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ON THE EXISTENCE AND GLOBAL BIFURCATION OF PERIODIC SOLUTIONS TO PLANAR DIFFERENTIAL DELAY EQUATIONS

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ABSTRACT. This paper is concerned with periodic solutions to a one-parameter family of planar differential delay equations. The concept of slowly oscillating periodic solution is extended to this setting and we state an existence theorem of this type of solutions. Indeed, we state the existence of an unbounded continuum of such solutions.

1. Introduction.

In the current years the scalar differential-delay equation

$$\dot{u}(t) = -\alpha u(t) + \alpha f(u(t-1)), \qquad \alpha > 0, \tag{1.1}$$

which by a suitable time scaling is equivalent to

$$\dot{v}(t) = -v(t) + f(v(t-\alpha)), \tag{1.2}$$

has becoming an object of special interest of research. Besides its mathematical significance it has been proposed as a model for a variety of phenomena in several areas of science. Hadeler [7], who mention some similar versions of (1.1), Mallet-Paret & Nussbaum [10–12] and references therein give a good sample of the profusion of recent works in this subject. Indeed, in [10,11] the authors began the study of the more general case where the time-lag depends on the state, that is, $\alpha = \alpha(v(t))$ in the equation (1.2).

In this paper we continue the study initiated in [16] in order to extend the results related to (1.1) to the planar equation

$$\dot{x}(t) = -\alpha x(t) + \alpha F(x(t-1)), \qquad \alpha > 0, \tag{1.3}$$

where $x = \mathrm{col}(x_1, x_2) \in \mathbb{R}^2$, $F = \mathrm{col}(F_1, F_2)$ is a map from $\mathbb{R}^2$ to itself. Grafton [6] deals with a two dimensional differential-delay equation, but there special features of a second order equation are exploited.

The main results are stated in the section 2. The Theorem 2.1 gives an infinite sequence of local Hopf bifurcations of periodic solutions to (1.3).

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A first fundamental step in this work is to define for the nonlinearity F, a counterpart of the negative feedback condition, usually assumed in the scalar case, $uf(u) < 0$, if $u \neq 0$. This is contained in the hypothesis (H1).

Following the approach of Mallet-Paret & Nussbaum [12], we define a closed convex set $K_\alpha \subset C = C([-1,0],\mathbb{R}^2)$ and a return map $A : K_\alpha \to K_\alpha$, which is built by following the orbits of (1.3) with initial condition in K_α. Certain periodic solutions, the ***slowly spiralling periodic solutions***, are identified to fixed points of A. This is the counterpart of the so called slowly oscillating periodic solutions of the scalar setting.

By an application of a global bifurcation theorem, the Theorem 6.1, a crucial result due to Nussbaum [13], we accomplish the existence of an unbounded continuum of slowly spiralling periodic solutions of (1.3) in the Theorem 2.2.

The Theorem 2.3 is an auxiliary result for the Theorem 2.2, but it has interest in itself, since in cases where the solution $x = 0$ of (1.3) is asymptotically stable, it gives the existence of a uniform lower bound for the size of the basin of attraction, for α in compact intervals J_0.

The section 3 is devoted to the study of the location of the eigenvalues of the linear part of (1.3).

In section 4 we give some exponential estimates on the grow of the solutions of (1.3), which are useful in the proof of the Theorem 2.3

In section 5 we obtain the existence of slowly spiralling solutions of (1.3). We use the same approach as Hadeler & Tomiuk [8] in the scalar case. An essential tool for this is a fixed point principle attributed to Nussbaum, the Lemma 5.3, which depends in a crucial way on the concept of ejectivity, according to Browder [3].

Section 6 is dedicated to the proof of the theorem 2.2.

2. Statement of results.

Consider the delay differential equation

$$\dot{x}(t) = -\alpha x(t) + \alpha F(x(t-1)) \tag{2.1α}$$

where $x = \mathrm{col}(x_1, x_2) \in \mathbb{R}^2$, $F = \mathrm{col}(F_1, F_2)$ is a map from $\mathbb{R}^2$ to itself and α is a positive real parameter.

All over this paper F is assumed to be differentiable up to order two at the origin.

For $\varphi \in C$, $C := C([-1,0],\mathbb{R}^2)$ with the sup norm $\|\cdot\|$, there exists a unique function $x(t) = x(t;\alpha,\varphi)$ which is continuous in $[-1,\infty)$, continuously diferentiable in $[0,\infty)$, $x|_{[-1,0]} = \varphi$ and satisfies (2.1α) for $t \geq 0$.

Given a continuous function $x : [\tau - 1, \tau + a) \to \mathbb{R}^2$, $a > 0$, the notation x_t indicates the function $x_t(\theta) = x(t+\theta)$, $-1 \leq \theta \leq 0$, $\tau \leq t < \tau + a$.

Let us define $\delta_1 = \dfrac{\partial F_1}{\partial x_2}(0)$, $\delta_2 = \dfrac{\partial F_2}{\partial x_1}(0)$. At various stages, some or all of the following hypoteses on F will be required:

(H1) $x_2 F_1(x) > 0$, if $x_2 \neq 0$; $x_1 F_2(x) < 0$, if $x_1 \neq 0$ and $\delta := \sqrt{-\delta_1\delta_2} > 1$.
(H2) There exists $M > 0$ such that $|F(x)| \leq M$, $x \in \mathbb{R}^2$.
(H3) $x \cdot F(x) < |x|^2$, if $0 < |x| \leq M$.
(H4) $|\delta_1 + \delta_2| < 2$.

Where "$\cdot$" in (H3) denotes the scalar product of the $\mathbb{R}^2$.

Remark 2.1. The condition $x_2F_1(x) > 0$, $x_2 \neq 0$, and $x_1F_2(x) < 0$, $x_1 \neq 0$, implies $\delta_1, -\delta_2 \geq 0$ and imposes that a point in any quadrant is carried by F to the next one, in the clockwise sense; this forces the orbits of (2.1α) to rotate around the origin and turns plausible the existence of periodic solutions; we can say that (H1), without the assumption on δ, is a planar version of the negative feedback condition, current in the literature of the scalar case. We easily see that, if $F \in C^2$ in a neighborhood of $(0,0)$, (H4) implies the inequality set in (H3), for $x \neq 0$ in this neighborhood. Moreover, the expansion of F up to the second order and the condition (H3) give

$$[Ax_1 + Bx_2]x_1x_2 + O(|x|^4) < x_1^2 + x_2^2 - (\delta_1 + \delta_2)x_1x_2,$$

where

$$A = \frac{1}{2}\frac{\partial^2 F_2}{\partial {x_1}^2}(0) + \frac{\partial^2 F_1}{\partial x_1 \partial x_2}(0),$$

$$B = \frac{1}{2}\frac{\partial^2 F_1}{\partial {x_2}^2}(0) + \frac{\partial^2 F_2}{\partial x_1 \partial x_2}(0).$$

Therefore, $|\delta_1 + \delta_2| \leq 2$, and a sufficient condition for $|\delta_1 + \delta_2| < 2$ is $A \neq \pm B$.

Remark 2.2. The hypotheses (H1) and (H2) imply the existence of a constant $\Omega > 1$ such that $|F(x)| \leq \Omega|x|$, $x \in \mathbb{R}^2$.

Remark 2.3. Suppose the hypotheses (H2) and (H3) are fulfilled and let $\xi > 0$. Then, there exists $\varepsilon > 0$ such that, if $\xi \leq |x| \leq M$, then $|y - F(x)| < \varepsilon$ implies $x \cdot y < |x|^2$. In fact, if this is not the case, there are sequences, $\xi \leq x_n \leq M$, y_n, $n = 1, 2, \ldots$, with $x_ny_n \geq |x_n|^2$, $|y_n - F(x_n)| < \frac{1}{n}$ and $x_n \to x_0$, as $n \to \infty$. Therefore, $y_n \to F(x_0)$ and, finally, $x_0 \cdot F(x_0) \geq |x_0|^2$, which contradicts (H3).

Remark 2.4. For the study of periodic solutions of (2.1α), the hypotesis (H2) can be replaced by the more general condition:

$$(\overline{\mathrm{H2}}) \qquad F(\bar{B}_M) \subset \bar{B}_M$$

where $\bar{B}_M = \{x \in \mathbb{R}^2 \mid |x| \leq M\}$, for some $M > 0$. In fact, if this is the case, defining

$$\bar{F}(x) = \begin{cases} F(x), & \text{for } |x| \leq M \\ F(Mx/|x|), & \text{for } |x| > M, \end{cases}$$

it follows that $\bar{F}$ satisfies (H2) and, if $x(t)$ is a periodic solution of

$$\dot{x}(t) = -\alpha x(t) + \alpha \bar{F}(x(t-1)),$$

then $|x(t)| \leq M$; so $x(t)$ is a periodic solution of (2.1α).

Remark 2.5. The function $F(x) = (2\arctan x_2, -2\arctan x_1)$. satisfies the hypoteses (H1) – (H4)

Definition 2.1. A solution $x(t) = (x_1(t), x_2(t))$ of (2.1α) is a *slowly spiralling periodic solution* if there are real numbers q_1, q_2, q_3, q_4 with $q_j > q_{j-1} + 1$, $j = 2, 3, 4$, such that

(1) $x_1(-1) = 0$, $x_1(t) > 0$, if $t \in (-1, q_2)$ and $x_1(t) < 0$, if $t \in (q_2, q_4)$.
(2) $x_2(t) > 0$, if $t \in (-1, q_1) \cup (q_3, q_4)$ and $x_2(t) < 0$, if $t \in (q_1, q_3)$.
(3) $x(t) = x(t + q_4 + 1)$, $t \in \mathbb{R}$.

The Figure 2.1 shows a slowly spiralling periodic solution. This term is suggested by the fact that the orbit encircles the origin, crossing the coordinate semi-axis in times spaced by intervals greater than the time-lag. It is inspirated in the concept of *slowly oscillating* periodic solution from the scalar case. The period of a slowly spiralling solution is obviously greater than four.

FIGURE 2.1

Remark 2.6. The hypotheses (H3) and (H4) play a fundamental role in the lemma 6.4, which implies that, for some $\bar{\alpha} > 0$, there is no slowly spiralling periodic solutions of (1.3), for $0 < \alpha < \bar{\alpha}$. This is an essential requirement to state a global Hopf bifurcation for the equation (1.3).

In this paper the problem of finding slowly spiralling solutions is reduced to that of finding fixed points of a return map from a subset of C into itself.

We define in the sequel, two important closed convex sets $K, K_\alpha \subset C$. If $x(t)$ is a slowly spiralling periodic solution of (2.1α), then $\varphi = x|_{[-1,0]} \in K$, where

$$K = \{\varphi \in C \mid \varphi_1(-1) = 0,\ \varphi_1, \varphi_2 \geq 0\}, \tag{2.2}$$

moreover,

$$\frac{d}{dt}e^{\alpha t}x(t) = \alpha e^{\alpha t}F(x(t-1)) = \alpha e^{\alpha t}F(x(t+q_4)), \quad -1 \leq t \leq 0.$$

It follows from the periodicity of $x(t)$ that $\varphi \in K_\alpha$, where

$$K_\alpha = \{\varphi \in K \mid e^{\alpha\theta}\varphi_j(\theta) \text{ is nondecreasing in } [-1, 0],\ j = 1, 2\} \tag{2.3}$$

We state now the main results of this work.

Theorem 2.1. *If* (H1) *is satisfied, there exists a sequence* $0 < \alpha_0 < \alpha_1 < \ldots \to +\infty$,

$$\alpha_0 = \frac{b_0}{\sqrt{\delta^2 - 1}}, \quad 0 < b_0 := \arcsin 1/\delta < \pi/2, \tag{2.4}$$

such that the equation (2.1α) *has a Hopf bifurcation at* $\alpha = \alpha_k, k = 0, 1, 2, \ldots$.

Theorem 2.2. *Let us assume the hypoteses (H1)-(H4), and let*

$$\Sigma = \{(\alpha, \varphi) \in (0, \infty) \times K \mid x(t; \alpha, \varphi) \text{ is a slowly spiralling periodic solution}\}.$$

Then the following conditions are satisfied:

(1) *There exists $\bar{\alpha} > 0$ such that if $(\alpha, \varphi) \in \Sigma$, then $\alpha \geq \bar{\alpha}$ and $||\varphi|| \leq M$.*
(2) *The closure $\overline{\Sigma}$ of Σ in $(0, \infty) \times K$ is $\overline{\Sigma} = \Sigma \cup (\alpha_0, 0)$*
(3) *Let $\Sigma_0 \subseteq \overline{\Sigma}$ be the connected component of $\overline{\Sigma}$ containing $(\alpha_0, 0)$. Then Σ_0 is an unbounded subset of $(0, \infty) \times K$, more specifically, for each $\alpha > \alpha_0$, there exists a slowly spiralling periodic solution $x(t; \alpha, \varphi)$ of (2.1α) such that $(\alpha, \varphi) \in \Sigma_0$ and $|x(t; \alpha, \varphi)| \leq M$, for all t.*

Remark 2.6. It will be clear later that only (H1) and (H2) are needed for the existence of slowly spiralling periodic solutions, for $\alpha > \alpha_0$. The additional hypoteses are required for the assertions (1) and (2).

Theorem 2.3. *Let $J_0 \subset (0, \alpha_0)$ be compact. Then, there exist positive constants δ, L, and η, such that, if $\alpha \in J_0$ and $||\varphi|| < \delta$, the solution $x(t; \alpha, \varphi)$ of (2.1α) satisfies*

$$|x(t; \alpha, \varphi)| \leq Le^{-\eta t}||\varphi||, \quad t \geq 0.$$

Theorem 2.2 is proved in Section 6. The proof of Theorem 2.1 is a straightforward consequence of a description of the locus of the eigenvalues of the linearized equation of (2.1α), as given in Section 3. Theorem 2.3 is an auxiliary result for the Theorem 2.2, however, as we have pointed out in the section 1, it has interest in itself.

3. The eigenvalues of the linearized equation.

We state below properties of the eigenvalues of the linear part of (2.1α) which are relevant for this paper. For a general study of this subject see [2] and [15]. A linearization of (2.1α) near the origin, with F satisfying (H1), gives:

$$\dot{x}(t) = -\alpha x(t) + \alpha B x(t-1), \qquad B = \begin{pmatrix} 0 & \delta_1 \\ \delta_2 & 0 \end{pmatrix}. \tag{3.1}$$

The characteristic equation of (3.1) is:

$$(\lambda + \alpha)^2 e^{2\lambda} = -\delta^2 \alpha^2 \tag{3.2}$$

If λ is a characteristic root of (3.2) (eigenvalue of (3.1)), its complex conjugate $\bar{\lambda}$ is. Moreover, $\lambda = 0$ is the unique real eigenvalue, and it occurs for $\alpha = 0$; therefore, we can restrict our study to the upper semi-plane $\Im(\lambda) > 0$.

Lemma 3.1. *There is no root of (3.2) in the lines $a + ib$ with fixed $b = (k + 1/2)\pi$, $k = 0, 1, \ldots$, and $a \in \mathbb{R}$.*

Proof. For $\lambda = a + i(k + 1/2)\pi$, $k = 0, 1, \ldots$, the equation (3.2) leads to

$$\begin{cases} e^{2a}[(a + \alpha)^2 - (k + 1/2)^2\pi^2] = \delta^2\alpha^2 \\ (a + \alpha)(2k + 1)\pi = 0, \end{cases}$$

and this system is incompatible. □

The Lemma 3.1 asserts that the characteristic roots $\lambda \in \mathbb{C}_+ := \{\lambda \in \mathbb{C} \mid \Im(\lambda) > 0\}$ are in the strips:

$$S_k = \{\lambda \in \mathbb{C}_+ \mid (k - 1/2)\pi < \Im(\lambda) < (k + 1/2)\pi\}, \quad k = 0, 1, 2, \ldots$$

Lemma 3.2. *In each strip S_k, $k \geq 0$, the characteristic equation is*

$$(\lambda + \alpha)e^{\lambda} = (-1)^k i\delta\alpha \tag{3.3}$$

Proof. The equation (3.2) is equivalent to

$$(\lambda + \alpha)e^{\lambda} = \pm i\delta\alpha$$

Taking $\lambda = a + ib$, one obtains:

$$\begin{cases} (a+\alpha)\cos b - b\sin b = 0 \\ (a+\alpha)\sin b + b\cos b = \pm\delta\alpha e^{-a}. \end{cases}$$

The first equation implies $(a+\alpha)\sin b$ and $b\cos b$ have the same sign. The Lemma is proved noticing that, in S_k, the sign of $b\cos b$ is $(-1)^k$. □

Lemma 3.3. *The only pure imaginary characteristic roots are $\lambda_k = ib_k \in S_k$, where $\sin b_k = (-1)^k 1/\delta$, and the corresponding values $\alpha = \alpha_k$ are*

$$\alpha_k = \frac{b_k}{\sqrt{\delta^2 - 1}} \qquad k = 0, 1, \ldots$$

Proof. Taking $\lambda = ib$ in (3.3), we obtain $\alpha = b\tan b$ and $\alpha\sin b + b\cos b = (-1)^k\delta\alpha$. So that, $\sin b = (-1)^k/\delta$ and $\cos b = (-1)^k\sqrt{\delta^2-1}/\delta$, for every $k \geq 0$. □

Remark 3.1. The sequence (α_k) satisfies $0 < \alpha_0 < \alpha_1 < \alpha_2 < \ldots \to +\infty$, as $k \to +\infty$, since $b_k = b_0 + k\pi$, $k \geq 1$, and $0 < b_0 = \arcsin 1/\delta < \pi/2$. The value

$$\alpha_0 = \frac{b_0}{\sqrt{\delta^2 - 1}} \tag{3.4}$$

has special importance in this paper.

Lemma 3.4. *Each strip S_k, $k \geq 0$, contains exactly one root of (3.2), and it is simple.*

Proof. Taking $\lambda = a + ib \in S_k$, $k \geq 0$ in (3.3), we obtain :

$$\begin{cases} a + \alpha = b\tan b \\ (a+\alpha)\sin b + b\cos b = (-1)^k\delta\alpha e^{-a}, \end{cases} \tag{3.5}$$

wich gives

$$\beta := \delta\alpha e^{\alpha} = (-1)^k b e^{b\tan b}\sec b,$$

so that

$$\frac{d\beta}{db} = (-1)^k e^{b\tan b}\sec b[(1 + b\tan b)^2 + b^2] > 0.$$

Furthermore, $\beta = \beta(b) \to 0$, as either $b \to 0_+$ or $b \to (n\pi + \pi/2)_+$, and $\beta(b) \to +\infty$, as $b \to (n\pi + \pi/2)_-$, $n = 0, 1, 2, \ldots$. So that, $b = \Im(\lambda)$ is a strictly increasing function of $\alpha \in (0, \infty)$. Consequently, $a = \Re(\lambda)$ is a function of $\alpha \in (0, \infty)$ and

$$\frac{da}{d\alpha} = \frac{a(1 + a + \alpha) + b^2}{\alpha[(1 + a + \alpha)^2 + b^2]}.$$

The last part of the lemma is a consequence of

$$\frac{d}{d\lambda}[(\lambda + \alpha)e^{\lambda} - (-1)^k\delta\alpha i] = e^{\lambda}(1 + \lambda + \alpha) \neq 0. \quad \square$$

Lemma 3.5. *In each strip S_k, $k \geq 0$, the roots $\lambda = \lambda(\alpha)$ of* (3.2) *satisfy $\Re(\lambda) < \ln\delta$ and $\lim_{\alpha\to\infty}\lambda = \ln\delta + i(k\pi + \pi/2)$, $k = 0, 1, 2, \ldots$.*

Proof. It follows from (3.5) that

$$e^a(1 + \frac{a}{\alpha}) = (-1)^k\delta\sin b.$$

By Lemma 3.4, $\lim_{\alpha\to\infty}(-1)^k\delta\sin b = \delta$ and, moreover, $e^a(1+a/\alpha) > e^a$ for α large. So that, $a = a(\alpha)$ is bounded as $\alpha \to \infty$ and

$$\delta = \lim_{\alpha\to\infty} e^a(1 + \frac{a}{\alpha}) = \lim_{\alpha\to\infty} e^a,$$

which means that, $a = \Re(\lambda) \to \ln\delta$ as $\alpha \to \infty$.

The other assertions result from Lemma 3.4. □

Lemma 3.6. *If $\lambda_k = a_k + ib_k \in S_k$, $k = 0, 1, 2, \ldots$ are roots of* (3.2)*, for some $\alpha > 0$, then $a_0 > a_1 > \ldots \to -\infty$.*

Proof. After the translation $\lambda + \alpha = \mu$, (3.2) becomes

$$\mu^2 e^{2\mu} = -\beta^2, \tag{3.6}$$

where $\beta = \alpha\delta e^{2\alpha}$. Since $\alpha \in \mathbb{R}$, the lemma will be proved if the specified ordering occurs for the real parts of the roots of (3.6). Taking $\mu = a + ib$ in (3.6), one obtains:

$$(a^2 + b^2)e^{2a} = \beta^2. \tag{3.7}$$

For $\beta = \pi$, (3.7) defines b as function of a, $a \in (-\infty, 0]$, with

$$\frac{db}{da} = -(\pi^2 e^{-2a} + a)/b < 0,$$

thus, $a_0 > a_1 = 0 > a_2 > \ldots > a_k > \ldots$, where $\mu_k = a_k + ib_k$ is the unique root of (3.6), with $\beta = \pi$, in S_k, $k = 0, 1, \ldots$.

Suppose temporarily this ordering is false for some value of $\beta > 0$. Since each a_k depends continuously on β, there is some $\beta = \bar{\beta}$ for which (3.6) has the roots $\mu_k \in S_k$ and $\mu_l \in S_l$, with $a_k = a_l = a$, $k > l$. But (3.7) implies $b_k = b_l$, a contradiction. □

The foregoing lemmas assure that, if $\delta > 1$ and α increases from 0 to $+\infty$, once upon it reaches each α_k, $k = 0, 1, \ldots$, given in lemma 3.3, a unique pair of conjugate eigenvalues $\lambda_k = \lambda(\alpha_k)$ and $\bar{\lambda}_k$ crosses transversally the imaginary axis , from the left to the right.

The Figure 3.1 shows the locus of the characteristic roots in the several strips for $\delta = \sqrt{2}$. When α varies from 0 to $+\infty$, the real part of λ varies from $-\infty$ to $\ln\sqrt{2}$, with the λ_k's on the indicated curves, obeying the ordering of the Lemma 3.6.

Since all the eigenvalues of the linear part of (2.1α), $0 < \alpha < \alpha_0$, have negative real parts, the following lemma is a standard result in the stability theory of differential-delay equations.

FIGURE 3.1

Lemma 3.7. *Assume the hypothesis* (H1) *and* $0 < \alpha < \alpha_0$. *There is a number* $\nu > 0$ *such that, for any* $\varepsilon > 0$, *there exists* $T_\varepsilon \geq 0$ *such that, if* $||\varphi|| < \nu$, *the solution* $x(t;\alpha,\varphi)$ *of* (2.1α) *satisfies*

$$\sup_{t \geq T_\varepsilon} |x(t;\alpha,\varphi)| < \varepsilon.$$

Theorem 2.1 is a consequence of the Hopf Bifurcation Theorem for retarded differential equations, according to [9], because the above lemmas guarantee the following hypotheses:

(1) The pure imaginary eigenvalues $\lambda_k = \lambda_k(\alpha_k)$, $k = 0, 1, \ldots$ of (3.1) are simple and, for any k, if $\lambda_j \neq \lambda_k, \bar{\lambda}_k$, then $\lambda_j \neq m\lambda_k$, for any integer m.

(2) $\frac{d}{d\alpha}[\Re(\lambda_k)(\alpha)]_{\alpha=\alpha_k} \neq 0$.

The periodic solutions which appear at $\alpha = \alpha_k$ have period close to $2\pi/b_k$, with b_k defined by

$$b_k = [\arcsin(-1)^k 1/\delta] \in (k\pi - \pi/2\,,\, k\pi + \pi/2) \cap (0\,,\,\infty),\ k = 0, 1, \ldots \tag{3.8}$$

Remark 3.2. Although this is not our main concern, we state a result, without proof, for $\alpha < 0$:

If $1 < \delta < \sqrt{\pi^2+1}$, there exists a sequence (α_k), $0 > \alpha_1 > \alpha_2 > \ldots \to -\infty$, such that the equation (2.1α) has a Hopf Bifurcation in $\alpha = \alpha_k$,

$$\alpha_k := -c_k/\sqrt{\delta^2-1}; \quad c_k = [\arcsin(-1)^{k+1}1/\delta] \in (k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}), \quad k = 1, 2, \ldots.$$

The proof can be done by studying the location of the roots of (3.2), with $\alpha < 0$.

4. Uniform exponential estimates.

In this section we study a one-paremeter family of linear diferential delay equations, which includes the linearization of (2.1α) near the origin. We obtain uniform exponential estimates for the solutions increase, for α in compact sets. When the real parts of the characteristic roots are negative and bounded away from zero, therefore, $x = 0$ is an asymptotically stable solution of (2.1α), we assure the existence of a uniform lower bound for the basin of attraction, for α in compact sets.

Consider the linear autonomous equation

$$\dot{y}(t) = Ay(t) + By(t-\alpha), \tag{4.1}$$

$y \in \mathbb{R}^n$, $\alpha > 0$, which after a rescaling of time can be written as

$$\dot{x}(t) = \alpha Ax(t) + \alpha Bx(t-1) \tag{4.2}$$

It is known that the solution $x_\alpha(\varphi)(t) \in C([-1,+\infty),\mathbb{R}^n)$ of (4.2), $x_\alpha(\varphi)|_{[-1,0]} = \varphi$, satisfies

$$|x_\alpha(\varphi)(t)| \leq a(\alpha)e^{\alpha bt}|\varphi|, \qquad t \geq 0 \tag{4.3}$$

where $a(\alpha) = 1 + \alpha|B|$, $b = |A| + |B|$, with $|\cdot|$ denoting a norm in $\mathbb{R}^n$ as well as a matrix norm.

The characteristic equation of (4.2) is:

$$\det H_\alpha(\lambda) = 0, \quad H_\alpha(\lambda) := \lambda I - \alpha A - \alpha Be^{-\lambda} \tag{4.4}$$

where I is the $n \times n$ identity matrix.

Since

$$\det H_\alpha(\lambda) = \lambda^n + p_1(e^{-\lambda})\lambda^{n-1} + \ldots + p_{n-1}(e^{-\lambda})\lambda + p_n(e^{-\lambda}), \tag{4.5}$$

with $p_i(s)$ being polinomials of degree less or equal to i, $i = 1, 2, \ldots, n$, there exists an infinity of characteristic roots with arbitrarily large negative real parts, see [15, Th.1]. We state this more precisely as an extension of the Lemma 4.1 in [9, Chap.1], as follows:

Lemma 4.1. *Let (λ_j) be a sequence of roots of (4.4) such that $|\lambda_j| \to +\infty$ as $j \to +\infty$. Then $\Re\lambda_j \to -\infty$ as $j \to +\infty$ and there exists only a finite number of roots of (4.4) to the right of any line $\Re\lambda = a$, $a \in \mathbb{R}$.*

Lemma 4.2. *Let $J_0 \subset (0,+\infty)$ be compact and suppose the roots of (4.4), for $\alpha \in J_0$, are simple. Then $\{\Re\lambda \mid \det H_\alpha(\lambda) = 0, \alpha \in J_0\}$ is upper bounded.*

Proof. According to Lemma 4.1 we can define

$$\beta_\alpha := \max\{\Re\lambda \mid \det H_\alpha(\lambda) = 0\}. \tag{4.6}$$

Since J_0 is compact, it is sufficient to prove that β_α depends continuously on $\alpha > 0$.

Let us take $\tilde{\alpha} > 0$ and prove that β_α is continuous in $\alpha = \tilde{\alpha}$. If $b \in \mathbb{R}$ satisfies

$$F_b = \{\lambda \mid \det H_{\tilde{\alpha}}(\lambda) = 0\ , \Re\lambda > b\} \neq \emptyset,$$

F_b is finite and $\beta_{\tilde{\alpha}} = \max\{\Re\lambda \mid \lambda \in F_b\}$.

Since the roots of (4.4) are simple and the right side of (4.2) is continuously differentiable in α, we have $\Re\lambda_j(\alpha)$, $j = 1, 2, \ldots, m$, continuous in $\alpha \in [\tilde{\alpha} - \delta, \tilde{\alpha} + \delta]$, for some $\delta > 0$, see [9, Chap.7]. Consequently, β_α is continuous in $\alpha \in [\tilde{\alpha} - \delta, \tilde{\alpha} + \delta]$. $\square$

If $x_\alpha^1(t), x_\alpha^2(t), \dots, x_\alpha^n(t)$ are solutions of the equation (4.2) for some $\alpha > 0$, the matrix $Y_\alpha(t) = (x_\alpha^1(t), \dots, x_\alpha^n(t))$ is called a solution matrix of (4.2). For $t \geq -1$, $X_\alpha(t)$ is said to be the Fundamental Matrix of (4.2) if it is the solution matrix of (4.2) such that $X_\alpha(t)|_{[-1,0]} \equiv \Psi(t)$, where

$$\Psi(t) = \begin{cases} 0 & t < 0 \\ I & t = 0, \end{cases}$$

0 and I are the zero and identity $n \times n$ matrices, respectively. $X_\alpha(t)$ is of bounded variation in compact subsets of $[-1, \infty)$ and the following exponential estimate holds:

$$|X_\alpha(t)| \leq a(\alpha) e^{\alpha b t} |\Psi|, \quad t \geq 0, \tag{4.7}$$

where $a(\alpha) = 1 + \alpha|B|$, $b = |A| + |B|$.

The results below are stated in [9, chap. 7] for the scalar case, and request some elementary facts from the theory of Laplace Transforms. The estimate (4.7) guarantees the existence of the Laplace transform, $\mathcal{L}(x)(\lambda)$, of solutions $x(t)$ to (4.2), for $\Re(\lambda) > \alpha b$.

For any real number c, we denote

$$\int_{(c)} = \lim_{T \to +\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} .$$

Theorem 4.1. *For each $\alpha > 0$, the fundamental matrix $X_\alpha(t)$ of (4.2) satisfies*

$$\mathcal{L}(X_\alpha)(\lambda) = H_\alpha^{-1}(\lambda)$$

for any $\lambda \in \mathbb{C}$ such that $\Re\lambda > c_\alpha = \max\{\alpha b, \beta_\alpha\}$, with β_α and b given in (4.6) and (4.3), respectively. Moreover, for any constant $c > c_\alpha$,

$$X_\alpha(t) = \int_{(c)} H_\alpha^{-1}(\lambda) e^{\lambda t} d\lambda, \quad t > 0.$$

Corollary 4.1. *Let $J_0 \subset (0, \infty)$ bounded, with $\gamma_0 := \sup\{\Re\lambda \mid \det H_\alpha(\lambda) = 0,\ \alpha \in J_0\}$ finite. Then, for every $\alpha \in J_0$,*

$$\mathcal{L}(X_\alpha)(\lambda) = H_\alpha^{-1}(\lambda), \quad \Re\lambda > \bar{c}$$

and

$$X_\alpha(t) = \int_{(c)} H_\alpha^{-1}(\lambda) e^{\lambda t} d\lambda, \quad t > 0$$

where c is a real number greater than $\bar{c} := \max\{\gamma_0, (|A| + |B|) \sup J_0\}$.

Theorem 4.2. *Let J_0 and γ_0 be as in the Corollary 4.1. Then, for any $\gamma > \gamma_0$, there exists $K = K(\gamma) > 0$ such that the fundamental matrix $X_\alpha(t)$ of (4.2) satisfies*

$$|X_\alpha(t)| \leq K e^{\gamma t}, \quad t \geq 0,\ \alpha \in J_0. \tag{4.8}$$

Proof. First of all we prove that , for $\alpha \in J_0$ and $\gamma > \gamma_0$,

FIGURE 4.1

$$X_\alpha(t) = \int_{(\gamma)} H_\alpha^{-1}(\lambda) e^{\lambda t} d\lambda, \quad t > 0. \tag{4.9}$$

Consider the integral of $H_\alpha^{-1}(\lambda)e^{\lambda t}$ along the boudary $L_1 M_1 L_2 M_2$ of the box B in the complex plane, see Figure 4.1, in the indicated direction, where

$$L_1 = \{c + i\tau \mid -T \le \tau \le T\}, \quad M_1 = \{\sigma + iT \mid \gamma \le \sigma \le c\},$$

$$L_2 = \{\gamma + i\tau \mid -T \le \tau \le T\}, \quad M_2 = \{\sigma - iT \mid \gamma \le \sigma \le c\}.$$

Since $H_\alpha(\lambda)$ is invertible and $H_\alpha^{-1}(\lambda)e^{\lambda t}$ is analitic in $\Re\lambda > \gamma_0$, we have

$$\int_B H_\alpha^{-1}(\lambda) e^{\lambda t} d\lambda = 0.$$

To prove (4.9), it suffices to show that

$$\int_{M_1} H_\alpha^{-1}(\lambda) e^{\lambda t} d\lambda, \int_{M_2} H_\alpha^{-1}(\lambda) e^{\lambda t} d\lambda \longrightarrow 0, \quad \text{as } T \to +\infty.$$

It follows from (4.4) that

$$(|\lambda| - \alpha|A| - \alpha e^{-\Re\lambda}|B|)|H_\alpha^{-1}(\lambda)| \le 1, \tag{4.10}$$

for every λ such that $\Re\lambda > \gamma_0$, $\alpha \in J_0$.

If $\lambda \in M_1$ and $\alpha \in J_0$, taking $\bar\alpha = \sup J_0$,

$$|\lambda| - \alpha|A| - \alpha e^{-\Re\lambda}|B| \ge (\sigma^2 + T^2)^{1/2} - \bar\alpha\left(|A| + e^{-\gamma}|B|\right).$$

Then, under this condition, there exists $T_0 \ge 0$, independent of α, such that, for $T \ge T_0$,

$$|\lambda| - \alpha|A| - \alpha e^{-\Re\lambda}|B| > T/2, \qquad \alpha \in J_0,\ \lambda \in M_1, \tag{4.11}$$

and (4.10), (4.11) lead to

$$\left|\int_{M_1} H_\alpha^{-1}(\lambda)e^{\lambda t}d\lambda\right| \leq 2e^{ct}(c-\gamma)/T \to 0, \text{ as } T \to +\infty.$$

Similarly, one shows $\int_{M_2} H_\alpha^{-1}(\lambda)e^{\lambda t}d\lambda \to 0$, as $T \to +\infty$.

If $\lambda = \gamma + i\tau$ and $\alpha \in J_0$, we define

$$G_\alpha(\lambda) = H_\alpha^{-1}(\lambda) - (\lambda-\gamma_0)^{-1}I = (\lambda-\gamma_0)^{-1}[(\lambda-\gamma_0)I - H_\alpha(\lambda)]H_\alpha^{-1}(\lambda). \tag{4.12}$$

Let $T_0 > 0$ is chosen in such a way that $T \geq T_0$ implies (4.11), thus

$$\begin{aligned}\int_\gamma |G_\alpha(\lambda)|d\lambda = \lim_{T\to+\infty}\frac{1}{2\pi}\Bigg[&\int_{-T}^{-T_0}|G_\alpha(\gamma+i\tau)|d\tau+\\ &+\int_{-T_0}^{T_0}|G_\alpha(\gamma+i\tau)|d\tau+\int_{T_0}^{T}|G_\alpha(\gamma+i\tau)|d\tau\Bigg].\end{aligned} \tag{4.13}$$

Being $G_\alpha(\lambda)$ analytic in $\lambda \in L_2$ and continuous in α, there exists $K_1 > 0$ such that

$$\int_{-T_0}^{T_0}|G_\alpha(\gamma+i\tau)|d\tau \leq 2K_1T_0, \qquad \alpha \in J_0, \tag{4.14}$$

therefore, (4.10), (4.11), (4.12) imply

$$|G_\alpha(\gamma+i\tau)| \leq \frac{2}{\tau^2}\left(|\gamma_0| + \bar{\alpha}|A| + \bar{\alpha}|B|e^{-\gamma}\right), \quad |\tau| \geq T_0, \alpha \in J_0, \tag{4.15}$$

and (4.13), (4.14) lead to

$$\left|\int_{(\gamma)}|G_\alpha(\lambda)|d\lambda\right| \leq K_2, \quad \alpha \in J_0,$$

with $K_2 = \frac{1}{2\pi}\left[2K_1T_0 + \frac{4}{T_0}(|\gamma_0| + \bar{\alpha}|A| + \bar{\alpha}|B|e^{-\gamma})\right]$, independent of α and, consequently,

$$\left|\int_{(\gamma)} G_\alpha(\lambda)e^{\lambda t}d\lambda\right| \leq K_2e^{\gamma t}. \tag{4.16}$$

Standard arguments from complex integration theory lead to

$$\left|\int_{(\gamma)}(\lambda-\gamma_0)^{-1}e^{\lambda t}d\lambda\right| \leq 4\pi e^{\gamma t}, \tag{4.17}$$

and (4.16), (4.17), together with (4.12), gives

$$|X_\alpha(t)| = \left|\int_{(\gamma)} H_\alpha^{-1}(\lambda)e^{\lambda t}d\lambda\right| \leq Ke^{\gamma t},$$

for $t \geq 0$, $\alpha \in J_0$, with the constant $K = K(\gamma)$ independent of α. □

Theorem 4.3. *Let $J_0 \subset (0, \infty)$ and γ_0 as in the Corollary 4.1. Then, for each $\gamma > \gamma_0$, there exists a constant $K = K(\gamma) > 0$, independent of α, such that the solution $x_\alpha(\varphi)(t)$ of the equation (4.2) satisfies*

$$x_\alpha(\varphi)(t)| < Ke^{\gamma t}|\varphi|, \quad t > 0, \alpha \in J_0. \tag{4.18}$$

So that, if $\gamma_0 < 0$, the condition $\gamma_0 < \gamma < 0$ implies the origin is exponentially stable.

Proof. The solution $x_\alpha(\varphi)(t)$ of the equation (4.2) can be represented by

$$x_\alpha(\varphi)(t) = X_\alpha(t)\varphi(0) + B\int_{-1}^{0} X_\alpha(t-\theta-1)\varphi(\theta)d\theta, \quad t \geq 0. \tag{4.19}$$

This and the exponential boundedness of $X_\alpha(t)$ in Theorem 4.2 assure the result. □

An exponential decay like (4.18) is transferred, locally, to the nonlinear equation

$$\dot{x}(t) = \alpha Ax(t) + \alpha Bx(t-1) + \alpha f(x(t), x(t-1)), \tag{4.20}$$

where $x = \mathrm{col}(x_1, \ldots, x_n)$, A, B are $n \times n$ matrices and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable at 0, $f(0) = 0$, $f'(0) = 0$,and $\alpha > 0$. We will prove that, although the global character of the stability is lost, there is a uniform lower bound for the basin of attraction of the zero solution of (4.20), for α in compact sets. Let $x_\alpha(\varphi, f)(t)$ be the solution of (4.20) with initial condition $\varphi \in C$, defined in its maximal interval of existence.

Theorem 4.4. *Let $J_0 \subset (0, \infty)$ and γ_0 given in the Corollary 4.1, with $\gamma_0 < 0$. Then there are constants $\delta, L, \eta > 0$ such that the solution $x_\alpha(\varphi, f)(t)$ of the equation (4.20) satisfies*

$$|x_\alpha(\varphi, f)(t)| \leq Le^{-\eta t}||\varphi||, \quad t \geq 0, \alpha \in J_0, ||\varphi|| < \delta$$

Proof. Denoting by $[-1, b_+)$ the maximal interval of existence of $x_\alpha(\varphi, f)$, the variation-of-constants formula gives

$$x_\alpha(\varphi, f)(t) = x_\alpha(\varphi)(t) + \alpha\int_0^t X_\alpha(t-s)f(x(s), x(s-1))ds, \tag{4.21}$$

$0 \leq t \leq b_+$, where $x_\alpha(\varphi)$ is the solution of (4.2) with initial condition φ.

It follows from (4.8) and Theorem 4.18 that, for $\gamma_0 < \gamma < 0$ and $\bar{\alpha} = \sup J_0$,

$$|x_\alpha(\varphi, f)(t)| \leq Ke^{\gamma t}||\varphi|| + \bar{\alpha}K\int_0^t e^{\gamma(t-s)}|f(x(s), x(s-1))|ds. \tag{4.22}$$

Given $0 < \varepsilon_0 < \dfrac{-\gamma}{\bar{\alpha}K(1+e^{-\gamma})}$, let $\delta_0 > 0$, such that $|f(u,v)| \leq \varepsilon_0(|u| + |v|)$, if $|u| + |v| \leq \delta_0$. Let

$$||\varphi|| < \delta = \min\left\{\frac{\delta_0}{2}, \frac{\delta_0}{2Ke^{\bar{\alpha}K\varepsilon_0(1+e^{-\gamma})}}\right\}$$

and suppose there is a $t_0 \in (0, b_+)$ so that $|x_\alpha(\varphi, f)(t_0)| = \delta_0$, $|x_\alpha(\varphi, f)(t)| < \delta_0$, $0 \leq t < t_0$. Then, (4.22) implies

$$e^{-\gamma t}|x_\alpha(\varphi, f)(t)| \leq K||\varphi|| + \bar{\alpha}K\varepsilon_0 \int_0^t e^{-\gamma s}|x(s)|ds + \bar{\alpha}K\varepsilon_0 \int_0^t e^{-\gamma s}|x(s-1)|ds,$$

for $0 \leq t \leq t_0$, and

$$e^{-\gamma t}|x_\alpha(\varphi, f)(t)| \leq K||\varphi|| + \bar{\alpha}K\varepsilon_0[1 + e^{-\gamma}] \int_{-1}^t e^{-\gamma s}|x(s)|ds.$$

Consequently,

$$|x_\alpha(\varphi, f)(t)| \leq Le^{-\eta t}||\varphi||, \quad 0 \leq t \leq t_0 \tag{4.23}$$

where $L = Ke^{\bar{\alpha}K\varepsilon_0}[1 + e^{-\gamma}]$ and $\eta = -\gamma - \bar{\alpha}K\varepsilon_0[1 + e^{-\gamma}]$.

But, the choice of δ implies $|x_\alpha(\varphi, f)(t)| \leq \delta_0/2$, $0 \leq t \leq t_0$, a contradiction. □

Proof of the Theorem 2.3.

The equation (2.1α) is as (4.20), being $-A$ the 2×2-matrix identity and B given in (3.1).

Moreover, the analysis of the characteristic equation of (3.1) done in section 3 shows that $\gamma_0 < 0$, for each compact set $J_0 \subset (0, \alpha_0)$. The result is now a straightforward consequence of the Theorem 4.4. □

5. Existence of slowly spiralling periodic solutions.

In order to accomplish the existence of slowly spiralling periodic solutions of (2.1α), for α greater than $\alpha_0 > 0$, we need the concept of ejectivity, according to Browder [3]. However, the global Hopf bifurcation depends on some uniformty assumptions on the parameter. That is why we formulate the following stronger definition of ejectivity. The existence of slowly spiralling solutions can be stated under weaker conditions than those of the theorem 5.1, see [16], for instance. We keep the present formulation because the aim of this section is a preparation for the results of section 6.

Definition 5.1. Let X a topological space and $\mathcal{H} = \{ H_\alpha, \alpha \in \Omega \}$ a family of continuous functions $H_\alpha : X \to X$. The point $x_0 \in X$ is said to be a *uniform ejective point* of $\mathcal{H}$, if there exists an open neighborhood, $x_0 \in W \subset X$, such that, for every $x \in W - \{x_0\}$ and $\alpha \in \Omega$, there is an integer $n = n(\alpha, \varphi)$ for which ${H_\alpha}^n\varphi \notin W$.

Remark 5.1. For the constant family $H_\alpha = H$, $\alpha \in \Omega$, the above definition reduces to that of Browder's *ejective point*.

The Lemma 5.2 below gives a sufficient condition for the ejectivity of the origin for certain return maps H_α defined by families of autonomous equations. Our proof differs from that given by Táboas in [16], since there the ejectivity is not required to be uniform.

Consider the autonomous equations:

$$\dot{x}(t) = \alpha L x_t + \alpha f(x_t) \tag{5.1}$$

$$\dot{y}(t) = \alpha L y_t, \tag{5.2}$$

where $L : C \longrightarrow \mathbb{R}^2$ is a linear continuous map, f is completely continuous, continuously diferentiable, $f(0) = 0$, $f'(0) = 0$ and $\alpha > 0$. Let us keep the notations $x_t(\cdot;\alpha,\varphi)$ and $y_t(\cdot;\alpha,\varphi)$ for the solutions of (5.1) and (5.2), with initial condition $\varphi \in C$.

If $T_\alpha(t)$, $\alpha > 0$, $t \geq 0$, is the solution map, $T_\alpha(t)\varphi = y_t(.;\alpha,\varphi)$, $\varphi \in C$,, for any eigenvalue $\lambda = \lambda(\alpha)$ of (5.2), the space C is decomposed as $C = P_\lambda \oplus Q_\lambda$, with P_λ and Q_λ invariant under $T_\alpha(t)$, and the eigenspace P_λ finite dimensional, see [9, Chap. 7].

Let Π_λ be the projection with range P_λ given by this decomposition,

$$\Pi_\lambda \varphi = \Phi_\lambda b \tag{5.3}$$

where $\Phi_\lambda = ({\varphi_1}^\lambda, \dots, {\varphi_d}^\lambda)$ is a basis of P_λ and b is a d-vector.

Lemma 5.1. *Let $J_0 \subset (\alpha_0, \infty)$ be compact and suppose that, for each $\alpha \in J_0$, there exists an eigenvalue $\tilde{\lambda} = \tilde{\lambda}(\alpha)$ of (5.2) such that $\Re\tilde{\lambda} > 0$. Then, for any $\alpha \in J_0$, there is a positive definite quadratic form in C, $V_\alpha(\varphi) = b^T D_\alpha b$, for some matrix D_α, where the vector b is defined by $\Pi_{\tilde{\lambda}}\varphi = \Phi_{\tilde{\lambda}} b$, which satisfies the following property:*

For any $p > 0$, there is a $\delta_0 > 0$ such that, for any δ, $0 < \delta \leq \delta_0$, if φ satisfies $||\varphi|| \leq \delta$ and $V_\alpha(\varphi) \geq p^2\delta^2$, then

$$\dot{V}_\alpha(\varphi) := \liminf_{t\to 0_+} \frac{1}{t}[V_\alpha(x_t(\cdot;\alpha,\varphi) - V_\alpha(\varphi)] > \frac{1}{2\gamma_\alpha} V_\alpha(\varphi) > 0$$

for some constant $\gamma_\alpha > 0$. Moreover, δ_0 is independent of α and the scalar function $V_\alpha(\varphi)$ is continuous in $(\alpha,\varphi) \in J_0 \times C$.

The proof of this lemma is an adaptation of Lemma 3.1 by Chow and Hale in [4]. The continuity of $V_\alpha(\varphi)$ and the fact of δ_0 being independent of α follow from the continuity in α of the involved quantities, according to [9, Chap. 7].

Lemma 5.2. *Let $J_0 \subset (\alpha_0, \infty)$ be compact and suppose the following hypoteses hold:*

1. *For every $\alpha \in J_0$, there exists a eigenvalue $\tilde{\lambda} = \tilde{\lambda}(\alpha)$ of (5.2) with $\Re\tilde{\lambda} > 0$.*
2. *For every $\alpha \in J_0$, there exists a convex and closed set $S_\alpha \subset C$, $0 \in S_\alpha$, and a continuous function $\tau_\alpha : S_\alpha - \{0\} \to [\rho, \infty)$, for some $\rho > 0$, such that the function $A_\alpha : S_\alpha \to C$, given by*
$$A_\alpha(\varphi) = x_{\tau_\alpha(\varphi)}(\cdot;\alpha,\varphi), \quad \varphi \neq 0$$
$$A_\alpha(0) = 0,$$
is completely continuous and $A_\alpha S_\alpha \subset S_\alpha$.
3. *For any a, $0 < a < M$,*
$$\inf\Big\{ ||\Pi_{\tilde{\lambda}(\alpha)} x_t(\cdot;\alpha,\varphi)|| \mid \alpha \in J_0,\ \varphi \in S_\alpha,\ ||\varphi|| \geq a,\ 0 \leq t \leq \tau_\alpha(\varphi)\Big\} > 0$$
4. *For any open set G, $0 \in G \subset C$, there exists a neighborhood V of 0 in C such that $x_t(\cdot;\alpha,\varphi) \in G$, for $\alpha \in J_0$, $\varphi \in V \cap S_\alpha$, $\varphi \neq 0$, $0 \leq t \leq \tau_\alpha(\varphi)$.*

Then, 0 is a uniform ejective fixed point of $\{A_\alpha,\ \alpha \in J_0\}$.

Proof. For any $\alpha \in J_0$, let us take an eigenvalue $\tilde{\lambda}(\alpha)$ of (5.2) with positive real part, denote $\Pi_\alpha = \Pi_{\tilde{\lambda}(\alpha)}$ and define $b(t)$ such that

$$\Pi_\alpha \frac{x_t}{||x_t||} = \Phi_{\tilde{\lambda}(\alpha)} b(t)$$

where $x_t = x_t(\cdot;\alpha,\varphi)$, $\varphi \in S_\alpha - \{0\}$.

Considering the function $V_\alpha(\varphi)$ of Lemma 5.1, it follows that

$$V_\alpha\left(\frac{x_t}{||x_t||}\right) = b(t)^\top D_\alpha b(t) \geq |b(t)|^2\beta_\alpha^2,$$

with $\beta_\alpha^2 := \min\{b^\top D_\alpha b \mid |b| = 1\}$. Therefore,

$$(5.4) \qquad ||\Pi_\alpha \frac{x_t}{||x_t||}||^2 \leq \frac{||\Phi_{\bar\lambda(\alpha)}||^2}{\beta_\alpha^2} V_\alpha\left(\frac{x_t}{||x_t||}\right) \leq \mu^2 V_\alpha\left(\frac{x_t}{||x_t||}\right)$$

where $\mu^2 = \dfrac{\max_{\alpha\in J_0}||\Phi_{\bar\lambda(\alpha)}||^2}{\min_{\alpha\in J_0}\beta_\alpha^2}$ is well defined, because $\Phi_{\bar\lambda(\alpha)}$ and β_α are continuous.

Letting $x_t = x(\cdot;\alpha,\varphi)$, it follows from (5.4) and hypotesis (3) that

$$\inf\left\{V_\alpha\left(\frac{x_t}{|||x_t||}\right) \mid \alpha \in J_0, \ , \ \varphi \in S_\alpha, \ ||\varphi|| = a, \ 0 \leq t \leq \tau_\alpha(\varphi)\right\} = \Upsilon^2 > 0.$$

If p is fixed, $0 < p < \Upsilon$, and δ_0, independent of α, is like in Lemma 5.1, let us define, for each $\alpha \in J_0$, and $0 < \delta \leq \delta_0$

$$G_\alpha(\delta) = \{\varphi \in C \mid ||\varphi|| < \delta, \ V_\alpha(\varphi) < p^2\delta^2\}.$$

It follows from Lemma 5.1 that $G_\alpha(\delta)$ is open, $0 \in G_\alpha(\delta)$ and there exists an open set O_δ such that $O_\delta \subset G_\alpha(\delta)$, for any $\alpha \in J_0$.

The hypotesis (4) ensures the existence of a neighborhood $0 \in U \subset O_{\delta_0}$ such that $x_t(\cdot;\alpha,\varphi) \in O_{\delta_0}$, if $\alpha \in J_0$, $\varphi \in U \cap S_\alpha - \{0\}$ and $t \in [0\,,\,\tau_\alpha(\varphi)]$. This and the Lemma 5.1 imply $V_\alpha(x_t(\cdot;\alpha,\varphi))$ is strictly increasing in $t \in [0,\,\tau_\alpha(\varphi)]$ for $\alpha \in J_0$, $\varphi \in U \cap (S_\alpha - \{0\})$, in such a way that the derivative $\dot V_\alpha(x_t)$ is bounded away from zero, that is,

$$\dot V_\alpha(x_t) > \frac{1}{2\gamma_\alpha} V_\alpha(x_t) \geq \frac{1}{2\gamma_\alpha} V_\alpha(\varphi).$$

If $A_\alpha(\varphi) = x_{\tau_\alpha(\varphi)}(\cdot;\alpha,\varphi)$ still belongs to $U \cap S_\alpha$, we can repeat this reasoning to ensure that $V_\alpha(x_t)$ behaves in the same way in $\tau_\alpha(\varphi)$, $\tau_\alpha(\varphi) + \tau_\alpha(A_a(\varphi))$, and so on.

Suppose temporarily that $A_\alpha^n(\varphi) \in U \cap S_\alpha$, $n = 1, 2, \ldots$. Then, $V_\alpha(x_t)$ is strictly increasing in t, $V_\alpha(\varphi) < p^2\delta_0^2$ and $\dot V_\alpha(x_t)$ is bounded away from zero for $t \geq 0$. Hence, there exist a first t_0 such that $V_\alpha(x_{t_0}) = p^2\delta_0^2$ and a greatest n_0 such that $A_\alpha^{n_0}\varphi = x_T$, with $T \leq t_0$.

Since $x_{t_0} \notin O_{\delta_0}$, the choice of U implies $A_\alpha^{n_0}(\varphi) \notin U \cap S_\alpha$, a contradiction. Therefore,

$$\alpha \in J_0, \ \varphi \in U \cap (S_\alpha - 0) \quad \Rightarrow \quad A_\alpha^n(\varphi) \notin U \cap S_\alpha,$$

for some n, and the proof is over if one takes W from Definition 5.1 as $W = U \cap S_\alpha$. $\square$

The ejectivity is essential in our approach to the existence of nontrivial periodic solutions of (2.1α), by way of the Lemma 5.3, due to Nussbaum [14]. The uniformty in α will be used in proving a global continuation of a Hopf bifurcation.

Lemma 5.3. *Let X be a Banach space, $S \subset X$ a closed, bounded, convex, infinite dimensional set, $A : S - \{0\} \to S$ a completely continuous map and $0 \in S$ an ejective point of A. Then, A has a fixed point $x_0 \in S - \{0\}$.*

Let us retake the closed convex sets K and K_α given in (2.2) and (2.3). Our first aim is to construct in K_α a return map A_α associed to the equation (2.1α) satisfying the hypoteses of Lemma 5.3. From now on, we assume the hypoteses (H1) and (H2) hold all over this section.

Lemma 5.4. *If* (H1), (H2) *hold,* $\alpha > 0$, *and* $||\varphi|| \le M$, *then the solution* $x(t;\alpha,\varphi)$ *of* (2.1α) *remains in* $\bar{B}_M = \{x \in \mathbb{R}^2 \mid |x| \le M\}$, *for* $t \ge 0$.

Proof. From the equation (2.1α) one obtains

$$x(t)\cdot \dot{x}(t) = -\alpha|x(t)|^2 + \alpha F(x(t-1))\cdot x(t),$$

where " $\cdot$ " denotes usual inner product in $\mathbb{R}^2$. Being M a bound for $|F|$, this equation, together with the Cauchy-Schwarz inequality, gives $\frac{d}{dt}|x(t;\alpha,\varphi)| \le 0$, whenever $|x(t;\alpha,\varphi)| \ge M$. □

It follows from the proof of Lemma 5.4 that $|x(t;\alpha,\varphi)| \le M$, $t \in \mathbb{R}$, for any periodic solution $x(t;\alpha,\varphi)$ of (2.1α).

It is convenient, sometimes, to denote $x = (x_1, x_2)$ and rewrite (2.1α) as:

$$\dot{x}_1(t) = -\alpha x_1(t) + \alpha F_1(x(t-1)), \tag{5.5}$$

$$\dot{x}_2(t) = -\alpha x_2(t) + \alpha F_2(x(t-1)). \tag{5.6}$$

Defining the quadrants

$$Q_1 = \{x \in \mathbb{R}^2 \mid x_1 \ge 0,\, x_2 \ge 0\}, \quad Q_2 = \{x \in \mathbb{R}^2 \mid x_1 \ge 0,\, x_2 \le 0\},$$

the lemmas 5.5–5.9 are proven in [1] and, with minor adaptations, in [16].

Lemma 5.5. *For any* $\alpha > 0$, *there is a continuous map* $t_{1\alpha} : K_\alpha - \{0\} \to [0,\infty)$ *with* $x(t_{1\alpha}(\varphi);\alpha,\varphi) \in \partial Q_1$, $x_1(t_{1\alpha}(\varphi);\alpha,\varphi) > 0$, *and* $x(t;\alpha,\varphi) \in \mathrm{int}Q_2$, *if* $t_{1\alpha}(\varphi) < t < t_{1\alpha}(\varphi) + \eta$, *for some* $\eta > 0$ *sufficiently small.*

Lemma 5.6. *If G is open, $0 \in G \subset C$, there is a neighborhood $U \subset C$ of 0 such that $x_t(\cdot;\alpha,\varphi) \in G$, for any $\alpha > 0$, $\varphi \in U \cap K_\alpha$, $\varphi \ne 0$, and $0 \le t \le t_{1\alpha}(\varphi) + 1$.*

For any $\alpha > 0$, given $\varphi \in K_\alpha - \{0\}$, let $\psi = x_{t_{1\alpha}(\varphi)+1}(\cdot;\alpha,\varphi)$. So that, by (2.1$\alpha$),

$$(e^{\alpha\theta}\psi(\theta))' = \alpha e^{\alpha\theta} F(x(t_{1\alpha}(\varphi)+\theta;\alpha,\varphi)), \qquad \theta \in [-1,0]. \tag{5.7}$$

Since $x(t_{1\alpha}(\varphi)+\theta;\alpha,\varphi) \in Q_1$ for $-1 \le \theta \le 0$, (5.7) and (H1) imply $e^{\alpha\theta}\psi_1(\theta)$ is nondecreasing and $e^{\alpha\theta}\psi_2(\theta)$ is nonincreasing. If $\mathcal{R}$ is the rotation

$$\mathcal{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $\mathcal{R}K_\alpha := \{\mathcal{R}\varphi \mid \varphi \in K_\alpha\}$, then $\psi \in \mathcal{R}K_\alpha$ and the map $A_{1\alpha} : K_\alpha \to \mathcal{R}K_\alpha$ is given by

$$\begin{aligned} A_{1\alpha}(\varphi) &= x_{t_{1\alpha}(\varphi)+1}(\cdot;\alpha,\varphi), \qquad \varphi \ne 0 \\ A_{1\alpha}(0) &= 0 \end{aligned} \tag{5.8}$$

Lemma 5.7. *For each $\alpha > 0$, the map $A_{1\alpha}$ is completely continuous.*

The rotational symmetry of our hypotheses ensures we might repeat our arguments to define continuous maps $\varphi \to t_{j\alpha}(\varphi)$, from $\mathcal{R}^{j-1}(K_\alpha - \{0\})$ to $[0, +\infty)$, $j = 2, 3, 4$, and the completely continuous maps,

$$A_{j\alpha} : \mathcal{R}^{j-1} K_\alpha \to \mathcal{R}^j K_\alpha, \qquad j = 2, 3, 4.$$

Now we define the map $A_\alpha : K_\alpha \to \mathcal{R}^4 K_\alpha = K_\alpha$ by

$$\begin{aligned} A_\alpha(\varphi) &= (A_{4\alpha} \circ A_{3\alpha} \circ A_{2\alpha} \circ A_{1\alpha})(\varphi), \qquad \varphi \in K_\alpha - \{0\} \\ A_\alpha(0) &= 0. \end{aligned} \tag{5.9}$$

For any $\varphi \in K_\alpha - \{0\}$, if $\tau_\alpha(\varphi)$ is the first time that $A_\alpha(\varphi) = x_{\tau_\alpha(\varphi)}(\cdot; \alpha, \varphi)$, then

$$\tau_\alpha(\varphi) = t_{1\alpha}(\varphi) + t_{2\alpha}(A_{1\alpha}\varphi) + t_{3\alpha}(A_{2\alpha}A_{1\alpha}\varphi) + t_{4\alpha}(A_{3\alpha}A_{2\alpha}A_{1\alpha}\varphi) + 4 \tag{5.10}$$

and this implies the continuity of the function $\tau_\alpha : K_\alpha - \{0\} \to [4, \infty)$.

The Figure 5.1 helps to understand the map A_α.

FIGURE 5.1

Notice that, from the definition of $t_{1\alpha}$, $\psi = x_{t_{1\alpha}(\varphi)+1}(\cdot; \alpha, \varphi)$ satisfies $\psi_1(\theta) > 0$ for $-1 \le \theta \le 0$, $\psi_2(-1) = 0$, $\psi_2(\theta) < 0$ for $-1 < \theta \le 0$. Furthermore, the solution $x(t; \alpha, \varphi)$, $\varphi \in K_\alpha$, rotates around the origin, in the clockwise sense, crossing transversally two successive semi-axis at instants spaced by a lag greater than one. If $\varphi = (\varphi_1, \varphi_2)$ is a fixed point of A_α, giving rise to a nontrivial periodic solution of (2.1α), one has $\varphi_1(\theta) > 0$, $-1 < \theta \le 0$, $\varphi_2(\theta) > 0$.

The Lemmas 5.6 and 5.7, with the symmetry of our hypotheses, imply immediately the next two lemmas.

Lemma 5.8. *Let G be an open neighborhood of zero in C. There exists a neighborhood $U \subset C$ of zero such that, if $\alpha > 0$, $\varphi \in U \cap K_\alpha$, $\varphi \neq 0$, then $x_t(\cdot, \alpha, \varphi) \in G$, for $0 \le t \le \tau_\alpha(\varphi)$.*

Lemma 5.9. *For each $\alpha > 0$, there is a continuous function $\tau_\alpha : K_\alpha - \{0\} \to [4, \infty)$ such that the map $A_\alpha : K_\alpha \to C$, given by*

$$A_\alpha(\varphi) = x_{\tau_\alpha(\varphi)}(\cdot; \alpha, \varphi),\ \varphi \neq 0$$
$$A_\alpha(0) = 0$$

is completely continuous and $A_\alpha K_\alpha \subset K_\alpha$.

Returning to the hypoteses of the Lemma 5.2, one sees that (2) and (4) follow from Lemmas 5.9 and 5.8, respectively, with $S_\alpha = K_\alpha$. If we take the compact $J_0 \subset (\alpha_0, \infty)$, $\alpha_0 = b_0/\sqrt{\delta^2 - 1}$, $0 < b_0 = \arcsin 1/\delta < \pi/2$, then (1) is satisfied, according to the analysis of the characteristic equation in the Section 3. It remains to prove that the projections on $P_{\bar\lambda(\alpha)}$ of $x_t(\cdot\,; \alpha, \varphi)$, with $\alpha \in J_0$, $||\varphi|| \geq a$, $t \in [0, \tau_\alpha(\varphi)]$, stay uniformly away from zero. For this purpose, the following lemma assures that $x_1(t_{1\alpha}(\varphi); \alpha, \varphi)$ has an upper bound independ of α, if $\alpha \in J_0$ and $||\varphi||$ is sufficiently large.

For a compact $J_0 \subset (0, \infty)$, let $\alpha_1 = \min J_0$, $\alpha_2 = \max J_0$.

Lemma 5.10. *Let $d > 0$ and $J_0 \subset (0\,, \infty)$, compact. There exists a constant $c > 0$ such that if $\alpha \in J_0$, $\varphi \in K_\alpha$, $||\varphi|| \geq d$, then $x_1(t_1; \alpha, \varphi) \geq c$ and $x_1(t; \alpha, \varphi) \geq ce^{-\alpha_2}$ for $t \in [t_1, t_1 + 1]$, where $t_1 = t_{1\alpha}(\varphi) \geq 0$ is defined in the* Lemma 5.5.

This lemma results from the three subsequent ones, which describe the evolution, in Q_1, of the orbits $x(\cdot; \alpha, \varphi)$ of (2.1α), $\alpha \in J_0$, $\varphi \in K_\alpha$, $||\varphi|| \geq d$, $0 \leq t \leq t_{1\alpha}(\varphi)$. The Figure 5.2 can be helpful in following the proofs.

FIGURE 5.2

Lemma 5.11. *Let $d > 0$ and $J_0 \subset (0, \infty)$ a compact set. There exists $k > 0$, $\sigma > 0$ such that if $\alpha \in J_0$, $\varphi \in K_\alpha$, $||\varphi|| \geq d$, $\varphi_1(0) \leq \sigma$, then $x_1(\tau_1; \alpha, \varphi) > \sigma$ and $x_2(t; \alpha, \varphi) \geq k$, $t \in [0, \tau_1]$, for some $\tau_1 \in (0, t_{1\alpha}(\varphi))$.*

Proof. Near $x = 0 \in \mathbb{R}^2$, the hypothesis (H1) imply

$$F_2(x) = \delta_2 x_1 + x_1 \left[a x_1 + b x_2 + O(|x|^2) \right].$$

Thus, for $0 < \eta < -\delta_2$, there is an open set $0 \in V \subset \mathbb{R}^2$ such that, if $x \in V$, $x_1 > 0$,

$$F_2(x) < (\delta_2 + \eta) x_1. \tag{5.11}$$

Let $\xi > 0$ such that $R = \{(x_1, x_2) \in \mathbb{R}^2 \mid -\xi < x_1, x_2 < \xi\} \subset V$. Defining

$$k := \min\left\{\frac{d}{4e^{3\alpha_2}}, \xi\right\}, \tag{5.12}$$

there exists σ' such that, if $x \in Q_1$, $x_2 > k$,

$$F_1(x_1, x_2) > \max\left\{\sigma', \frac{\sigma'}{\alpha_1}\right\}. \tag{5.13}$$

Moreover, given $0 < \varepsilon < \frac{d}{4e^{3\alpha_2}}$, there is a $\delta > 0$ such that

$$-\varepsilon \leq F_2(x_1, x_2) < 0, \tag{5.14}$$

if $x \in Q_1$ and $x_1 \leq \delta$.

Defining

$$\sigma := \min\left\{\frac{\delta(\varepsilon)}{e^{\alpha_2}}, \frac{\sigma'}{2}, k, \frac{1}{2e^{\alpha_2}}\right\}, \tag{5.15}$$

if $\varphi \in K_\alpha$, $||\varphi|| \geq d$, $\varphi_1(0) \leq \sigma$, one has $\varphi_2(0) \geq \frac{d}{2e^{\alpha_2}} > k$.

Since $x_2(t; \alpha, \varphi)$ is decreasing in Q_1, there exists $\bar{t}$, $0 < \bar{t} < t_{1\alpha}(\varphi)$, for which $x_2(\bar{t}; \alpha, \varphi) = k$ and $x_2(t; \alpha, \varphi) > k$, $0 \leq t < \bar{t}$.

Suppose for a moment that $x_1(t) \leq \sigma$, $t \in [0, \bar{t}]$. So that (5.6) and (5.14) imply

$$x_2(t) \geq [\varphi_2(0) + \varepsilon]e^{-\alpha t} - \varepsilon, \qquad t \in [0, \bar{t}]. \tag{5.16}$$

Since the right side of (5.16) is greater than k for $t = 2$, it follows that $\bar{t} > 2$. But (5.5) and (5.16) imply

$$\dot{x}_1(t) \geq -\alpha\sigma + \alpha \max\left\{\sigma', \frac{\sigma'}{\alpha_1}\right\} \geq \sigma \qquad t \in [0, \bar{t}]$$

and, therefore, $x_1(t) > \sigma$, for some $t \in [0, \bar{t}]$, a contradition. □

Lemma 5.12. *Let $d > 0$, $J_0 \subset (0, \infty)$ a compact set, k and σ as in (5.12) and (5.15). If the solution $x(t; \alpha, \varphi)$ of (2.1α), $\alpha \in J_0$, $\varphi \in K_\alpha$, is such that $x_1(\tau_1; \alpha, \varphi) > \sigma$, $x_2(\tau_1; \alpha, \varphi) \geq k$ for some $\tau_1 \geq 0$, then there exists τ_2, $\tau_1 < \tau_2 < t_{1\alpha}(\varphi)$, such that $0 < x_2(\tau_2; \alpha, \varphi) < k$ and $x_1(t; \alpha, \varphi) > \sigma e^{-\alpha_2}$, $t \in [\tau_1, \tau_2]$.*

Proof. Since $x_1(\tau_1; \alpha, \varphi) > \sigma$, it follows from (5.5) that

$$x_1(t; \alpha, \varphi) > \sigma e^{-\alpha_2}, \qquad t \in [\tau_1, \tau_1 + 1].$$

If $0 < x_2(t; \alpha, \varphi) < k$ for some $t \in [\tau_1, \tau_1 + 1]$, we let $\tau_2 = t$. If $x_2(t; \alpha, \varphi) > k$, for all $t \in [\tau_1, \tau_1 + 1]$, let $\tilde{t}$, $\tau_1 + 1 < \tilde{t} < t_{1\alpha}(\varphi)$ such that $x_2(\tilde{t}; \alpha, \varphi) = k$ and $x(t; \alpha, \varphi) < k$, $\tilde{t} < t < t_{1\alpha}(\varphi)$. Then (5.5), (5.13) and (5.15) imply

$$x_1(t) > \sigma e^{-\alpha_2}, \qquad t \in [\tau_1 + 1, \tilde{t} + 1].$$

Consequently, there exists τ_2 according to the lemma. □

Lemma 5.13. *Let $d > 0$, $J_0 \subset (0,\infty)$ a compact set, k and σ as in (5.12) and (5.15). There exists $c > 0$ such that, if the solution $x(t;\alpha,\varphi)$ of (2.1α), $\alpha \in J_0$, $\varphi \in K_\alpha$, satisfies $x_1(\tau_2;\alpha,\varphi) > \sigma e^{-\alpha_2}$, $0 < x_2(\tau_2;\alpha,\varphi) < k$, for some $\tau_2 \geq 0$, then $x_1(t_{1\alpha}(\varphi);\alpha,\varphi) \geq c$.*

Proof. If $x_1(t;\alpha,\varphi) > \sigma e^{-\alpha_2}$, $t \in (t_1-\delta,t_1)$, $t_1 = t_{1\alpha}(\varphi)$, for some $\delta > 0$ we take $c = \sigma e^{-\alpha_2}$. If this is not the case, let $\bar{t} = \bar{t}(\alpha,\varphi)$, $\tau_2 < \bar{t} < t_1$, the last instant t for which $x_1(t;\alpha,\varphi) = \sigma e^{-\alpha_2}$.

If $t_1 \leq \bar{t}+1$, we can take $c = \sigma e^{-2\alpha_2}$ since, by (5.5), $x_1(t_1;\alpha,\varphi) \geq \sigma e^{-\alpha_2}$.

In the case $t_1 > \bar{t}+1$, (5.6) and (5.11) imply, for $t \in [\bar{t}+1, t_1+1]$,

$$x_2(t;\alpha,\varphi) \leq e^{-\alpha(t-\bar{t}-1)}x_2(\bar{t}+1) + \alpha(\delta_2+\eta)e^{-\alpha t}\int_{\bar{t}+1}^{t} e^{\alpha s}x_1(s-1;\alpha,\varphi)ds.$$

Since $e^{\alpha t}x_1(t)$ increases in Q_1,

$$x_2(t;\alpha,\varphi) \leq e^{-\alpha(t-\bar{t}-1)}\left[k + \alpha_1(\delta_2+\eta)\sigma e^{-\alpha_2}(t-\bar{t}-1)\right],$$

and notice that the right side of this inequality vanishes in $t = \theta$, where

$$\theta = \frac{ke^{\alpha_2}}{\alpha_1(-\delta_2-\eta)\sigma} + \bar{t} + 1. \tag{5.17}$$

According to (5.5),

$$x_1(t;\alpha,\varphi) \geq \sigma e^{-\alpha_2}e^{-\alpha(t-\bar{t})}, \qquad t \in [\bar{t}, t_1+1], \tag{5.18}$$

and, since $t_1 \leq \theta$, it follows from (5.17) and (5.18) that $x_1(t_1;\alpha,\varphi) \geq c$, where $c := \sigma\exp\left(-2 + \frac{k\exp\alpha_2}{\alpha_1(\delta_2+\eta)\sigma}\alpha_2\right)$.

If $t_1 \leq \bar{t}+1$, (5.18) imply $x_1(t;\alpha,\varphi) \geq \sigma e^{-2\alpha_2}$. □

Proof of the Lemma 5.10.

According to (5.5), it suffices to prove the first inequality. If $\varphi_2(0) > 0$, this follows from Lemmas 5.10–5.12. If $\varphi_2(0) = 0$, then $\varphi_1(0) \geq de^{-\alpha_2}$ and $x_1(t_1;\alpha,\varphi) \geq de^{-2\alpha_2}$. □

Letting $y = \mathrm{col}\,(y_1,y_2)$, the linearized equation of (2.1α) at $x = 0$ is

$$\dot{y}(t) = -\alpha y(t) + \alpha By(t-1), \qquad B := \begin{pmatrix} 0 & \delta_1 \\ \delta_2 & 0 \end{pmatrix}. \tag{5.19}$$

Let G_α be the infinitesimal generator of semigroup $\{T_\alpha(t),\, t \geq 0\}$ defined by

$$T_\alpha(t)\varphi = y_t(\cdot;\alpha,\varphi), \quad \varphi \in C,\ t \geq 0.$$

If $\alpha > \alpha_0$, there is a unique eigenvalue $\tilde{\lambda} = \tilde{\lambda}(\alpha)$ in the strip S_0 such that $\Re\tilde{\lambda}(\alpha) > 0$ and it is simple. Therefore, the generalized eigenspace $P_{\tilde{\lambda}}$ of $\tilde{\lambda}$, $\mathcal{N}(\tilde{\lambda}I - G_\alpha)$, is one-dimensional and the projection operator with range $P_{\tilde{\lambda}}$ is

$$\pi_{\tilde{\lambda}}\varphi = \{\varphi_1(0) - i\mu\varphi_2(0) + \alpha\int_{-1}^{0} e^{-\tilde{\lambda}(\xi+1)}\left[\delta_1\varphi_2(\xi) - i\delta_2\mu\varphi_1(\xi)\right]d\xi\}\rho_\alpha,$$

where $\rho_\alpha(\theta) := e^{\tilde{\lambda}\theta}\mu$, $\theta \in [-1,0]$, and μ is a non trivial solution of $H_\alpha(\tilde{\lambda})\mu = 0$, $H_\alpha(\tilde{\lambda}) := (\tilde{\lambda}I - \alpha A - \alpha Be^{-\tilde{\lambda}})$.

Lemma 5.14. *Let d be a constant, $0 < d < M$, and $J_0 \subset (\alpha_0, \infty)$ a compact set. Then*

$$m = \inf\{||\pi_{\bar\lambda} x_t(\cdot;\alpha,\varphi)|| \mid \alpha \in J_0,\ \varphi \in K_\alpha,\ d \le ||\varphi|| \le M,\ 0 \le t \le t_{1\alpha}(\varphi)+1\} > 0.$$

Proof. Suppose for a moment $m = 0$. Then there exists sequences $\alpha_n \in J_0$, $\varphi_n \in K_{\alpha_n}$, $d \le ||\varphi_n|| \le M$, $t_n \in [0, t_{1\alpha_n}(\varphi_n)+1]$, $n = 1,2,\ldots$, such that $x_n(t) = (x_{1n}(t), x_{2n}(t)) := x(t;\alpha_n,\varphi_n)$, $\bar\lambda_n = \bar\lambda(\alpha_n)$ and

$$\begin{aligned} ||\pi_{\bar\lambda_n} x_{nt_n}|| &= |x_{1n}(t_n) - i\mu x_{2n}(t_n) + \\ &\alpha_n \int_{-1}^{0} e^{-\bar\lambda_n(\xi+1)}[\delta_1 x_{2n}(t_n+\xi) - i\delta_2\mu x_{1n}(t_n+\xi)]d\xi| \longrightarrow 0, \end{aligned} \tag{5.20}$$

as $n \to \infty$. Taking $\lambda_n = a_n + b_n i$, with $a_n > 0$, $0 < b_n < \frac{\pi}{2}$, we have

$$\begin{aligned} ||\pi_{\bar\lambda_n} x_{nt_n}|| = |x_{1n}(t_n) + \alpha_n &\int_{-1}^{0} e^{-a_n(\xi+1)}[\delta_1 x_{2n}(t_n+\xi)\cos b_n(\xi+1) - \\ &- \delta_2\mu x_{1n}(t_n+\xi)\sin b_n(\xi+1)]d\xi + i\,\{-\mu x_{2n}(t_n) + \\ &+ \alpha_n \int_{-1}^{0} e^{-a_n(\xi+1)}[-\delta_2\mu x_{1n}(t_n+\xi)\cos b_n(\xi+1) - \\ &- \delta_1\ x_{2n}(t_n+\xi)\ \sin b_n(\xi+1)]d\xi\}\,|. \end{aligned}$$

Since $d \le ||\varphi_n|| \le M$, the proof of the Lemma 5.4 implies $||x_n(t)|| \le M$, $t \ge -1$, $n = 1,2,\ldots$. Then, by taking a subsequence, if necessary, we can assume $x_n(t_n) \to L = (L_1, L_2)$ and $\alpha_n \to \alpha$. Lemmas 5.11–5.13 imply $(L_1, L_2) \ne (0,0)$.

If $L_1 > 0$, $L_2 \ge 0$, we have $\Re\pi_{\bar\lambda_n} x_{nt_n}$ positive and bounded away from zero, for n large. If $L_1 = 0$, $L_2 > 0$, then $\Im\pi_{\bar\lambda_n} x_{nt_n}$ is negative and bounded away from zero for n large. Both cases lead to a contradiction with (5.20).

Suppose now $L_2 < 0$. Surely, $L_1 > 0$, since $t_{1\alpha_n}(\varphi_n) \le t_n \le t_{1\alpha_n}(\varphi_n)+1$, for n large and, by the Lemma 5.10, $x_1(t_n;\alpha_n,\varphi_n) \ge ce^{-\alpha_2}$.

If n is large, (5.20) imply $x_{1n}(t_n) > \frac{L_1}{2}$, $x_{2n}(t_n) < \frac{L_2}{2}$, and the curve in $\mathbb{C}$, given by

$$p_n(\xi) = \delta_1 x_{2n}(t_n+\xi) - i\delta_2\mu x_{1n}(t_n+\xi), \qquad \xi \in [-1,0]$$

is in the sector $\{re^{i\theta} \mid 0 \le \theta \le \pi,\ r \ge 0\}$ and $e^{-\bar\lambda_n(\xi+1)}$, $\xi \in [-1,0]$, is in the sector $\{z = re^{i\theta} \mid r \ge \frac{1}{\delta},\ \frac{\pi}{2} < -B \le \theta \le 0\}$, for any $B \in (0, \frac{\pi}{2})$. Defining

$$\widetilde\theta_n = \max_{-1\le\xi\le0} \arg\left[e^{-\bar\lambda_n(\xi+1)} p_n(\xi)\right],$$

it might be seen that the curve $e^{-\bar\lambda(\xi+1)}p_n(\xi)$, $\xi \in [-1,0]$, is in the semi-plane determined by the line with slope $\widetilde\theta_n$, which contains the positive x real semi-axis .

If $0 \le \widetilde\theta_n \le \frac{\pi}{2}$, then $||\pi_{\bar\lambda_n} x_{nt_n}|| \not\to 0$.

If $\frac{\pi}{2} < \widetilde\theta_n < \pi$, defining $\omega_n = \frac{\pi}{2} - \widetilde\theta_n$, one arrives to the same result. $\square$

Notice that the Lemma 5.14 remains valid if K_α is replaced by $R^j K_\alpha$, and $[0, t_{1\alpha}(\varphi)+1]$ by $[t_{j\alpha}(\varphi)+1, t_{(j+1)\alpha}(\varphi)+1]$, $j = 1,2,3$.

Lemma 5.15. *For every compact $J_0 \subset (\alpha_0, \infty)$ and every constant $0 < a < M$,*

$$\Upsilon = \inf\left\{ \|\Pi_{\bar\lambda(\alpha)} x_t(\cdot;\alpha,\varphi)\| \mid \alpha \in J_0,\ \varphi \in K_\alpha,\ \|\varphi\| \geq a,\ 0 \leq t \leq \tau_\alpha(\varphi) \right\} > 0$$

Proof. Let $0 < a < M$. The Lemma 5.14 implies

$$\inf\left\{ \|\pi_{\bar\lambda} x_t(\cdot;\alpha,\varphi)\| \mid \alpha \in J_0,\ \varphi \in K_\alpha,\ \|\varphi\| \geq a,\ 0 \leq t \leq t_{1\alpha}(\varphi) + 1 \right\} > 0.$$

The Lemma 5.4 and the existence of $d_1 > 0$ according to Lemma 5.10 imply

$$M \geq \|x_{t_{1\alpha}(\varphi)+1}(\cdot;\alpha,\varphi)\| \geq d_1,$$

for all $\alpha \in J_0$, $\varphi \in K_\alpha$, $\|\varphi\| \geq a$. Then, by the Lemma 5.14,

$$\inf\left\{ \|\pi_{\bar\lambda} x_t(\cdot;\alpha,\varphi)\| \mid \alpha \in J_0,\ \varphi \in K_\alpha,\ \|\varphi\| \geq a,\ 0 \leq t \leq t_{2\alpha}(\varphi) + 1 \right\} > 0.$$

The result follows now by successive applications of the same reasoning to the intervals $[t_{2\alpha}(\varphi) + 1, t_{3\alpha}(\varphi) + 1]$, $[t_{3\alpha}(\varphi) + 1, \tau_\alpha(\varphi)]$. □

As an immediate consequence of the above lemmas, the hypotheses of the Lemma 5.2 are satisfied, with respect to the map A_α defined in (5.9). So that we can state:

Theorem 5.1. *Let $\alpha_0 := b_0/\sqrt{\delta^2 - 1}$, $0 < b_0 = (\arcsin 1)/\delta < \pi/2$. If (H1), (H2) are satisfied, then the equation (2.1α), $\alpha > \alpha_0$, has a nonconstant periodic solution, $x(\cdot;\alpha,\varphi)$, $\varphi \in K_\alpha$, with period $T > 4$ and $\varphi \in K_\alpha$.*

Proof. According to the Lemma 5.4, if $S_\alpha := K_\alpha \cap \bar B_M$, $A_\alpha S_\alpha \subset S_\alpha$, $\alpha > \alpha_0$. It is now a matter of routine to show that all the hypotheses of the Lemma 5.3 are fulfilled for $A_\alpha|_{S_\alpha} : S_\alpha \to S_\alpha$, $\alpha > \alpha_0$. □

6. The global continuation.

The purpose of this section is to prove the Theorem 2.2, which is motived by the existence of periodic solutions of the equation (2.1α), according to Theorem 5.1, and by the local Hopf bifurcation at $\alpha = \alpha_0$. Although it is deeply inspirated in the results by Mallet-Paret & Nussbaum [12] for the scalar case, it presents some features and difficulties inherent to the planar problem.

The main tool we need is an abstract global bifurcation theorem due to Nussbaum [13], the Theorem 6.1 below.

Definition 6.1. Let X be a topological space and $A : X \to X$ a continuous map with a fixed point x_0. We say x_0 is an *attractive fixed point* of A if there exists an open neighborhood U of x_0 such that, for every open neighborhood V of x_0, there exists an integer $n = n(V)$ such that $A^j(x) \in V$, for all $j \geq n$ and $x \in U$.

Theorem 6.1. *Let Y be a Banach space, $S \subset Y$ a closed, convex set, 0 an extreme point of S, $S \neq \{0\}$ and $J = (a, \infty)$, $a \geq -\infty$. Suppose $F : J \times S \to S$ satisfies the following hypotheses:*

(1) *$F(\alpha, 0) = 0$, for $\alpha \in J$,*
(2) *F is completely continuous,*
(3) *There exists $\alpha_0 \in J$ such that, for every compact $J_0 \subset J - \{\alpha_0\}$, there exists $\varepsilon = \varepsilon(J_0) > 0$ such that $F(\alpha, x) \neq x$, for $\alpha \in J_0$, $0 < |x| \leq \varepsilon$,*
(4) *There exists an open interval $I_0 \ni \alpha_0$ such that 0 is a atractive point of $F(\alpha, \cdot)$, for $\alpha \in I_0$, $\alpha < \alpha_0$, and 0 is an ejective point of $F(\alpha, \cdot)$, for $\alpha \in I_0$, $\alpha > \alpha_0$, or conversely,*
(5) *There exists $\alpha_1 \in J$ such that $F(\alpha, x) = x$, with $x \neq 0$ implies $\alpha \geq \alpha_1$.*

Define the set E by

$$E = \{(\alpha, x) \in J \times S \mid x \neq 0 \text{ and } F(\alpha, x) = x\}$$

and denote by E_0 the connected component of the clousure $\overline{E}$ of E, which contains $(\alpha_0, 0)$. Then E_0 is not empty and unbounded.

According to the results of section 5, $x(t; \alpha, \varphi)$ is a slowly spiralling periodic solution of (2.1α) if, and only if, $\varphi \in K_\alpha$ is a fixed point of the map $A_\alpha : K_\alpha \to C$, defined in the Lemma 5.2. So, we can represent the set Σ in the Theorem 2.2 as

$$\Sigma = \{(\alpha, \varphi) \in (0, \infty) \times K \mid \varphi \in K_\alpha - \{0\},\ A_\alpha(\varphi) = \varphi\} \tag{6.1}$$

Remark 6.1. Under the hypotesis (H1) and (H2) one has:

(1) $\Sigma \subset (0, \infty) \times \bar{B}_M$,
(2) For every $\alpha > \alpha_0$, there is $\varphi \in K \cap \bar{B}_M$, $\varphi \neq 0$, such that $(\alpha, \varphi) \in \Sigma$.

We construct now a function F satisfying the hypotheses of Theorem 6.1.

Let $\Gamma \subset \mathbb{R} \times C$ given by

$$\Gamma = \{(\alpha, \varphi) \in (0, \infty) \times K \mid \varphi \in K_\alpha\} \tag{6.2}$$

and $A : \Gamma \to K$, given by

$$A(\alpha, \varphi) = A_\alpha(\varphi). \tag{6.3}$$

It is not difficult to prove that $A(\alpha, \varphi)$ is completely continuous.

Lemma 6.1. *There exists a continuous retraction $\rho : (0, \infty) \times K \to \Gamma$, such that*

$$\rho(\alpha, \varphi) = (\alpha, \rho_\alpha(\varphi))$$

for all $(\alpha, \varphi) \in (0, \infty) \times K$, where $\rho_\alpha^{-1}(0) = \{0\}$.

Proof. The function $h_{\alpha\beta} : K_\alpha \to K_\beta$, given by

$$h_{\alpha\beta}(\varphi)(\theta) = e^{(\alpha-\beta)\theta}\varphi(\theta), \quad -1 \leq \theta \leq 0,$$

for any $\alpha, \beta > 0$, is a homeomorphism with inverse $h_{\beta\alpha}$.

Let $H_{\alpha\beta} : K \to K$ the homeomorphism given by the extension of $h_{\alpha\beta}$ to K:

$$H_{\alpha\beta}(\varphi)(\theta) = e^{(\alpha-\beta)\theta}\varphi(\theta).$$

Because $K_0 = \{\varphi \in K \mid \varphi_j(\theta) \text{ is nondecreasing}, j = 1,2\}$, the map $\rho_0 : K \to K_0$ such that

$$\rho_0(\varphi)(\theta) = \operatorname{col}\left(\max_{-1\leq s\leq\theta}\varphi_1(s), \max_{-1\leq s\leq\theta}\varphi_2(s)\right)$$

is a continuous retraction from K to K_0 and $\rho_0^{-1}(0) = \{0\}$.

Defining $\rho_\alpha : K \to K_\alpha$ by

$$\rho_\alpha(\varphi) = h_{0\alpha}(\rho_0(H_{\alpha 0}(\varphi))),$$

it follows that ρ_α is a continuous retraction from K to K_α and $\rho_\alpha^{-1}(0) = \{0\}$. □

Let us define now an extension of A, $A^* : (0,\infty) \times K \to K$, by

$$A^*(\alpha,\varphi) = A(\rho(\alpha,\varphi)), \tag{6.4}$$

and $A^*_\alpha : K \to K$, by

$$A^*_\alpha(\varphi) = A^*(\alpha,\varphi). \tag{6.5}$$

One easily sees that φ is a fixed point of A^*_α if, and only if, it is a fixed point of A_α, hence

$$\Sigma = \{(\alpha,\varphi) \in (0,\infty) \times K \mid A^*_\alpha(\varphi) = \varphi \text{ and } \varphi \neq 0\}.$$

Remark 6.2. The continuity of A and $\rho_\alpha(\varphi)$ lead to $\overline{\Sigma} \subset \Sigma \cup [(0,\infty) \times \{0\}] \cup [\{0\} \times K]$.

Lemma 6.2. . *A^* is completely continuous.*

Proof. The map ρ in Lemma 6.1 takes bounded subsets of $(0,\infty)\times K$ to bounded subsets of Γ. So, the lemma follows from the continuity of ρ and the complete continuity of A. □

Lemma 6.3. *Assume that F satisfies hypoteses (H.1) and (H.2). Then 0 is attractive point of A^*_α for $0 < \alpha < \alpha_0$ and it is ejective point for $\alpha > \alpha_0$.*

Proof. If $0 < \alpha < \alpha_0$, the Lemm 3.7 implies that 0 is a attractive point of A^*_α, with the neighborhood U of the Definition 6.1 being $U = \{\varphi \in C \mid ||\varphi|| < \nu\}$.

If $\alpha > \alpha_0$,the ejective of origem for A_α according to the Lemma 5.2 as well the inclusion $A^*_\alpha(K) \subset K_\alpha$ and $A^{*n}_\alpha(\varphi) = A^n_\alpha(\rho_\alpha(\varphi))$ for every integer $n \geq 1$ guarantee the afirmation. □

Lemma 6.4. *Assume F satisfies the hypotheses $(H1)$-$(H4)$. There exist constants $\xi > 0$ and $\alpha' > 0$ such that for every slowly spiralling periodic solution $x(t) = x(t;\alpha,\varphi)$ of (2.1α), $0 < \alpha < \alpha'$,*

$$|x(\bar{t})| := \max_{-\infty<t<\infty} |x(t)| > \xi.$$

Proof. Suppose the assertion is not true. So, there are $\alpha_n > 0$, $n = 1,2,\dots$, $\alpha_n \to 0$, and slowly spiralling solutions $(x_n(t))$ of $(2.1\alpha_n)$ $n = i,2,\dots$, such that

$$|x_n(t_n)| := \max_{-\infty<t<\infty} |x_n(t)| \to 0, \text{ as } n \to \infty.$$

Furthermore, with $x_n(t_n) \neq 0$, $n = 1, 2, \ldots$ and $(|x_n(t_n)|)$, (α_n) decreasing.

According to remark 2.2,

$$\dot{x}_n(t_n) = \alpha_n[-x_n(t_n) + F(x_n(t_n - 1))] = O(\alpha_n |x_n(t_n)|), \text{ as } n \to \infty.$$

Furthermore,

$$F(x_n(t_n - 1)) = \mathrm{col}(\delta_1 x_{2n}(t_n - 1), \delta_2 x_{1n}(t_n - 1)) + o|x_n(t_n)|), \text{ as } n \to \infty, \tag{6.6}$$

where,

$$x_{in}(t_n - 1) = x_{in}(t_n) - \dot{x}_{in}(\tau_n) = x_{in}(t_n) + O(\alpha_n |x_n(t_n)|), \text{ as } n \to \infty, \; i = 1, 2. \tag{6.7}$$

Taking subsequences, if necessary, (6.6) and (6.7) imply

$$F(x_n(t_n - 1)) = \mathrm{col}(\delta_1 x_{2n}(t_n), \delta_2 x_{1n}(t_n)) + O(\alpha_n |x_n(t_n)|) + o(|x_n(t_n)|), \text{ as } n \to \infty \tag{6.8}$$

From the equation (2.1α) it follows,

$$\frac{1}{\alpha_n}[x_n(t_n) \cdot \dot{x}_n(t_n)] = -|x_n(t_n)|^2 + x_n(t_n) \cdot F(x_n(t_n - 1)), \tag{6.9}$$

so that,

$$\begin{aligned} \frac{1}{2\alpha_n}\frac{d}{dt}|x_n(t_n)|^2 &= -|x_n(t_n)|^2 + (\delta_1 + \delta_2) x_{1n}(t_n) x_{2n}(t_n) + \\ &\quad + O(\alpha_n |x_n(t_n)|^2) + o(|x_n(t_n)|^2), \text{ as } n \to \infty. \end{aligned} \tag{6.10}$$

Consequently,

(6.11)

$$\frac{1}{2\alpha_n}\frac{d}{dt}|x_n(t_n)|^2 \leq \left[\left(-1 + \frac{|\delta_1 + \delta_2|}{2}\right)|x_n(t_n)|^2 + O(\alpha_n |x_n(t_n)|^2) + o(|x_n(t_n)|^2)\right],$$

as $n \to \infty$. Since $|\delta_1 + \delta_2| < 2$, (6.11) implies $\frac{d}{dt}|x_n(t_n)|^2 < 0$, for n sufficiently large, and this is a contradition. □

Lemma 6.5. *Assume F satisfies $(H1)$-$(H4)$. For some $\bar{\alpha} > 0$, there is no slowly spiralling periodic solution to* (2.1α), $0 < \alpha < \bar{\alpha}$.

Proof. Let ξ as in Lemma 6.3 and $\varepsilon > 0$ as in remark 2.3. If $\nu = \nu(\varepsilon) > 0$ is chosen in such a way that $x, y \in \bar{B}_M$, $|x - y| < \nu$ imply $|F(x) - F(y)| < \varepsilon$, let $0 < \bar{\alpha} < \min\{\nu/2M, \alpha'\}$, where α' is defined in Lemma 6.3.

We claim that there is no slowly spiralling periodic solution $x(t)$ of (2.1α), for $0 < \alpha < \overline{\alpha}$. Actually, if we suppose the contrary, let $\bar{t}$ as in Lemma 6.3. Thus,

$$|x(\bar{t}) - x(\bar{t}-1)| = \alpha \left| - \int_{\bar{t}-1}^{\bar{t}} x(s)ds + \int_{\bar{t}-1}^{t} F(x(s-1))ds \right| \leq 2M\alpha < 2M\bar{\alpha} < \nu.$$

Consequentelly, $|F(x(\bar{t})) - F(x(\bar{t}-1))| < \varepsilon$ and this implies (see Remark 2.2) $x(\bar{t}) \cdot F(x(\bar{t}-1)) < |x(\bar{t})|^2$.

Therefore,

$$\frac{1}{2\alpha}\frac{d}{dt}|x(\bar{t})|^2 = -|x(\bar{t})|^2 + F(x(\bar{t}-1)) \cdot x(\bar{t}) < 0,$$

which is a contradition. □

Remark 6.3. The previous lemma gives a sharper inclusion than that of the remark 6.2: $\overline{\Sigma} \subset \Sigma \cup [(0,\infty) \times \{0\}]$.

Lemma 6.6. *Assume F satisfies hypoteses $(H1)$, $(H2)$. Then, for any compact interval $J_0 \subset (0,\infty) - \{\alpha_0\}$, there exists $\varepsilon = \varepsilon(J_0) > 0$, such that $A^\star_\alpha(\varphi) \neq \varphi$, for $\alpha \in J_0$, $0 < |\varphi| \leq \varepsilon$.*

Proof. For $J_0 \subset (\alpha_0, \infty)$, this is a consequence of the uniform ejectivity of 0, for the family $\{A^\star_\alpha,\ \alpha \in J_0\}$, assured by Lemma 5.2. For $J_0 \subset (0, \alpha_0)$, the assertion follows from Theorem 2.3. □

Proof of the Theorem 2.2.

The Lemma 6.5 imply the assertion (1). The assertion (2) is consequence of the Remark 6.3 and Lemma 6.6.

Finally, the function $A^\star : (0,\infty) \times K \to K$ such that $A^\star(\alpha,\varphi) = A(\rho(\alpha,\varphi))$ satisfies all the hypoteses of Theorem 6.1 and gives (3). □

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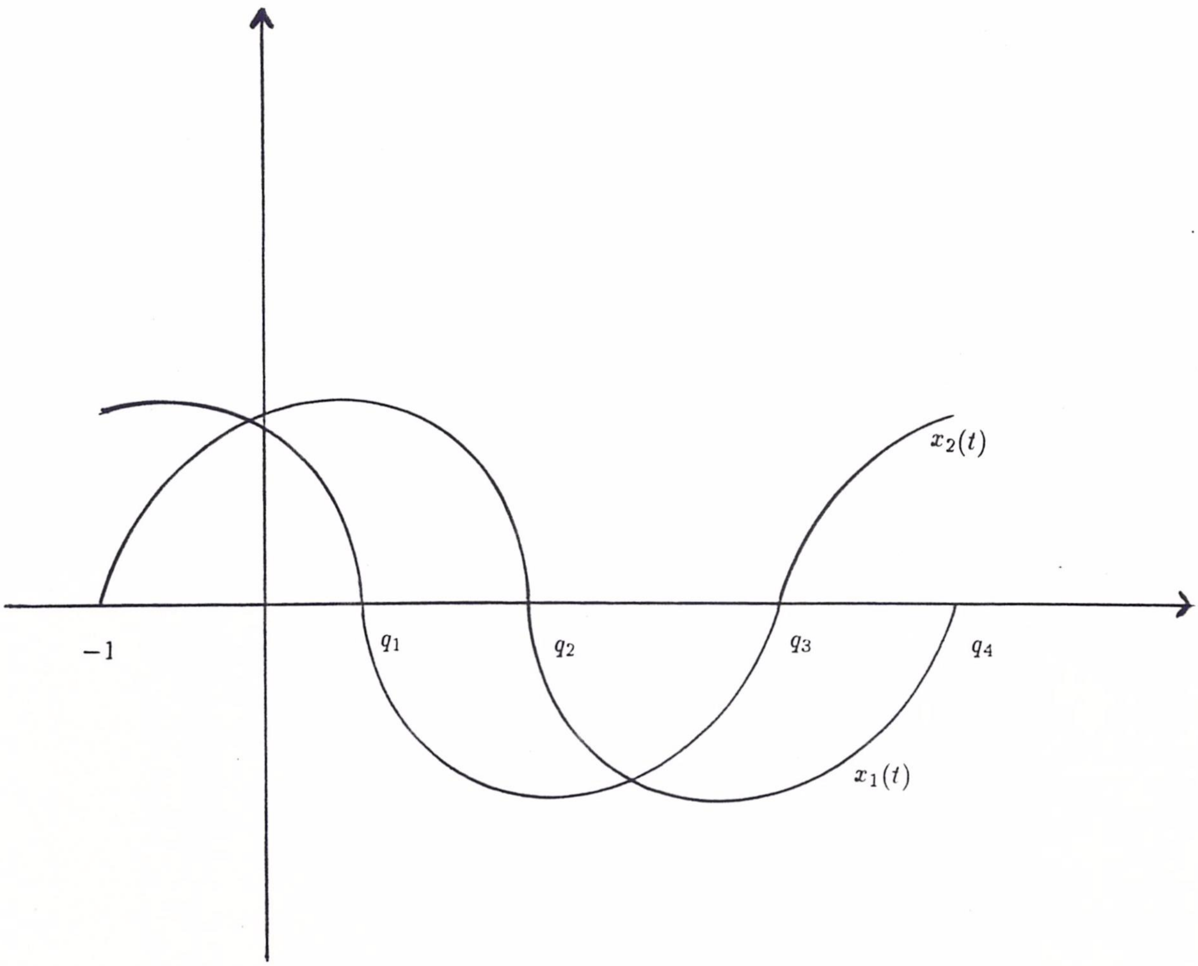


FIGURE 2.1

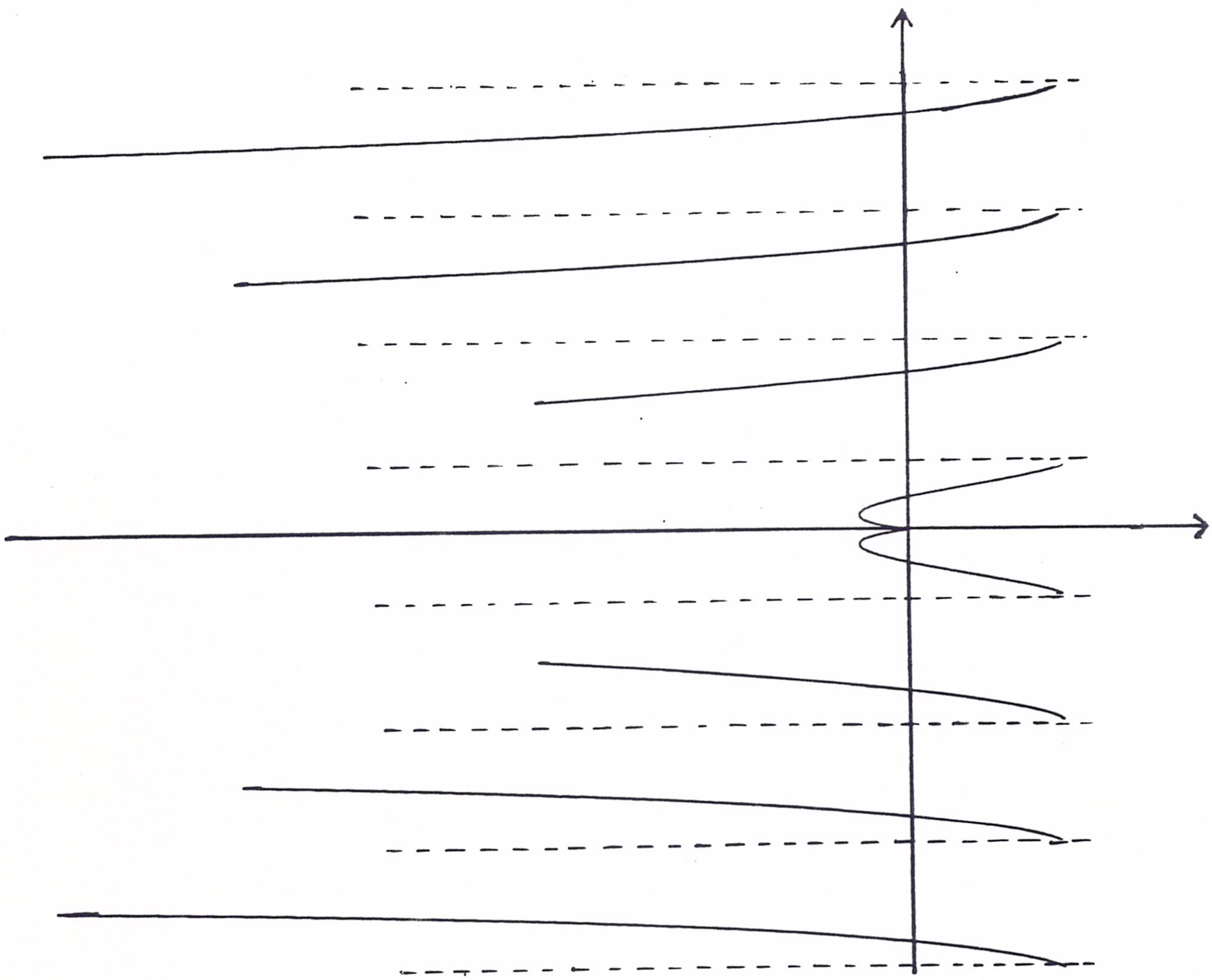


FIGURE 3.1

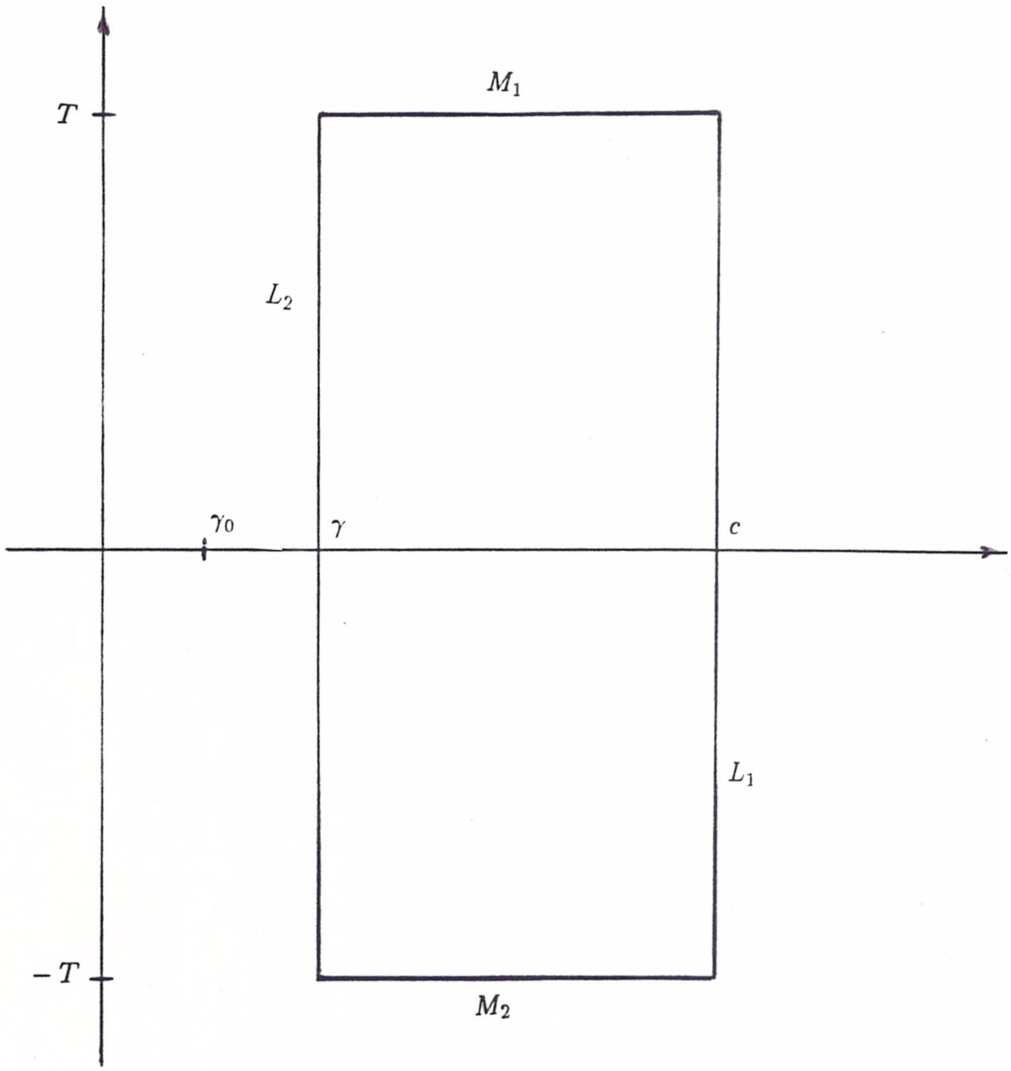
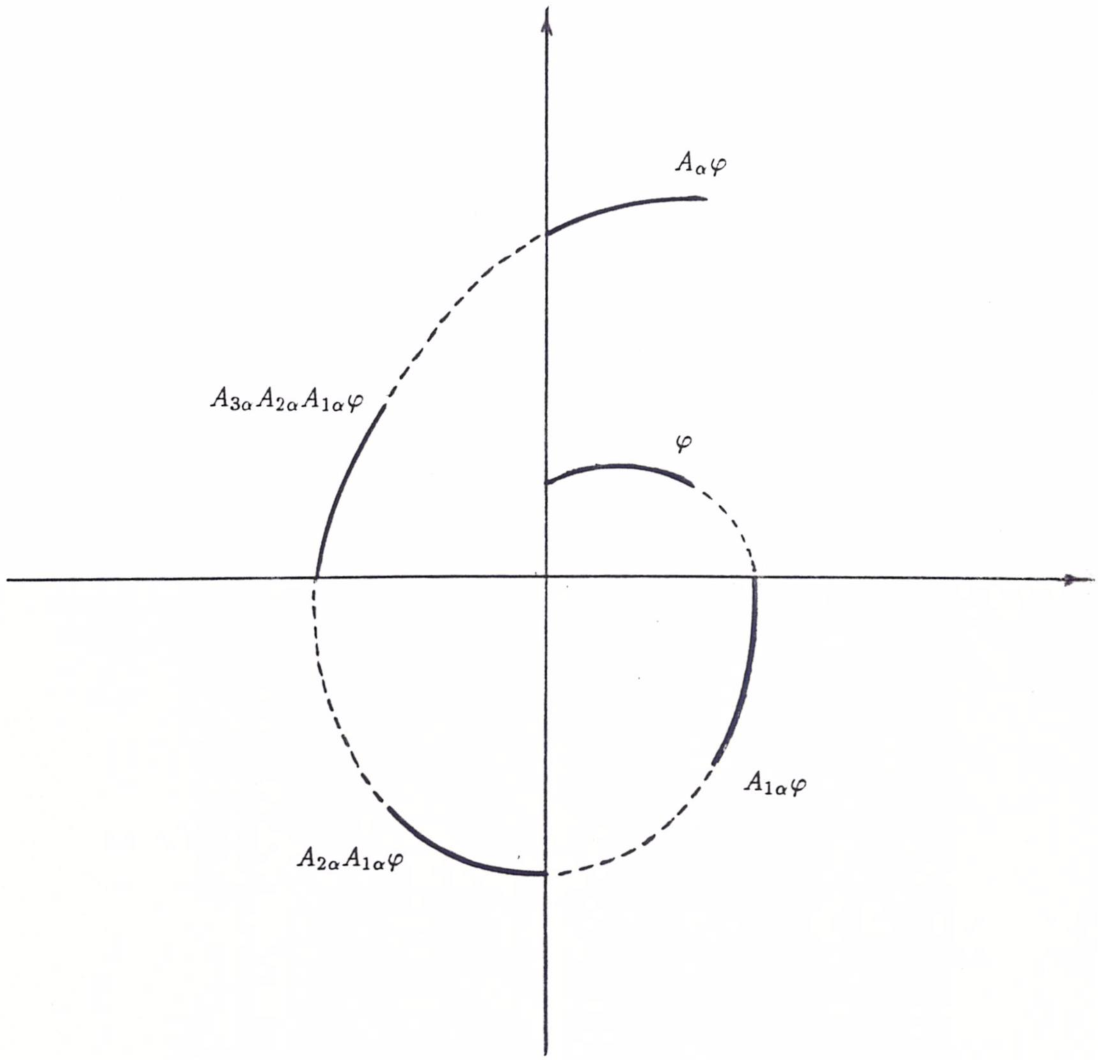


FIGURE 4.1


$A_\alpha \varphi$
$A_{3\alpha} A_{2\alpha} A_{1\alpha} \varphi$
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$A_{1\alpha} \varphi$
$A_{2\alpha} A_{1\alpha} \varphi$

FIGURE 5.1

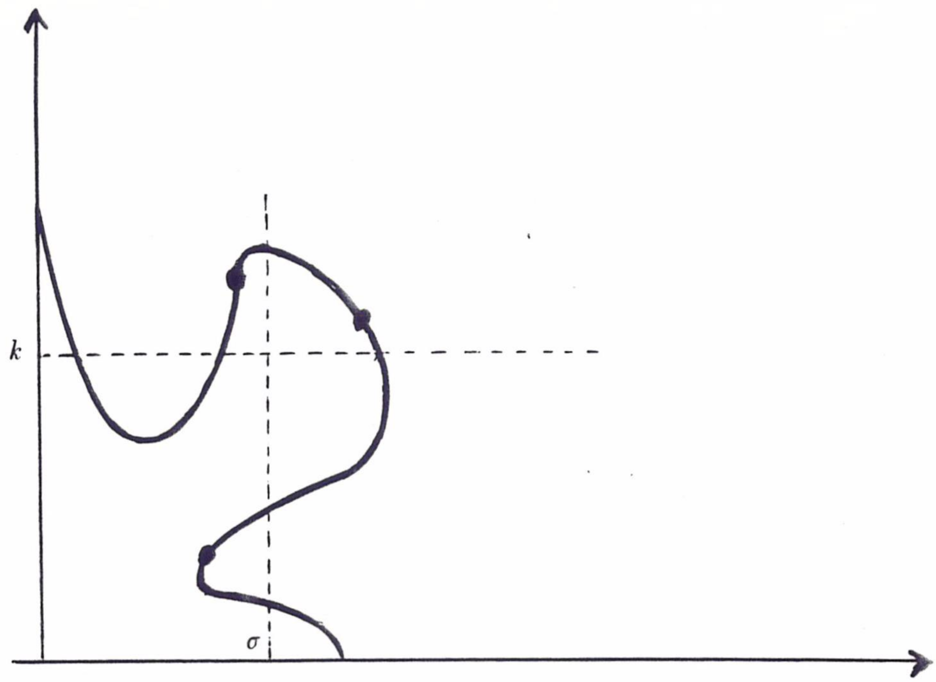

FIGURE 5.2

N O T A S D O I C M S C

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