Instituto de Ciências Matemáticas de São Carlos

ISSN - 0103-2577

Semiregular Surfaces with a single Triple-point

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Nº 006

N O T A S D O I C M S C
Série Matemática

São Carlos
maio / 1993

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There are two well-known surfaces which have a single triple-point. Steiner's surface is the image of a semiregular map and has six non-immersive points or crosscaps. Its domain is the projective plane $\mathbf{P}^2$. When given as the zero-set of the function

$$f(x, y, x) = x^2y^2 + y^2z^2 + z^2x^2 + xyz$$

the surface has tetrahedral symmetry and the singular set consists of an interval on each of the three axes. These intervals are lines of double-points which intersect in a triple-point at the origin and are terminated at each end by a crosscap. Boy's surface is an immersion of $\mathbf{P}^2$. Its singular set is a bouquet of three circles and we can assume that these also contain intervals of the coordinate axes connected by 270° circular arcs .

Here we exhibit a family of 18 surfaces with a single triple-point which includes the Steiner and Boy surfaces; one of the others (which we have designated 2_B) has been investigated by the second author and David Mond [M], [M-M]. They are built from the basic ingredients in the Steiner and Boy surfaces: the singular set consists of an interval of each axis some of which may be connected by arcs and the rest are terminated by crosscaps. The four cases are shown in figure 1. Although none of our surfaces is given as the image of a map we shall refer to them as *semiregular* surfaces and to their abstract topological type as their *domain*. We also discuss transitions between the surfaces that can be achieved by anihilating pairs of crosscaps by a surgery. There is a unique path from the Steiner to the Boy surface.

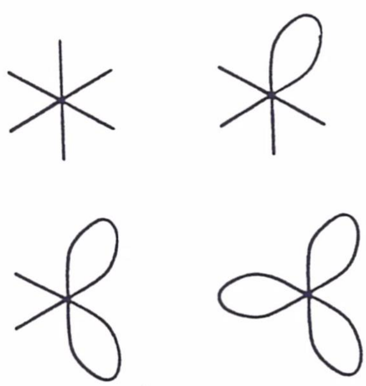


figure 1

1. Surfaces with six crosscaps.

Since all our surfaces must contain a triple-point, we begin by constructing this. The vertices $\{a, b, c, d, e, f\}$ of a regular octahedron determine three squares that intersect in a triple-point as shown in figure 2(a). We will call this basic object a ***tri-square***. If we add the four triangles abf, cdf, ace, bde to these squares we obtain the familiar heptahedron—a polyhedral representation of the Steiner surface (see [H-CV] p.302). The six crosscaps in the surface appear at the vertices. To construct other surfaces with six crosscaps we can add a pair of small triangles at each vertex of the tri-square to form the crosscaps. Each triangular face of the octahedron forming the convex hull will meet 0, 1, 2 or 3 crosscaps and contain this number of small triangles. We represent the four cases by the shaded patterns in figure 3. The tri-square and its six crosscaps is not closed and needs to be completed by attaching a disc to each boundary component. The boundary is visible in figure 3 as the junction between the shaded and unshaded regions. We will refer to this unfinished surface as a *partial* surface.

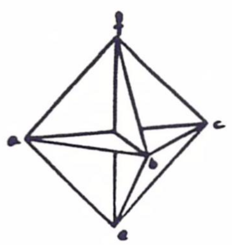


figure 2(a)

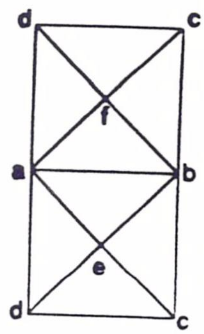


figure 2(b)

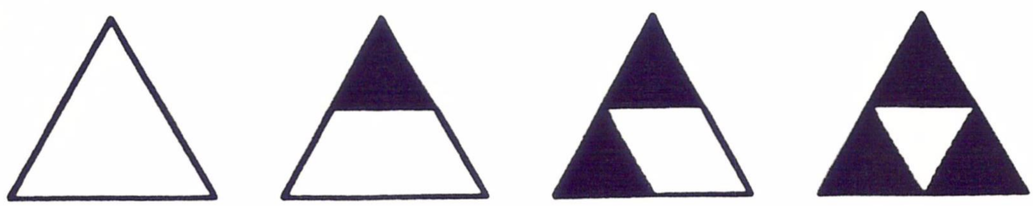


figure 3

From this viewpoint it is easy to see that enumerating the surfaces with one triple-point and six crosscaps is equivalent to the combinatorial problem of enumerating octahedra with particular patterns. To facilitate this we shall introduce a labelling of the vertices of a partial surface. Choose one of the two positions of the heptahedron as a reference position. Compare each vertex $(a, \cdots, f)$ of the partial surface with its corresponding vertex in the reference heptahedron and assign it the label '+' if the crosscaps agree and '−' otherwise. The 2^6 labellings of the tri-square correspond to the 64 partial surfaces. When symmetry is taken into account many of these will be regarded as the same. The symmetry on the tri-square is that of a regular octahedron; the symmetries of the hepthedron are the reflections and rotations of a regular tetrahedron. We denote these two symmetry groups by O_h and T_d respectively. Any symmetry of the heptahedron acting on a labelled tri-square permutes the vertices but leaves their corresponding labels unchanged. The other symmetries in O_h, such as a reflection in the plane of one of the squares, interchange the two forms of the heptahedron and so swap the signs in a labelling.

Definition. Two labellings λ_1 and λ_2 of the vertices of the tri-square are *equivalent* if there is some $\sigma \in O_h$ which carries λ_1 to λ_2 such that for every vertex v in λ_1 we have

$$\begin{aligned} \mathrm{sign}(\sigma(v)) &= \mathrm{sign}(v) : \text{ if } \sigma \in T_d \\ \mathrm{sign}(\sigma(v)) &= -\mathrm{sign}(v) : \text{ if } \sigma \notin T_d. \end{aligned}$$

It is straight forward to check that under this equivalence relation the 64 partial surfaces fall into seven orbits.

Proposition. *The seven orbits of partial surfaces are generated by the labellings:*

(A) $(+,+,+,+,+,+)$
(B) $(-,-,+,+,-,+)$
(C) $(+,+,-,-,+,-)$
(D) $(+,+,+,+,-,-)$
(E) $(-,+,-,+,+,-)$
(F) $(+,+,+,+,-,+)$
(G) $(+,+,-,-,+,+)$.

These are indicated schematically in figure 4. The vertices of the tri-square are identified as shown in figure 2(b). The boundaries are marked by heavy lines. Note that the surfaces 6_A, 6_D, 6_E, 6_F and 6_G are self-dual in the sense that interchanging the plus and minus signs produces a surface in the same orbit whereas 6_B and 6_C are reciprocal in this respect.

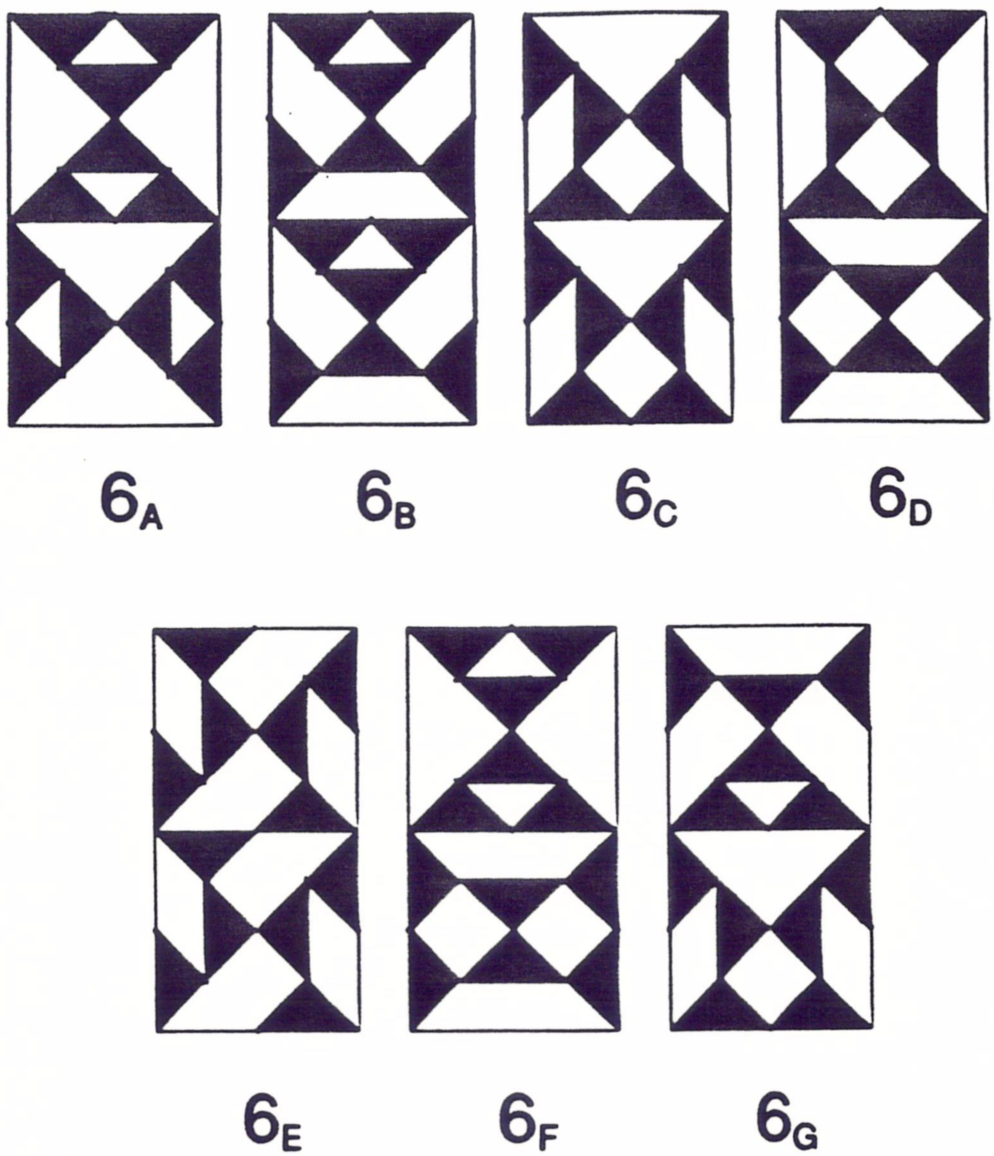


figure 4

	$\alpha(F)$	$\mu(F)$	$\mathrm{Sym}(F)$	$[G:\mathrm{Sym}(F)]$	$\tau(F)$	orientable	$\chi(\hat{F})$	domain
6_A	0	4	T_d	2	1		1	P^2 (Steiner)
6_B	3	3	D_{2v}	4	2	✓	0	torus
6_C	3	1	D_{2v}	4	2		-2	$P^2 \# P^2 \# P^2 \# P^2$
6_D	0	2	D_{3v}	6	2		-1	$P^2 \# P^2 \# P^2$
6_E	1	1	D_2	12	4		-2	$P^2 \# P^2 \# P^2 \# P^2$
6_F	1	3	C_{2v}	12	4		0	Klein bottle
6_G	2	2	C_s	24	7		-1	$P^2 \# P^2 \# P^2$
4_A	1	3	C_s	2	2		1	P^2
4_B	2	2	C_s	2	2		0	klein bottle
4_C	2	4	C_s	2	2	✓	2	sphere
4_D	1	3	C_s	2	2		1	P^2
4_E	1	1	C_2	2	2		-1	$P^2 \# P^2 \# P^2$
4_F	1	3	C_2	2	2		1	P^2
4_G	0	2	C_1	4	4		0	klein bottle
2_A	1	1	C_2	1	1		0	klein bottle
2_B	1	3	C_2	1	1	✓	2	sphere
2_C	0	2	C_1	2	1		1	P^2
0_A		1	D_3	1	0		1	P^2 (Boy)

TABLE 5

Data concerning these seven surfaces are collected in the first seven rows of table 5. They are as follows.

- Consider the subsurface of a partial surface formed by two crosscaps at opposite vertices and the two squares that they meet. The domain for this subsurface will be either an annulus or a Möbius band. In the first case we say that the crosscaps are *well-aligned.* The number of well-aligned pairings is denoted by $\alpha(F)$. It is clear that any Möbius bands will make the surface non-orientable.
- The number of boundary components of the partial surface F is denoted by $\mu(F)$. A closed surface $\hat{F}$ can be formed from F by attaching $\mu(F)$ discs without introducing any extra singularities.

• The tri-square has three faces, six vertices and twelve edges so its Euler characteristic (χ) is $3+6-12=-3$. It is unchanged when the triangles forming the crosscaps are added to make a partial surface. The closed surface $\hat{F}$ formed by attaching $\mu(F)$ discs has Euler characteristic $\chi(\hat{F})=\mu(F)-3$. This identifies the domain.

• The stabiliser of a partial surface F is a subgroup of O_h denoted by Sym(F). All except one of the surfaces have mirror symmetry; 6_B is cheiral. The notation used for the groups is as follows [H]. The letters $C$and D indicate cyclic and dihedral systems of rotation (as usual).

C_1 :	asymmetric.
C_s :	a single mirror plane.
C_n :	a single axis of n-fold rotation.
C_{nv} :	n mirror planes meeting in a single axis of n-fold rotation. The 'v' indicates that the mirrors are vertical.
D_n :	n 2-fold axes lying in a plane that is perpendicular to an n-fold axis.
D_{nv} :	n mirror planes meeting in an n-fold axis together with n 2-fold axes lying in a plane perpendicular to the principal axis and interleaved with the mirror planes.

In the notation of [C-M] these are $[1]^+$, $[1]$, $[n]^+$, $[n]$, $[2,n]^+$ and $[2^+,2n]$ while O_h and T_d are $[3,4]$ and $[3,3]$ respectively.

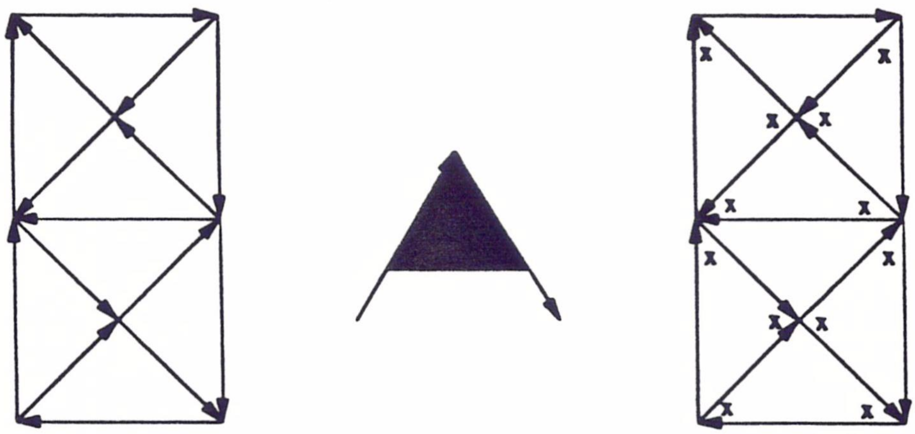
• The index $[O_h : \mathrm{Sym}(F)]$ is the size of the orbit of F. Thus these numbers sum to 64.

• The column headed $\tau(F)$ lists the number of transitivity classes of edges of the tri-square that connect two crosscaps. This will be used in the next section: it is a measure of the number of different ways that the number of crosscaps can be reduced.

Theorem. *There are seven topologically distinct semiregular closed surfaces having one triple-point and six crosscaps. Only one of them is orientable.*

Proof. The numbers $\alpha(F)$ and $\chi(\hat{F})$ are topological invariants and these are sufficient to distinguish the surfaces.

To show that only one surface is orientable assume that we have a partial surface constructed as above that is orientable. Then we can assign an orientation to each edge of the tri-square as follows: imagine you are sitting on the edge of a square with the positive normal to your right. Orient the edge in the direction you are facing. The boundary of each square will be consistently oriented. Without loss of generality, we can orient each of the squares as shown in figure 6(a), say. When we add a crosscap triangle, the surface can be orientable only if the orientations are as shown in figure 6(b). In figure 6(c) all the positions where it is impossible to place a crosscap and retain an orientable surface have been marked 'X', adding crosscaps in the positions that remain produces surface 6_B.



(a) (b) (c)

figure 6

2. Surfaces with fewer crosscaps.

Besides adding crosscaps to the tri-square we can also join two adjacent vertices by a *bridge* as shown in figure 7(a). The bridge has a '+'-shaped cross-section and we can assume that the singular line which connects the two vertices follows a 270°circular arc. This arc lies in the plane of one of the squares of the tri-square and we can add a compressing disc so that part of the bridge can be thought of as a natural extension of this square. In figures 8 and 10 and table 5 this compression has been performed to eliminate one boundary component. When a single bridge has been add edwe can think of the singular set as composed of two arcs; with two bridges it is a single arc. We can still determine whether or not the crosscaps that terminate an arc are well-aligned so $\alpha(F)$ is still defined.

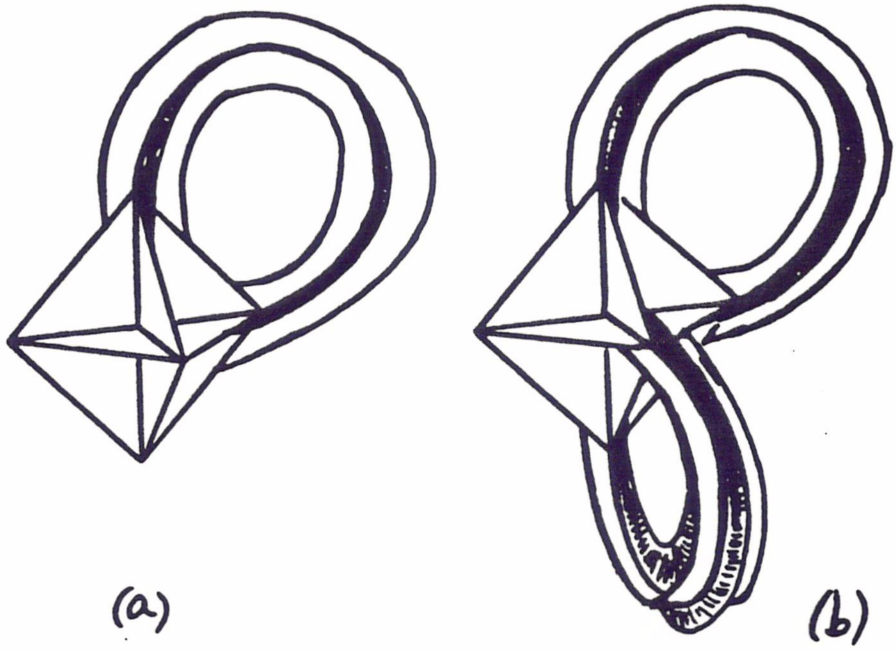
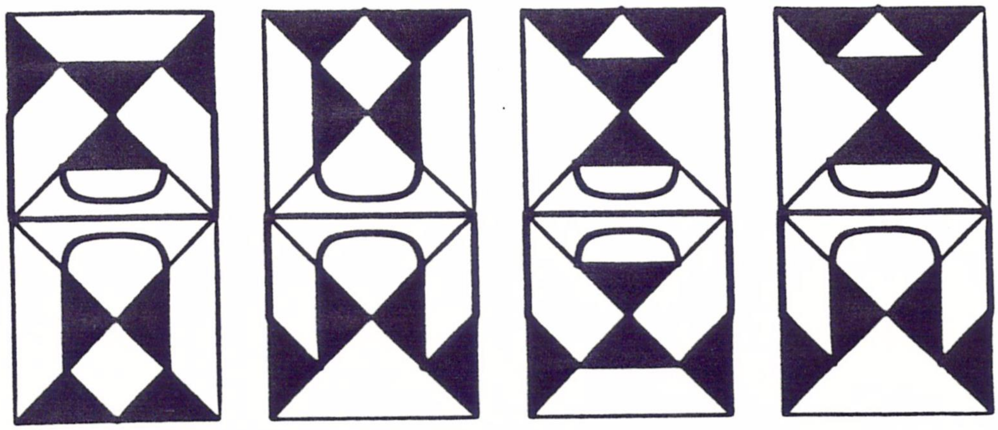


figure 7

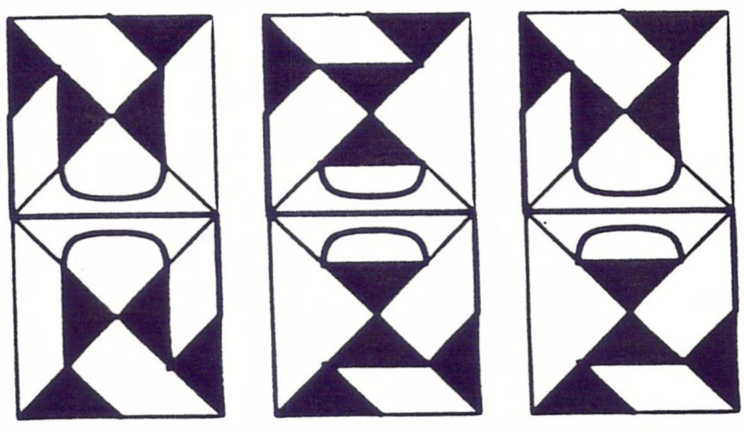
There are two possible approaches we can use to find the surfaces with four crosscaps and one bridge. One is two select two adjacent vertices of the tri-square and connect them by a bridge as shown in figure 7(a) and then analyse the 2^4 possible ways of assigning crosscaps to the other four vertices. This is similar to the previous method. The symmetry group of the primary object has been reduced from the O_h of the tri-square to C_{2v}. It is impossible to arrange the crosscaps so that the partial surface has both rotational and reflection symmetries. Therefore the only possibilities for Sym(F) are C_1, C_s and C_2; and the possible orbit sizes—given by $[C_{2v}\text{:Sym}(F)]$—will be either 2 or 4.

Again, it turns out, there are seven partial surfaces. These are indicated schematically in figure 8. The bridge is shown connecting vertices a and b and appears across the middle of each diagram. Attaching discs completes the closed surfaces. Other data concerning the surfaces appears in table 5.

The alternative method of producing these surfaces is to take a partial surface with six crosscaps, choose an edge of the tri-square to be replaced by a bridge and remove the crosscaps at each end. This operation is called *reduction.* There is a choice of twelve edges for each of the partial surface s, but clearly edges in the same transitivity class under Sym(F) will produce the same result. So, in fact, each surface can produce at most $\tau(F)$ distinct 4-crosscap surfaces. Interestingly, all seven possibilities can be derived from 6_G. Table 9 lists the results of the reductions. An entry in row i and column j is a representative in a transitivity class of edges of the tri-square such that replacing that edge in surface 6_j by a bridge produces surface 4_i. The significance of an asterisk will be explined below.



4_A 4_B 4_C 4_D



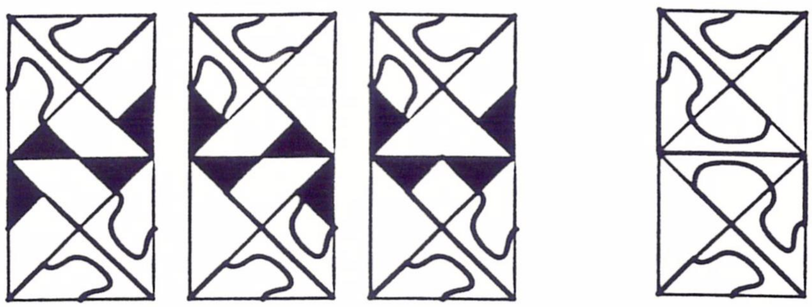
4_E 4_F 4_G

figure 8

	6_A	6_B	6_C	6_D	6_E	$6_F'$	6_G
4_A				ab°	ae		ab°
4_B			ab°			bc°	cd
4_C		ab°				ab°	ce
4_D	ab°					ae	cd°
4_E		bc			ab		ae°
4_F			bc		bc		af
4_G				af	af	af	bc

TABLE 9

The surfaces with two crosscaps and two bridges can be obtained either by reducing a 4-crosscap surface or by analysing the 2^2 ways of adding a pair of crosscaps to the basic form shown in figure 7(b). Since this basic form is cheiral, none of the partial surfaces can have mirror symmetry. The three possibilities are shown in figure 10 and the connections between the 4-crosscap and 2-crosscap surfaces are listed in table 11. The four transitivity classes of edges for 4_G are all shown even though there are only three surfaces for them to reduce to. Replacing the final pair of crosscaps with a bridge produces the Boy surface.



2_A 2_B 2_C 0_A

figure 10

	4_A	4_B	4_C	4_D	4_E	4_F	4_G
2_A	cf	ce*			cf		de*
2_B	ce		cf*			ce	df*
2_C		cf	ce	ce*	ce*	cf*	ce/cf

TABLE 11

Theorem.

(1) There are seven topologically distinct semiregular closed surfaces having one triple-point, four crosscaps and one bridge.

(2) There are three topologically distinct semiregular closed surfaces having one triple-point, two crosscaps and two bridges.

Proof. In the second case, the Euler characteristic is sufficient to distinguish the three cases. In case (1) even $\alpha(F)$ and$\chi(\hat{F})$ together do not suffice: 4_A, 4_D and 4_F are still not distinguished. In surface 4_F the crosscaps at the ends of the short singular line are well-aligned, but in the other cases they are not. In any case, the reduction products separate the three cases: 4_A has two derivatives one of which is orientable, 4_D has only a single non-orientable product.

3. Surgery relationships.

In some circumstances, replacing two adjacent crosscaps in a partial surface by a bridge can be realised by a surgery on the closed surface known as a hyperbolic confluence. This results in the mutual anihilation of two crosscaps . The asterisks in tables 9 and 11 indicate that such a surgery is possible. When we consider which surfaces are related by such surgeries we find that they split into two families (see figure 12). Membership is determined by the parity ofthe Euler characteristic.

This observation is easily explained. The surgery is analogous to the more common one of removing two discs and connecting the newly formed boundary components by a cylinder. A hyperbolic confluence can be achieved by removing two crosscaps to leave two figure-8 shaped boundary components and connecting them by a cylinder having a figure-8 cross-section.

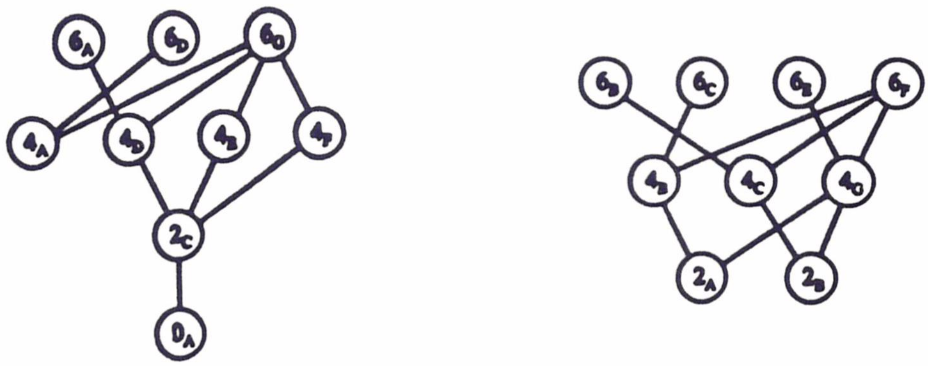


figure 12

All the surfaces described above are incompressible in the sense that any loop not meeting the singular set bounds a disc in the surface. However, the surfaces produced by surgery do not have this property. Each lobe of the figure-8 can carry a compressing loop. In the case sabove, at least one compression is always possible and the compressing disc is the obvious one used to fill in the surplus boundary component in figure 7(a). A second compression may also be possible.

None of these changes affects the Euler characteristic modulo 2. The surgery reduces it by two; a compression increases it by 2. Therefore, in our examples, the reductions that can be realised through surgery followed by compression will leave the Euler characteristic unchanged (for a single compression) or increase it by two.

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The first author is grateful to the SERC for financial support.

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The second author is grateful to the CNPq of Brazil for financial support.

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