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1 Introduction

Unitary groups over local rings in which 2 is invertible have been studied by James in [5]. In this paper, we extend some of the results of James on the generators and commutator subgroups to strongly semilocal rings, without assuming 2 invertible, and in the setting of parameter forms as introduced by Bak in [1] (Theorems **4.6**, **4.7**, **4.9** and **4.12**). These results also generalize results of Ishibashi ([4]) in the strongly semilocal case. A ring (not necessarily commutative) is called strongly semilocal if modulo its Jacobson radical it is a finite product of division rings (see **2.4**). In section 2 we give some results on the structure of strongly semilocal rings and in section 3, some results on orthogonal decompositions of unitary spaces over strongly semilocal rings. The mains results are in section 4.

2 Strongly Semilocal Rings

Let A be a (not necessarily commutative) ring with the identity 1. A is called a *strongly semilocal ring* if there exist finitely many two-sided ideals $\{\mathcal{P}_i\}$ such that each proper right ideal is contained in at least one $\mathcal{P}_i$ and each proper left ideal is contained in at least one $\mathcal{P}_j$.

We may assume that all $\{\mathcal{P}_i\}$ are maximal two-sided ideals by dropping nonmaximal ones. Then each $\mathcal{P}_i$ is maximal as a right ideal and also as a left ideal. Further the $\{\mathcal{P}_i\}$ are pairwise comaximal, i.e. $\mathcal{P}_i + \mathcal{P}_j = A$ if $i \neq j$. Therefore the Chinese Remainder Theorem holds for the set $\{\mathcal{P}_i\}$. From now on, by a maximal ideal we mean a maximal two-sided ideal.

Strongly semilocal rings are called semilocal rings in [4]. The usual terminology (see [7]) is to call a ring semilocal if modulo its Jacobson radical it is a semisimple ring.

Proposition 2.1 *Let A be a strongly semilocal ring. Then*
(i) *if A is commutative, then A is semilocal;*
(ii) $\mathrm{rad}A = \cap\mathcal{P}_i$, *where* rad *means the Jacobson radical;*
(iii) *A is semilocal;*
(iv) *for any two-sided ideal I of A, A/I is strongly semilocal.*

Proof - (i) follows from the definition. For (ii), the inclusion $\mathrm{rad}A \subseteq \cap\mathcal{P}_i$ is clear. Let x be an element of $\cap\mathcal{P}_i - \mathrm{rad}A$. There exists a maximal left ideal $\mathcal{M}$ of A such that $x \notin \mathcal{M}$, so $Ax + \mathcal{M} = A$. Since $\mathcal{M} \subseteq \mathcal{P}_i$ for some i, and $x \in \cap\mathcal{P}_i$, we have $A = Ax + \mathcal{M} \subseteq \mathcal{P}_i$, a contradiction. Hence we have, as claimed, $\mathrm{rad}A = \cap\mathcal{P}_i$. We prove (iii). Suppose that there exists a maximal ideal $\mathcal{P}$ of A containing two distinct maximal left ideals $\mathcal{M}_1$ and $\mathcal{M}_2$. Without loss of generality we can assume that exists $y \in \mathcal{M}_1 - \mathcal{M}_2$. Then we have $A = Ay + \mathcal{M}_2 \subseteq \mathcal{P}$. So, if A is a strongly semilocal ring, then A has a finite number of maximal left ideals and (iii) follows from Proposition (20.1) in [7]. Finally, (iv) follows of the definition and the fact that if $\mathcal{M}$ is a proper left (or right) ideal of A/I, then $\mathcal{M} + I$ is a proper left (or right) ideal of A. ■

Remark 2.2 Semilocal does not imply strongly semilocal. For example, a matrix algebra over a field is semilocal and clearly not strongly semilocal. In particular a finite algebra over a commutative semilocal ring is not necessarily strongly semilocal.

The following converse of (iv) holds:

Proposition 2.3 *Let A be a ring and I be a two-sided ideal of A contained in* $\mathrm{rad}A$. *Then A/I strongly semilocal implies A strongly semilocal.*

Proof - Let $\varphi : A \to A/I$ be the natural projection. Let $\{\mathcal{P}_i\}$ be a finite set of maximal ideals of A/I satisfing the conditions for A/I to be strongly semilocal. Clearly $\{\varphi^{-1}(\mathcal{P}_i)\}$ is a set of maximal ideals of A. Let $\mathcal{M}$ be a proper left (or right) ideal of A. Then $\varphi(\mathcal{M})$ is a left (or right) ideal of A/I. If $\varphi(\mathcal{M})$ is proper, then $\varphi(\mathcal{M}) \subseteq \mathcal{P}_i$ for some i, and $\mathcal{M} \subseteq \varphi^{-1}(\mathcal{P}_i)$ for some i. If $\varphi(\mathcal{M}) = 0$ in A/I, we have $\mathcal{M} \subseteq I$ in A. Suppose $\varphi(\mathcal{M}) = A/I$. Then $\mathcal{M} + I = A$. So, there exist $x \in \mathcal{M}$ and $y \in I$ such that $1 = x + y$. Since $I \subseteq \text{rad}A$, it follows that x is a unit in A and $\mathcal{M}$ is not proper. Thus $\varphi(\mathcal{M})$ cannot be the full ring A/I. It follows that the set of two sided ideals $\{\varphi^{-1}(\mathcal{P}_i), I\}$ satisfies the conditions for A to be strongly semilocal. ■

A ring A is semilocal if and only if $A/\text{rad}A \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$, where the D_i are division rings for $1 \leq i \leq r$. For strongly semilocal rings we get:

Theorem 2.4 *Let A be a ring and* rad$A$ *the Jacobson radical of A. Then A is strongly semilocal if and only if $A/\text{rad}A \cong D_1 \times \cdots \times D_r$ for some integer r and division rings D_i, $1 \leq i \leq r$.*

Proof - One sees easily that the ring $D_1 \times \cdots \times D_r$ is strongly semilocal if the D_i are division rings. Thus a ring A such that $A/\text{rad}A \cong D_1 \times \cdots \times D_r$, is strongly semilocal by **2.3**.

Now assume that A is strongly semilocal and let $\{\mathcal{P}_i\}$, $1 \leq i \leq r$, be a corresponding set of maximal ideals of A. Since the Chinese Remainder Theorem holds for $\{\mathcal{P}_i\}$ and $\text{rad}A = \cap_{i=1}^{r} \mathcal{P}_i$, we have $A/\text{rad}A \cong A/\mathcal{P}_1 \times \cdots \times A/\mathcal{P}_r$. We claim that $A/\mathcal{P}_i$ is a division ring for each $i = 1, \ldots, r$. Take $x \in A/\mathcal{P}_i$ such that $x \neq 0$. If x is not invertible in $A/\mathcal{P}_i$, then $(A/\mathcal{P}_i)x$ is a proper left ideal of $A/\mathcal{P}_i$. Choose $y \in A$ with $y \equiv x \bmod \mathcal{P}_i$. Then $Ay + \mathcal{P}_i$ is a proper left ideal of A. As A is strongly semilocal, there exists $j \in \{1, \ldots, r\}$ such that $Ay + \mathcal{P}_i \subseteq \mathcal{P}_j$. This is an absurd, since $\mathcal{P}_i \subset Ay + \mathcal{P}_i$ and $\mathcal{P}_i, \mathcal{P}_j$ are maximal. Hence x is invertible in $A/\mathcal{P}_i$ and $A/\mathcal{P}_i$ is a division ring, as required. ■

Let A^* the group of units of A. As an immediate consequence of **2.4** we have:

Corollary 2.5 (i) *If A is a strongly semilocal ring with maximal ideals $\{\mathcal{P}_i\}$, then $A^* = A - \cup\mathcal{P}_i$.*
(ii) *A finite direct product of strongly semilocal rings is strongly semilocal.*

Proof - (i) is immediate and (ii) follows from the fact that the radical of a direct product is a direct product of the radicals (see Exerc. 12, § 4 in [7]). ∎

The following result, about matrices over strongly semilocal rings, will be used later.

Proposition 2.6 *Let A be a strongly semilocal ring with maximal ideals $\mathcal{P}_1, \ldots, \mathcal{P}_r$. If $B \in M_2(A)$ is such that $B \equiv B_i \bmod M_2(\mathcal{P}_i)$, with $B_i \in GL_2(A/\mathcal{P}_i)$, for all $i = 1, \ldots, r$, then $B \in GL_2(A)$.*

Proof - By **2.4** we have $\tilde{A} = A/\mathrm{rad}A \cong \prod_{i=1}^{r}(A/\mathcal{P}_i)$, with $A/\mathcal{P}_i$ division rings. So $M_2(\tilde{A}) \cong \prod_{i=1}^{r} M_2(A/\mathcal{P}_i)$ and $GL_2(\tilde{A}) \cong \prod_{i=1}^{r} GL_2(A/\mathcal{P}_i)$. Since $B \equiv B_i \bmod M_2(\mathcal{P}_i)$ with $B_i \in GL_2(A/\mathcal{P}_i)$, $1 \leq i \leq r$, we have $\tilde{B} = (B_1, \ldots, B_r) \in GL_2(\tilde{A})$. By Exercise 21, § 4 of [7], there exists $B' \in GL_2(A)$ such that $\widetilde{B'} = \tilde{B}$, i.e., $B' \equiv B_i \bmod M_2(\mathcal{P}_i)$ for all $i = 1, \ldots, r$. As $B \in GL_2(A)$ if and only if $BB'^{-1} \in GL_2(A)$, we can assume $\tilde{B} = \tilde{1}$ in $GL_2(\tilde{A})$. Put $B = (b_{ij})$, $1 \leq i, j \leq 2$. We have $b_{ij} \in \mathrm{rad}A$, $i \neq j$, and $b_{ii} \equiv 1 \bmod \mathrm{rad}A$. It follows that $b_{ii} \in A^*$ and $1 - b_{ij}x \in A^*, i \neq j, 1 \leq i, j \leq 2$ for all $x \in A$. For $C = (c_{ij}) \in M_2(A)$, with $c_{ij} = -b_{ii}^{-1} b_{ij} b_{jj}^{-1}$, $i \neq j$ and $c_{ii} = b_{ii}^{-1}$, $i = 1, 2$, we have

$$BC = \begin{pmatrix} 1 + b_{12}c_{21} & 0 \\ 0 & 1 + b_{21}c_{12} \end{pmatrix},$$

with $(1 + b_{ij}c_{ji}) \in A^*$, $i \neq j$, and $B \in GL_2(A)$, as required. ∎

3 Unitary Spaces and Orthogonal Decompositions

A *unitary ring* (A, λ, Γ) is a triple consisting of a ring A (not necessarily commutative) with identity 1 and an involution $^-$, a central unit λ of A with $\lambda\bar{\lambda} = 1$ and an additive subgroup Γ of A such that:

(i) $S_{-\lambda}(A) = \{a - \lambda\bar{a};\ a \in A\} \subseteq \Gamma \subseteq \{a \in A;\ a + \lambda\bar{a} = 0\} = S^{-\lambda}(A)$

(ii) $ax\bar{a} \in \Gamma$ for all $x \in \Gamma$ and $a \in A$.

For a finitely generated projective right A-module M, let $\mathrm{Sesq}(M)$ be the set of sesquilinear forms on M,

$$\mathrm{Sesq}^{-\lambda}(M) = \{k \in \mathrm{Sesq}(M);\ \ k + \lambda\overline{k} = 0\},$$

where $\overline{k}(x,y) = \overline{k(y,x)},\ \ x,y \in M$, and let

$$\mathrm{Sesq}_{\Gamma}^{-\lambda}(M) = \{k \in \mathrm{Sesq}^{-\lambda}(M);\ \ k(x,x) \in \Gamma \text{ for all } x \in M\}.$$

A *unitary module* is a pair $(M,[k])$ where M is a finitely generated projective right A-module and $[k]$ is the class of $k \in \mathrm{Sesq}(M)$ (modulo $\mathrm{Sesq}_{\Gamma}^{-\lambda}(M)$). The associated even λ-hermitian form $k + \lambda\overline{k}$ will be denoted by $<,>$, i.e., $<x,y> = k(x,y) + \lambda\overline{k(y,x)}$ for all $x,y \in M$. Thus $<x,y> = \lambda\overline{<y,x>}$ for all $x,y \in M$. The pair $(M,[k])$ is a *unitary space* if $<,>$ is nonsingular.

Let $(M,[k])$ be a unitary module. We define a "quadratic function" $[q] : M \to A/\Gamma$ by $[q](x) = k(x,x) + \Gamma,\ \ x \in M$. Let $[a]$ be the class of $a \in A$ modulo Γ. Since $\overline{a}xa \in \Gamma$ for all $x \in \Gamma$ and $a \in A$, we can define $\overline{a}[x]a = [\overline{a}xa]$ and we have

$$[q](xa) = \overline{a}[q](x)a$$

for all $x \in M$ and $a \in A$. The quadratic function $[q]$ and the even form $<,>$ are related through the formula

$$[q](x+y) - [q](x) - [q](y) = [<x,y>].$$

This means that

$$<x,y> \equiv k(x+y,x+y) - k(x,x) - k(y,y) \bmod \Gamma,$$

for all $x,y \in M$.

A *morphism of unitary modules* $\varphi : (M_1,[k_1]) \to (M_2,[k_2])$ is an A-homomorphism of right A-modules $\varphi : M_1 \to M_2$ preserving the quadratic functions and the even λ-hermitian forms. A morphism of unitary modules is said to be an *isometry* if φ is an isomorphism. The *unitary group* of all isometries of a unitary module $(M,[k])$ onto itself is denoted by $\mathcal{U}(M,[k])$ or simply $\mathcal{U}(M)$.

From now on in this section, and in the following one, we will assume that (A,λ,Γ) is a unitary ring with A strongly semilocal. All A-modules considered will be free right A-modules. Since a strongly semilocal ring is semilocal, each left (or right) unit in A is a unit and the rank of a finitely generated free A-module is well defined.

Let $\mathcal{P}$ be a maximal ideal of A with $\mathcal{P} \neq \overline{\mathcal{P}}$. Take $A' = A/\mathcal{P} \times A/\overline{\mathcal{P}}$. The involution of A', induced by the involution of A, is given by $\overline{(a,b)} = (\overline{b}, \overline{a})$, for all $(a,b) \in A'$ and we have $A/\overline{\mathcal{P}} \cong (A/\mathcal{P})^{op}$, hence $A' \cong D \times D^{op}$, D a division ring with the involution $(a, b^o) \mapsto (b, a^o)$. We call a ring $D \times D^{op}$ with the involution $(a, b^o) \mapsto (b, a^o)$ a *double division ring.* Furthermore, since $(1,0)$ is a central element of A' such that $(1,0) + \overline{(1,0)} = (1,1)$, we have $S_{-\lambda}(A') = \Gamma(A') = S^{-\lambda}(A')$, where $\lambda = (\lambda_1, \lambda_2)$ is the image of λ in A' and $\Gamma(A')$ is the image of Γ. By (6.7), I in [6], any unitary space over A' has an orthogonal basis $\{e_1, \dots, e_n\}$ such that $[q](e_i) \equiv (1,0) \bmod S_{-\lambda}(A') = S^{-\lambda}(A')$.

Let (A, λ, Γ) be a unitary ring with A strongly semilocal and $\{\mathcal{P}_i, \quad i = 1, \dots, r\}$ be the maximal ideals of A. Let $\widetilde{A} = A/\mathrm{rad}A$. By **2.4**, $(\widetilde{A}, \widetilde{\lambda}, \widetilde{\Gamma})$ has a decomposition, as a product of unitary rings, of the form

$$(\widetilde{A}, \widetilde{\lambda}, \widetilde{\Gamma}) = (A_1, \lambda_1, \Gamma_1) \times \cdots \times (A_s, \lambda_s, \Gamma_s),$$

where each A_i is a division ring or a double division ring.

Proposition 3.1 *Let $(M, [k])$ be a unitary space over (A, λ, Γ) with* $\mathrm{rank}_A M = n$.

(i) *If $n = 2m$, then $(M, [k]) \simeq \perp_{i=1}^{m} (N_i, [k_i])$, where $\mathrm{rank}_A N_i = 2$ and N_i has a basis $\{x_{i1}, x_{i2}\}$ with $< x_{i1}, x_{i2} > = 1$ for each $i = 1, \dots, m$.*

(ii) *If $n = 2m + 1$, then $(M, [k]) \simeq \perp_{i=1}^{m} (N_i, [k_i]) \perp (N_n, [k_n])$, where $(N_i, [k_i])$, $1 \leq i \leq m$ are as in* (i) *and $N_n = x_n A$ with $< x_n, x_n > \in A^*$.*

Proof - Let $\widetilde{A} = A/\mathrm{rad}A$ and suppose $n = 2$. If $\widetilde{M} = M/(M\,\mathrm{rad}A)$ has a basis $\{\widetilde{x_1}, \widetilde{x_2}\}$ with $<\widetilde{x_1}, \widetilde{x_2}> = \widetilde{1}$ in $\widetilde{A}$ then, lifting $\widetilde{x_1}, \widetilde{x_2}$ to $x_1, x_2 \in M$, we have $M = x_1 A \oplus x_2 A$ and $< x_1, x_2 > \in A^*$. Thus $(M, [k])$ has a basis $\{x_1', x_2'\}$ with $< x_1', x_2' > = 1$. By the Redution Theorem (4.6.1), II, [6] and **2.4**, we may assume that $A = A_1 \times \cdots \times A_s$ with A_i division rings or double division rings. We have a corresponding decomposition

$$(M, [k]) = (M_1, [k_1]) \times \cdots \times (M_s, [k_s]).$$

Since it suffices to prove **3.1** for each $(M_i, [k_i])$, we can assume that A is a division ring or a double division ring. If A is a division ring, the Proposition follows from (6.5.3), I, [6]. If A is a double division ring and $n = 2$, then by (6.7), I in [6], $(M, [k])$ has an orthogonal basis $\{e_1, e_2\}$ such that $k(e_i, e_i) \equiv (1, 0) \bmod S^{-\lambda}(A)$. Since $(1,0) \equiv (0, \lambda_2) \bmod S^{-\lambda}(A)$, we can assume that $k(e_1, e_1) = (1, 0)$ and $k(e_2, e_2) = (0, 1)$. Now, $\{e_1 + e_2, e_2\}$ is a basis of $(M, [k])$ with $< e_1 + e_2, e_2 > = (1, 1) = 1$ in A and the Proposition follows from this and (6.7), I, [6]. ■

Remark 3.2 Replacing division rings or double division rings by simple rings in the proof of **3.1** and using (9.5), I, [6], one can obtain similar orthogonal decompositions for unitary spaces over semilocal rings.

We now discuss the existence of an orthogonal basis for a unitary space. By the Reduction Theorem, a unitary space $(M, [k])$ over A has an orthogonal basis if and only if $(\widetilde{M}, [\widetilde{k}])$ has an orthogonal basis over $\widetilde{A}$. Let (D, λ, Γ) be a unitary ring with D a division ring and $(M, [k])$ a unitary space over D. The space $(M, [k])$ has an orthogonal basis if $(M, <, >)$, the even λ-hermitian space associated to $[k]$, has an orthogonal basis.

Proposition 3.3 *With the above notations, $(M, [k])$ has an orthogonal basis if and only if D satisfies one of the following conditions:*
(1) *The involution of D is not trivial.*
(2) *The involution of D is trivial (D is a field), $\lambda = 1$ and $2 \neq 0$ in D.*

Proof - The part "if" is immediate from (6.2.4), I, [6]. Now, assume that D does not satisfy the conditions (1) and (2). Then the involution of D is trivial (D is a field), $\lambda = -1$ or $2 = 0$ in D. Since $(M, [k])$ is a unitary space over D, from (3.1.1), I, [6], we have $< x, x > \in S_\lambda(D) = \{a + \lambda\bar{a};\ \ a \in D\}$ for all $x \in M$. But $S_\lambda(D) = (1 + \lambda)D$ and λ is a central element of D with $\lambda\bar{\lambda} = \lambda^2 = 1$, i.e., $\lambda = \pm 1$. If $\lambda = -1$ then $S_\lambda(D) = 0$ and $< x, x > = 0$ for all $x \in M$. If $\lambda = 1$ and $2 = 0$ in D, we also have $S_\lambda(D) = 0$. Thus in both cases, $(M, [k])$ does not have an orthogonal basis. ■

The next proposition is now immediate.

Proposition 3.4 *Let (A, λ, Γ) be a unitary ring with A strongly semilocal and $(\widetilde{A}, \widetilde{\lambda}, \widetilde{\Gamma}) = (A_1, \lambda_1, \Gamma_1) \times \cdots \times (A_s, \lambda_s, \Gamma_s)$, where each A_i is a division ring or a double division ring. A unitary space over A has an orthogonal* *basis if and only if A_i satisfies* (1) *or* (2) *in* **3.3** *for all A_i which are division rings.*

Let $(M,[k])$ be an unitary space over A with $\operatorname{rank}_A M = 2$. By **3.1** the space M has a basis $\{x_1, x_2\}$ with $<x_1, x_2> = 1$. So $<,>$ is represented by the matrix $\begin{pmatrix} a_{11} & 1 \\ \lambda & a_{22} \end{pmatrix}$, where $a_{ii} = <x_i, x_i>$, $i = 1,2$. Since

$$\begin{pmatrix} 1 & -\overline{a_{11}} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & 1 \\ \lambda & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1-\overline{a_{11}}a_{22} \\ \lambda & a_{22} \end{pmatrix},$$

we have $1-\overline{a_{11}}a_{22} \in A^*$ and $\lambda - a_{11}a_{22} = \lambda(1-\overline{a_{11}}a_{22}) \in A^*$.

We may assume that $[q](x_1) \cap A^* \neq \emptyset$: the fact that $[q](x_1) \cap A^* \neq \emptyset$ is equivalent with the existence of $B = (b_{ij}) \in GL_2(A)$ such that

$$\alpha_1 = \overline{b_{11}}k_{11}b_{11} + \overline{b_{11}}b_{12} + \overline{b_{12}}k_{22}b_{12} \in A^*$$

and

$$\alpha_2 = \overline{b_{11}}a_{11}b_{21} + \overline{b_{11}}b_{22} + \lambda\overline{b_{12}}b_{21} + \overline{b_{12}}a_{22}b_{22} \in A^*,$$

where $a_{ii} = <x_i, x_i>$ and $k_{ii} = k(x_i, x_i)$, since $x_1' = x_1b_{11} + x_2b_{12}$ and $x_2' = x_1b_{21} + x_2b_{22}$ will be a new basis of M, with $\alpha_1 \in [q](x_1') \cap A^*$ and $<x_1', x_2'> \in A^*$.

Lemma 3.5 *Let $(M,[k])$ be an unitary space over A with* $\operatorname{rank}_A M = 2$. *Let $\mathcal{P}$ be a maximal ideal of A. There exists $B = (b_{ij}) \in M_2(A)$, B invertible modulo $M_2(\mathcal{P})$ and $M_2(\overline{\mathcal{P}})$, such that $\alpha_1\alpha_2 \notin \mathcal{P} \cup \overline{\mathcal{P}}$ (α_1, α_2 as above).*

Proof - If $k_{11} \notin \mathcal{P}\cup\overline{\mathcal{P}}$ (or $k_{22} \notin \mathcal{P}\cup\overline{\mathcal{P}}$), we take $b_{ii} = 1$ and $b_{ij} = 0$, $i \neq j$ (or $b_{ii} = 0$ and $b_{ij} = 1$, $i \neq j$), $1 \leq i,j \leq 2$. So we can assume k_{11}, $k_{22} \in \mathcal{P}\cup\overline{\mathcal{P}}$. If k_{11}, $k_{22} \in \mathcal{P}\cap\overline{\mathcal{P}}$, then $b_{11} = b_{22} = b_{12} = 1$ and $b_{21} = 0$ are as required. In the case that $k_{11} \notin \mathcal{P}\cap\overline{\mathcal{P}}$ or $k_{22} \notin \mathcal{P}\cap\overline{\mathcal{P}}$, changing the index if necessary, we may assume $k_{11} \notin \mathcal{P}\cap\overline{\mathcal{P}}$ and, without loss of generality we can assume $k_{11} \in \mathcal{P}-\overline{\mathcal{P}}$. We claim that there exists $b \in \overline{\mathcal{P}}-\mathcal{P}$ such that $1+\bar{b}a_{22} \notin \overline{\mathcal{P}}$ and, in this case $b_{11} = b_{22} = 1$, $b_{21} = 0$ and $b_{12} = b$ are as required. The claim is equivalent with the existence of $b \in \mathcal{P}-\overline{\mathcal{P}}$ such that $1+ba_{22} \notin \overline{\mathcal{P}}$. If $1+ba_{22} \in \overline{\mathcal{P}}$ for all $b \in \mathcal{P}-\overline{\mathcal{P}}$ we take $b \in A-\overline{\mathcal{P}}$. If $b \in \mathcal{P}$, then $1+ba_{22} \in \overline{\mathcal{P}}$. If $b \notin \mathcal{P}$, since $\mathcal{P}+\overline{\mathcal{P}} = A$, we have $b = b_1+b_2$ with $b_1 \in \mathcal{P}-\overline{\mathcal{P}}$ and $b_2 \in \overline{\mathcal{P}}-\mathcal{P}$ and $1+ba_{22} \equiv 1+b_1a_{22} \bmod \overline{\mathcal{P}}$. Thus $1+ba_{22} \in \overline{\mathcal{P}}$ for all $b \in A-\overline{\mathcal{P}}$. As $k_{11} \in \mathcal{P}-\overline{\mathcal{P}}$, we have $b = -\overline{a_{11}} \in A-\overline{\mathcal{P}}$ and $1-\overline{a_{11}}a_{22} \in \overline{\mathcal{P}}$ a contradiction, since $1-\overline{a_{11}}a_{22}$ is a unit. ∎

Proposition 3.6 *Let $(M,[k])$ be a unitary space over A with* $\mathrm{rank}_A M = 2$. *Then $(M,[k])$ has a basis $\{x_1', x_2'\}$ such that $<x_1', x_2'> = 1$, $[q](x_1') \cap A^* \neq \emptyset$ and $(\lambda - <x_1', x_1'><x_2', x_2'>) \in A^*$.*

Proof - By the remark before Lemma **3.5**, it suffices to find $B \in GL_2(A)$ such that $\alpha_1\alpha_2 \in A^*$. Let $\{\mathcal{P}_i;\ i \in J\}$ be the maximal ideals of A. We define a permutation * on J by $i^* = j$ if and only if $\overline{\mathcal{P}_i} = \mathcal{P}_j$. Let J' be a set of representatives of J modulo the action of *. By Lemma **3.5** there exists, for every $i \in J'$, $B_i = ((b_i)_{jl}) \in M_2(A)$ such that B_i is invertible modulo $M_2(\mathcal{P}_i)$ and $M_2(\overline{\mathcal{P}_i})$,

$$\alpha_{i1} = \overline{(b_i)_{11}}k_{11}(b_i)_{11} + \overline{(b_i)_{11}}(b_i)_{12} + \overline{(b_i)_{12}}k_{22}(b_i)_{12} \notin \mathcal{P}_i \cup \overline{\mathcal{P}_i}$$

and

$$\alpha_{i2} = \overline{(b_i)_{11}}a_{11}(b_i)_{21} + \overline{(b_i)_{11}}(b_i)_{22} + \lambda\overline{(b_i)_{12}}(b_i)_{21} + \overline{(b_i)_{12}}a_{22}(b_i)_{22} \notin \mathcal{P}_i \cup \overline{\mathcal{P}_i}.$$

We take, for each $i \in J'$, $\xi_i \in A$ such that $\xi_i \equiv 1 \bmod (\mathcal{P}_i)$ and $(\overline{\mathcal{P}_i})$, and $\xi_i \equiv 0 \bmod (\overline{\mathcal{P}_j})$, $j \neq i, i^*$. Putting $b_{11} = \sum_{i \in J'} \xi_i (b_i)_{11}$, $b_{12} = \sum_{i \in J'} \xi_i (b_i)_{12}$, $b_{21} = \sum_{i \in J'} \xi_i (b_i)_{21}$ and $b_{22} = \sum_{i \in J'} \xi_i (b_i)_{22}$ it is easy to see that $\alpha_1\alpha_2 \equiv \alpha_{i1}\alpha_{i2} \bmod (\mathcal{P}_i)$ and $(\overline{\mathcal{P}_i})$, and $B \equiv B_i \bmod M_2(\mathcal{P}_i)$ and $M_2(\overline{\mathcal{P}_i})$ for all $i \in J'$. Thus $\alpha_1\alpha_2 \in A - \cup_{i \in J}\mathcal{P}_i = A^*$ and $B \in GL_2(A)$ by **2.6**. ■

Let $(M,[k])$ be a unitary space over A with $\mathrm{rank}_A M = 1$. Assume $|A/\mathcal{P}| \geq 3$ for every maximal ideal $\mathcal{P}$ of A such that $\mathcal{P} \neq \overline{\mathcal{P}}$. By **3.1**, $M = xA$ with $<x, x> \in A^*$. Let $(\tilde{A}, \tilde{\lambda}, \tilde{\Gamma}) = (A_1, \lambda_1, \Gamma_1) \times \cdots \times (A_s, \lambda_s, \Gamma_s)$, where each A_i is a division ring or a double division ring, and $\tilde{k} = k(\tilde{x}, \tilde{x}) = (k_1, \ldots, k_s)$, with $k_i \in A_i$, $i = 1, \ldots, s$. If A_i is a division ring, then $A_i = A/\mathcal{P}$ for some maximal ideal $\mathcal{P}$ of A with $\mathcal{P} = \overline{\mathcal{P}}$ and $k_i \equiv k \bmod \mathcal{P}$. Since $<x, x> = k + \lambda\overline{k}$ and $\mathcal{P} = \overline{\mathcal{P}}$, we have $k_i \neq 0$ in A_i, i.e. $k_i \in A_i^*$. If A_i is a double division ring, then $A_i = A/\mathcal{P} \times A/\overline{\mathcal{P}}$, for some maximal ideal $\mathcal{P}$ of A with $\mathcal{P} \neq \overline{\mathcal{P}}$, and $k_i = (a_1, a_2)$ where $a_1 \equiv k \bmod \mathcal{P}$ and $a_2 \equiv k \bmod \overline{\mathcal{P}}$. If $a_i \neq 0$ for $i = 1, 2$, then $k_i \in A_i^*$. Since $<x, x> \in A^*$, we have a_1, a_2 are not both zero. If one of them is zero, changing $\mathcal{P}$ by $\overline{\mathcal{P}}$ if necessary, we may assume $a_1 = 0$ and $a_2 \neq 0$, i.e. $k \in \mathcal{P} - \overline{\mathcal{P}}$. Since $|A/\overline{\mathcal{P}}| \geq 3$ and $\mathcal{P} \neq \overline{\mathcal{P}}$, there exists $a \in A$ such that $a(k - a) \notin \mathcal{P} \cup \overline{\mathcal{P}}$. For such element a and $b = \lambda(\overline{k} - \overline{a})$ we have $k_i \equiv (b, a) \bmod \Gamma_i$ and $(b, a) \in A_i^*$. Thus, there exists $\tilde{u} \in (\tilde{A})^*$ such that

$\tilde{k} \equiv \tilde{u} \bmod \tilde{\Gamma}$. Now, it is easy to see that $k(x,x) \equiv u \bmod \Gamma$ for some $u \in A^*$, i.e. $[q](x) \cap A^* \neq \emptyset$.

Proposition **3.1** and the above discussion show that, for every unitary space of rank n over (A, λ, Γ), one has a decomposition

$$(M, [k]) \simeq \perp_{i=1}^{m} (x_{i1}A + x_{i2}A) \quad \text{if} \quad n = 2m$$

and

$$(M, [k]) \simeq \perp_{i=1}^{m} (x_{i1}A + x_{i2}A) \perp x_n A \quad \text{if} \quad n = 2m+1,$$

with $<x_{i1}, x_{i2}> = 1$, $(\lambda - <x_{i1}, x_{i1}><x_{i2}, x_{i2}>) \in A^*$, $[q](x_{i1}) \cap A^* \neq \emptyset$ for all $i = 1, \ldots, m$ and $<x_n, x_n> \in A^*$. Further, if n is odd and $|A/\mathcal{P}| \geq 3$, for every maximal ideal $\mathcal{P}$ of A such that $\mathcal{P} \neq \overline{\mathcal{P}}$, then $[q](x_n) \cap A^* \neq \emptyset$. We call the corresponding basis of M a *special basis* of M.

4 Generators for the Unitary Group

Throughout this section we will assume that A is a strongly semilocal ring with $|A/\mathcal{P}| \geq 3$, for all maximal ideals $\mathcal{P}$, and that $(M, [k])$ is a unitary space over (A, λ, Γ), with hyperbolic rank ≥ 1 and $\text{rank}_A M \geq 3$. We fix a splitting $M = H \perp N$ where H is a hyperbolic plane with basis $\{u, v\}$, i.e. $<u, v> = 1$ and $[q](u) = [q](v) = 0$, and $(N, [k])$ is a unitary subspace of $(M, [k])$ with $\text{rank}_A N \geq 1$.

We now introduce special elements of the unitary group. We define $\Delta \in \mathcal{U}(M)$ by $\Delta(u) = v$, $\Delta(v) = u\overline{\lambda}$ and $\Delta(z) = z$ for all $z \in N$. For each unit ε in A, we define $\phi(\varepsilon) \in \mathcal{U}(M)$ by $\phi(\varepsilon)(u) = u\varepsilon$, $\phi(\varepsilon)(v) = v\overline{\varepsilon}^{-1}$ and $\phi(\varepsilon)(z) = z$ for all $z \in N$. For $c \in \Gamma$ we define the transvection $T_c(u)$ in $\mathcal{U}(M)$ by

$$T_c(u)(z) = z + uc\overline{\lambda}<u, z>, \quad z \in M.$$

For $x \in N$ and $a \in A$ with $a \equiv k(x,x) \bmod \Gamma$, the *Eichler transvection* $E_{u,a,x}$ in $\mathcal{U}(M)$ is defined by

$$E_{u,a,x}\,(z) = z + u<x, z> - x\overline{\lambda}<u, z> - u\overline{\lambda}a<u, z>, \quad z \in M.$$

We have $T_c(v) = \Delta T_c(u) \Delta^{-1}$ and $E_{v,a,x} = \Delta E_{u,a,x} \Delta^{-1}$ and the following identities can be easily verified.

(1) $[\phi(\varepsilon),\Delta] = \phi(\varepsilon)\Delta\phi(\varepsilon)^{-1}\Delta^{-1} = \phi(\varepsilon\overline{\varepsilon}), \quad \phi(\varepsilon)^{-1} = \phi(\varepsilon^{-1})$

(2) $T_c(u)T_d(u) = T_{c+d}(u), \quad T_c(u)^{-1} = T_{\overline{c}}(u)$

(3) $T_c(ua) = T_{ac\overline{a}}(u), \quad a \in A$

(4) $\phi(\varepsilon)T_c(u)\phi(\varepsilon^{-1}) = T_{\varepsilon c\overline{\varepsilon}}(u)$

(5) $\phi(\varepsilon)T_d(v)\phi(\varepsilon^{-1}) = T_{\overline{\varepsilon}^{-1}c\varepsilon^{-1}}(v)$

(6) $\varphi E_{u,a,x}\varphi^{-1} = E_{\varphi(u),a,\varphi(x)}, \quad \forall\ \varphi \in \mathcal{U}(M)$

(7) $T_c(u)E_{u,a,x}T_c(u)^{-1} = E_{u,a,x}$

(8) $E_{u,a,x}E_{u,a',y} = E_{u,a+a'+<x,y>,x+y} = E_{u,k(x+y,x+y),x+y}T_c(u),$
$c = k(y,x) - \lambda\overline{k(y,x)} \in \Gamma$

(9) $(E_{u,a,x})^{-1} = E_{u,<x,x>-a,-x} = E_{u,a,-x}T_c(u), \quad c = a - \lambda\overline{a} \in \Gamma$

(10) $[E_{u,a,x}, E_{u,a',y}] = T_c(u), \quad c = <y,x> - <x,y> \in \Gamma$

(11) $E_{ur,a,x} = E_{u,ra\overline{r},x\overline{r}}, \quad r \in A$

(12) $\phi(\varepsilon)E_{u,a,x}\phi(\varepsilon^{-1}) = E_{u,\varepsilon a\overline{\varepsilon},x\overline{\varepsilon}}$

(13) $\phi(\varepsilon)E_{v,a,x}\phi(\varepsilon^{-1}) = E_{v,\overline{\varepsilon}^{-1}a\varepsilon^{-1},x\varepsilon^{-1}}$

Let x be an unimodular element of $(M,[k])$, i.e. $< M, x > \doteq A$. We can write $x = u\alpha' + v\beta' + z'$ with α', $\beta' \in A$ and $z' \in N$ such that $A\alpha' + A\beta' + < N, z' > = A$. For every $w \in N$ and $a \in [q](w)$ we have

$$E_{u,a,w}(x) = u\alpha + v\beta' + z,$$

with $\alpha = \alpha' + < w, z' > -\overline{\lambda}a\beta'$ and $z = z' - w\overline{\lambda}\beta'$.

From now on, we fix a special basis $\{x_1, \ldots, x_n\}$ for N. We put $a_{ij} = < x_i, x_j >$ and $k_{ij} = k(x_i, x_j)$ for all $1 \le i,j \le n$, so $a_{ij} = k_{ij} + \lambda\overline{k_{ji}}$ and $a_{ij} = \lambda\overline{a_{ji}}, \ \ 1 \le i,j \le n$.

For $w = \sum_{i=1}^{n} x_i w_i \in N$ we have

$$k(w,w) \equiv \sum_{i=1}^{n} \overline{w_i}k_{ii}w_i + \sum_{1\le i<j\le n} \overline{w_i}a_{ij}w_j \bmod \Gamma.$$

Let $a = \sum_{i=1}^{n} \overline{w_i} k_{ii} w_i + \sum_{1 \leq i < j \leq n} \overline{w_i} a_{ij} w_j \in [q](w)$, where $k_{ii} \in [q](x_i) \cap A^*$ if i is odd, it is possible since $\{x_1, \ldots, x_n\}$ is a special basis of N. We have

(14) $\alpha = \alpha' + \sum_{i=1}^{n} \overline{w_i} < x_i, z' > -\overline{\lambda} \sum_{i=1}^{n} \overline{w_i} k_{ii} w_i \beta' - \overline{\lambda} \sum_{1 \leq i < j \leq n} \overline{w_i} a_{ij} w_j \beta'$,

with $A\alpha' + A {<} x_1, z' {>} + \cdots + A {<} x_n, z' {>} + A\beta' = A$ (since x is unimodular).

We now want to show that there exists $w = \sum_{i=1}^{n} x_i w_i \in N$, as above, such that α is a unit in A. We start with some lemmas.

Lemma 4.1 *Let $\mathcal{P}$ be a maximal ideal of A with $|A/\mathcal{P}| \geq 3$. There exist $w_1, \ldots, w_n \in A$ such that $\alpha \notin \mathcal{P}$.*

Proof - If $\alpha' \not\equiv 0 \ \ (\mathcal{P})$, we take $w_i = 0, \ \ 1 \leq i \leq n$, so we may assume that $\alpha' \equiv 0 \ \ (\mathcal{P})$. If $\beta' \equiv 0 \ \ (\mathcal{P})$ there exists $i \in \{1, \ldots, n\}$ such that $< x_i, z' > \not\equiv 0 \ \ (\mathcal{P})$ (since x is unimodular). We take $w_i = 1$ and $w_j = 0$ for $j \neq i$. Thus we may assume that $\alpha' \equiv 0 \ \ (\mathcal{P})$ and $\beta' \not\equiv 0 \ \ (\mathcal{P})$. Since $k_{11} \in A^*$, putting $w_i = 0$ for $1 < i \leq n$ in (14), we get

$$\alpha \equiv \overline{w_1} {<} x_1, z' {>} - \overline{\lambda}\, \overline{w_1} k_{11} w_1 \beta' \ \ (\mathcal{P}).$$

If $c_1 = {<} x_1, z' {>} \equiv 0 \ \ (\mathcal{P})$, we take $w_1 = 1$. If $c_1 \not\equiv 0 \ \ (\mathcal{P})$ and $\mathcal{P} = \overline{\mathcal{P}}$ we take $w_1 \in A$ such that $w_1 \not\equiv 0 \ \ (\mathcal{P})$ and $c_1 - \overline{\lambda} k_{11} w_1 \beta' \not\equiv 0 \ \ (\mathcal{P})$ (such a w_1 exists since $|A/\mathcal{P}| \geq 3$). If $\mathcal{P} \neq \overline{\mathcal{P}}$, using the Chinese Remainder Theorem, we take $w_1 \in A$ such that $w_1 \equiv 0 \ \ (\mathcal{P})$ and $w_1 \not\equiv 0 \ \ (\overline{\mathcal{P}})$. ∎

Lemma 4.2 *Let $\mathcal{P}$ be a maximal ideal of A with $|A/\mathcal{P}| \geq 3$. There exist $w_1, \ldots, w_n \in A$ such that $\alpha \notin \mathcal{P} \cup \overline{\mathcal{P}}$.*

Proof - The case $\mathcal{P} = \overline{\mathcal{P}}$ is the Lemma **4.1**, so we can assume $\mathcal{P} \neq \overline{\mathcal{P}}$. If $\alpha' \notin \mathcal{P} \cup \overline{\mathcal{P}}$, we take $w_i = 0, \ \ 1 \leq i \leq n$. Without loss of generality we may assume $\alpha' \equiv 0 \ \ (\mathcal{P})$. Let $c_i = {<} x_i, z' {>}, \ \ 1 \leq i \leq n$.

(a) We first assume that there exists $i \in \{1, \ldots, n\}$ such that

$$(\alpha', c_i, \beta') \not\equiv (0,0,0) \ \ (\mathcal{P}) \text{ and } (\overline{\mathcal{P}}).$$

Observe that $c_i = {<} x_i, z' {>}$, where x_i is an element of a basis of a subspace N_j of N with $\mathrm{rank}_A N_j \leq 2$. If $\mathrm{rank}_A N_j = 1$, then $i = n$ is odd. Taking $w_1 = \cdots = w_{n-1} = 0$ in (14) we have

$$\alpha = \alpha' + \overline{w_n} c_n - \overline{\lambda}\, \overline{w_n} k_{nn} w_n \beta',$$

where $k_{nn} \in [q](x_n) \cap A^*$ and the proof is in this case similar to the proof of Lemma (4.2) in [4]. If $\mathrm{rank}_A N_j = 2$, without loss of generality, we can assume $n = 2$. So

$$(15)\ \alpha \equiv \alpha' + \overline{w_1}c_1 + \overline{w_2}c_2 - \overline{\lambda}\,\overline{w_1}k_{11}w_1\beta' - \overline{\lambda}\,\overline{w_2}k_{22}w_2\beta' - \overline{\lambda}\,\overline{w_1}w_2\beta',$$

modulo $(\mathcal{P})$ and $(\overline{\mathcal{P}})$, and with $(\alpha', c_1, \beta') \not\equiv (0,0,0)\quad (\mathcal{P})$ and $(\overline{\mathcal{P}})$ or $(\alpha', c_2, \beta') \not\equiv (0,0,0)\quad (\mathcal{P})$ and $(\overline{\mathcal{P}})$. Again, we may assume $(\alpha', c_1, \beta') \not\equiv (0,0,0)\quad (\mathcal{P})$ and $(\overline{\mathcal{P}})$. But, we cannot assume that k_{11} is a unit. We will consider two cases:

Case 1 - $c_1 \not\equiv 0\ (\mathcal{P})$.

If $\alpha' \not\equiv 0\quad (\overline{\mathcal{P}})$, then $w_2 = 0$ and $w_1 \in \mathcal{P} - \overline{\mathcal{P}}$ are as required. Assume $\alpha' \equiv 0\quad (\overline{\mathcal{P}})$. Let $k_{11} \in \mathcal{P} \cap \overline{\mathcal{P}}$. If $c_1 \not\equiv 0\quad (\overline{\mathcal{P}})$, $w_1 = 1$ and $w_2 = 0$ are as required. If $c_1 \equiv 0\quad (\overline{\mathcal{P}})$, we have $c_1 \in \overline{\mathcal{P}} - \mathcal{P}$ and $\beta' \not\equiv 0\quad (\overline{\mathcal{P}})$. For $w_2 \in \mathcal{P} - \overline{\mathcal{P}}$ we get

$$\alpha \equiv \begin{cases} \overline{w_1}c_1 + \overline{w_2}c_2 & (\mathcal{P}) \\ -\overline{\lambda}\,\overline{w_1}w_2\beta' & (\overline{\mathcal{P}}). \end{cases}$$

We take $w_1 \in \overline{\mathcal{P}} - \mathcal{P}$ if $c_2 \not\equiv 0\quad (\mathcal{P})$ and $w_1 = 1$ if $c_2 \equiv 0\quad (\mathcal{P})$. If $k_{11} \notin \overline{\mathcal{P}}$ or $k_{11} \in \overline{\mathcal{P}} - \mathcal{P}$ and $c_1 \not\equiv 0\quad (\overline{\mathcal{P}})$, then $(c_1, k_{11}\beta') \not\equiv (0,0)\quad (\mathcal{P})$ and $(\overline{\mathcal{P}})$. For $w_2 = 0$ in (15) we get

$$\alpha \equiv \overline{w_1}(c_1 - \overline{\lambda}k_{11}w_1\beta')\quad (\mathcal{P}) \text{ and } (\overline{\mathcal{P}})$$

and we can find $w_1 \in A$ with the property $\alpha \notin \mathcal{P} \cup \overline{\mathcal{P}}$, as in case (i) of Proposition (4.2) in [4]. Let $k_{11} \in \overline{\mathcal{P}} - \mathcal{P}$ and $c_1 \equiv 0\quad (\overline{\mathcal{P}})$. If $c_2 \not\equiv 0\quad (\mathcal{P})$ then $w_1 \in \overline{\mathcal{P}} - \mathcal{P}$ and $w_2 \in \mathcal{P} - \overline{\mathcal{P}}$ are as required. If $c_2 \equiv 0\quad (\mathcal{P})$, for $w_2 \in \mathcal{P} - \overline{\mathcal{P}}$ in (15) we get

$$\alpha \equiv \begin{cases} \overline{w_1}(c_1 - \overline{\lambda}k_{11}w_1\beta') & (\mathcal{P}) \\ \overline{w_1}(-\overline{\lambda}w_2\beta') & (\overline{\mathcal{P}}). \end{cases}$$

Now, it is enough to find $w_1 \in A$ such that $w_1 \notin \mathcal{P} \cup \overline{\mathcal{P}}$ and $c_1 - \overline{\lambda}k_{11}w_1\beta' \not\equiv 0\quad (\mathcal{P})$. Such a w_1 exists since $|A/\mathcal{P}| \geq 3$.

Case 2 - $c_1 \equiv 0\quad (\mathcal{P})$.

In this case we have $\beta' \not\equiv 0\quad (\mathcal{P})$ and

$$(16)\quad \alpha \equiv \begin{cases} \overline{w_2}c_2 - \overline{\lambda}\,\overline{w_1}k_{11}w_1\beta' - \overline{\lambda}\,\overline{w_2}k_{22}w_2\beta' - \overline{\lambda}\,\overline{w_1}w_2\beta' \ (\mathcal{P}) \\ \alpha' + \overline{w_1}c_1 + \overline{w_2}c_2 - \overline{\lambda}\,\overline{w_1}k_{11}w_1\beta' - \overline{\lambda}\,\overline{w_2}k_{22}w_2\beta' - \overline{\lambda}\,\overline{w_1}w_2\beta' \ (\overline{\mathcal{P}}) \end{cases}$$

If $\alpha' \not\equiv 0\ (\overline{\mathcal{P}})$, then for $w_1 \in \mathcal{P} - \overline{\mathcal{P}}$ in (16), we have

$$\alpha \equiv \begin{cases} \overline{w_2}c_2 - \overline{\lambda}\,\overline{w_2}k_{22}w_2\beta' - \overline{\lambda}\,\overline{w_1}w_2\beta' \ (\mathcal{P}) \\ \alpha' + \overline{w_2}c_2 - \overline{\lambda}\,\overline{w_2}k_{22}w_2\beta' \ (\overline{\mathcal{P}}) \end{cases}$$

If $c_2 \not\equiv 0\ (\mathcal{P})$ (or $c_2 \in \mathcal{P} \cap \overline{\mathcal{P}}$), then $w_2 \in \mathcal{P} - \overline{\mathcal{P}}$ (or $w_2 \in \overline{\mathcal{P}} - \mathcal{P}$) is as required. We may assume $c_2 \in \mathcal{P} - \overline{\mathcal{P}}$. It suffices to find $w_2 \in \overline{\mathcal{P}} - \mathcal{P}$ such that $\alpha' + \overline{w_2}c_2 \not\equiv 0\ (\overline{\mathcal{P}})$. Suppose that no such w_2 exists. Then $\alpha' + \overline{w'}c_2 \equiv 0\ (\overline{\mathcal{P}})$ for all $w' \in \overline{\mathcal{P}} - \mathcal{P}$, or equivalently $\alpha' + w'c_2 \equiv 0\ (\overline{\mathcal{P}})$ for all $w' \in \mathcal{P} - \overline{\mathcal{P}}$. Using that $\alpha', c_2 \in \mathcal{P} - \overline{\mathcal{P}}$, we obtain $w' \equiv 1\ (\overline{\mathcal{P}})$ for all $w' \in \mathcal{P} - \overline{\mathcal{P}}$. Let $w' \in A - \overline{\mathcal{P}}$. If $w' \in \mathcal{P}$, then $w' \equiv 1\ (\overline{\mathcal{P}})$. If $w' \notin \mathcal{P}$, we have $w' \notin \mathcal{P} \cup \overline{\mathcal{P}}$ and since $\mathcal{P} + \overline{\mathcal{P}} = A$, we can write $w' = w_1' + w_2'$ with $w_1' \in \mathcal{P} - \overline{\mathcal{P}}$ and $w_2' \in \overline{\mathcal{P}} - \mathcal{P}$. So $w' \equiv w_1' \equiv 1\ (\overline{\mathcal{P}})$ and we get $|A/\overline{\mathcal{P}}| = 2$, a contradiction. Hence, there exists $w_2 \in \overline{\mathcal{P}} - \mathcal{P}$ such that $\alpha' + \overline{w_2}c_2 \not\equiv 0\ (\overline{\mathcal{P}})$.

Now we assume that $\alpha' \equiv 0\ (\overline{\mathcal{P}})$ in (16). If $c_1 \not\equiv 0\ (\overline{\mathcal{P}})$, we can find the required elements as in Case 1, when $\alpha' \equiv 0\ (\overline{\mathcal{P}})$, changing $\mathcal{P}$ by $\overline{\mathcal{P}}$. If $c_1 \equiv 0\ (\overline{\mathcal{P}})$ we have $\beta' \notin \mathcal{P} \cup \overline{\mathcal{P}}$ and

$$\alpha \equiv \overline{w_2}c_2 - \overline{\lambda}\,\overline{w_1}k_{11}w_1\beta' - \overline{\lambda}\,\overline{w_2}k_{22}w_2\beta' - \overline{\lambda}\,\overline{w_1}w_2\beta' \ (\mathcal{P}) \text{ and } (\overline{\mathcal{P}}).$$

If $c_2 \not\equiv 0\ (\mathcal{P})$, we do as in Case 1, replacing the index 1 by 2. If $c_2 \in \mathcal{P} - \overline{\mathcal{P}}$, then $w_2 \in \overline{\mathcal{P}} - \mathcal{P}$ and $w_1 \in \mathcal{P} - \overline{\mathcal{P}}$ are as required. Thus we may assume that

$$\alpha \equiv -\overline{\lambda}\,\overline{w_1}k_{11}w_1\beta' - \overline{\lambda}\,\overline{w_2}k_{22}w_2\beta' - \overline{\lambda}\,\overline{w_1}w_2\beta' \ (\mathcal{P}) \text{ and } (\overline{\mathcal{P}}).$$

Since $\lambda \in A^*$ and $\beta' \notin \mathcal{P} \cup \overline{\mathcal{P}}$, it is enough to find w_1, $w_2 \in A$ such that $\overline{w_1}k_{11}w_1 + \overline{w_2}k_{22}w_2 + \overline{w_1}w_2 \notin \mathcal{P} \cup \overline{\mathcal{P}}$. If $k_{11} \notin \mathcal{P} \cup \overline{\mathcal{P}}$ (or $k_{22} \notin \mathcal{P} \cup \overline{\mathcal{P}}$), then for $w_1 = 1$ and $w_2 = 0$ (or $w_1 = 0$ and $w_2 = 1$) we have $\alpha \not\equiv 0\ (\mathcal{P})$ and $(\overline{\mathcal{P}})$. If $k_{ii} \in \mathcal{P} \cup \overline{\mathcal{P}}$, $i = 1, 2$, then, without loss of generality, we may assume that $k_{11} \in \mathcal{P}$. If $k_{11} \in \mathcal{P} - \overline{\mathcal{P}}$, then $w_1 = 1$ and $w_2 \in \overline{\mathcal{P}} - \mathcal{P}$ are as required. If $k_{11} \in \mathcal{P} \cap \overline{\mathcal{P}}$ and $k_{22} \in \mathcal{P}$ (or $k_{22} \in \overline{\mathcal{P}}$), we take $w_1 = 1$ and $w_2 \notin \mathcal{P} \cup \overline{\mathcal{P}}$ such that $\overline{w_2}k_{22} + 1 \not\equiv 0\ (\overline{\mathcal{P}})$ (or $\overline{w_2}k_{22} + 1 \not\equiv 0\ (\mathcal{P})$). This can be done since $|A/\mathcal{P}| \geq 3$ and $|A/\overline{\mathcal{P}}| \geq 3$, and the proof of Case 2 is completed.

(b) There is no $i \in \{1,\ldots,n\}$ such that

$$(\alpha', c_i, \beta') \not\equiv (0,0,0) \ \ (\mathcal{P}) \text{ and } (\overline{\mathcal{P}}).$$

Since $A\alpha' + Ac_1 + \cdots + Ac_n + A\beta' = A$, there exist $i, j \in \{1,\ldots,n\}$, $i \neq j$, such that

$$(\alpha', c_i, \beta')\begin{cases} \not\equiv (0,0,0) \ (\mathcal{P}) \\ \equiv (0,0,0) \ (\overline{\mathcal{P}}) \end{cases} \text{and } (\alpha', c_j, \beta')\begin{cases} \equiv (0,0,0) \ (\mathcal{P}) \\ \not\equiv (0,0,0) \ (\overline{\mathcal{P}}) \end{cases}$$

This implies $\alpha', \beta' \in \mathcal{P} \cap \overline{\mathcal{P}}$, $c_i \in \overline{\mathcal{P}} - \mathcal{P}$ and $c_j \in \mathcal{P} - \overline{\mathcal{P}}$. Taking $w_l = 0$ for $l \neq i,\ j$ and $w_i = w_j = 1$ in (14) we have $\alpha \not\equiv 0 \ \ (\mathcal{P})$ and $(\overline{\mathcal{P}})$ as required. ■

Proposition 4.3 *Let $(M, [k])$ be a unitary space over A, with $M = H \perp N$, $H = uA + vA$ a hyperbolic plane and $\operatorname{rank}_A N \geq 1$. If $x = u\alpha' + v\beta' + z'$, α', $\beta' \in A$ and $z' \in N$, is an unimodular element of M, then there exist $w \in N$ and $a \in [q](w)$ such that the coefficient of u in $E_{u,a,w}(x)$ is a unit of A.*

Proof - It is enough to show that there exists $w_1, \ldots, w_n \in A$ such that $\alpha \in A^*$, where α is as in (14). We consider the permutation * on J and J' as in the proof of Proposition **3.6**. For each $i \in J'$, by Lemma **4.2**, there exist $w_{i1}, \ldots, w_{in} \in A$ such that

$$\alpha_i = \alpha' + \sum_{r=1}^{n} \overline{w_{ir}} c_r - \overline{\lambda} \sum_{r=1}^{n} \overline{w_{ir}} k_{rr} w_{ir} \beta' - \overline{\lambda} \sum_{1 \leq r < s \leq n} \overline{w_{ir}} a_{rs} w_{is} \beta' \notin \mathcal{P}_i \cup \overline{\mathcal{P}_i}.$$

Take, for each $i \in J'$, $\xi_i \in A$ such that $\xi_i \equiv 1 \ \ (\mathcal{P}_i)$ and $(\overline{\mathcal{P}_i})$ and $\xi_i \equiv 0 \ \ (\mathcal{P}_j)$ for $j \neq i, i^*$. Putting $w_r = \sum_{i \in J'} \xi_i w_{ir}$, $1 \leq r \leq n$, it is easy to see that $\alpha \equiv \alpha_i \ \ (\mathcal{P}_i)$ and $(\overline{\mathcal{P}_i})$ for all $i \in J'$. Thus $\alpha \in A - \cup_{i \in J} \mathcal{P}_i = A^*$ as required. ■

Proposition **4.3** does not hold in general for semilocal rings, as showed by the following example.

Example 4.4 Let $A = M_2(\mathbf{F}_2)$ with the involution

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

Let $\lambda = 1 \in A$. We have

$$S_\lambda(A) = \{X + \overline{X};\ X \in A\} = \left\{ \begin{pmatrix} a+d & 0 \\ 0 & a+d \end{pmatrix};\ a, d \in \mathbf{F}_2 \right\}.$$

Over $(A, \lambda, S_\lambda(A))$, we consider the unitary space $M = H \perp N$, $H = uA + vA$ a hyperbolic plane and $(N, [k])$ the unitary subspace $N = xA;\ k(x,x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $<x, x> = 1$.

We have $[q](x) \cap A^* = \emptyset$. In fact if $[q](x) \cap A^* \neq \emptyset$, then $k(x,x)$ is congruent to a unit modulo $S_\lambda(A)$, i.e., there exists $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A$ such that $ad + bc = 1$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod S_\lambda(A)$, so $\begin{pmatrix} 1+a & b \\ c & d \end{pmatrix} \in S_\lambda(A)$. We have $(1+a) = d$ and $b = c = 0$. Thus we get $ad = 1 = a + d$ in $\mathbf{F}_2$, which is absurd and, as claimed, $[q](x) \cap A^* = \emptyset$.

The element $v \in M$ is unimodular and we have

$$E_{u,a,w}(v) = u(-\overline{\lambda}a) + v - \overline{\lambda}w$$

for any $w \in N$ and $a \in [q](w)$. Thus there is no $w \in N$ such that the coefficient of u in $E_{u,a,w}(v)$ is a unit of A.

Let $E(M)$ be the normal subgroup of $\mathcal{U}(M)$ generated by all transvections $E_{u,a,x}$ with $x \in N$ and $a \in [q](w)$.

An ideal I of A is called *involutary* if $\overline{I} = I$ (in particular I is two-sided). For such an ideal I, let $E(M, I)$ be the normal subgroup of $\mathcal{U}(M)$ generated by the transvections $E_{u,a,x}$ with $a \in [q](x)$ and $x \in NI = \{\sum y_i a_i;\ y_i \in N,\ a_i \in I\}$ and by $\mathcal{U}(M, I) = \{\varphi \in \mathcal{U}(M);\ \varphi \equiv 1 \bmod I\}$. Clearly $\mathcal{U}(M, I)$ is a normal subgroup of $\mathcal{U}(M)$ and $E(M, I) \subseteq \mathcal{U}(M, I)$. Let $\mathcal{U}(H, I) = \{\varphi \in \mathcal{U}(M, I);\ \varphi \text{ fixes } N\}$.

Since $T_c(u) = E_{u,-c,0}$ for all $c \in \Gamma$, we have $T_c(u) \in E(M, I)$ for all $c \in \Gamma \cap I$.

Lemma 4.5 *If $(M, [k])$ is a unitary space over A with $\operatorname{rank}_A M = 2$ and $\{x_1, x_2\}$ is a special basis of M, then $[q](x_1 - x_2 a_{11}) \cap A^* \neq \emptyset$, where $a_{11} = <x_1, x_1>$.*

Proof - Take $k = \overline{a_{11}}k_{22}a_{11} - k_{11} \in [q](x_1 - x_2a_{11})$, where $k_{ii} = k(x_i, x_i)$ for $i = 1,2$. We want to show that $k \equiv u \bmod \Gamma$ for some $u \in A^*$. Let $(\tilde{A}, \tilde{\lambda}, \tilde{\Gamma}) = (A_1, \lambda_1, \Gamma_1) \times \cdots \times (A_s, \lambda_s, \Gamma_s)$, where each A_i is a division ring or a double division ring, and let $\tilde{k} = (k_1, \ldots, k_s)$ with $k_i \in A_i$, $i = 1, \ldots, s$. If $k_i \in A_i^*$ for all $i = 1, \ldots, s$, then $k \in A^*$. Suppose $k_i \notin A_i^*$ for some $i = 1, \ldots, s$. If A_i is a division ring, then $A_i = A/\mathcal{P}$ for some maximal ideal $\mathcal{P}$ of A with $\mathcal{P} = \overline{\mathcal{P}}$ and $k_i = 0$ in A_i, i,e, $\overline{a_{11}}k_{22}a_{11} - k_{11} = k \in \mathcal{P}$. As $\mathcal{P} = \overline{\mathcal{P}}$, we have $\overline{a_{11}}a_{22}a_{11} - a_{11} = k + \lambda\overline{k} \in \mathcal{P}$, where $a_{22} = <x_2, x_2>$. But $\overline{a_{11}}a_{22} - 1 \in A^*$ (since $\{x_1, x_2\}$ is a special basis of M). So $a_{11} \in \mathcal{P}$ and $k_{11} \equiv \overline{a_{11}}k_{22}a_{11} \equiv 0 \bmod \mathcal{P}$, a contradiction since $k_{11} \in A^*$. Thus, if $k_i \notin A_i^*$ for some $i = 1, \ldots, s$, then A_i is a double division ring, i.e. $A_i = A/\mathcal{P} \times A/\overline{\mathcal{P}}$ for some maximal ideal $\mathcal{P}$ of A with $\mathcal{P} \neq \overline{\mathcal{P}}$ and $k_i = (a_1, a_2) \in A_i$ with $a_1 \equiv k \bmod \mathcal{P}$ and $a_2 \equiv k \bmod \overline{\mathcal{P}}$. If $k \in \mathcal{P} \cap \overline{\mathcal{P}}$ (i.e. $a_1 = a_2 = 0$), then $(\overline{a_{11}}a_{22} - 1)a_{11} = k + \lambda\overline{k} \in \mathcal{P} \cap \overline{\mathcal{P}}$ and as $\overline{a_{11}}a_{22} - 1 \in A^*$, we have $a_{11} \in \mathcal{P} \cap \overline{\mathcal{P}}$. Thus $k_{11} = \overline{a_{11}}k_{22}a_{11} - k \in \mathcal{P} \cap \overline{\mathcal{P}}$, a contradition since $k_{11} \in A^*$. Then a_1, a_2 are not both zero and without loss of generality, we can assume that $a_1 = 0$ and $a_2 \neq 0$. Now we can complete the proof of **4.5** by the same arguments as given after Proposition **3.6**. ■

Theorem 4.6 *Let (A, λ, Γ) be an unitary ring with A strongly semilocal such that $|A/\mathcal{P}| \geq 3$ for all maximal ideals $\mathcal{P}$ of A. If $(M, [k])$ is a unitary space over A as assumed in Proposition* **4.3**, *then $\mathcal{U}(M, I)$ is generated by $\mathcal{U}(H, I)$ and $E(M, I)$.*

Proof - Let $\{x_1, \ldots, x_n\}$ be a special basis of N and let $\{y_1, \ldots, y_n\}$ the dual basis. Let $\varphi \in \mathcal{U}(M, I)$. We assume that $\varphi(x_i) = x_i$ for $1 \leq i \leq m - 1$ (if $\varphi(x_1) \neq x_1$ take $m = 1$). Since $\varphi \equiv 1 \bmod I$, we can write

$$\varphi(x_m) = u\alpha' + v\beta' + z',$$

where $\alpha', \beta' \in I$ and $z' \equiv x_m \bmod I$. We claim that there exist $w \in N$, $a \in [q](w)$ such that:

(i) $E_{u,a,w}\varphi(x_m) = u\alpha + v\beta' + z$, with $\alpha \in A^*$, $\alpha \equiv <w, z'> \bmod I$ and $z \equiv x_m \bmod I$.

(ii) if $m > 1$ then $E_{u,a,w}(x_i) = x_i$ for $1 \leq i \leq m - 1$.

To prove the claim we consider the following two cases:

Case 1 - m is odd.

If $m > 1$, then $N = N_0 \perp N_0^\perp$ where N_0 is the unitary subspace of N generated by $\{x_1, \dots, x_{m-1}\}$. Since $<x_i, \varphi(x_m)> = 0$ for $1 \le i \le m-1$, we have $z' \in N_0^\perp$. Thus $\varphi(x_m)$ is an unimodular element of $M_0 = H \perp N_0^\perp$ and by **4.3**, there exist $w \in N_0^\perp \subseteq N$ and $a \in [q](w)$ satisfying (i) and (ii). If $m = 1$ the claim follows from Proposition **4.3**.

Case 2 - m is even.

In this case, $N = N_1 \perp N_1^\perp$, where N_1 is the unitary subspace of N generated by $\{x_{m-1}, x_m, \dots, x_n\}$ and $z' \in N_1$. It is enough to find $w \in N_1$, $a \in [q](w)$, such that $\alpha = \alpha' + <w, z'> - \overline{\lambda} a \beta' \in A^*$ and $<x_{m-1}, w> = 0$. Writing $w = \sum_{i=m-1}^{n} x_i w_i$ and putting $w_{m+1} = \dots = w_n = 0$, we may assume that $m = 2$, i.e., $N_1 = x_1 A + x_2 A$ with $< x_1, x_2 > = 1$, $k_{11} \in [q](x_1) \cap A^*$ and $(\lambda - a_{11} a_{22}) \in A^*$ where $a_{ii} = < x_i, x_i >$, $i = 1, 2$. Since $<x_1, w> = 0$, we have $w = (x_1 - x_2 a_{11}) w_1$ and $[q](w) = \overline{w_1}[q](x_1 - x_2 a_{11}) w_1$. Take $a \in [q](x_1 - x_2 a_{11}) \cap A^*$, which there exists by **4.5**. It follows from $< x_1, z' > = < x_1, \varphi(x_2) > = 1$, that $z' = x_1 z_1 + x_2(1 - a_{11} z_1)$, for some $z_1 \in A$, and $<w, z'> = \overline{w_1}(1 - \overline{a_{11}} a_{22})(1 - a_{11} z_1)$. Thus we get

$$\alpha = \alpha' + \overline{w_1} c - \lambda\, \overline{w_1} a w_1 \beta',$$

with $a \in A^*$ and $c = (1 - \overline{a_{11}} a_{22})(1 - a_{11} z_1)$. Observe that the condition $\varphi(x_2)$ unimodular implies $(\alpha', c, \beta') \not\equiv (0,0,0)$ $(\mathcal{P})$ for every maximal ideal $\mathcal{P}$ of A. In fact, if there exists a maximal ideal $\mathcal{P}$ of A such that $(\alpha', c, \beta') \equiv (0,0,0)$ $(\mathcal{P})$, then $a_{11} z_1 \equiv 1$ $(\mathcal{P})$ and $a_{22} \equiv < z', z' > \equiv \lambda z_1$ $(\mathcal{P})$. Thus $a_{11} a_{22} \equiv \lambda a_{11} z_1 \equiv \lambda$ $(\mathcal{P})$ which is absurd since $\lambda - a_{11} a_{22} \in A^*$. Hence $(\alpha', c, \beta') \not\equiv (0,0,0)$ $(\mathcal{P})$ and $w_1 \in A$ such that $\alpha \in A^*$ can be found as in the proof of Proposition **4.3** above. This completes the proof of the claim.

Let now $w \in N$ and $a \in [q](w)$ be such that $E_{u,a,w}\varphi(x_m) = u\alpha + v\beta' + z$, where $\alpha \in A^*$, $z = z' - \overline{\lambda} w \beta'$ (so $z \equiv x_m \bmod I$) and $E_{u,a,w}\varphi$ fixes $x_1, \dots, x_{m-1}$ (if $m > 1$). We have $< w, x_m > -\alpha \equiv < w, x_m > -\alpha' - < w, z' > + \overline{\lambda} a \beta' \equiv$ $< w, x_m - z' > \equiv 0 \bmod I$. Taking $a_1 \in [q]((z - x_m)\alpha^{-1})$, we get

$$E_{v,a_1,(z-x_m)\alpha^{-1}} E_{u,a,w}\varphi(x_m) = u\alpha + v\beta + x_m,$$

for some $\beta \in I$. It is easy to see that $\beta\alpha^{-1} \in I \cap \Gamma$ and

$$E_{v,c,0}E_{v,a_1,(z-x_m)\alpha^{-1}}E_{u,a,w}\varphi(x_m) = u\alpha + x_m,$$

where $c = \beta\alpha^{-1}$. Let $a_2 \in [q](y_m\overline{(<w, x_m> -\alpha)})$. Then

$$\psi_m = (E_{u,a,w})^{-1}E_{u,a_2,y_m\overline{(<w,x_m>-\alpha)}}E_{v,c,0}E_{v,a_1,(z-x_m)\alpha^{-1}}E_{u,a,w}$$

is an isometry of M such that, in this expression, all factors other than $E_{u,a,w}$ and $(E_{u,a,w})^{-1}$ are in $E(M,I)$. Thus $\psi_m \in E(M,I)$. Further $\psi_m\varphi(x_m) = x_m$ and $\psi_m\varphi$ fixes $x_1,\ldots,x_{m-1}$ (note that $<z - x_m, x_i> = 0,\ 1 \leq i \leq m-1$). By induction on m, we conclude that there exists ψ in $E(M,I)$ such that $\psi\varphi$ belongs to $\mathcal{U}(H,I)$. ■

Let $\Phi(M,I) = \{\phi(\varepsilon);\ \ \varepsilon \in A^* \text{ with } \varepsilon \equiv 1 \bmod I\}$.

Theorem 4.7 *Let (A,λ,Γ) and $(M,[k])$ be as in* **4.6**. *Then $\mathcal{U}(M,I)$ is generated by $E(M,I)$, $\Phi(M,I)$ and all $\theta \in \mathcal{U}(M,I)$, such that θ is the identity in $H \perp (x_1A + \cdots + x_{n-1}A)$.*

Proof - Let $\varphi \in \mathcal{U}(M,I)$. By **4.6** there exists $\psi \in E(M,I)$ such that $\psi\varphi$ is the identity in N. Hence $\psi\varphi(H) \subseteq H$. Write $\psi\varphi(u) = u\alpha' + v\beta'$, with $\alpha' \equiv 1 \bmod I$ and $\beta' \in I$. If n is odd, $\psi\varphi(u)$ is an unimodular element of $H \perp x_nA$, where $a_{nn} \in A^*$. By **4.3** there exist $w \in x_nA$ and $a \in [q](w)$ such that $E_{u,a,w}\psi\varphi(u) = u\alpha + v\beta' - \overline{\lambda}w\beta'$ with $\alpha \in A^*$, $\alpha \equiv 1 \bmod I$ and $\beta' \in I$. If n is even, $\psi\varphi(u)$ is an unimodular element of $H \perp (x_{n-1}A + x_nA)$ where $<x_{n-1}, x_n> = 1$, $(\lambda - a_{(n-1)(n-1)}a_{nn}) \in A^*$ and $k_{(n-1)(n-1)} \in [q](x_{n-1}) \cap A^*$. To simplify the notation, we assume $n = 2$.

Take $w \in (x_1A + x_2A)$ with $<x_1, w> = 0$. So $w = (x_1 - x_2a_{11})w'$ for some $w' \in A$. For $a' \in [q](x_1 - x_2a_{11}) \cap A^*$ (see **4.5**), we have

$$E_{u,a,w}\psi\varphi(u) = u\alpha + v\beta' - \overline{\lambda}w\beta',$$

where $a = \overline{w'}a'w' \in [q](w)$ and $\alpha = \alpha' - \overline{\lambda}\ \overline{w'}a'w'\beta'$. Since $a' \in A^*$ and $(\alpha',\beta') \not\equiv (0,0)\ \ (\mathcal{P})$ for all maximal ideals $\mathcal{P}$ of A, we can find $w' \in A$ such that $\alpha \in A^*$ as in Proposition **4.3**. Thus there exist $w \in N$ and $a \in [q](w)$ such that $E_{u,a,w}\psi\varphi(u) = u\alpha + v\beta' - \overline{\lambda}w\beta'$, with $\alpha \in A^*$, $\alpha \equiv 1 \bmod I$,

$\beta' \in I$ and $< x_i, w > = 0$ for $1 \leq i \leq n-1$. For $a_1 \in [q](-\overline{\lambda} w \beta')$ we have $E_{v,a_1,-\overline{\lambda}w\beta'} \in E(M,I)$ and

$$E_{v,a_1,-\overline{\lambda}w\beta'} E_{u,a,w} \psi\varphi(u) = u\alpha + v\beta$$

for some $\beta \in I$. So $c = \beta\alpha^{-1} \in I \cap \Gamma$ and $E_{v,c,0} \in E(M,I)$ is such that $\rho = (E_{u,a,w})^{-1} E_{v,c,0} E_{v,a_1,-\overline{\lambda}w\beta'} E_{u,a,w} \psi \in E(M,I)$ and $\phi(\alpha^{-1})\rho\varphi$ fixes $u, x_1, \ldots, x_{n-1}$. Hence, we can write

$$\phi(\alpha^{-1})\rho\varphi(v) = u\gamma' + v + z,$$

with $\gamma' \in I$, $z \in N$; $z \equiv 0 \bmod I$ and $< x_i, z > = 0$, $1 \leq i \leq n-1$. For $a_2 \in [q](\lambda z)$, we have $E_{u,a_2,\lambda z} \in E(M,I)$ and $E_{u,a_2,\lambda z}\phi(\alpha^{-1})\rho\varphi(v) = u\gamma + v$, for some $\gamma \in I$. It is easy to see that $\overline{\gamma} \in I \cap \Gamma$. Then $\lambda\gamma \equiv \overline{\gamma} \bmod \Gamma$ belongs to $I \cap \Gamma$, $E_{u,\lambda\gamma,0} \in E(M,I)$ and $E_{u,\lambda\gamma,0} E_{u,a_2,\lambda z}\phi(\alpha^{-1})\rho\varphi(v) = v$. Since $< x_i, z > = 0$, $1 \leq i \leq n-1$, we have

$$\theta = E_{u,\lambda\gamma,0} E_{u,a_2,\lambda z}\phi(\alpha^{-1})\rho\varphi \in \mathcal{U}(M,I)$$

and θ fixes $u, v, x_1, \ldots, x_{n-1}$, which proves the theorem. ∎

Lemma 4.8 *Consider* (A, λ, Γ) *and* $(M, [k])$ *as in* **4.6**. *Then*

$$E(M,I) \subseteq [\mathcal{U}(M), \mathcal{U}(M,I)\ ].$$

Proof - Since $|A/\mathcal{P}| \geq 3$ for all maximal ideals $\mathcal{P}$ of A, there exists $\varepsilon \in A^*$ such that $\overline{\varepsilon} - 1 \in A^*$. Let $\xi \in A^*$ with $\xi(\overline{\varepsilon} - 1) = 1$. For $x \in NI$ and $a \in [q](x)$ we have $E_{u,\overline{\xi}a\xi,x\xi} \in E(M,I) \subseteq \mathcal{U}(M,I)$ and by formula (12) we get

$$[\phi(\varepsilon), E_{u,\overline{\xi}a\xi,x\xi}] = E_{u,a,x}E_{u,c,0}$$

with $c = \overline{\xi}a - \lambda\overline{\overline{\xi}a} \in \Gamma \cap I$. Take $y', z' \in N$ such that $< y', z' > = 1$ (such y', z' exist since $(N, [k])$ is a unitary space). So, for $y = y'\xi$ and $z = z'a \in NI$, we have $c = < y, z > - < z, y >$. By formula (10) we get $E_{u,c,0} = [E_{u,a_1,y}, E_{u,a_2,z}] \in [E(M),\ E(M,I)] \subseteq [\mathcal{U}(M), \mathcal{U}(M,I)]$, for some $a_1 \in [q](y)$ and $a_2 \in [q](z)$. Hence $E_{u,a,x} \in [\mathcal{U}(M), \mathcal{U}(M,I)]$ as required. ∎

If $M = M' \perp xA$ for some unimodular element x in M, then, for each central unit $a \in A$ with $a\overline{a} = 1$, the linear map τ_x^a defined by $\tau_x^a(x) = xa$ and

$\tau_x^a = 1$ in M' is an isometry on M with $(\tau_x^a)^{-1} = \tau_x^{\bar{a}}$. If $a = -1$, we write τ_x instead of τ_x^a. If there exists $a' \in [q](x) \cap A^*$ with $a' = \lambda\overline{a'}$, then τ_x is also given by the formula

$$\tau_x(z) = z - xa'^{-1} < x, z > .$$

Nevertheless, we observe that for the definition of τ_x we do not need that xA is a subspace of M, i.e. that $< x, x > \in A^*$.

Theorem 4.9 *If $N = (x_1A + \cdots + x_{n-1}A) \perp x_nA$ with $n \geq 1$ and if A is commutative or M has hyperbolic rank at least two, then*

$$E(M, I) = [\mathcal{U}(M), \mathcal{U}(M, I)].$$

Proof - By **4.8** it is enough to check that $[\mathcal{U}(M), \mathcal{U}(M, I)] \subseteq E(M, I)$. By Theorem **4.7**, we know that $\mathcal{U}(M)$ is generated by $\{E(M), \phi(\varepsilon), \tau_{x_n}^a\}$ and $\mathcal{U}(M, I)$ is generated by $\{E(M, I), \phi(\eta), \tau_{x_n}^b\}$, where $\varepsilon, \eta \in A^*$, a, b are central units in A with $a\bar{a} = b\bar{b} = 1$ and $\eta \equiv b \equiv 1 \bmod I$. Since $[\mathcal{U}(M), E(M, I)] \subseteq E(M, I)$, it suffices to show that $[\mathcal{U}(M), \phi(\eta)] \subseteq E(M, I)$ and $[\mathcal{U}(M), \tau_{x_n}] \subseteq E(M, I)$. Now, $[E_{u,a,x}, \phi(\eta)] = E_{u,a(1-\bar{\eta})+\lambda(\eta-1)\bar{a}\,\bar{\eta},x(1-\bar{\eta})}$ is in $E(M, I)$ and $[E_{u,a,x}, \tau_{x_n}^b] = E_{u,<x,x-\tau_{x_n}^b(x)>,x-\tau_{x_n}^b(x)} \in E(M, I)$. Then $[E(M), \phi(\eta)] \subseteq E(M, I)$ and $[E(M), \tau_{x_n}] \subseteq E(M, I)$. We now check that $[\{\phi(\varepsilon), \tau_{x_n}^a\}, \{\phi(\eta), \tau_{x_n}^b\}] \subseteq E(M, I)$. Since a, b are central we have $[\tau_{x_n}^a, \tau_{x_n}^b] = 1 \in E(M, I)$ and $[\tau_{x_n}^a, \phi(\eta)] = [\phi(\varepsilon), \tau_{x_n}^b] = 1 \in E(M, I)$ is obvious. We finally show that $[\phi(\varepsilon), \phi(\eta)] \in E(M, I)$: if A is commutative $[\phi(\varepsilon), \phi(\eta)] = 1$ is in $E(M, I)$. Suppose $M = H \perp H' \perp N'$, or equivalently, $N = H' \perp N'$ with $H' = u'A + v'A$ a hyperbolic plane. Let $\phi'(\varepsilon)$ be defined on H' as $\phi(\varepsilon)$ on H. We have the relation

$$E_{u,0,v'\lambda} E_{v,0,u'(\varepsilon-1)} \phi'(\varepsilon) E_{u,0,v'\lambda} = E_{v,0,u'(1-\varepsilon^{-1})} \phi(\varepsilon)$$

for all $\varepsilon \in A^*$. For $\eta \in A^*$, with $\eta \equiv 1 \bmod I$, we obtain $\phi(\eta) \equiv \phi'(\eta) \bmod E(M, I)$. So $[\phi(\varepsilon), \phi(\eta)] \equiv [\phi(\varepsilon), \phi'(\eta)] \bmod E(M, I)$ and $[\phi(\varepsilon), \phi'(\eta)] = 1$. It follows that $[\phi(\varepsilon), \phi(\eta)] \in E(M, I)$. ∎

Lemma 4.10 *Let x be a vector in N such that there exists $a \in [q](x) \cap A^*$ with $a = \lambda\bar{a}$. Then*

$$E_{v,a,x} E_{u,\bar{a}^{-1},x\lambda a^{-1}} E_{v,a,x} = \Delta\phi(-a)\tau_x.$$

Proof - Immediate. ∎

Lemma 4.11 *If there exists at least one vector x in N such that there exists $a \in [q](x) \cap A^*$ with $a = \lambda\bar{a}$ and a central in A, then $\phi(\varepsilon\bar{\varepsilon})$ lies in $E(M)$ for all $\varepsilon \in A^*$.*

Proof - By the previous lemma, we know that $\psi = \Delta\phi(-a)\,\tau_x$ is in $E(M)$. For $\varepsilon \in A^*$, $[\phi(\varepsilon), \psi] = \phi(\varepsilon\bar{\varepsilon})$ (since a is central). So $\phi(\varepsilon\bar{\varepsilon})$ is in $E(M)$. ∎

Theorem 4.12 *Assume that $|A/\mathcal{P}| > 4$ for all maximal ideals $\mathcal{P}$ of A and that N contains a vector x as in* **4.11**. *Then*

$$E(M, I) = [E(M),\ E(M, I)].$$

Proof - Since $E(M,I)$ is a normal subgroup of $\mathcal{U}(M)$, the inclusion $[E(M),\ E(M,I)] \subseteq E(M,I)$ is clear. We check the converse. It follows from $|A/\mathcal{P}| > 4$ for all maximal ideals $\mathcal{P}$ of A, that there exists $\varepsilon \in A^*$ such that $\varepsilon\bar{\varepsilon} - 1$ is a unit. Take $\xi \in A^*$ such that $\xi(\varepsilon\bar{\varepsilon} - 1) = 1$. For $E_{u,a,x}$ in $E(M,I)$ we have $E_{u,\bar{\xi}a\xi,x\xi} \in E(M,I)$ and $[\phi(\varepsilon\bar{\varepsilon}), E_{u,\bar{\xi}a\xi,x\xi}] = E_{u,a,x}E_{u,c,0}$, where $c = \overline{\xi}a - \lambda\overline{\overline{\xi}a}$ is in $\Gamma \cap I$. As in the proof of Lemma **4.8**, we have $E_{u,c,0} \in [E(M),\ E(M,I)]$ and from **4.11**, we have $\phi(\varepsilon\bar{\varepsilon}) \in E(M)$. Thus, we get $E(M,I) \subseteq [E(M),\ E(M,I)]$, as required. ∎

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References

[1] A. BAK, K-Theory of Forms, *Ann. Math. Studies* 98, Princeton University Press, Princeton, 1981.

[2] H. BASS, Unitary Algebraic K-Theory, *Lecture Notes in Math.* 343, Springer-Verlag, (1973), 57 - 265.

[3] I. DIAS, On the Orthogonal Group over an LG-ring, *Communications in Algebra* 19(2), (1991), 443 - 454.

[4] H. ISHIBASHI, Unitary Groups with Excellent S and Entire E(u,L), *J. Algebra* 76, (1982), 442 - 458.

[5] D.G. JAMES, Unitary Groups over Local Rings, *J. Algebra* 52, (1978), 354 - 363.

[6] M.-A. KNUS, Quadratic and Hermitian Forms over Rings, Springer-Verlag, Heidelberg, New York, 1991.

[7] T.Y. LAM, A First Course in Noncommutative Rings, *Graduate Texts in Math.* 131, Springer-Verlag, Heidelberg, New York, 1991.

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