varepsilon > 0$ is sufficiently small. Let k_0 be a positive integer number such that $\frac{1}{k_0} < \varepsilon$. We will show that (N_k,f_k) belongs to Σ_ρ if $k > k_0$.

For a fixed $k > k_0$, if $(\tilde{N},\, f+\frac{1}{k}f_0)$ belongs to Σ_ρ then $(N_k,f_k) \in \Sigma_\rho$. On the other hand, if $(\tilde{N},\, f+\frac{1}{k}f_0) \notin \Sigma_\rho$ then $\rho(\tilde{N},\, f+\frac{1}{k}f_0)=0$. In this case we have:

$$\begin{aligned}
\rho(N_k, f_k) &= \rho(\bar{N} + \frac{1}{k}Q^0_{m+1}, f + \frac{1}{k}f_0) \\
&= \delta\left(Q_{m+1} + \frac{1}{k}Q^0_{m+1}, (Kf + \frac{1}{k}Kf_0)^2\right) \\
&\quad + \bar{\delta}\left(Q_3, \dots, Q_{m-1},\ f + \frac{1}{k}f_0\right) \\
&= \delta\left(Q_{m+1}, (Kf + \frac{1}{k}Kf_0)^2\right) \\
&\quad + \bar{\delta}\left(Q_3, \dots, Q_{m-1}, f + \frac{1}{k}f_0\right) + \frac{1}{k}\delta\left(Q^0_{m+1}, (Kf + \frac{1}{k}Kf_0)^2\right) \\
&= \rho\left(\bar{N},\ f + \frac{1}{k}f_0\right) + \frac{1}{k}q(\frac{1}{k}) \neq 0\,.
\end{aligned}$$

Therefore $(N_k, f_k) \in \Sigma_\rho$.

¿From the inequality

$$\|(N_k, f_k) - (\bar{N}, f_0)\| \leq \frac{1}{k}\ \left(\|Q^0_{m+1}\| + \|f_0\|\right)$$

if follows that (N_k, f_k) converge to $(\bar{N}, f)$ in $\mathcal{H}^s_m \times \mathcal{D}^s_m$, as $k \to \infty$. Therefore $\overline{\Sigma}_\rho = \mathcal{H}^s_m \times \mathcal{D}^s_m$.

The continuity of $\rho(\bar{N}, f)$ implies that Σ_ρ is open.

Using the same ideas, one can analyze the case $m \geq 3$, m odd, to obtain the following similar result:

THEOREM 4.8 *Suppose the hypotheses (H_1), (H_2), (H_4) are satisfied where $m \geq 3$ is odd. If there exist $f_0 \in \mathcal{D}^s_m$ and $Q^0_m \in \mathcal{L}^m(X, Z)$ such that, the homogeneous polynomial $x \in X \mapsto Q^0_m(x^m) \in Z$ belongs to $\mathcal{H}^s_m$ and $\nu(Q^0_m, Kf_0) \neq 0$, then*

$$\Sigma_\rho \stackrel{\text{def}}{=} \left\{(\bar{N}, f) \in \mathcal{H}^s_m \times \mathcal{D}^s_m \,/\, \rho(\bar{N}, f) \neq 0\right\}$$

is open and dense in $\mathcal{H}^s_m \times \mathcal{D}^s_m$.

5. EXAMPLES

EXAMPLE 5.1 Consider the nonlinear scalar equation:

$$\ddot{x} + x = [p + \varepsilon H(t)]x + g(x) + h(x, t, p, \varepsilon) + \mu f(t) \ , \tag{5.1}$$

where, $m \geq 2$ is a fixed integer and

(a) p, μ and ε are small parameters;

(b) $f,\ H : \mathbb{R} \mapsto \mathbb{R}$ are continuous, $\frac{2\pi}{m}$-periodic, even functions;

(c) $h : \mathbb{R}^4 \mapsto \mathbb{R}$ is a C^{m+3} function, which is odd in x, $\frac{2\pi}{m}$-periodic and even in t and it is of order $o(|px| + |\varepsilon x|)$ as $(x, p, \varepsilon) \to (0, 0, 0)$;

(d) $g : \mathbb{R} \mapsto \mathbb{R}$ is an odd C^{m+3} function with $g'(0) = 0$ and $g^{(3)}(0) \neq 0$.

The equation (5.1) can be put into the abstract form (2.1) by taking $X \stackrel{\text{def}}{=} C^2_{2\pi}(\mathbb{R}, \mathbb{R})$, $Z \stackrel{\text{def}}{=} C_{2\pi}(\mathbb{R}, \mathbb{R})$ with the C^2 and C^0 topologies, respectively, and $(Lx)(\cdot) \stackrel{\text{def}}{=} \ddot{x}(\cdot) + x(\cdot) : X \mapsto Z$, $N(x, p, \varepsilon)(\cdot) \stackrel{\text{def}}{=} [pA + \varepsilon B]x(\cdot) + \alpha_3 x^3(\cdot) + R(x, p, \varepsilon)(\cdot)$ where $R(x, p, \varepsilon)(\cdot) \stackrel{\text{def}}{=} g(x(\cdot)) - \alpha_3 x^3(\cdot) + h(x(\cdot), (\cdot), p, \varepsilon)$, $A, B \in \mathcal{L}(X, Z)$; $Ax(\cdot) = x(\cdot)$, $Bx(\cdot) = H(\cdot)x(\cdot)$ and $\alpha_3 \stackrel{\text{def}}{=} \frac{1}{3!}g^{(3)}(0)$.

Let $\Gamma : O(2) \mapsto \mathcal{L}(X)$ and $\tilde{\Gamma} : O(2) \mapsto \mathcal{L}(Z)$ be defined by $\tilde{\Gamma}(\phi)z(\cdot) \stackrel{\text{def}}{=} z(\cdot - \phi)$, $\tilde{\Gamma}_\tau z(\cdot) \stackrel{\text{def}}{=} z(-\cdot)$, $\Gamma(\phi) = \tilde{\Gamma}(\phi)/X$ and $\Gamma_\tau = \tilde{\Gamma}_\tau/X$.

Then the hypotheses (H_1), (H_2) and (H_3) are satisfied with $s = 0$, $u_1(\cdot) = w_1(\cdot) = \cos(\cdot)$ and $u_2(\cdot) = w_2(\cdot) = \sin(\cdot)$.

The projections $P \in \mathcal{L}(X)$ and $Q \in \mathcal{L}(Z)$ are defined by:

$$Qf(\cdot) \stackrel{\text{def}}{=} \frac{1}{\pi} \sum_{j=1}^{2} \int_0^{2\pi} f(s) w_j(s) ds \cdot w_j(\cdot) \ \text{ and } \ P = Q/X \ .$$

Since $\alpha_0 = \frac{1}{8}g^{(3)}(0) \neq 0$, from Theorem 3.7 follows that equation (5.1) has small 2π-periodic solutions $x(t)$ which are either harmonic or subharmonic. In the last case there exists j, $j = 0, \dots, m-1$ such that $x(t - \phi_j)$ is an even function of t, where $\phi_j = j\frac{\pi}{m}$.

In the special case, $f(t) \stackrel{\text{def}}{=} H(t) = 1 + \cos mt$ and $h(t, x, p, \varepsilon) \stackrel{\text{def}}{=} (p + \varepsilon \cos mt)\varepsilon x^3$ the bifurcation surfaces are given by:

$$p = -\tfrac{3}{2}\varepsilon - \tfrac{13}{6}\alpha_3\mu^2 + o(|\varepsilon| + \mu^2) \quad \text{and}$$

$$p = -\tfrac{1}{2}\varepsilon - \tfrac{25}{6}\alpha_3\mu^2 + o(|\varepsilon| + \mu^2) \quad \text{for} \quad m = 2\,,$$

$$p = -\varepsilon - \tfrac{3093}{1024}\alpha_3\mu^2 + o(|\varepsilon| + \mu^2) \quad \text{for} \quad m = 3$$

and,

$$p = -\varepsilon - 3\alpha_3\left[1 + \tfrac{1}{2(m^2-1)^2}\mu^2\right] + o(|\varepsilon| + \mu^2) \quad \text{for} \quad m > 3\,.$$

Consider now equation (5.1) with $\varepsilon = 0$. Concerning to the genericity described in §4, to use Theorem 4.7 we consider the function $f_0(t) = 1 + \cos mt$, which belongs to $\mathcal{D}^s_m$ for m even and the homogeneous polynomial $Q^0_{m+1}(x(\cdot)) = x^{m+1}(\cdot) : X \mapsto Z$ which belongs to $\mathcal{H}^s_m$. After some calculation one obtains

$$\delta(Q^0_{m+1}, (Kf_0)^2) = \frac{m}{(m-1)2^{m-1}} \neq 0\,.$$

Therefore, for most pairs (g, f) such that g and f satisfy (d) and (b) respectively, the only small solutions of (5.1) with $\varepsilon = 0$ and $\mu \neq 0$ are either harmonic or subharmonic solutions with phases $\phi_j = j\frac{\pi}{m}$, $j = 0, \ldots, m-1$. That is, if $x(t)$ is a small 2π-periodic solution, either it is even harmonic or there exists j, $j = 0, \ldots, m-1$ such that $x(t \dot{-} \phi_j)$ is an even function of t.

With some adaptations, one can establish similar results in the case m odd.

EXAMPLE 5.2 Consider the nonlinear system of equations:

$$\dot{W} = A_0W + [p + \varepsilon H(t)]AW + g(W) + h(W, t, p, \varepsilon) + \mu F(t) \tag{5.2}$$

where $m \geq 2$ is a fixed integer, and

(a) $W \in \mathbb{R}^{2n}, p, \mu$ and ε are small real parameters;

(b) $A_0 = \operatorname{diag}(A_1^0, \ldots, A_n^0)$, $A_1^0 = \begin{pmatrix} 0 & \frac{1}{2} \\ -2 & 0 \end{pmatrix}$, $A_k^0 = \begin{pmatrix} 0 & w_k \\ -w_k & 0 \end{pmatrix}$ where w_k is a positive real number which is not integer, $k = 2, \ldots, n$;

(c) $F : \mathbb{R} \to \mathbb{R}^{2n}$ and $H : \mathbb{R} \to \mathcal{L}(\mathbb{R}^{2n})$ are continuous, $\frac{2\pi}{m}$-periodic and H is even;

(d) $A = \operatorname{diag}(A_1, \ldots, A_n)$ where $A_k = \begin{pmatrix} 0 & -\frac{1}{6} \\ \frac{11}{6} & 0 \end{pmatrix}$, $k = 1, \ldots, n$;

(e) $g : \mathbb{R}^{2n} \mapsto \mathbb{R}^{2n}$ is a C^{m+3} odd function with $g'(0) = 0$, $g^{(3)}(0) \neq 0$;

(f) $h : \mathbb{R}^{2n} \times \mathbb{R}^3 \mapsto \mathbb{R}^{2n}$, $(W, t, p, \varepsilon) \mapsto h(W, t, p, \varepsilon)$ is a C^{m+3} function, which is odd in W, even and $\frac{2\pi}{m}$-periodic in t and $h(W, t, p, \varepsilon) = o(|pW| + |\varepsilon W|)$ as $(W, p, \varepsilon) \mapsto (0, 0, 0)$.

(g) for $S \stackrel{\text{def}}{=} \operatorname{diag}(S_1, \ldots, S_n)$ where $S_k \stackrel{\text{def}}{=} \operatorname{diag}(1, -1)$, $k = 1, \ldots, n$, one has, $-Sg(W) = g(SW)$, $SH(-t) = H(t)S$, $-SF(-t) = F(t)$ and $-Sh(W, -t, p, \varepsilon) = h(SW, t, p, \varepsilon)$.

The equation (5.2) can be put into the abstract form (2.1) by defining $X \stackrel{\text{def}}{=} C^1_{2\pi}(\mathbb{R}, \mathbb{R}^{2n})$, $Z \stackrel{\text{def}}{=} C_{2\pi}(\mathbb{R}, \mathbb{R}^{2n})$ with the topologies C^1 and C^0 respectively,

$$(LW)(\cdot) \stackrel{\text{def}}{=} \dot{W}(\cdot) - A_0 W(\cdot) : X \mapsto Z ,$$

$N(W, p, \varepsilon)(\cdot) \stackrel{\text{def}}{=} [pA + \varepsilon B]W(\cdot) + Q_3(W^3(\cdot)) + R(W, p, \varepsilon)$ where $R(W, p, \varepsilon)(\cdot) = g(W(\cdot)) - Q_3(W^3(\cdot)) + h(W(\cdot), (\cdot), p, \varepsilon)$, $B \in \mathcal{L}(X, Z)$, $BW(\cdot) = H(\cdot)AW(\cdot)$ and $Q_3 = \frac{1}{3!} g^{(3)}(0)$.

Let $\tilde{\Gamma} : 0(2) \mapsto \mathcal{L}(Z)$ and $\Gamma : 0(2) \mapsto \mathcal{L}(X)$ be defined by $\tilde{\Gamma}(\phi)W(\cdot) \stackrel{\text{def}}{=} W(\cdot - \phi)$, $\tilde{\Gamma}_\tau W(\cdot) \stackrel{\text{def}}{=} SW(-\cdot)$, $\forall W(\cdot) \in Z$, $\Gamma(\phi) \stackrel{\text{def}}{=} \tilde{\Gamma}(\phi)/_X$ and $\Gamma_\tau \stackrel{\text{def}}{=} \tilde{\Gamma}_\tau/_X$. The hypotheses (H_1), (H_2) and (H_3) are satisfied for systems (5.2) with $s = 1$ and

$$u_1(t) \stackrel{\text{def}}{=} (\cos t, -2\sin t, 0, \ldots, 0) ,$$

$$u_2(t) \stackrel{\text{def}}{=} (\sin t, 2\cos t, 0, \ldots, 0) ,$$

$$w_1(t) \stackrel{\text{def}}{=} \left(\cos t, \frac{-\sin t}{2}, 0, \ldots, 0\right) ,$$

$$w_2(t) \stackrel{\text{def}}{=} \left(\sin t, \frac{\cos t}{2}, 0, \ldots, 0\right) .$$

The projections $P \in \mathcal{L}(X)$ and $Q \in \mathcal{L}(Z)$ can be defined by:

$$Pf(\cdot) \stackrel{\text{def}}{=} \frac{1}{5\pi} \sum_{j=1}^{2} \int_0^{2\pi} (f(s), u_j(s))ds \cdot u_j(\cdot)$$

and

$$Qf(\cdot) \stackrel{\text{def}}{=} \frac{4}{5\pi} \sum_{j=1}^{2} \int_0^{2\pi} (f(s), w_j(s))ds \cdot w_j(\cdot) ,$$

where $(\cdot,\cdot)$ is the usual inner product of $\mathbb{R}^{2n}$.

If $\alpha_1 = \frac{4}{5\pi} \int_0^{2\pi} (Q_3({u_1}^3(s)) \cdot w_2(s))\, ds \neq 0$ the system has harmonic and subharmonic solutions with phases $\phi_j = j\frac{\pi}{m}$, $j = 0, \ldots, m-1$.

Let $m \geq 2$ a fixed even integer. Consider equation (5.2) with $\varepsilon = 0$ and $h(W,t,p,0) = 0$. Concerning to genericity described in §4, to use Theorem (4.7), we consider the function $F_0(t) \stackrel{\text{def}}{=} ((1-m)\sin mt, 2 + 2(1-m)\cos mt, 0, \ldots, 0)$ which belongs to $\mathcal{D}_m^1$ space and the homogeneous polynomial:

$$Q^0_{m+1}(W(\cdot)) = \mathrm{col}\left(W_2^{m+1}(\cdot), W_1^{m+1}(\cdot), \ldots, W_{2n}^{m+1}(\cdot), W_{2n-1}^{m+1}(\cdot)\right) : X \mapsto Z ,$$

where

$$W(\cdot) = (W_1(\cdot), \ldots, W_{2n}(\cdot)) ,$$

which belongs to $\mathcal{H}_m^1$. After some calculations one obtains

$$\delta(Q^0_{m+1}, (Kf_0)^2) = -\frac{m(m+1)}{5 \cdot 2^{m-2}} \neq 0 .$$

Therefore, for most pairs (g, F) such that (e) and (c) are satisfied, the only small solutions of equation (5.2) with $\varepsilon = 0$ and $\mu \neq 0$ are either harmonic or subharmonic with phases $\phi_j = j\frac{\pi}{m}$, $j = 0, \ldots, m-1$. That is, if $W(t) = (W_1(t), \ldots, W_{2n}(t))$ is a small 2π-periodic solution either it is harmonic or there exists j, $j = 0, \ldots, m-1$ such that $W(t)$ is subharmonic and the components $W_{2k}(t-\phi_j)$ and $W_{2k-1}(t-\phi_j)$ of $W(t-\phi_j)$ are, respectively, even and odd function, for $k = 1, \ldots, n$.

In the above example the assumptions on the linear part can be considerably weakened.

EXAMPLE 5.3 Let $m \geq 2$ be a fixed integer. We analyze the existence and bifurcation of classicals solutions of the boundary value problem on the disc $B = \{w \in \mathbb{R}^2 / |w| < 1\}$, defined by:

$$\begin{cases} (\Delta + \lambda)u = pu + g(u,p) + h(u,p) + \mu f(w), \ w \in B \\ \\ u(w) = 0, \ w \in \partial B ; \end{cases} \tag{5.3}$$

where:

(a) p and μ are small parameters;

(b) $f : \overline{B} \to \mathbb{R}$ is Hölder continuous with index $\alpha \in (0,1)$;

(c) $f(R_{\frac{2\pi}{m}} w) = f(w)$ and $f(Sw) = f(w)$ where S is the matrix $\operatorname{diag}(1,-1)$;

(d) $g : \mathbb{R} \to \mathbb{R}$ is a C^{m+3} odd function with $g'(0) = 0$ and $g^{(3)}(0) \neq 0$;

(e) $h : \mathbb{R}^2 \mapsto \mathbb{R}$, $(u,p) \mapsto h(u,p)$ is a C^{m+3} function which is odd in u and of order $o(|pu|)$ as $(u,p) \to (0,0)$;

(f) $\lambda = \xi^2$, where ξ is the first positive zero of the Bessel function $J_1(x)$.

Let $C^{k+\alpha}(\overline{B})$ be the space of the functions $u : B \mapsto \mathbb{R}$ having partial derivatives up to order k, which can be continuously extended up to $\overline{B}$, and the k-th derivatives are uniformly Holder continuous with index α.

Consider in $C^{k+\alpha}(\overline{B})$ the norm

$$\|u\|_{k+\alpha} = \|u\|_k + \sup_{|\beta|=k} \left[D^\beta u\right]_\alpha ,$$

where

$$\|u\|_k = \sum_{j=0}^{k} \sup_{|\beta|=j} \sup_{w\in\overline{B}} |D^\beta u(w)|$$

and

$$\left[D^\beta u\right]_\alpha = \sup_{\substack{w\neq\nu \\ w,\nu\in\overline{B}}} \frac{|u(w)-u(\nu)|}{|w-\nu|^\alpha} .$$

The boundary value problem (5.3) can be put into the abstract form (2.1) by taking $X \stackrel{\text{def}}{=} C_0^{2+\alpha}(\overline{B}) = \{u \in C^{2+\alpha}(\overline{B}) \,/ u(w) = 0 \text{ if } w \in \partial B\}$ and $Z \stackrel{\text{def}}{=} C^\alpha(\overline{B})$ with norms $\|\ \|_{2+\alpha}$ and $\|\ \|_\alpha$ respectively,

$$(Lu)(\cdot) \stackrel{\text{def}}{=} (\Delta + \lambda)u(\cdot) : X \mapsto Z ,$$

$N(u,p)(\cdot) = pu(\cdot) + \alpha_3 u^3(\cdot) + R(u,p)(\cdot) : B_r \times \Lambda \subset X \times \mathbb{R} \mapsto Z$, where $\alpha_3 = \frac{1}{3!} g^{(3)}(0)$, $B_r = B(0,r) \subset X$; Λ is a small neighborhood of the origin in $\mathbb{R}$ and $R(u,p)(\cdot) = g(u(\cdot)) - \alpha_3 u(\cdot)^3 + h(u(\cdot),p)$.

Defining $\Gamma : O(2) \mapsto \mathcal{L}(X)$ and $\tilde{\Gamma} : O(2) \mapsto \mathcal{L}(Z)$ by $\tilde{\Gamma}(\phi)u(w) \stackrel{\text{def}}{=} u(R_\phi w)$, $\tilde{\Gamma}_\tau u(w) \stackrel{\text{def}}{=} u(Sw)$, $\Gamma(\phi) \stackrel{\text{def}}{=} \tilde{\Gamma}(\phi)/_X$ and $\Gamma_\tau \stackrel{\text{def}}{=} \tilde{\Gamma}_\tau/_X$ we see that hypotheses (H_1), (H_2) and (H_3) are satisfied, with $s = 0$, $u_1(\bar{r},\theta) = w_1(\bar{r},\theta) =$

$J_1(\sqrt{\lambda}\bar{r})\cos\theta$ and $u_2(\bar{r},\theta) = w_2(\bar{r},\theta) = J_1(\sqrt{\lambda}\bar{r})\sin\theta$, in polar coordinates and $\ker L = \operatorname{span}\{u_1, u_2\}$.

The projections $P \in \mathcal{L}(X)$ and $Q \in \mathcal{L}(Z)$ are defined by

$$Qf(\cdot) \stackrel{\text{def}}{=} \frac{1}{\ell}\sum_{j=1}^{2}\int_{\overline{B}} f(w)u_j(w)dw \cdot w_j(\cdot)\,,$$

and $P = Q/_X$, where $\ell = \pi\int_0^1 rJ_1^2(\sqrt{\lambda}r)dr$.

Since $\alpha_0 = \frac{3\pi\alpha_3}{4\ell}\int_0^1 rJ_1^4(\sqrt{\lambda}r)dr \neq 0$, it follows that, for $\lambda = (p,\mu)$ in a small neighborhood of origin in $\mathbb{R}^2$, there exist small 2π-periodic solutions of (5.3), with phases $\phi_j = j\frac{\pi}{m}$, $j = 0,\ldots,m-1$.

In polar coordinates these solutions have the form $x(r,\phi_j,\lambda)(\bar{r},\theta) = r\Gamma(\phi_j)u_1(\bar{r},\theta) + v^*(r,\phi_j,\lambda)(\bar{r},\theta)$. They are $\frac{2\pi}{m}$-periodic if $r = 0$ and 2π-periodic in θ if $r \neq 0$.

For the special case where $f(w) = f(\bar{r},\theta) \stackrel{\text{def}}{=} 4 + (\bar{r}^2 - 1) - Jm(\sqrt{\nu}\bar{r})\cos m\theta(\nu - \lambda)$, with $Jm(\sqrt{\nu}) = 0$ (Jm is the Bessel functions of order m) the bifurcation equations are given by:

$$p = -\lambda_0(\phi_j)\mu^2 + o(u^2),\ \phi_j = j\frac{\pi}{2},\ j = 0,1$$

in the case $m = 2$, and

$$p = \left(\frac{\eta^2}{4\alpha_0} - \lambda_0\right)\mu^2 + o(\mu^2)$$

in the case $m \geq 3$, where

$$\lambda_0(0) = \frac{3\pi\alpha_3}{\ell}\int_0^1 rJ_1^2(\sqrt{\lambda}\cdot r)\left[(1-r^2)J_2(\sqrt{\nu}\cdot r) + (r^2-1)^2 + \frac{1}{2}J_2^2(\sqrt{\lambda}\cdot r)\right]dr,$$

$$\lambda_0(\frac{\pi}{2}) = \frac{3\pi\alpha_3}{\ell}\int_0^1 rJ_1^2(\sqrt{\lambda}\cdot r)\left[(r^2+1)^2 + (r^2-1)J_2(\sqrt{\nu}\cdot r) + \frac{1}{2}J_2^2(\sqrt{\gamma}\cdot r)\right]dr,$$

$$\lambda_0 = \frac{3\pi\alpha_3}{\ell}\int_0^1 rJ_1^2(\sqrt{\lambda}\cdot r)\left[(r^2-1) + \frac{1}{2}J_m^2(\sqrt{\nu}\cdot r)\right]dr$$

and

$$\eta = \begin{cases} 0 \text{ if } m > 3\,, \\ \frac{-3\pi\alpha_3}{\ell}\int_0^1 rJ_3(\sqrt{\gamma}\cdot r)J_1^3(\sqrt{\lambda}\cdot r)dr \text{ if } m = 3\,. \end{cases}$$

Consider now the problem (5.3), with m even. The function $f_0(w) \stackrel{\text{def}}{=} f(w) : \overline{B} \mapsto \mathbb{R}$ and the homogeneous polynomial $Q^0_{m+1}(u)(w) \stackrel{\text{def}}{=} u^{m+1}(w)$: $X \mapsto Z$ are respectively in spaces $\mathcal{D}^0_m$ and $\mathcal{H}^0_m$ and they satisfy:

$$\delta(Q^0_{m+1},(Kf_0)^2) = \frac{m(m+1)\pi}{2^{m-1}\ell}\int_0^1 r(r^2+1)Jm(\sqrt{\nu}\cdot r)J_1^m(\sqrt{\lambda}\cdot r)dr > 0\,.$$

It follows that, for most pairs (g, f), satisfying the conditions (d) and (b), the boundary value problem (5.3) with $\mu \neq 0$ in $C_0^{2+\alpha}(\overline{B})$ has only solutions $u(\bar{r}, \theta)$ which are either harmonic and even or subharmonic solutions with phases $\phi_j = j\frac{\pi}{m}$, such that, $u(\bar{r}, \theta - \phi_j)$ are even for $j = 0, \ldots, m-1$.

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