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# On a General Conley Index Continuation Principle for Singular Perturbation Problems


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# On a general Conley index continuation principle for singular perturbation problems

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**Resumo**

Em problemas de perturbação singular, freqüentemente tem-se uma família $(\pi_\epsilon)_{\epsilon\in]0,1]}$ de semifluxos definidos em um espaço $X$ e um 'fluxo limite singular' $\pi_0$ desta família o qual é definido somente em um subespaço $X_0$ de $X$. Além disso, $\pi_\epsilon$ converge para $\pi_0$ somente em um sentido 'singular'. Tal situação ocorre, por exemplo, em sistemas *fast-slow* de equações diferenciais ou em equações de evolução em domínios finos.

Neste artigo mostramos um princípio geral de continuação singular do índice de Conley o qual afirma que todo conjunto invariante isolado $K_0$ de $\pi_0$ pode ser continuado a uma família $\tilde{K}_\epsilon$ de conjuntos invariantes isolados de $\tilde{\pi}_\epsilon$ com o mesmo índice de Conley. Ilustramos este resultado de continuação com as equações das ondas amortecidas definidas em domínios finos. Isto estende alguns resultados de nosso trabalho anterior, *Conley index continuation and thin domain problems*, a aparecer.

# ON A GENERAL CONLEY INDEX CONTINUATION PRINCIPLE FOR SINGULAR PERTURBATION PROBLEMS

M. C. CARBINATTO AND K. P. RYBAKOWSKI

ABSTRACT. In singular perturbation problems one frequently has a family $(\pi_\epsilon)_{\epsilon\in]0,1]}$ of semiflows on a space $X$ and a 'singular limit flow' $\pi_0$ of this family which is defined only on a subspace $X_0$ of $X$. Moreover, $\pi_\epsilon$ converges to $\pi_0$ only in some 'singular' sense. Such a situation occurs, for example, in fast-slow systems of differential equations or in evolution equations on thin spatial domains.

In this paper we prove a general singular Conley index continuation principle stating that every isolated invariant set $K_0$ of $\pi_0$ can be continued to a nearby family $K_\epsilon$ of isolated invariant sets of $\pi_\epsilon$ with the same Conley index. We illustrate this continuation result with damped wave equations on thin domains. This extends some results of our previous work, *Conley index continuation and thin domain problems*, to appear.

## 1. INTRODUCTION

The Conley index is a well known tool used in the analysis of dynamical systems, both in finite and in infinite dimensional spaces. Its usefulness is largely due to the fact that it enjoys a so-called continuation property, which can roughly be described as follows:

Let $X$ be a (metric) space and $(\pi_\epsilon)_{\epsilon\in[0,1]}$ be a family of flows (or semiflows) on $X$ such that $\pi_\epsilon$ converges to $\pi_0$ on $X$ as $\epsilon \to 0^+$. This means that whenever $x_\epsilon \to x_0$ and $t_\epsilon \to t_0$ as $\epsilon \to 0^+$ then $x_\epsilon\pi_\epsilon t_\epsilon \to x_0\pi_0 t_0$. Let $N \subset X$ be an isolating neighborhood of an isolated invariant set $K_0$ of $\pi_0$. Then, under some additional compactness assumptions, for every $\epsilon > 0$ small enough the set $N$ is an isolating neighborhood of an isolated invariant set $K_\epsilon$ of $\pi_\epsilon$ and the Conley indices of $K_\epsilon$ and $K_0$ are the same.

This result is commonly used when we want to deduce information about the more complicated invariant sets $K_\epsilon$, $\epsilon > 0$, from the simpler invariant set $K_0$.

In the situation described above the flow $\pi_0$ is a 'regular limit' of the flows $\pi_\epsilon$, $\epsilon \in ]0, 1]$.

On the other hand, in singular perturbation problems one frequently has a family $(\pi_\epsilon)_{\epsilon\in]0,1]}$ of flows on the space $X$, a family $(\rho_\epsilon)_{\epsilon\in]0,1]}$ of metrics on $X$ and one also has a 'singular limit flow' $\pi_0$ which is defined only on a subspace $X_0$ of $X$. Moreover, $\pi_\epsilon$ converges to $\pi_0$ only in some restricted sense with respect to the above family of metrics. Such a situation occurs, for example, in evolution equations on thin spatial domains, which we now briefly describe.

Let $M$ and $N$ be positive integers and $\Omega$ be an arbitrary nonempty smooth bounded domain in $\mathbb{R}^M \times \mathbb{R}^N$. Write $(x, y)$ for a generic point of $\mathbb{R}^M \times \mathbb{R}^N$. Given $\epsilon > 0$ squeeze $\Omega$ by the factor $\epsilon$ in the $y$-direction to obtain the squeezed domain $\Omega_\epsilon$. More precisely, let $T_\epsilon\colon \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}^M \times \mathbb{R}^N$, $(x, y) \mapsto (x, \epsilon y)$, be the *squeezing operator* and define

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$\Omega_\epsilon := \{ (x, \epsilon y) \mid (x, y) \in \Omega \}$. Consider the following reaction-diffusion equation on $\Omega_\epsilon$:

$$(\tilde{E}_\epsilon) \qquad \begin{aligned} &u_t = \Delta u - \gamma u + f(\epsilon, x, y, u), && t > 0,\ (x, y) \in \Omega_\epsilon, \\ &\partial_{\nu_\epsilon} u = 0, && t > 0,\ (x, y) \in \partial\Omega_\epsilon, \end{aligned}$$

where $\nu_\epsilon$ is the exterior normal vector field on $\partial\Omega_\epsilon$, $\gamma > 0$ is a constant and $f$ is a given (scalar valued) nonlinearity. We are interested in what happens to the dynamics of this equation as $\epsilon \to 0^+$. This problem was first considered by Hale and Raugel in [5] for special so-called ordinate set domains and later, for general domains, by Prizzi and Rybakowski in [11, 12] and also by the present authors in [1].

In order to describe some of the results obtained we first use the transformation $u \mapsto u \circ T_\epsilon$ to transfer problem $(\tilde{E}_\epsilon)$ to the equivalent problem

$$(E_\epsilon) \qquad \begin{aligned} &u_t = \Delta_x u + (1/\epsilon^2)\Delta_y u - \gamma u + f(\epsilon, x, \epsilon y, u), && t > 0,\ (x, y) \in \Omega, \\ &\nu_1 \cdot \nabla_x u + (1/\epsilon^2)\nu_2 \cdot \nabla_y u = 0, && t > 0,\ (x, y) \in \partial\Omega. \end{aligned}$$

Here, $\nu = (\nu_1, \nu_2)$ is the exterior normal vector field on $\partial\Omega$.

If the nonlinearity $f$ is a $C^1$-function of its arguments and has appropriate polynomial growth in $u$, $\epsilon$ and $y$, then we can define, for every $\epsilon > 0$, the Nemitski operator

$$f_\epsilon \colon H^1(\Omega) \to L^2(\Omega)$$

$$f_\epsilon(u)(x, y) := f(\epsilon, x, \epsilon y, u(x, y)).$$

Then equation $(E_\epsilon)$ can be written in the abstract form

$$\dot{u} + A_\epsilon u = f_\epsilon(u),$$

where $A_\epsilon$ is the linear operator defined by

$$A_\epsilon u = -\Delta_x u - (1/\epsilon^2)\Delta_y u + \gamma u \in L^2(\Omega)$$

for $u \in H^2(\Omega)$ with $\nu_1 \cdot \nabla_x u + (1/\epsilon^2)\nu_2 \cdot \nabla_y u = 0$ on $\partial\Omega$. Equation $(E_\epsilon)$ defines a (local) semiflow $\pi_\epsilon$ on $H^1(\Omega)$. Note that the operator $A_\epsilon$ is generated, in the usual way, by the bilinear form

$$a_\epsilon \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$$

given by

$$a_\epsilon(u, v) := \int_\Omega \left(\gamma uv + \nabla_x u \cdot \nabla_x v + (1/\epsilon^2)\nabla_y u \cdot \nabla_y v\right) dx\, dy.$$

(Cf. formula $(F)$ below). On $X := H^1(\Omega)$ we define the norm $|\cdot|_\epsilon$ by

$$|u|_\epsilon^2 := a_\epsilon(u, u)$$

and the corresponding metric $\rho_\epsilon(u, v) := |u - v|_\epsilon$.

Now define the space

$$X_0 := H^1_s(\Omega) := \{ u \in H^1(\Omega) \mid \nabla_y u = 0 \}.$$

This is a closed linear subspace of $H^1(\Omega)$. The bilinear form given by the formula:

$$a_0(u, v) := \int_\Omega (\gamma uv + \nabla_x u \cdot \nabla_x v)\, dx\, dy, \quad u, v \in H^1_s(\Omega)$$

uniquely determines a densely defined selfadjoint linear operator $A_0 \colon D(A_0) \subset H^1_s(\Omega) \to L^2_s(\Omega)$ by the usual formula

$$(F) \qquad a_0(u, v) = \langle A_0 u, v \rangle_{L^2(\Omega)}, \quad \text{for } u \in D(A_0) \text{ and } v \in H^1_s(\Omega).$$

Here, $L^2_s(\Omega)$ is the closure of $H^1_s(\Omega)$ in the $L^2$-norm, so $L^2_s(\Omega)$ is a closed linear subspace of $L^2(\Omega)$. It is proved that the Nemitski operator $f_0$ defined by

$$f_0(u)(x,y) := f(0,x,0,u(x,y)),$$

maps the space $H^1_s(\Omega)$ into $L^2_s(\Omega)$. Consequently the abstract parabolic equation

$$(E_0) \qquad \dot{u} = -A_0 u + f_0(u)$$

defines a semiflow $\pi_0$ on the space $X_0 = H^1_s(\Omega)$. This is the singular limit semiflow of the family $\pi_\epsilon$. In fact, the family $(\pi_\epsilon)_{\epsilon\in]0,1]}$ converges to $\pi_0$ in the following restricted sense: whenever $\epsilon_n \to 0^+$, $t_n \to t_0$ and $\rho_{\epsilon_n}(u_n, u_0) \to 0$, where $t_n$, $t_0 \in [0,\infty[$, $u_n \in X$, $n \geq 1$, and $u_0 \in X_0$ then $\rho_{\epsilon_n}(u_n\pi_{\epsilon_n}t_n, u_0\pi_0 t_0) \to 0$ (cf. [11, 1]).

If the nonlinearity $f$ is *dissipative*, then each semiflow $\pi_\epsilon$, $\epsilon \in [0,1]$, possesses a global attractor $\mathcal{A}_\epsilon$. The family of attractors $(\mathcal{A}_\epsilon)_{\epsilon\in[0,1]}$ is upper-semicontinuous at $\epsilon = 0$ with respect to the family $|\cdot|_\epsilon$ of norms, i.e.

$$\lim_{\epsilon\to 0^+} \sup_{u\in\mathcal{A}_\epsilon} \inf_{v\in\mathcal{A}_0} |u-v|_\epsilon = 0.$$

(See [11, Theorem 5.10], which generalizes previous results of [5]).

Notice that each attractor $\mathcal{A}_\epsilon$ for $\epsilon \in [0,1]$ is a compact isolated $\pi_\epsilon$-invariant set with Conley index $\Sigma^0$. Therefore the above theorem implies that the invariant set $\mathcal{A}_0$ can be continued to a family of invariant sets with the same Conley index.

It was proved in our previous paper [1] that the latter statement holds true for an arbitrary isolated invariant set of the limit semiflow:

**Theorem C (Singular Conley index continuation principle).** *Let $K_0 \subset H^1_s(\Omega)$ be a compact isolated invariant set of the limit semiflow $\pi_0$. Then there is an $\epsilon_0 > 0$ and for each $\epsilon \in ]0,\epsilon_0]$ there is a compact isolated invariant set $K_\epsilon \subset H^1(\Omega)$ of the semiflow $\pi_\epsilon$ such that the Conley indices of $K_\epsilon$ and $K_0$ are the same. Moreover, the family $(K_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ is upper-semicontinuous at $\epsilon = 0$ with respect to the family $|\cdot|_\epsilon$ of norms, i.e.*

$$\lim_{\epsilon\to 0^+} \sup_{u\in K_\epsilon} \inf_{v\in K_0} |u-v|_\epsilon = 0.$$

For the precise assumptions on $f$ see [1]. By what was said above, it is clear that this result is not a mere consequence of the regular Continuation Principle described at the beginning of this section. The reader is referred to [1] for some applications of Theorem C to reaction-diffusion equations on thin domains.

For special ordinate set domains $\Omega$ of the form $\Omega = U \times ]0,1[$, where $U$ is a bounded domain in $\mathbb{R}^M$ with $M = 1$ or 2, Theorem C was also independently proved by Q. Huang in the recent preprint [8] (see [8, Theorem 6.1]). The author obtains his result as an application of an abstract Conley index continuation theorem (see [8, Theorem A]).

On the other hand, an inspection of the proof of Theorem C in [1] shows that we use only some general properties of the family $(\pi_\epsilon)_{\epsilon\in[0,1]}$ of semiflows like the restricted convergence of $(\pi_\epsilon)_{\epsilon\in]0,1]}$ to $\pi_0$ and a technical *admissibility* assumption. By carefully abstracting these properties we arrive at a general singular Conley index continuation principle (Theorem 4.1 below), which is the main result of this paper.

Roughly speaking, our admissibility assumption (similarly as that introduced in [13] for the construction of infinite-dimensional Conley index) covers semiflows whose solution operators are conditional $\alpha$-contractions. Therefore Theorem 4.1 is applicable not only to semiflows whose solution operators are compact maps (like the dynamical systems generated by ordinary differential equations in finite dimensions, parabolic partial differential equations on bounded domains or retarded functional differential equations with finite delays) but also to some classes of semiflows (like those generated by damped wave equations, neutral functional differential equations or infinite delay equations) whose solution operators are not compact. Thus, in particular, Theorem 4.1 is more general than Theorem A of [8], since the latter requires compact solution operators.

We illustrate Theorem 4.1 by considering semiflows generated by damped wave equations on thin domains. More precisely, let $\Omega$, $\Omega_\epsilon$, $\gamma$ and $f$ be as above and consider the following damped wave equation on $\Omega_\epsilon$:

$$(\tilde{D}_\epsilon) \qquad \begin{aligned} &u_{tt} = -\nu u_t + \Delta u - \gamma u + f(\epsilon, x, \epsilon y, u), && t > 0,\ (x,y) \in \Omega_\epsilon, \\ &\partial_{\nu_\epsilon} u = 0, && t > 0,\ (x,y) \in \partial\Omega_\epsilon. \end{aligned}$$

Here, $\nu > 0$ is a fixed constant.

Proceeding as above we can transfer equation $(\tilde{D}_\epsilon)$ to the following equivalent equation on the fixed domain $\Omega$:

$$(D_\epsilon) \qquad \begin{aligned} &u_{tt} = -\nu u_t + \Delta_x u + (1/\epsilon^2)\Delta_y u - \gamma u + f(\epsilon, x, y, u), && t > 0,\ (x,y) \in \Omega, \\ &\nu_1 \cdot \nabla_x u + (1/\epsilon^2)\nu_2 \cdot \nabla_y u = 0, && t > 0,\ (x,y) \in \partial\Omega. \end{aligned}$$

Set $X_\epsilon \equiv X := H^1(\Omega) \times L^2(\Omega)$ for $\epsilon > 0$ and $X_0 := H^1_s(\Omega) \times L^2_s(\Omega)$. Given $u \in X_\epsilon$ we will write $u_0$ (resp. $u_1$) for the first (resp. second) component of $u$, i.e. $u = (u_0, u_1)$. For $\epsilon > 0$ endow $X_\epsilon$ with the norm $|\cdot|_{X_\epsilon}$ defined by

$$|(u_0, u_1)|^2_{X_\epsilon} := |u_0|^2_\epsilon + |u_1|^2_{L^2(\Omega)}.$$

For $\epsilon \geq 0$ define $D(B_\epsilon) := D(A_\epsilon) \times H^1(\Omega) \subset X_\epsilon$ and let $B_\epsilon \colon D(B_\epsilon) \to X_\epsilon$ be the operator

$$B_\epsilon(u_0, u_1) := (-u_1, \nu u_1 + A_\epsilon u_0) \text{ for } (u_0, u_1) \in D(B_\epsilon).$$

Moreover, for all $\epsilon \geq 0$ define the map $g_\epsilon \colon X_\epsilon \to X_\epsilon$ by

$$g_\epsilon(u_0, u_1) = (0, f_\epsilon(u_0)) \text{ for } (u_0, u_1) \in X_\epsilon.$$

Then, for all $\epsilon > 0$, equation $(D_\epsilon)$ can be written in the equivalent abstract form

$$\dot{u} = -B_\epsilon u + g_\epsilon(u).$$

This equation defines a (local) semiflow (actually, a flow) $\pi_\epsilon$ on $X$. The limit flow of the family $(\pi_\epsilon)_{\epsilon \in ]0,1]}$ is given by the flow $\pi_0$ generated by the equation

$$\dot{u} = -B_0 u + g_0(u).$$

An application of our abstract Theorem 4.1 now gives the following

**Theorem D (Continuation principle for damped wave equations).** *Let $K_0 \subset X_0$ be a compact isolated invariant set of the limit semiflow $\pi_0$. Then there is an $\epsilon_0 > 0$ and for each $\epsilon \in ]0, \epsilon_0]$ there is a compact isolated invariant set $K_\epsilon \subset X$ of the semiflow $\pi_\epsilon$*

*such that the Conley indices of $K_\epsilon$ and $K_0$ are the same. Moreover, the family $(K_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ is upper-semicontinuous at $\epsilon = 0$ with respect to the family $|\cdot|_{X_\epsilon}$ of norms, i.e.*

$$\lim_{\epsilon\to 0^+} \sup_{u\in K_\epsilon} \inf_{v\in K_0} |u-v|_{X_\epsilon} = 0.$$

For the precise assumptions on $f$, see Examples 5.4 and 5.8 below. Let us only mention, that unlike in Theorem C, we assume, in Theorem D, subcritical growth of $f$ with respect to $u$. If $f$ has critical growth, then, under some additional conditions, the same result obtains. The proof is more difficult and is given in [3].

Let us remark that Theorem 4.1 can also be applied to both reaction-diffusion equations and the corresponding damped wave equations on curved thin domains introduced in the recent paper [10]. Furthermore, it is applicable to various fast-slow systems of ordinary differential equations and evolution equations. In particular, we can establish a continuation principle for singularly perturbed hyperbolic equations, like those studied in [7] (cf. [4]).

This paper is organized as follows. In the next section we review some basic concepts and results of the Conley index theory for semiflows defined on a metric space. In Section 3 we introduce the concepts of singular convergence and singular admissibility which are the key properties in the statement of our abstract continuation result. Section 4 is devoted to the proof of Theorem 4.1. Finally, an application to damped wave equations is presented in Section 5.

## 2. Conley index for semiflows on metric spaces

We will review some basic definitions and results concerning the Conley index theory for semiflows defined on a metric space. The reader is referred to [13] or [14] for the proofs of several of the results stated below and for further details on the subject.

Given a topological space $Z$ and an arbitrary set $Y$, we define the quotient space $Z/Y$ as follows: fix an arbitrary $p \notin Z$. Define the set $Z/Y$ as

$$Z/Y := (Z \setminus Y) \cup \{p\}$$

and the map $q\colon Z \to Z/Y$ as

$$q(z) = \begin{cases} z & \text{if } z \in Z \setminus Y, \\ p & \text{otherwise.} \end{cases}$$

We commonly write $[z]$ instead of $q(z)$. We also write $[Y]$ instead of $p$.

Call a subset $V$ of $Z/Y$ *open in* $Z/Y$ if and only if $q^{-1}(V)$ is open in $Z$. This defines a (quotient) topology on $Z/Y$. Note that if $Y \cap Z \neq \emptyset$, then $q$ is a (surjective) identification map. If $Y \cap Z = \emptyset$, then $Z/Y = Z \cup \{p\}$ and $V \subset Z/Y$ is open in $Z/Y$ if and only if $V \cap Z$ is open in $Z$.

Let us now recall the definition of a semiflow and a few related concepts.

Let $X$ be a topological space, let $D$ be an open subset of $[0, \infty[ \times X$ and $\pi\colon D \to X$ be a continuous map. We write $x\pi t := \pi(t, x)$ for $(t, x) \in D$. The map $\pi$ is called a *local semiflow on* $X$ if the following properties are satisfied:

1. For every $x \in X$ there is a number $\omega_x = \omega_x^\pi \in ]0, \infty]$ such that $(t, x) \in D$ if and only if $0 \leq t < \omega_x$.
2. $x\pi 0 = x$ for all $x \in X$.
3. If $(t, x) \in D$ and $(s, x\pi t) \in D$ then $(t + s, x) \in D$ and

$$x\pi(t + s) = (x\pi t)\pi s.$$

If $\omega_x = \infty$ for every $x \in X$, then $\pi$ is called a *global semiflow on* $X$.

Let $N$ be an arbitrary subset of $X$. We say that the local semiflow $\pi$ *does not explode* in $N$, if whenever $x\pi t \in N$ for all $t \in [0, \omega_x[$ then $\omega_x = \infty$.

Let $J$ be an arbitrary interval in $\mathbb{R}$. A map $\sigma\colon J \to X$ is called a *solution of* $\pi$ if for all $t \in J$ and $s \in [0, \infty[$ for which $t + s \in J$, it follows that $\sigma(t)\pi s$ is defined and $\sigma(t)\pi s = \sigma(t + s)$. If $0 \in J$ and $\sigma(0) = x$, we say that $\sigma$ is a *solution through* $x$. If $J = \mathbb{R}$ (respectively $J = ]-\infty, 0]$), then $\sigma$ is called a *full solution* (respectively *full left solution*) relative to $\pi$.

Suppose that $Y$ is a subset of $X$. We define the following subsets of $X$:

$$\begin{aligned}
A_\pi^+(Y) &:= \{\, x \in X \mid x\pi\,[0, \omega_x[\, \subset Y \,\},\\
A_\pi^-(Y) &:= \{\, x \in X \mid \exists \text{ solution } \sigma\colon\, ]-\infty, 0] \to X \text{ through } x \text{ with } \sigma(]-\infty, 0]) \subset Y \,\},\\
A_\pi(Y) &:= A_\pi^+(Y) \cap A_\pi^-(Y).
\end{aligned}$$

A subset $Y$ of $X$ is called *invariant* (respectively *positively invariant*, respectively *negatively invariant*) relative to $\pi$ if $Y = A_\pi(Y)$ (respectively $Y = A_\pi^+(Y)$, respectively $Y = A_\pi^-(Y)$).

Let $N$ be a closed subset of $X$ such that $K := A_\pi(N)$ is closed and $K \subset \operatorname{Int} N$. Then $N$ is called an *isolating neighborhood of $K$ relative to $\pi$* and $K$ is called an *isolated invariant set relative to $\pi$*.

Let $B \subset X$ be a closed set and $x \in \partial B$. The point $x$ is called a *strict egress* (respectively *strict ingress*, respectively *bounce-off*) of $B$, if for every solution $\sigma\colon\ [-\delta_1, \delta_2] \to X$ through $x$, with $\delta_1 \geq 0$ and $\delta_2 > 0$, the following properties hold:

1. there exists an $\epsilon_2 \in ]0, \delta_2[$ such that $\sigma(t) \notin B$ (respectively $\sigma(t) \in \operatorname{Int} B$, respectively $\sigma(t) \notin B$), for $t \in ]0, \epsilon_2]$;
2. if $\delta_1 > 0$, then there exists an $\epsilon_1 \in ]0, \delta_1[$ such that $\sigma(t) \in \operatorname{Int} B$ (respectively $\sigma(t) \notin B$, respectively $\sigma(t) \notin B$), for $t \in [-\epsilon_1, 0[$.

The set of all strict egress points (respectively strict ingress, respectively bounce-off) of the closed set $B$ will be denoted by $B^e$ (respectively $B^i$, respectively $B^b$). A closed set $B \subset X$ is called an *isolating block*, if $\partial B = B^e \cup B^i \cup B^b$ and $B^- := B^e \cup B^b$ is closed.

Let $N$ be a closed subset of $X$. Then $N$ is called *strongly $\pi$-admissible* if the following properties are satisfied:

1. the local semiflow $\pi$ does not explode in $N$;
2. whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in $X$ and $(t_n)_{n\in\mathbb{N}}$ is a sequence in $[0, \infty[$ such that $t_n \to \infty$ as $n \to \infty$ and $x_n\pi\,[0, t_n] \subset N$ for all $n \in \mathbb{N}$, then the sequence $(x_n\pi t_n)_{n\in\mathbb{N}}$ of endpoints has a convergent subsequence.

**Theorem 2.1.** *Let $K$ be an isolated $\pi$-invariant set and $N$ be a strongly $\pi$-admissible isolating neighborhood of $K$. Then there exists an open set $V$ such that $B := \operatorname{cl} V$ is an isolating block such that $K \subset V \subset B \subset N$ and $\partial V = \partial B$.*

*Proof.* By [14, Theorem I.5.1] there exists an isolating block $B'$ such that $K \subset \operatorname{Int} B' \subset B' \subset N$. Set $V := \operatorname{Int} B'$ and $B := \operatorname{cl} V$. It is easily verified that $B := \operatorname{cl} V$ is an isolating block such that $K \subset V \subset B \subset N$ and $\partial V = \partial B$. □

Let $N$, $Y$ be subsets of $X$ such that $Y \subset N$. The set $Y$ is called *$N$-positively invariant* relative to $\pi$, if whenever $x \in X$, $t \geq 0$ are such that $x\pi\,[0,t] \subset N$ and $x \in Y$, then $x\pi\,[0,t] \subset Y$.

**Definition 2.2.** *Let $N$ be a closed set in $X$ and $N_1$ and $N_2$ be closed subsets of $N$. The pair $(N_1, N_2)$ is called a* pseudo-index pair in $N$ *if:*

1. *$N_1$ and $N_2$ are $N$-positively invariant;*
2. *whenever $x \in N_1$ and $x\pi t_0 \notin N$ for some $t_0 \in [0, \infty[$, then there exists a $t' \in [0, t_0]$ such that $x\pi\,[0,t'] \subset N$ and $x\pi t' \in N_2$.*

*A pseudo-index pair $(N_1, N_2)$ in $N$ is called an* index pair in $N$ *if $A_\pi(N)$ is closed and $A_\pi(N) \subset \operatorname{Int}(N_1 \setminus N_2)$.*

An isolating block $B$ determines a special index pair in $B$, namely $(B, B^-)$.

Let $K$ be an isolated invariant set, $N$ be a strongly $\pi$-admissible isolating neighborhood of $K$ and $(N_1, N_2)$ be an index pair in $N$. Then the homotopy type of the pointed space $(N_1/N_2, [N_2])$ depends only on the semiflow $\pi$ and the isolated invariant set $K$ (see [14, Theorem I.10.1]). The *Conley index*, $h(\pi, K)$, of the isolated invariant set $K$ with respect to $\pi$ is defined to be the homotopy type of $(N_1/N_2, [N_2])$.

**Remark 2.3.** *If $N$ is a strongly $\pi$-admissible isolating neighborhood relative to $\pi$, we will sometimes write $h(\pi, N)$ to denote $h(\pi, A_\pi(N))$. This will not lead to confusion.*

## 3. Singular convergence and singular admissibility

In this section we introduce the basic concepts and preliminary results which will used in the next sections.

Let $(X_0, d_0)$ be a metric space. Let $\epsilon_0 > 0$ and for each $\epsilon \in \,]0, \epsilon_0]$ let $(Y_\epsilon, d_\epsilon)$ be a metric space and $\theta_\epsilon \in Y_\epsilon$ be a distinguished point of $Y_\epsilon$. The open ball in $Y_\epsilon$ of center in $v$ and radius $\beta > 0$ is denoted by $B_\epsilon(v, \beta)$. Analogously, $B_\epsilon[v, \beta]$ is the closed ball in $Y_\epsilon$ of center in $v \in Y_\epsilon$ and radius $\beta > 0$.

For each $\epsilon \in \,]0, \epsilon_0]$ define the set $Z_\epsilon := X_0 \times Y_\epsilon$. Endow $Z_\epsilon$ with the metric

$$\Gamma_\epsilon((u,v),(u',v')) := \max\{\, d_0(u,u'), d_\epsilon(v,v') \,\} \text{ for } (u,v),\ (u',v') \in Z_\epsilon.$$

**Definition 3.1.** *Given a subset $V$ of $X_0$, $\beta > 0$ and $\epsilon \in \,]0, \epsilon_0]$ define the 'inflated' subsets $]V[_{\epsilon,\beta}$ and $[V]_{\epsilon,\beta}$ of $Z_\epsilon$ as follows:*

$$]V[_{\epsilon,\beta} := \{\, (u,v) \in Z_\epsilon \mid u \in V \text{ and } v \in B_\epsilon(\theta_\epsilon, \beta) \,\},$$
$$[V]_{\epsilon,\beta} := \{\, (u,v) \in Z_\epsilon \mid u \in V \text{ and } v \in \operatorname{cl}_\epsilon B_\epsilon(\theta_\epsilon, \beta) \,\}.$$

**Lemma 3.2.** *Let $V$ be a subset of $X_0$. Then for every $\beta > 0$ and $\epsilon \in ]0, \epsilon_0]$*

$$\mathrm{cl}_\epsilon ]V[_{\epsilon,\beta} = [\mathrm{cl}\, V]_{\epsilon,\beta}.$$

*Proof.* Since $[\mathrm{cl}\, V]_{\epsilon,\beta}$ is closed in $Z_\epsilon$, we obtain that $\mathrm{cl}_\epsilon ]V[_{\epsilon,\beta} \subset [\mathrm{cl}\, V]_{\epsilon,\beta}$.

Conversely, let $(u, v) \in [\mathrm{cl}\, V]_{\epsilon,\beta}$. This implies that $u \in \mathrm{cl}\, V$ and $v \in \mathrm{cl}_\epsilon B_\epsilon(\theta_\epsilon, \beta)$. Therefore, there exist sequences $(u_n)_{n\in\mathbb{N}}$ in $V$ and $(v_n)_{n\in\mathbb{N}}$ in $B_\epsilon(\theta_\epsilon, \beta)$ such that

$$d_0(u_n, u) \to 0 \text{ and } d_\epsilon(v_n, v) \to 0 \text{ as } n \to \infty.$$

Thus $(u_n, v_n) \in ]V[_{\epsilon,\beta}$ and $\Gamma_\epsilon((u_n, v_n), (u, v)) \to 0$ as $n \to \infty$. □

Let $\pi_0$ be a local semiflow on $X_0$ and for every $\epsilon \in ]0, \epsilon_0]$ let $\pi_\epsilon$ denote a local semiflow on $Z_\epsilon$. For each $\epsilon \in ]0, \epsilon_0]$ given $w \in Z_\epsilon$ and $t \geq 0$ such that $w\pi_\epsilon t$ is defined, the components of $w\pi_\epsilon t$ are denoted by

$$(w\phi_\epsilon t, w\psi_\epsilon t),$$

where $w\phi_\epsilon t \in X_0$ and $w\psi_\epsilon t \in Y_\epsilon$.

The next two concepts contain the main assumptions required in Theorem 4.1.

**Definition 3.3.** *Under the above notation, we say that the family $(\pi_\epsilon)_{\epsilon\in]0,\epsilon_0]}$ of local semiflows* converges singularly to *the local semiflow $\pi_0$ if whenever $(\epsilon_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are sequences of positive numbers such that*

$$\epsilon_n \to 0, \ t_n \to t_0 \textit{ as } n \to \infty, \textit{ for some } t_0 \in [0, \infty[$$

*and whenever $u_0 \in X_0$ and $w_n \in Z_{\epsilon_n}$ are such that*

$$\Gamma_{\epsilon_n}(w_n, (u_0, \theta_{\epsilon_n})) \to 0 \textit{ as } n \to \infty$$

*and $u_0\pi_0 t_0$ is defined, then there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $w_n\pi_{\epsilon_n} t_n$ is defined and*

$$\Gamma_{\epsilon_n}(w_n\pi_{\epsilon_n} t_n, (u_0\pi_0 t_0, \theta_{\epsilon_n})) \to 0 \textit{ as } n \to \infty.$$

**Definition 3.4.** *Let $\beta$ be a positive number and $N$ be a closed subset of $X_0$. We say that $N$ is a* singularly strongly admissible set with respect to $\beta$ and the family $(\pi_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *of local semiflows if the following conditions are satisfied:*

1. *$N$ is a strongly $\pi_0$-admissible set;*
2. *for each $\epsilon \in ]0, \epsilon_0]$ the set $[N]_{\epsilon,\beta}$ is strongly $\pi_\epsilon$-admissible;*
3. *whenever $(\epsilon_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are sequences of positive numbers such that*

$$\epsilon_n \to 0, \ t_n \to \infty \textit{ as } n \to \infty$$

*and whenever $w_n \in Z_{\epsilon_n}$ is such that $w_n\pi_{\epsilon_n}[0, t_n] \subset [N]_{\epsilon_n,\beta}$, then there exist a $u_0 \in N$ and a subsequence of the sequence $(w_n\pi_{\epsilon_n} t_n)_{n\in\mathbb{N}}$ of endpoints, denoted again by $(w_n\pi_{\epsilon_n} t_n)_{n\in\mathbb{N}}$, such that*

$$\Gamma_{\epsilon_n}(w_n\pi_{\epsilon_n} t_n, (u_0, \theta_{\epsilon_n})) \to 0 \textit{ as } n \to \infty.$$

Basic consequences of the above definitions are contained in the following

**Lemma 3.5.** *Suppose $(\pi_\epsilon)_{\epsilon\in]0,\epsilon_0]}$ is a family of local semiflows that converges singularly to the local semiflow $\pi_0$ and $N$ is a singularly strongly admissible set with respect to $\beta$ and the family $(\pi_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ of local semiflows.*

1. *Let* $(\epsilon_n)_{n\in\mathbb{N}}$ *and* $(t_n)_{n\in\mathbb{N}}$ *be sequences of positive numbers such that*
$$\epsilon_n \to 0,\ t_n \to \infty \text{ as } n \to \infty.$$
*For each* $n \in \mathbb{N}$, *let* $w_n \in Z_{\epsilon_n}$. *Suppose that* $w_n\pi_{\epsilon_n}[0,t_n]$ *is defined and included in* $[N]_{\epsilon_n,\beta}$. *If* $(w_n^0\pi_{\epsilon_n^0}t_n^0)_{n\in\mathbb{N}}$ *is a subsequence of* $(w_n\pi_{\epsilon_n}t_n)_{n\in\mathbb{N}}$ *such that*
$$\Gamma_{\epsilon_n}\left(w_n^0\pi_{\epsilon_n^0}t_n^0,(u_0,\theta_{\epsilon_n^0})\right) \to 0 \text{ as } n\to\infty,$$
*for some* $u_0 \in N$, *then* $u_0 \in A^-_{\pi_0}(N)$.
2. *Let* $(\epsilon_n)_{n\in\mathbb{N}}$ *and* $(t_n)_{n\in\mathbb{N}}$ *be sequences of positive numbers such that* $\epsilon_n \to 0$ *and* $t_n\to\infty$ *as* $n\to\infty$. *Let* $u_0\in X_0$ *and for each* $n\in\mathbb{N}$, *let* $w_n \in Z_{\epsilon_n}$. *Suppose that*
$$\Gamma_{\epsilon_n}(w_n,(u_0,\theta_{\epsilon_n}))\to 0 \text{ as } n\to\infty.$$
*For each* $n\in\mathbb{N}$ *suppose that* $w_n\pi_{\epsilon_n}[0,t_n]$ *is defined and included in* $[N]_{\epsilon_n,\beta}$. *Then* $u_0\pi_0 t$ *is defined for all* $t\geq 0$ *and* $u_0\in A^+_{\pi_0}(N)$.

*Proof.* Condition 3 of Definition 3.4 implies that there exists a subsequence of the sequence $(w_n^0\pi_{\epsilon_n^0}t_n^0)_{n\in\mathbb{N}}$, denoted by $(w_n^1\pi_{\epsilon_n^1}t_n^1)_{n\in\mathbb{N}}$, with $t_n^1\geq 1$, such that
$$\Gamma_{\epsilon_n^1}\left(w_n^1\pi_{\epsilon_n^1}(t_n^1-1),(u_{-1},\theta_{\epsilon_n^1})\right)\to 0 \text{ as } n\to\infty,$$
for some $u_{-1}\in N$. Proceeding recursively, for every $k\geq 1$ we obtain a subsequence $(w_n^k\pi_{\epsilon_n^k}t_n^k)_{n\in\mathbb{N}}$ of $(w_n^{k-1}\pi_{\epsilon_n^{k-1}}t_n^{k-1})_{n\in\mathbb{N}}$ with $t_n^k\geq k$ such that
$$\Gamma_{\epsilon_n^k}\left(w_n^k\pi_{\epsilon_n^k}(t_n^k-k),(u_{-k},\theta_{\epsilon_n^k})\right)\to 0 \text{ as } n\to\infty. \tag{1}$$
Moreover, $u_{-k}\in N$ for all $k\in\mathbb{N}$.

We claim that $u_{-k}\pi_0 t$ is defined for all $t\in[0,k]$. Suppose $u_{-k}\pi_0 t$ is not defined for some $t\in[0,k]$. Then there is a $t_0\in[0,k]$ with $u_{-k}\pi_0t_0$ defined and $u_{-k}\pi_0t_0\notin N$. Since $(\pi_\epsilon)_{\epsilon\in]0,\epsilon_0]}$ is a family of local semiflows that converges singularly to the local semiflow $\pi_0$, formula (1) implies that
$$\Gamma_{\epsilon_n^k}\left(w_n^k\pi_{\epsilon_n^k}(t_n^k-k)\pi_{\epsilon_n^k}t_0,(u_{-k}\pi_0t_0,\theta_{\epsilon_n^k})\right)\to 0 \text{ as } n\to\infty.$$
Notice that $w_n^k\pi_{\epsilon_n^k}(t_n^k-k)\pi_{\epsilon_n^k}t_0=w_n^k\pi_{\epsilon_n^k}(t_n^k-k+t_0)$. Thus
$$d_0\left(w_n^k\phi_{\epsilon_n^k}(t_n^k-k+t_0),u_{-k}\pi_0t_0\right)\to 0 \text{ as } n\to\infty.$$
This implies that $w_n^k\phi_{\epsilon_n^k}(t_n^k-k+t_0)\notin N$ for all $n$ large enough. Since $t_n^k-k+t_0<t_n^k$ we obtain a contradiction. This proves our claim.

For $t\in[-k,0]$ define
$$\sigma_k(t):=u_{-k}\pi_0(t+k).$$
We have that $\sigma_k$ is a solution of $\pi_0$ in $N$. We claim that $\sigma_k(0)=u_0$ for all $k\in\mathbb{N}$. In fact, since $(\pi_\epsilon)_{\epsilon\in]0,\epsilon_0]}$ is a family of local semiflows that converges singularly to the local semiflow $\pi_0$, formula (1) implies that
$$\Gamma_{\epsilon_n^k}\left(w_n^k\pi_{\epsilon_n^k}(t_n^k-k)\pi_{\epsilon_n^k}k,(u_{-k}\pi_0k,\theta_{\epsilon_n^k})\right)\to 0 \text{ as } n\to\infty.$$
Therefore
$$d_0\left(w_n^k\phi_{\epsilon_n^k}t_n^k,u_{-k}\pi_0k\right)\to 0 \text{ as } n\to\infty.$$

On the other, hand since for every $k \in \mathbb{N}$, the sequence $(w_n^k \phi_{\epsilon_n^k} t_n^k)_{n\in\mathbb{N}}$ is a subsequence of $(w_n^0 \pi_{\epsilon_n^0} t_n^0)_{n\in\mathbb{N}}$, we have

$$d_0\left(w_n^k \phi_{\epsilon_n^k} t_n^k, u_0\right) \to 0 \text{ as } n \to \infty.$$

This implies that $\sigma_k(0) = u_0$. This completes the proof of our claim.

Notice that if $k < k'$, then $\sigma_k$ and $\sigma_{k'}$ coincide on $[-k, 0]$. Thus $\sigma(t) := \sigma_k(t)$ for $t \in [-k, 0]$ defines a solution of $\pi_0$ on $]-\infty, 0]$ with $\sigma(]-\infty, 0]) \subset N$ and $\sigma(0) = u_0$. This completes the proof of assertion 1.

We turn to the proof of part 2 of the lemma. Let $0 < \omega \leq \infty$ be such that $u_0\pi_0\tau$ is defined if and only if $\tau \in [0, \omega[$. Let $\tau \in [0, \omega[$. Since $(\pi_\epsilon)_{\epsilon\in]0,\epsilon_0]}$ is a family of local semiflows that converges singularly to the local semiflow $\pi_0$, we have that $w_n\pi_{\epsilon_n}\tau$ is defined for all $n \in \mathbb{N}$ large enough and

$$\Gamma_{\epsilon_n}\left(w_n\pi_{\epsilon_n}\tau, (u_0\pi_0\tau, \theta_{\epsilon_n})\right) \to 0 \text{ as } n \to \infty. \tag{2}$$

Let $t \in [0, \omega[$. Since $t_n \to \infty$ as $n \to \infty$, it follows that $t < t_n$ for all $n$ large enough and so $w_n\phi_{\epsilon_n}[0, t]$ is included in $N$ for all $n$. Therefore, formula (2) implies that $u_0\pi_0[0, t] \subset N$. It follows that $u_0\pi_0[0, \omega[ \subset N$ and so $\omega = \infty$. This concludes the proof of the lemma. □

We will also need the following technical result:

**Lemma 3.6.** *Suppose $(\pi_\epsilon)_{\epsilon\in]0,\epsilon_0]}$ is a family of local semiflows that converges singularly to the local semiflow $\pi_0$ and $N$ is a singularly strongly admissible set with respect to $\beta$ and $(\pi_\epsilon)_{\epsilon\in[0,\epsilon_0]}$. Let $(\epsilon_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be sequences of positive numbers such that*

$$\epsilon_n \to 0 \text{ and } t_n \to \infty \text{ as } n \to \infty.$$

*Moreover let $(w_n)_{n\in\mathbb{N}}$ be a sequence such that $w_n \in Z_{\epsilon_n}$ and*

$$w_n\pi_{\epsilon_n}[0, t_n] \subset [N]_{\epsilon_n,\beta}$$

*for every $n \in \mathbb{N}$. Then there exist a $u_0 \in A_{\pi_0}(N)$ and a subsequence of $(w_n\pi_{\epsilon_n}(t_n/2))_{n\in\mathbb{N}}$, again denoted by $(w_n\pi_{\epsilon_n}(t_n/2))_{n\in\mathbb{N}}$, such that*

$$\Gamma_{\epsilon_n}\left(w_n\pi_{\epsilon_n}(t_n/2), (u_0, \theta_{\epsilon_n})\right) \to 0 \text{ as } n \to \infty.$$

*Proof.* For each $n \in \mathbb{N}$ define $s_n := t_n/2$. It follows that $s_n \to \infty$ as $n \to \infty$ and

$$w_n\pi_{\epsilon_n}[0, s_n] \subset w_n\pi_{\epsilon_n}[0, t_n] \subset [N]_{\epsilon_n,\beta}.$$

Therefore condition 3 of Definition 3.4 implies that there exist a $u_0 \in N$ and a subsequence of $(w_n\pi_{\epsilon_n}s_n)_{n\in\mathbb{N}}$, denoted again by $(w_n\pi_{\epsilon_n}s_n)_{n\in\mathbb{N}}$, such that

$$\Gamma_{\epsilon_n}\left(w_n\pi_{\epsilon_n}s_n, (u_0, \theta_{\epsilon_n})\right) \to 0 \text{ as } n \to \infty.$$

The first part of Lemma 3.5 implies that $u_0 \in A^-_{\pi_0}(N)$. Define

$$y_n := w_n\pi_{\epsilon_n}s_n.$$

Then $\Gamma_{\epsilon_n}(y_n, (u_0, \theta_{\epsilon_n})) \to 0$ as $n \to \infty$. Moreover $y_n\pi_{\epsilon_n}[0, s_n] \subset w_n\pi_{\epsilon_n}[0, t_n]$ the latter set being included in $[N]_{\epsilon_n,\beta}$. Hence Lemma 3.5.2 implies that $u_0 \in A^+_{\pi_0}(N)$. This completes the proof. □

## 4. The singular Conley index continuation principle

In this section we state and prove the main result of this paper, a general Conley index continuation principle for singular perturbation problems:

**Theorem 4.1.** *Suppose that there exists an $\eta_0 > 0$ such that for all $\epsilon \in ]0, \epsilon_0]$ and all $\epsilon \in ]0, \eta_0]$ the set $\mathrm{cl}_\epsilon B_\epsilon(\theta_\epsilon, \eta)$ is contractible.*

*Let $\beta$ be a positive number. Suppose $(\pi_\epsilon)_{\epsilon \in ]0,\epsilon_0]}$ is a family of local semiflows that converges singularly to the local semiflow $\pi_0$ and $N$ is a singularly strongly admissible set with respect to $\beta$ and $(\pi_\epsilon)_{\epsilon \in [0,\epsilon_0]}$. Assume that $N$ is an isolating neighborhood for $\pi_0$.*

*Then for every $\eta \in ]0, \tilde{\eta}_0]$, where $\tilde{\eta}_0 < \min\{\eta_0, \beta\}$, there exists an $\epsilon^c = \epsilon^c(\eta) > 0$ such that for every $\epsilon \in ]0, \epsilon^c]$ the set $[N]_{\epsilon,\eta}$ is a strongly admissible isolating neighborhood relative to $\pi_\epsilon$ and*

$$h(\pi_\epsilon, [N]_{\epsilon,\eta}) = h(\pi_0, N). \tag{3}$$

The proof of this result is a careful abstraction of the Conley index continuation theorem for reaction-diffusion equations on thin domains given in our previous work [1]. We need a number of preliminary results.

**Proposition 4.2.** *Fix $\eta \in ]0, \beta]$ arbitrarily. For all sufficiently small $\epsilon > 0$ the set $[N]_{\epsilon,\eta}$ is a strongly $\pi_\epsilon$-admissible isolating neighborhood for $\pi_\epsilon$.*

*Proof.* Since $\eta \in ]0, \beta]$ the strong admissibility is an immediate consequence of Definition 3.4.

Suppose that there is a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive numbers converging to zero such that for each $n \in \mathbb{N}$ the set $[N]_{\epsilon_n,\eta}$ is not an isolating neighborhood for $\pi_{\epsilon_n}$. Then for each $n \in \mathbb{N}$ there exists a solution $\sigma_n : \mathbb{R} \to Z_{\epsilon_n}$ of $\pi_{\epsilon_n}$ such that

$$\sigma_n(\mathbb{R}) \subset [N]_{\epsilon_n,\eta} \text{ and } \sigma_n(0)\pi_{\epsilon_n} n \in \partial [N]_{\epsilon_n,\eta}.$$

For each $n \in \mathbb{N}$ define $t_n := 2n$. Therefore $t_n \to \infty$ as $n \to \infty$ and $\sigma_n(0)\pi_{\epsilon_n}[0, t_n] \subset [N]_{\epsilon_n,\eta} \subset [N]_{\epsilon_n,\beta}$. Lemma 3.6 implies that there exist a $u_0 \in A_{\pi_0}(N)$ and a subsequence of $(\sigma_n(0)\pi_{\epsilon_n}(t_n/2))_{n\in\mathbb{N}}$ which is denoted again by $(\sigma_n(0)\pi_{\epsilon_n}(t_n/2))_{n\in\mathbb{N}}$ such that

$$\Gamma_{\epsilon_n}(\sigma_n(0)\pi_{\epsilon_n}(t_n/2), (u_0, \theta_{\epsilon_n})) \to 0 \text{ as } n \to \infty.$$

Thus

$$d_0(\sigma_n(0)\phi_{\epsilon_n} n, u_0) \to 0 \text{ and } d_{\epsilon_n}(\sigma_n(0)\psi_{\epsilon_n} n, \theta_{\epsilon_n}) \to 0 \text{ as } n \to \infty.$$

Therefore $d_{\epsilon_n}(\sigma_n(0)\psi_{\epsilon_n} n, \theta_{\epsilon_n}) < \eta$ for all $n$ large enough. Hence $\sigma_n(0)\psi_{\epsilon_n} n \in B_{\epsilon_n}(\theta_{\epsilon_n}, \eta)$ and so $\sigma_n(0)\psi_{\epsilon_n} n \notin \partial\, \mathrm{cl}_{\epsilon_n} B_{\epsilon_n}(\theta_{\epsilon_n}, \eta)$ for all $n$ large enough. This implies that $\sigma_n(0)\phi_{\epsilon_n} n \in \partial N$ for all $n$ large enough and so $u_0 \in \partial N$. However this contradicts the fact that $N$ is an isolating neighborhood for $\pi_0$. □

**Proposition 4.3.** *Assume that $K_0 := A_{\pi_0}(N) = \emptyset$. Fix $\eta \in ]0, \beta]$ arbitrarily. For all sufficiently small $\epsilon > 0$ the set $K_{\epsilon,\eta} := A_{\pi_\epsilon}([N]_{\epsilon,\eta})$ is empty.*

*Proof.* Otherwise, we find an $\eta \in ]0, \beta]$ and a sequence $(\epsilon_n)_{n\in\mathbb{N}}$, of positive numbers converging to zero such that $K_{\epsilon_n,\eta} \neq \emptyset$ for each $n \in \mathbb{N}$. Therefore, for every $n \in \mathbb{N}$ there exists a full solution $\sigma_n : \mathbb{R} \to [N]_{\epsilon_n,\eta}$ of $\pi_{\epsilon_n}$. For each $n \in \mathbb{N}$ define $t_n := n$. Therefore $t_n \to \infty$ as $n \to \infty$ and $\sigma_n(0)\pi_{\epsilon_n}[0, t_n] \subset [N]_{\epsilon_n,\eta} \subset [N]_{\epsilon_n,\beta}$. Lemma 3.6 implies that there exists a $u_0 \in A_{\pi_0}(N)$. However this is a contradiction. □

Thus Theorem 4.1 holds if $K_0 = \emptyset$.

Assume from now on that $K_0 \neq \emptyset$. Since $N$ is a strongly admissible isolating neighborhood for $\pi_0$ and $K_0 = A_{\pi_0}(N) \neq \emptyset$ is an isolated invariant set of $\pi_0$, we can apply Theorem 2.1 and choose an open set $U_0$ in $X_0$ such that $N_0 := \operatorname{cl} U_0$ is an isolating block for $\pi_0$ and such that $K_0 \subset U_0 \subset N_0 \subset N$ and $\partial U_0 = \partial N_0$.

Let $\alpha\colon [0,\infty[ \to [1,2[$ be a monotone increasing $C^\infty$-diffeomorphism. Let $s^+\colon N_0 \to \mathbb{R} \cup \{\infty\}$ be given by

$$s^+(u) := \sup\{\, t \geq 0 \mid u\pi_0 t \text{ is defined and } u\pi_0\,[0,t] \subset N_0\,\},$$

and define the function $t^+\colon U_0 \to \mathbb{R} \cup \{\infty\}$ by

$$t^+(u) := \sup\{\, t \geq 0 \mid u\pi_0 t \text{ is defined and } u\pi_0[0,t] \subset U_0\,\}.$$

Moreover, define the function $F\colon X_0 \to [0,1]$ by

$$F(u) := \min\{\, 1, \mathrm{d}(u, A^-_{\pi_0}(N_0))\,\},$$

where $\mathrm{d}(u, A^-_{\pi_0}(N_0))$ is the distance (in $X_0$) of the point $u$ from the set $A^-_{\pi_0}(N_0)$. Finally, we define $g^-\colon N_0 \to \mathbb{R}$ as follows:

$$g^-(u) := \sup\{\, \alpha(t)F(u\pi_0 t) \mid t \in [0, s^+(u)]\,,\ \text{if } s^+(u) < \infty \text{ and } t \in [0,\infty[\,,\ \text{if } s^+(u) = \infty\,\}.$$

Whenever $(u_n)_{n\in\mathbb{N}}$ is a sequence in $N_0$ with $g^-(u_n) \to 0$ as $n \to \infty$, then, by admissibility, there is a subsequence of $(u_n)_{n\in\mathbb{N}}$ converging to an element of $A^-_{\pi_0}(N_0)$. Given $a > 0$, $b > 0$ define

$$V(a,b) := \{\, u \in U_0 \mid g^-(u) < a,\ t^+(u) > b\,\}.$$

Note that $V(a,b)$ is open in $X_0$, $K_0 \subset V(a,b)$ and admissibility implies that we can choose $a_0 > 0$ and $b_0 > 0$ such that $\operatorname{cl} V(a_0, b_0) \subset U_0$.

Given $a > 0$, $b > 0$, $\epsilon > 0$ and $\alpha > 0$ define

$$V_{\epsilon,\alpha}(a,b) := ]V(a,b)[_{\epsilon,\alpha}\,.$$

**Lemma 4.4.** *Fix positive numbers $a$, $b$, $\alpha$, $\delta$, $\eta$ and $M$ such that $a \leq a_0$, $b \geq b_0$ and $\eta \leq \beta$. Then*

$$K(\epsilon, \eta, a, b) := A_{\pi_\epsilon}([\operatorname{cl} V(a,b)]_{\epsilon,\eta}) \subset V_{\epsilon,\alpha}(\delta, M)$$

*for all sufficiently small $\epsilon > 0$.*

*Proof.* If the lemma is not true then there are positive numbers $a$, $b$, $\alpha$, $\delta$, $\eta$ and $M$ such that $a \leq a_0$, $b \geq b_0$ and $\eta \leq \beta$ and a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive numbers converging to zero such that for all $n \in \mathbb{N}$

$$K(\epsilon_n, \eta, a, b) \not\subset V_{\epsilon_n,\alpha}(\delta, M).$$

For each $n \in \mathbb{N}$, there is a solution $\sigma_n\colon \mathbb{R} \to [\operatorname{cl} V(a,b)]_{\epsilon_n,\eta}$ of $\pi_{\epsilon_n}$ with

$$\sigma_n(0)\pi_{\epsilon_n} n \notin V_{\epsilon_n,\alpha}(\delta, M).$$

Since $\sigma_n(0)\pi_{\epsilon_n}[0,2n] \subset [\operatorname{cl} V(a,b)]_{\epsilon_n,\eta} \subset [N]_{\epsilon_n,\eta} \subset [N]_{\epsilon_n,\beta}$ and $2n \to \infty$ as $n \to \infty$, Lemma 3.6 implies that there exist a $u_0 \in A_{\pi_0}(N)$ and a subsequence of $(\sigma_n(0)\pi_{\epsilon_n} n)_{n\in\mathbb{N}}$ which is denoted again by $(\sigma_n(0)\pi_{\epsilon_n} n)_{n\in\mathbb{N}}$ such that

$$\Gamma_{\epsilon_n}(\sigma_n(0)\pi_{\epsilon_n} n, (u_0, \theta_{\epsilon_n})) \to 0 \text{ as } n \to \infty.$$

This implies that $d_0(\sigma_n(0)\phi_{\epsilon_n}n, u_0) \to 0$ as $n \to \infty$ and $d_{\epsilon_n}(\sigma_n(0)\psi_{\epsilon_n}n, \theta_{\epsilon_n}) < \alpha$ for all $n$ large enough. Since $g^-(u_0) = 0$ and $t^+(u_0) = \infty$, it follows that $u_0 \in V(\delta, M)$. Therefore $\sigma_n(0)\phi_{\epsilon_n}n \in V(\delta, M)$ for all $n$ large enough. Thus $\sigma_n(0)\pi_{\epsilon_n}n \in V_{\epsilon_n,\alpha}(\delta, M)$ for all $n$ large enough which is a contradiction. □

Let $\epsilon \in ]0, \epsilon_0]$ and $\nu > 0$ be arbitrary. Given $w \in ]U_0[_{\epsilon,\nu}$, define

$$t^+_{\epsilon,\nu}(w) := \sup\{\, t \geq 0 \mid w\pi_\epsilon t \text{ is defined and } w\pi_\epsilon[0,t] \subset ]U_0[_{\epsilon,\nu} \,\}.$$

**Lemma 4.5.** *Fix $\nu > 0$. Let $\epsilon_n > 0$, $w_n \in ]U_0[_{\epsilon_n,\nu}$, $n \in \mathbb{N}$, and $u \in U_0$ be such that $\epsilon_n \to 0$ and*

$$\Gamma_{\epsilon_n}(w_n, (u, \theta_{\epsilon_n})) \to 0 \text{ as } n \to \infty.$$

*Then*

$$t^+_{\epsilon_n,\nu}(w_n) \to t^+(u) \text{ as } n \to \infty.$$

*Proof.* Suppose first that $t^+(u) < \infty$ and let $C \in ]0, \infty]$ be arbitrary with $t^+(u) < C$. Since $u\pi_0 t^+(u) \in \partial U_0 = \partial N_0$ and $N_0$ is an isolating block for $\pi_0$, there exists an $s \in \mathbb{R}$ with $C > s > t^+(u)$ such that $u\pi_0 s \notin N_0$. Since $(\pi_\epsilon)_{\epsilon\in]0,\epsilon_0]}$ converges singularly to $\pi_0$, we have

$$\Gamma_{\epsilon_n}(w_n\pi_{\epsilon_n}s, (u\pi_0 s, \theta_{\epsilon_n})) \to 0 \text{ as } n \to \infty.$$

Therefore

$$d_0(w_n\phi_{\epsilon_n}s, u\pi_0 s) \to 0 \text{ as } n \to \infty.$$

Thus $w_n\phi_{\epsilon_n}s \notin N_0$ for all $n$ large enough. Hence $w_n\pi_{\epsilon_n}s \notin [N_0]_{\epsilon_n,\nu}$ and so $t^+_{\epsilon_n,\nu}(w_n) < s < C$ for all $n$ large enough.

Now let $C$ be arbitrary with $t^+(u) > C > 0$. Then $u\pi_0[0, C] \subset U_0$. We claim that $w_n\pi_{\epsilon_n}[0, C] \subset ]U_0[_{\epsilon_n,\nu}$ for all $n$ large enough. Suppose that this is not true. Then we may assume that for every $n \in \mathbb{N}$

$$w_n\pi_{\epsilon_n}[0, C] \not\subset ]U_0[_{\epsilon_n,\nu}$$

and so there exists a $t_n \in [0, C]$ such that

$$w_n\pi_{\epsilon_n}t_n \notin ]U_0[_{\epsilon_n,\nu}\,.$$

Again we can assume that there exists a $t \in [0, C]$ such that $t_n \to t$ as $n \to \infty$. The fact that $(\pi_\epsilon)_{\epsilon\in]0,\epsilon_0]}$ converges singularly to $\pi_0$ implies

$$\Gamma_{\epsilon_n}(w_n\pi_{\epsilon_n}t_n, (u\pi_0 t, \theta_{\epsilon_n})) \to 0 \text{ as } n \to \infty.$$

This implies that

$$d_0(w_n\phi_{\epsilon_n}t_n, u\pi_0 t) \to 0 \text{ and } d_{\epsilon_n}(w_n\psi_{\epsilon_n}t_n, \theta_{\epsilon_n}) \to 0 \text{ as } n \to \infty.$$

Therefore $w_n\phi_{\epsilon_n}t_n \in U_0$ and $d_{\epsilon_n}(w_n\psi_{\epsilon_n}t_n, \theta_{\epsilon_n}) < \nu$ for $n$ all large enough. This implies that $w_n\pi_{\epsilon_n}t_n \in ]U_0[_{\epsilon_n,\nu}$ for all $n$ large enough which is a contradiction. This proves our claim, which, in turn, implies that $t^+_{\epsilon_n,\nu}(w_n) > C$ for all sufficiently large $n$. The lemma is proved. □

**Definition 4.6.** *Fix positive real numbers $M$, $M'$, $\nu$, $\eta$, $\rho$, $a$ and $b$. For all positive numbers $\epsilon \in ]0, \epsilon_0]$, $\alpha$ and $\delta$ define the following subsets of $Z_\epsilon$:*

$$N_1(\alpha, \delta, \epsilon) := [\operatorname{cl} V(a, b)]_{\epsilon,\eta} \cap \operatorname{cl}_\epsilon\{\, w \mid \text{there exist a } \overline{w} \in V_{\epsilon,\alpha}(\delta, M) \text{ and a } t \geq 0 \text{ such that } \overline{w}\pi_\epsilon[0,t] \subset ]U_0[_{\epsilon,\beta} \text{ and } \overline{w}\pi_\epsilon t = w \,\},$$

$$\begin{aligned}
N_2(\alpha,\delta,\epsilon) &:= N_1(\alpha,\delta,\epsilon) \cap \{\, w \in ]U_0[_{\epsilon,\nu} \mid t^+_{\epsilon,\nu}(w) \le M' \,\},\\
\hat{E}(\epsilon) &:= [\{\, u \in U_0 \mid t^+(u) \le 5M \,\} \cap \operatorname{cl} V(a,b)]_{\epsilon,\eta}\,,\\
\hat{C}(\epsilon,\alpha,\delta) &:= \{\, w \in ]U_0[_{\epsilon,\nu} \mid t^+_{\epsilon,\nu}(w) \le 4M \,\} \cap N_1(\alpha,\delta,\epsilon),\\
E(\epsilon,\alpha,\delta) &:= [\{\, u \in U_0 \mid t^+(u) \le 3M \,\} \cap \operatorname{cl} V(\delta,M)]_{\epsilon,\alpha}\,,\\
C(\epsilon,\rho,\alpha,\delta) &:= \{\, w \in ]U_0[_{\epsilon,\alpha} \mid t^+_{\epsilon,\alpha}(w) \le 2M \,\} \cap \operatorname{cl}_\epsilon V_{\epsilon,\rho}(\delta,M).
\end{aligned}$$

**Remark 4.7.** *If $\alpha \le \eta$, $\alpha \le \beta$, $\delta \le a$ and $M \ge b$ then clearly $V_{\epsilon,\alpha}(\delta,M) \subset N_1(\alpha,\delta,\epsilon)$. Moreover, if $a \le a_0$, $b \ge b_0$, $\eta \le \beta$ and $\eta < \nu$ then $[\operatorname{cl} V(a,b)]_{\epsilon,\eta} \subset ]U_0[_{\epsilon,\nu}$ and whenever $w$ lies in $K(\epsilon,\eta,a,b)$, the largest invariant set in $[\operatorname{cl} V(a,b)]_{\epsilon,\eta}$, then, by the definition of $t^+_{\epsilon,\nu}$, we have $t^+_{\epsilon,\nu}(w) = \infty$. In particular, $K(\epsilon,\eta,a,b) \cap N_2(\alpha,\delta,\epsilon) = \emptyset$.*

**Proposition 4.8.** *Assume that $M > b$, $M' > b$, $a \le a_0$, $b \ge b_0$ and $\eta < \nu < \beta$.*

1. *For all sufficiently small positive $\alpha$, $\delta$ and $\epsilon$:*
   (a) *$(N_1(\alpha,\delta,\epsilon), N_2(\alpha,\delta,\epsilon))$ is a pseudo-index pair in $[\operatorname{cl} V(a,b)]_{\epsilon,\eta}$;*
   (b) *the following inclusions are satisfied:*
   $$E(\epsilon,\alpha,\delta) \subset \hat{C}(\epsilon,\alpha,\delta) \subset \hat{E}(\epsilon);$$
2. *For every $\alpha > 0$ and all sufficiently small positive $\rho$, $\delta$ and $\epsilon$*
   $$C(\epsilon,\rho,\alpha,\delta) \subset E(\epsilon,\alpha,\delta).$$

*Proof.* It is clear that $N_1(\alpha,\delta,\epsilon)$ is a closed subset of $Z_\epsilon$. We will prove that $N_2(\alpha,\delta,\epsilon)$ is also a closed subset of $Z_\epsilon$. Let $(w_n)_{n\in\mathbb{N}}$ be a sequence in $N_2(\alpha,\delta,\epsilon)$ such that

$$\Gamma_\epsilon(w_n, w_0) \to \text{ as } n \to \infty.$$

Therefore $w_0 = (u_0, v_0) \in N_1(\alpha,\delta,\epsilon)$. We need to show that

$$w_0 \in ]U_0[_{\epsilon,\nu} \text{ and } t^+_{\epsilon,\nu}(w_0) \le M'.$$

Since $w_0 \in N_1(\alpha,\delta,\epsilon)$, it follows that $w_0 \in [\operatorname{cl} V(a,b)]_{\epsilon,\eta}$. Thus

$$u_0 \in \operatorname{cl} V(a,b) \subset U_0 \text{ and } v_0 \in \operatorname{cl}_\epsilon B_\epsilon(\theta_\epsilon,\eta) \subset B_\epsilon(\theta_\epsilon,\nu).$$

Therefore $w_0 \in ]U_0[_{\epsilon,\nu}$. The lower-semicontinuity of the function $t^+_{\epsilon,\nu}$ (cf. [14, Proposition I.5.2]) implies $t^+_{\epsilon,\nu}(w_0) \le M'$.

The next step is to show that $N_1(\alpha,\delta,\epsilon)$ is $[\operatorname{cl} V(a,b)]_{\epsilon,\eta}$-positively invariant relative to $\pi_\epsilon$. Let $w \in N_1(\alpha,\delta,\epsilon)$ and $s \ge 0$ such that $w\pi_\epsilon[0,s] \subset [\operatorname{cl} V(a,b)]_{\epsilon,\eta}$. Hence there exist a sequence $(w_n)_{n\in\mathbb{N}}$ in $V_{\epsilon,\alpha}(\delta,M)$ and a sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers such that

$$w_n\pi_\epsilon[0,t_n] \subset ]U_0[_{\epsilon,\beta}$$

and $\overline{w}_n := w_n\pi_\epsilon t_n$ is such that

$$\Gamma_\epsilon(\overline{w}_n, w) \to 0 \text{ as } n \to \infty.$$

We claim that $\overline{w}_n\pi_\epsilon[0,s] \subset ]U_0[_{\epsilon,\beta}$ for all $n$ large enough. Since $w\pi_\epsilon[0,s] \subset [\operatorname{cl} V(a,b)]_{\epsilon,\eta}$, it follows that for all $\tau \in [0,s]$

$$w\phi_\epsilon\tau \in \operatorname{cl} V(a,b) \subset U_0 \text{ and } w\psi_\epsilon\tau \in \operatorname{cl}_\epsilon B_\epsilon(\theta_\epsilon,\eta).$$

Since $\eta < \nu < \beta$, it follows that $w\psi_\epsilon\tau \in B_\epsilon(\theta_\epsilon,\beta)$. Thus $w\pi_\epsilon[0,s]$ is a subset of $]U_0[_{\epsilon,\beta}$. Since $\pi_\epsilon$ is continuous, there exists an $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\overline{w}_n\pi_\epsilon[0,s] \subset ]U_0[_{\epsilon,\beta}\,.$$

Therefore

$$w_n\pi_\epsilon [0, t_n + s] \subset ]U_0[_{\epsilon,\beta} \text{ for all } n \geq n_0. \tag{4}$$

Let $\tau \in [0, s]$ be arbitrary. Then

$$\Gamma_\epsilon (w_n\pi_\epsilon(t_n + \tau), w\pi_\epsilon\tau) = \Gamma_\epsilon (\overline{w}_n\pi_\epsilon\tau, w\pi_\epsilon\tau) \to 0 \tag{5}$$

as $n \to \infty$. It follows from (4) and (5) that

$$w\pi_\epsilon\tau \in \operatorname{cl}_\epsilon\{ z \mid \exists \overline{z} \in V_{\epsilon,\alpha}(\delta, M) \text{ and } \exists t \geq 0 \text{ such that } \overline{z}\pi_\epsilon[0, t] \subset ]U_0[_{\epsilon,\beta} \text{ and } \overline{z}\pi_\epsilon t = z \}.$$

Therefore $w\pi_\epsilon [0, s] \subset N_1(\alpha, \delta, \epsilon)$.

To verify that $N_2(\alpha, \delta, \epsilon)$ is $[\operatorname{cl} V(a, b)]_{\epsilon,\eta}$-positively invariant relative to $\pi_\epsilon$, let $w \in N_2(\alpha, \delta, \epsilon)$ and $s \geq 0$ such that $w\pi_\epsilon [0, s] \subset [\operatorname{cl} V(a, b)]_{\epsilon,\eta}$. Therefore

$$w\pi_\epsilon [0, s] \subset N_1(\alpha, \delta, \epsilon).$$

Let $\tau \in [0, s]$ be arbitrary. Then

$$w\phi_\epsilon\tau \in \operatorname{cl} V(a, b) \subset U_0 \text{ and } w\psi_\epsilon\tau \in \operatorname{cl}_\epsilon B_\epsilon(\theta_\epsilon, \eta).$$

Since $\eta < \nu$, it follows that $w\psi_\epsilon\tau \in B_\epsilon(\theta_\epsilon, \nu)$. Thus $w\pi_\epsilon\tau \in ]U_0[_{\epsilon,\nu}$. Since $t^+_{\epsilon,\nu}(w) \leq M'$, we have $t^+_{\epsilon,\nu}(w\pi_\epsilon\tau) \leq M'$. Therefore

$$w\pi_\epsilon [0, s] \subset N_2(\alpha, \delta, \epsilon).$$

To conclude the proof of the first part of proposition, suppose that part 1 does not hold. Then for some choice of the constants $M$, $M'$, $\nu$, $\eta$, $a$ and $b$ as above there are sequences $(\delta_n)_{n\in\mathbb{N}}$, $(\alpha_n)_{n\in\mathbb{N}}$ and $(\epsilon_n)_{n\in\mathbb{N}}$ of positive numbers converging to zero and a sequence $(w_n)_{n\in\mathbb{N}}$, with $w_n = (u_n, v_n)$, such that

$$w_n \in N_1(\alpha_n, \delta_n, \epsilon_n) \cap \partial_{\epsilon_n} [\operatorname{cl} V(a, b)]_{\epsilon_n,\eta} \setminus N_2(\alpha_n, \delta_n, \epsilon_n) \text{ for all } n \in \mathbb{N} \tag{6}$$

or

$$w_n \in \hat{C}(\epsilon_n, \alpha_n, \delta_n) \setminus \hat{E}(\epsilon_n) \text{ for all } n \in \mathbb{N} \tag{7}$$

or

$$w_n \in E(\epsilon_n, \alpha_n, \delta_n) \setminus \hat{C}(\epsilon_n, \alpha_n, \delta_n) \text{ for all } n \in \mathbb{N}. \tag{8}$$

In the first two cases for each $n \in \mathbb{N}$ we have $w_n \in N_1(\alpha_n, \delta_n, \epsilon_n)$, so there exist a $\overline{w}_n = (\overline{u}_n, \overline{v}_n) \in V_{\epsilon_n,\alpha_n}(\delta_n, M)$ and a $t_n \geq 0$ such that

$$\overline{w}_n\pi_{\epsilon_n} [0, t_n] \subset ]U_0[_{\epsilon_n,\beta}$$

and $\tilde{w}_n := \overline{w}_n\pi_{\epsilon_n}t_n$ is such that

$$\Gamma_{\epsilon_n} (\tilde{w}_n, w_n) < 2^{-n}.$$

Since $\overline{u}_n \in V(\delta_n, M)$, it follows that $g^-(\overline{u}_n) < \delta_n \to 0$, we may assume, taking subsequence if necessary, that $d_0(\overline{u}_n, u_0) \to 0$ for some $u_0 \in A^-_{\pi_0}(N_0)$. Since $d_{\epsilon_n}(\overline{v}_n, \theta_{\epsilon_n}) \leq \alpha_n \to 0$ as $n \to \infty$ we have

$$\Gamma_{\epsilon_n}(\overline{w}_n, (u_0, \theta_{\epsilon_n})) \to 0 \text{ as } n \to \infty. \tag{9}$$

We claim that there is a subsequence of $(\overline{w}_n\pi_{\epsilon_n}t_n)_{n\in\mathbb{N}}$ which we will denote again by $(\overline{w}_n\pi_{\epsilon_n}t_n)_{n\in\mathbb{N}}$ and there is a $\tilde{u}_0 \in A^-_{\pi_0}(N_0)$ such that

$$\Gamma_{\epsilon_n}(\overline{w}_n\pi_{\epsilon_n}t_n, (\tilde{u}_0, \theta_{\epsilon_n})) \to 0 \text{ as } n \to \infty.$$

First assume that the sequence $(t_n)_{n\in\mathbb{N}}$ is bounded. By taking subsequence, if necessary, we may assume that there exists a $t \in [0,\infty[$ such that $t_n \to t$ as $n \to \infty$. Since $(\pi_\epsilon)_{\epsilon\in]0,\epsilon_0]}$ converges singularly to $\pi_0$, formula (9) implies that

$$\Gamma_{\epsilon_n}(\overline{w}_n\pi_{\epsilon_n}t_n, (u_0\pi_0 t, \theta_{\epsilon_n})) \to 0 \text{ as } n \to \infty.$$

Since $A^-_{\pi_0}(N_0)$ is an $N_0$-positively invariant set relative to $\pi_0$, it follows that

$$\tilde{u}_0 := u_0\pi_0 t \in A^-_{\pi_0}(N_0).$$

Suppose that $(t_n)_{n\in\mathbb{N}}$ is an unbounded sequence. Then we may assume that $t_n \to \infty$ as $n \to \infty$. Recall that $\overline{w}_n\pi_{\epsilon_n}[0,t_n] \subset ]U_0[_{\epsilon_n,\beta} \subset [N_0]_{\epsilon_n,\beta} \subset [N]_{\epsilon_n,\beta}$. Since $N$ is singularly strongly admissible with respect to $\beta$ and $(\pi_\epsilon)_{\epsilon\in[0,\epsilon_0]}$, condition 3 of Definition 3.4 implies that there exist a subsequence of $(\overline{w}_n\pi_{\epsilon_n}t_n)_{n\in\mathbb{N}}$, denoted again by $(\overline{w}_n\pi_{\epsilon_n}t_n)_{n\in\mathbb{N}}$, and there is a $\tilde{u}_0 \in N$ such that

$$\Gamma_{\epsilon_n}(\overline{w}_n\pi_{\epsilon_n}t_n, (\tilde{u}_0, \theta_{\epsilon_n})) \to 0 \text{ as } n \to \infty.$$

Lemma 3.5.1 implies that $\tilde{u}_0 \in A^-_{\pi_0}(N_0)$. This concludes the proof of our claim. It follows that

$$\Gamma_{\epsilon_n}(w_n, (\tilde{u}_0, \theta_{\epsilon_n})) \to 0 \text{ as } n \to \infty.$$

Thus

$$d_{\epsilon_n}(v_n, \theta_{\epsilon_n}) < \eta \text{ and } d_{\epsilon_n}(v_n, \theta_{\epsilon_n}) < \nu \text{ for all } n \text{ large enough.} \tag{10}$$

Since $w_n \in [\operatorname{cl} V(a,b)]_{\epsilon_n,\eta}$, we also have $u_n \in \operatorname{cl} V(a,b) \subset U_0$. Thus $w_n \in ]U_0[_{\epsilon_n,\nu}$ for all $n$ large enough and so Lemma 4.5 and the continuity of of $t^+$ imply that $t^+_{\epsilon_n,\nu}(w_n) \to t^+(\tilde{u}_0)$ and $t^+(u_n) \to t^+(\tilde{u}_0)$ as $n \to \infty$.

Suppose that (6) holds. Then relations (6) and (10) imply that $u_n \in \partial \operatorname{cl} V(a,b) \subset U_0$, for all $n$ large enough and so $\tilde{u}_0 \in \partial \operatorname{cl} V(a,b) \subset U_0$. Since $t^+_{\epsilon_n,\nu}(w_n) > M'$ for all $n$ we also conclude that $t^+(\tilde{u}_0) \geq M' > b$. Since $g^-(\tilde{u}_0) = 0 < a$, we see that $\tilde{u}_0 \in V(a,b)$. However $V(a,b) \cap \partial \operatorname{cl} V(a,b) = \emptyset$. This contradiction proves part 1(a).

Suppose now that (7) holds. Then $t^+_{\epsilon_n,\nu}(w_n) \leq 4M$ for all $n$ and so $t^+(\tilde{u}_0) \leq 4M < 5M$. We thus conclude that $t^+(u_n) < 5M$ for all $n$ large enough. Since $u_n \in \operatorname{cl} V(a,b)$ and $d_{\epsilon_n}(v_n, \theta_{\epsilon_n}) < \eta$ for all $n$ large enough, all this clearly implies that $w_n \in \hat{E}(\epsilon_n)$ for all $n$ large enough, a contradiction proving the second inclusion in 1(b).

Finally assume that (8) holds. Therefore for all $n \in \mathbb{N}$

$$u_n \in \{\, u \in U_0 \mid t^+(u) \leq 3M \,\} \cap \operatorname{cl} V(\delta_n, M) \text{ and } v_n \in \operatorname{cl}_{\epsilon_n} B_{\epsilon_n}(\theta_{\epsilon_n}, \alpha_n). \tag{11}$$

Hence there exists a $\overline{u}_n \in V(\delta_n, M)$ such that $d_0(u_n, \overline{u}_n) < 2^{-n}$. Since $g^-(\overline{u}_n) \to 0$, we may again assume that there is a $u_0 \in A^-_{\pi_0}(N_0)$ such that $d_0(\overline{u}_n, u_0) \to 0$ as $n \to \infty$. Notice that $v_n \in B_{\epsilon_n}[\theta_{\epsilon_n}, \alpha_n]$. Since $\alpha_n \to 0$ as $n \to \infty$, we have

$$\Gamma_{\epsilon_n}(w_n, (u_0, \theta_{\epsilon_n})) \to 0 \text{ as } n \to \infty.$$

For all $n$ large enough we have $g^-(\overline{u}_n) < \delta_n < a$ and $t^+(\overline{u}_n) > M > b$ so $\overline{u}_n \in V(a,b)$. This implies that $u_0 \in \operatorname{cl} V(a,b) \subset U_0$. Since $t^+(u_n) \leq 3M$, the continuity of $t^+$ implies that $t^+(u_0) \leq 3M$. Notice that $u_n \in U_0$ and $d_{\epsilon_n}(v_n, \theta_{\epsilon_n}) < \nu$ for all $n$ large enough. Therefore $w_n \in ]U_0[_{\epsilon_n,\nu}$ for all $n$ large enough. Lemma 4.5 now shows that $t^+_{\epsilon_n,\nu}(w_n) \to t^+(u_0)$ as $n \to \infty$. Hence we can assume that for every $n$ large enough, $t^+_{\epsilon_n,\nu}(w_n) \leq 4M$.

Since $N_1(\alpha_n, \delta_n, \epsilon_n)$ is closed, we obtain from Remark 4.7 and Lemma 3.2 that, for all $n$ large enough,

$$w_n \in [\operatorname{cl} V(\delta_n, M)]_{\epsilon_n,\alpha_n} = \operatorname{cl}_{\epsilon_n} ]V(\delta_n, M)[_{\epsilon_n,\alpha_n} = \operatorname{cl}_{\epsilon_n} V_{\epsilon_n,\alpha_n}(\delta_n, M) \subset N_1(\alpha_n, \delta_n, \epsilon_n).$$

This implies that $w_n \in \hat{C}(\epsilon_n, \alpha_n, \delta_n)$. This contradiction completes the proof of part 1 of the proposition.

If the second part of the proposition is not true then there exist numbers $M > b$, $\alpha > 0$ and sequences $(\delta_n)_{n\in\mathbb{N}}$, $(\rho_n)_{n\in\mathbb{N}}$ and $(\epsilon_n)_{n\in\mathbb{N}}$ of positive numbers converging to zero and there is a sequence $(w_n)_{n\in\mathbb{N}}$, with $w_n = (u_n, v_n)$, such that

$$w_n \in C(\epsilon_n, \rho_n, \alpha, \delta_n) \text{ and } w_n \notin E(\epsilon_n, \alpha, \delta_n).$$

Therefore for every $n \in \mathbb{N}$ we have $w_n \in ]U_0[_{\epsilon_n,\alpha}$, $t^+_{\epsilon_n,\alpha}(w_n) \leq 2M$ and there exists a $\overline{w}_n = (\overline{u}_n, \overline{v}_n) \in V_{\epsilon_n,\rho_n}(\delta_n, M)$ such that

$$\Gamma_{\epsilon_n}(\overline{w}_n, w_n) < 2^{-n}.$$

Therefore $\overline{u}_n \in V(\delta_n, M)$. Hence for every $n \in \mathbb{N}$,

$$g^-(\overline{u}_n) < \delta_n \text{ and } t^+(\overline{u}_n) > M$$

and so there exists a subsequence of $(\overline{u}_n)_{n\in\mathbb{N}}$, denoted again by $(\overline{u}_n)_{n\in\mathbb{N}}$, and $u_0 \in A^-_{\pi_0}(N_0)$ such that $d_0(\overline{u}_n, u_0) \to 0$ as $n \to \infty$. Since $d_{\epsilon_n}(\overline{v}_n, \theta_{\epsilon_n}) < \rho_n \to 0$, we have

$$\Gamma_{\epsilon_n}(\overline{w}_n, (u_0, \theta_{\epsilon_n})) \to 0 \text{ as } n \to \infty.$$

For all $n$ large enough $g^-(\overline{u}_n) < \delta_n < a$. Therefore $\overline{u}_n \in V(a, b)$ for all large enough. This implies $u_0 \in \operatorname{cl} V(a, b) \subset U_0$. Since $w_n \in ]U_0[_{\epsilon_n,\alpha}$ and $\Gamma_{\epsilon_n}(w_n, (u_0, \theta_{\epsilon_n})) \to 0$ as $n \to \infty$, Lemma 4.5 implies that $t^+_{\epsilon_n,\alpha}(w_n) \to t^+(u_0)$ as $n \to \infty$. Therefore $t^+(u_0) \leq 2M$. This inequality and the continuity of $t^+$ imply that $t^+(u_n) \leq 3M$ for all $n$ large enough. Lemma 3.2 implies that

$$w_n \in \operatorname{cl}_{\epsilon_n} V_{\epsilon_n,\rho_n}(\delta_n, M) = [\operatorname{cl} V(\delta_n, M)]_{\epsilon_n,\rho_n}.$$

Therefore

$$u_n \in \operatorname{cl} V(\delta_n, M) \text{ and } v_n \in \operatorname{cl}_{\epsilon_n} B_{\epsilon_n}(\theta_{\epsilon_n}, \rho_n) \subset B_{\epsilon_n}[\theta_{\epsilon_n}, \rho_n].$$

Since $\rho_n \to 0$ as $n \to \infty$, it follows that for all $n$ large enough,

$$u_n \in \operatorname{cl} V(\delta_n, M) \text{ and } d_{\epsilon_n}(v_n, \theta_{\epsilon_n}) < \alpha.$$

Therefore we have shown that $w_n \in E(\epsilon_n, \alpha, \delta_n)$ for all $n$ large enough, a contradiction. The proposition is proved. □

**Lemma 4.9.** *Let $N_1$ and $N_2$ be closed subsets in $X_0$ such that $N_2 \subset N_1$.*

*Let $\iota_\epsilon \colon N_1/N_2 \to [N_1]_{\epsilon,\eta} / [N_2]_{\epsilon,\eta}$ be the map induced by the inclusion $u \mapsto (u, \theta_\epsilon)$ and $T_\epsilon \colon [N_1]_{\epsilon,\eta} / [N_2]_{\epsilon,\eta} \to N_1/N_2$ be the map induced by the projection onto the first factor.*

*Then for every $\epsilon \in ]0, \epsilon_0]$ and $\eta \in ]0, \eta_0]$ the maps $\iota_\epsilon$ and $T_\epsilon$ are homotopy inverses to each other in the category of pointed spaces. In particular, the pointed spaces $N_1/N_2$ and $[N_1]_{\epsilon,\eta} / [N_2]_{\epsilon,\eta}$ have the same homotopy type.*

*Proof.* The hypothesis implies that for each $\epsilon \in ]0, \epsilon_0]$ and for each $\eta \in ]0, \eta_0]$ there exists a continuous map $h_{\epsilon,\eta} \colon \operatorname{cl}_\epsilon B_\epsilon(\theta_\epsilon, \eta) \times [0, 1] \to \operatorname{cl}_\epsilon B_\epsilon(\theta_\epsilon, \eta)$ such that

$$h_{\epsilon,\eta}(v, 0) = v \text{ and } h_{\epsilon,\eta}(v, 1) = \theta_\epsilon.$$

Let $H_{\epsilon,\eta}\colon [N_1]_{\epsilon,\eta} / [N_2]_{\epsilon,\eta} \times [0,1] \to [N_1]_{\epsilon,\eta} / [N_2]_{\epsilon,\eta}$ be the map induced by the function

$$K_{\epsilon,\eta}\colon [N_1]_{\epsilon,\eta} \times [0,1] \to [N_1]_{\epsilon,\eta}, \ ((u,v),t) \mapsto (u, h_{\epsilon,\eta}(v,t)).$$

It is clear that $H_{\epsilon,\eta}$ is a homotopy between $I_{[N_1]_{\epsilon,\eta}/[N_2]_{\epsilon,\eta}}$ and $\iota_\epsilon \circ T_\epsilon$. Moreover, it is clear that $T_\epsilon \circ \iota_\epsilon = I_{N_1/N_2}$. This concludes the proof. □

Now we are able to complete the proof of Theorem 4.1.

Choose positive numbers $\nu$ and $M$ with $M > b_0$ and $\eta < \nu < \beta$. Moreover, choose positive numbers $\alpha' < \eta$, $\delta' < a_0$ and $\epsilon' \leq \epsilon_0$ such that for all positive $\alpha \leq \alpha'$, $\delta \leq \delta'$ and $\epsilon \leq \epsilon'$ all assertions of part 1 of Proposition 4.8 hold, where we fix $a := a_0$, $b := b_0$ and $M' := 4M$. Now choose positive numbers $\rho' < \alpha'$, $\delta'' \leq \delta'$ and $\epsilon'' \leq \epsilon'$ such that for all positive $\rho \leq \rho'$, $\delta \leq \delta''$ and $\epsilon \leq \epsilon''$ part 2 of Proposition 4.8 holds, where we fix $\alpha := \alpha'$.

Notice also that $2M > b_0$, $\rho' < \alpha'$ and $\alpha' < \beta$ ($\alpha' < \eta < \nu < \beta$). Therefore we are able to apply Proposition 4.8 with $M$ and $M'$ replaced by $2M$, $\eta := \rho'$, $\nu := \alpha'$, $a := \delta''$, $b := M$. Thus we obtain positive numbers $\alpha''' \leq \rho'$, $\delta''' < \delta''$ and $\epsilon''' \leq \epsilon''$ such that for all positive $\alpha \leq \alpha'''$, $\delta \leq \delta'''$ and $\epsilon \leq \epsilon'''$ the pair $(\tilde{N}_1(\alpha,\delta,\epsilon), \tilde{N}_2(\alpha,\delta,\epsilon))$ is a pseudo-index pair in $[\operatorname{cl} V(\delta'', M)]_{\epsilon,\rho'}$.

Here,

$$\tilde{N}_1(\alpha,\delta,\epsilon) = [\operatorname{cl} V(\delta'', M)]_{\epsilon,\rho'} \cap \operatorname{cl}_\epsilon\{\, w \mid \text{there exists a } \overline{w} \in V_{\epsilon,\alpha}(\delta, 2M) \text{ and a } t \geq 0 \text{ such that } \overline{w}\pi_\epsilon[0,t] \subset\, ]U_0[_{\epsilon,\beta} \text{ and } \overline{w}\pi_\epsilon t = w \,\}$$

and

$$\tilde{N}_2(\alpha,\delta,\epsilon) = \tilde{N}_1(\alpha,\delta,\epsilon) \cap \{\, w \in\, ]U_0[_{\epsilon,\alpha'} \mid t^+_{\epsilon,\alpha'}(w) \leq 2M \,\}.$$

Fix $\alpha := \alpha'''$, $\delta := \delta'''$ and write $\rho := \rho'$. We now conclude that for $\epsilon \in\, ]0, \epsilon'''[$

$$\begin{aligned} A_1 := \tilde{N}_1(\alpha,\delta,\epsilon) &\subset [\operatorname{cl} V(\delta'', M)]_{\epsilon,\rho} \subset A_2 := [\operatorname{cl} V(\delta'', M)]_{\epsilon,\alpha'} \\ &\subset A_3 := N_1(\alpha', \delta'', \epsilon) \subset A_4 := [\operatorname{cl} V(a_0, b_0)]_{\epsilon,\eta} \end{aligned}$$

and

$$\begin{aligned} B_1 := \tilde{N}_2(\alpha,\delta,\epsilon) &\subset [\operatorname{cl} V(\delta'', M)]_{\epsilon,\rho} \cap \{\, w \in\, ]U_0[_{\epsilon,\alpha'} \mid t^+_{\epsilon,\alpha'}(w) \leq 2M \,\} \\ &= C(\epsilon, \rho, \alpha', \delta'') \subset B_2 := E(\epsilon, \alpha', \delta'') \\ &\subset B_3 := \hat{C}(\epsilon, \alpha', \delta'') = N_2(\alpha', \delta'', \epsilon) \\ &\subset B_4 := \hat{E}(\epsilon). \end{aligned}$$

Notice that for each $i = 1,2,3,4$, $B_i \subset A_i$. Therefore we can consider the pointed spaces $A_i/B_i$. The two sequences of inclusions described above induce continuous maps

$$\Lambda_1\colon A_1/B_1 \to A_2/B_2, \ \Lambda_2\colon A_2/B_2 \to A_3/B_3, \text{ and } \Lambda_3\colon A_3/B_3 \to A_4/B_4.$$

Using Lemma 4.4 let us now choose a positive number $\epsilon^c \leq \epsilon'''$ such that

$$K(\epsilon, \eta, a_0, b_0) \subset V_{\epsilon,\alpha'}(\delta'', M) \cap V_{\epsilon,\alpha}(\delta, 2M)$$

for $\epsilon \in\, ]0, \epsilon^c]$. Then Remark 4.7 implies that $(A_1, B_1)$ and $(A_3, B_3)$ are index pairs for $K(\epsilon, \eta, a_0, b_0)$ and it follows from [14, Theorem I.9.4] that $\Lambda_2 \circ \Lambda_1$ is an isomorphism in the homotopy category of pointed spaces. The proof of Theorem 2.3 in [14] implies that

$$(\operatorname{cl} V(\delta'', M), \operatorname{cl} V(\delta'', M) \cap \{\, u \in U_0 \mid t^+(u) \leq 3M \,\})$$

and

$$(\operatorname{cl} V(a_0, b_0), \operatorname{cl} V(a_0, b_0) \cap \{\, u \in U_0 \mid t^+(u) \leq 5M \,\})$$

are index pairs for $K_0$. Since $\alpha' < \eta < \eta_0$ we conclude from Lemma 4.9 that there exist inclusion induced maps

$$\Lambda_4 \colon \operatorname{cl} V(\delta'', M)/(\operatorname{cl} V(\delta'', M) \cap \{\, u \in U_0 \mid t^+(u) \leq 3M \,\}) \to A_2/B_2$$

and

$$\Lambda_5 \colon \operatorname{cl} V(a_0, b_0)/(\operatorname{cl} V(a_0, b_0) \cap \{\, u \in U_0 \mid t^+(u) \leq 5M \,\}) \to A_4/B_4$$

which are isomorphism in the homotopy category of pointed spaces. Therefore, [14, Theorem I.9.4] allows us to conclude that $\Lambda_5^{-1} \circ \Lambda_3 \circ \Lambda_2 \circ \Lambda_4$ is also an isomorphism in the same category. Consequently $\Lambda_3 \circ \Lambda_2$ is an isomorphism. Now [14, Lemma I.12.4] implies that $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$ are all isomorphisms in the homotopy category of pointed spaces. We conclude that (3) also holds for the case $K_0 \neq \emptyset$. The proof of Theorem 4.1 is complete. □

Theorem 4.1 yields the following corollary:

**Corollary 4.10.** *For every $\eta \in \,]0, \tilde{\eta}_0]$, the family $(K_{\epsilon,\eta})_{\epsilon \in [0, \epsilon^c(\eta)]}$ of invariant sets, where $K_{\epsilon,\eta} := A_{\pi_\epsilon}([N]_{\epsilon,\eta})$, $\epsilon \in \,]0, \epsilon^c(\eta)]$, and $K_{0,\eta} = K_0 := A_{\pi_0}(N)$, is upper semicontinuous at $\epsilon = 0$ with respect to the family $(\Gamma_\epsilon)_{\epsilon \in [0, \epsilon^c(\eta)]}$ of metrics in the following sense:*

$$\lim_{\epsilon \to 0^+} \sup_{w \in K_{\epsilon,\eta}} \inf_{u \in K_0} \Gamma_\epsilon(w, (u, \theta_\epsilon)) = 0.$$

*Proof.* If the corollary is not true, then there are numbers $\eta \in \,]0, \tilde{\eta}_0]$ and $\varrho > 0$, a sequence $\epsilon_n \to 0^+$ and a sequence $w_n \in K_{\epsilon_n,\eta}$, $n \in \mathbb{N}$, such that

$$\inf_{u \in K_0} \Gamma_{\epsilon_n}(w_n, (u, \theta_{\epsilon_n})) > \varrho.$$

For every $n \in \mathbb{N}$ let $\sigma_n \colon \mathbb{R} \to [N]_{\epsilon_n,\eta}$ be a solution of $\pi_{\epsilon_n}$ with $\sigma_n(0)\pi_{\epsilon_n} n = w_n$. For each $n \in \mathbb{N}$ define $t_n := 2n$. Then $t_n \to \infty$ as $n \to \infty$ and $\sigma_n(0)\pi_{\epsilon_n}[0, t_n] \subset [N]_{\epsilon_n,\eta} \subset [N]_{\epsilon_n,\beta}$. Lemma 3.6 implies that there exist a $u_0 \in A_{\pi_0}(N)$ and a subsequence of $(\sigma_n(0)\pi_{\epsilon_n} n)_{n \in \mathbb{N}}$ which is denoted again by $(\sigma_n(0)\pi_{\epsilon_n} n)_{n \in \mathbb{N}}$ such that

$$\Gamma_{\epsilon_n}(\sigma_n(0)\pi_{\epsilon_n} n, (u_0, \theta_{\epsilon_n})) \to 0 \text{ as } n \to \infty.$$

This is a contradiction, which proves the corollary. □

## 5. Damped wave equations

In this section we will apply Theorem 4.1 to obtain a singular continuation principle for damped wave equations on thin domains. Damped wave equations, unlike parabolic equations, give rise to noncompactifying flows on a Banach space. This section is not self-contained, in particular, we use some results established in the recent paper [2]. For additional information on this subject the reader is referred to the seminal works [16, 9, 6].

Suppose $H$ is a linear space which is complete with respect to the scalar product $\langle \cdot, \cdot \rangle_H$ and let $A \colon D(A) \subset H \to H$ be a (densely defined) positive self-adjoint operator on $(H, \langle \cdot, \cdot \rangle_H)$ with $A^{-1} \colon H \to H$ compact. Let $\lambda_1 := \lambda_1(A) > 0$ be the first eigenvalue of $A$ and $H_1(A)$ be the linear space $D(A^{1/2})$. Endow $H_1(A)$ with the (complete) scalar product

$$\langle u, v \rangle_{H_1(A)} := \langle A^{1/2} u, A^{1/2} v \rangle_H.$$

Set $X(A) := H_1(A) \times H$. Given $u \in X(A)$ we will write $u_0$ (resp. $u_1$) for the first (resp. second) component of $u$, i.e. $u = (u_0, u_1)$. Endow $X(A)$ with the (complete) scalar product

$$\langle (u_0, u_1), (v_0, v_1) \rangle_{X(A)} := \langle u_0, v_0 \rangle_{H_1(A)} + \langle u_1, v_1 \rangle_H.$$

Write $|\cdot|_H$, $|\cdot|_{H_1(A)}$ and $|\cdot|_{X(A)}$ for the corresponding Hilbert space norms.

Let $\nu > 0$. Define $D(B_{\nu,A}) := D(A) \times H_1(A) \subset X(A)$ and let $B_{\nu,A} \colon D(B_{\nu,A}) \to X(A)$ be the operator

$$B_{\nu,A}(u_0, u_1) := (-u_1, \nu u_1 + A u_0) \text{ for } (u_0, u_1) \in D(B_{\nu,A}).$$

Now suppose that $f \colon H_1(A) \to H$ is a map which is Lipschitzian on bounded subsets of $H_1(A)$. Define the map $g = g_{A,f} \colon X(A) \to X(A)$ by

$$g_{A,f}(u_0, u_1) = (0, f(u_0)) \text{ for } (u_0, u_1) \in X(A).$$

It follows that $g_{A,f}$ is Lipschitzian on bounded subsets of $X(A)$. Consequently there exists a well-defined local semiflow (actually a local flow) $\pi_{\nu,A,f}$ generated by the mild solutions of the following *abstract damped wave equation*

$$\dot{u} = -B_{\nu,A} u + g_{A,f}(u).$$

Moreover, $\pi_{\nu,A,f}$ does not explode in bounded subsets of $X(A)$. This follows, e.g. from [15, Theorem 1] and its proof.

Next we will describe the precise set of assumptions which will allow us to obtain a continuation principle for certain 'singular' families of abstract damped wave equations. This, in turn, will imply a continuation principle for damped wave equations on thin domains. We start with the following

**Definition 5.1.** *Given a number $\epsilon_0 > 0$ we say that the families $(H^\epsilon, \langle \cdot, \cdot \rangle_{H^\epsilon})_{\epsilon \in [0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon \in [0,\epsilon_0]}$* satisfy condition (Spec) *if the following properties are satisfied:*

1. *for every $\epsilon \in [0, \epsilon_0]$, $(H^\epsilon, \langle \cdot, \cdot \rangle_{H^\epsilon})$ is a Hilbert space and $A_\epsilon \colon D(A_\epsilon) \subset H^\epsilon \to H^\epsilon$ is a densely defined positive self-adjoint operator on $(H^\epsilon, \langle \cdot, \cdot \rangle_{H^\epsilon})$ with ${A_\epsilon}^{-1} \colon H^\epsilon \to H^\epsilon$ compact. Write $H_1^\epsilon := H_1(A_\epsilon)$ and $X_\epsilon := X(A_\epsilon)$;*
2. *the linear spaces $H := H^\epsilon$ and $H_1 := H_1^\epsilon$ are independent of $\epsilon$ for $\epsilon \in ]0, \epsilon_0]$, $H^0$ is a linear subspace of $H$ and $H_1^0$ is a linear subspace of $H_1$;*
3. *there exists a constant $C \in ]0, \infty[$ such that*
$$|u|_{X_\epsilon} \le C |u|_{X_0} \text{ and } |u|_{X_0} \le C |u|_{X_\epsilon}$$
*for all $u \in X_0$ and all $\epsilon \in ]0, \epsilon_0]$;*
4. *for every $\epsilon \in ]0, \epsilon_0]$ let $(\lambda_{\epsilon,j})_{j \in \mathbb{N}}$ be the repeated sequence of eigenvalues of $A_\epsilon$ and $(w_{\epsilon,j})_{j \in \mathbb{N}}$ be a corresponding $H^\epsilon$-orthonormal sequence of eigenfunctions. Furthermore, let $(\lambda_{0,j})_{j \in \mathbb{N}}$ be the repeated sequence of eigenvalues of $A_0$.*
   *Whenever $(\epsilon_n)_{n \in \mathbb{N}}$ is a sequence of positive numbers converging to zero then*
   (a) *for every $j \in \mathbb{N}$,*
   $$\lambda_{\epsilon_n,j} \to \lambda_{0,j} \text{ as } n \to \infty;$$
   (b) *there is a subsequence of $(\epsilon_n)_{n \in \mathbb{N}}$, again denoted by $(\epsilon_n)_{n \in \mathbb{N}}$, and there is an $H^0$-orthonormal sequence of eigenfunctions $(w_{0,j})_{j \in \mathbb{N}}$ corresponding to $(\lambda_{0,j})_{j \in \mathbb{N}}$ such that*
   $$|w_{\epsilon_n,j} - w_{0,j}|_{H_1^{\epsilon_n}} \to 0 \text{ as } n \to \infty$$

*for all* $j \in \mathbb{N}$;

(c) *for every* $u \in H_1^0$,

$$\langle u, w_{\epsilon_n,j}\rangle_{H_1^{\epsilon_n}} \to \langle u, w_{0,j}\rangle_{H_1^0} \text{ as } n \to \infty$$

*for all* $j \in \mathbb{N}$.

The following example shows that the thin domain problems for reaction-diffusion equations and damped wave equations described in the Introduction provide families of Hilbert spaces and positive self-adjoint operators satisfying condition (Spec).

**Example 5.2.** *Let $\Omega$ be an arbitrary nonempty bounded domain in $\mathbb{R}^M \times \mathbb{R}^N$ with Lipschitz boundary and let $\epsilon > 0$ be arbitrary. As in the Introduction, write $(x, y)$ for a generic point of $\mathbb{R}^M \times \mathbb{R}^N$. Let $\gamma > 0$ be a fixed constant. Define the symmetric bilinear form*

$$a_\epsilon \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$$

*by*

$$a_\epsilon(u, v) := \int_\Omega \left(\gamma uv + \nabla_x u \cdot \nabla_x v + (1/\epsilon^2)\nabla_y u \cdot \nabla_y v\right) \, dx\, dy.$$

*The pair $(a_\epsilon, \langle\cdot,\cdot\rangle_{L^2(\Omega)})$ generates a linear operator $A_\epsilon \colon D(A_\epsilon) \subset H^1(\Omega) \to L^2(\Omega)$ such that $u \in D(A_\epsilon)$ if and only if there is a $w_u \in L^2(\Omega)$ for which*

$$a_\epsilon(u, v) = \langle w_u, v\rangle_{L^2(\Omega)} \text{ for all } v \in H^1(\Omega).$$

*The element $w_u \in L^2(\Omega)$ is then necessarily unique and we write $A_\epsilon(u) := w_u$.*

*Now define the "limit" space $H_s^1(\Omega)$ by*

$$H_s^1(\Omega) = \{\, u \in H^1(\Omega) \mid \nabla_y u = 0 \,\}.$$

*Note that $H_s^1(\Omega)$ is a closed linear subspace of $H^1(\Omega)$. Let us also define the space $L_s^2(\Omega)$ to be the closure of the set $H_s^1(\Omega)$ in $L^2(\Omega)$. It follows that $L_s^2(\Omega)$ is a Hilbert space under the scalar product of $L^2(\Omega)$. Now let $a_0 \colon H_s^1(\Omega) \times H_s^1(\Omega) \to \mathbb{R}$ be the "limit" bilinear form defined by*

$$a_0(u, v) := \int_\Omega (\gamma uv + \nabla u \cdot \nabla v) \, dx\, dy = \int_\Omega (\gamma uv + \nabla_x u \cdot \nabla_x v) \, dx\, dy.$$

*Denote by $A_0$ the operator defined, as above, by the pair $(a_0, \langle\cdot,\cdot\rangle_{L_s^2(\Omega)})$.*

*For $\epsilon > 0$ write $H^\epsilon := L^2(\Omega)$ and $\langle\cdot,\cdot\rangle_{H^\epsilon} := \langle\cdot,\cdot\rangle_{L^2(\Omega)}$. Moreover, define $H^0 := L_s^2(\Omega)$ and $\langle\cdot,\cdot\rangle_{H^0} := \langle\cdot,\cdot\rangle_{L_s^2(\Omega)}$. With these notations it follows from the results of* [11] *that for every $\epsilon_0 > 0$ the families $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfy condition* (Spec).

For the rest of the paper, let $\nu > 0$ be fixed arbitrarily. If the families $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfy condition (Spec) then we shall write $B_\epsilon := B_{\nu,A_\epsilon}$, $\epsilon \in [0, \epsilon_0]$.

The following concept plays an important role in the convergence result stated below.

**Definition 5.3.** *Given $\epsilon_0 > 0$ let $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfy condition* (Spec). *We say that the family $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ of maps satisfies hypothesis* (Conv) *if the following properties are satisfied:*

1. $f_\epsilon \colon H_1^\epsilon \to H^\epsilon$ *for every* $\epsilon \in [0, \epsilon_0]$;
2. $\lim_{\epsilon\to 0^+} |f_\epsilon(u) - f_0(u)|_{H^\epsilon} = 0$ *for every* $u \in H_1^0$;

3. *for every* $M \in [0,\infty[$ *there is an* $L = L_M \in [0,\infty[$ *such that*
$$|f_\epsilon(u) - f_\epsilon(v)|_{H^\epsilon} \le L|u-v|_{H_1^\epsilon}$$
*for all* $\epsilon \in [0,\epsilon_0]$ *and* $u$, $v \in H_1^\epsilon$ *satisfying* $|u|_{H_1^\epsilon}$, $|v|_{H_1^\epsilon} \le M$.

Families of maps satisfying hypothesis (Conv) can be obtained as follows:

**Example 5.4.** *Let* $\epsilon_0 > 0$ *be arbitrary and the families* $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ *and* $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *be as in Example 5.2. Let* $G\colon \mathbb{R}\times\mathbb{R}^M\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$, $(\epsilon,x,y,\xi)\mapsto G(\epsilon,x,y,\xi)$, *be a* $C^1$*-function for which there are constants* $\beta$, $\delta$ *and* $C \in [0,\infty[$ *such that for all* $(\epsilon,x,y,\xi) \in \mathbb{R}\times\mathbb{R}^M\times\mathbb{R}^N\times\mathbb{R}$ *the following estimates are satisfied:*

1. $|\partial_\epsilon G(\epsilon,x,y,\xi)| \le C(1+|\xi|^\beta)$;
2. $|\nabla_y G(\epsilon,x,y,\xi)| \le C(1+|\xi|^\beta)$;
3. $|\partial_\xi G(\epsilon,x,y,\xi)| \le C(1+|\xi|^\delta)$.

*If* $n := M+N > 2$ *then we also assume that* $\beta \le 2^*/2$ *and* $\delta \le (2^*/2)-1$, *where* $2^* := 2n/(n-2)$.

*For* $\epsilon > 0$ *define the map* $f_\epsilon$ *by*
$$f_\epsilon(u)(x,y) := G(\epsilon,x,\epsilon y,u(x,y)), \text{ for } u \in H^1(\Omega) \text{ and } (x,y)\in\Omega.$$
*Furthermore, for* $u \in H_s^1(\Omega)$ *define* $f_0(u)\colon \Omega\to\mathbb{R}$ *by*
$$f_0(u)(x,y) := G(0,x,0,u(x,y)) \text{ for } (x,y)\in\Omega.$$
*It is easy to prove (cf.* [1]*) that the family* $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *satisfies hypothesis* (Conv).

In the sequel, if the family $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ of maps satisfies hypothesis (Conv) then we will write $\pi_\epsilon := \pi_{\nu,A_\epsilon,f_\epsilon}$ for every $\epsilon\in[0,\epsilon_0]$.

We have the following convergence result, which is proved in [2].

**Theorem 5.5.** *Let* $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ *and* $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *satisfy condition* (Spec) *and suppose that* $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *satisfies hypothesis* (Conv). *Let* $(\epsilon_n)_{n\in\mathbb{N}}$ *be a sequence in* $]0,\epsilon_0]$ *with* $\epsilon_n\to 0$ *as* $n\to\infty$ *and* $(t_n)_{n\in\mathbb{N}}$ *be a sequence in* $[0,\infty[$ *with* $t_n\to t_\infty$ *as* $n\to\infty$, *for some* $t_\infty\in[0,\infty[$. *Moreover, let* $(u_n)_{u\in\mathbb{N}}$ *be a sequence in* $H_1\times H$ *such that*
$$|u_n - u_\infty|_{X_{\epsilon_n}} \to 0 \text{ as } n\to\infty$$
*for some* $u_\infty \in H_1^0\times H^0$. *Under these assumptions if* $u_\infty\pi_0 t_\infty$ *is defined then, for all* $n$ *large enough* $u_n\pi_{\epsilon_n}t_n$ *is defined and*
$$|u_n\pi_{\epsilon_n}t_n - u_\infty\pi_0 t_\infty|_{X_{\epsilon_n}} \to 0 \text{ as } n\to\infty. \quad \square$$

For every $\epsilon\in]0,\epsilon_0]$, the set $X_0 = H_1^0\times H^0$ is a closed subspace of $X_\epsilon$. For each $\epsilon\in]0,\epsilon_0]$, let $Q_\epsilon\colon X_\epsilon\to X_\epsilon$ be the $X_\epsilon$-orthogonal projection of $X_\epsilon$ onto $X_0$. Define $Y_\epsilon := (I-Q_\epsilon)X_\epsilon$ and endow $Y_\epsilon$ with the norm $|\cdot|_{X_\epsilon}$ restricted to $Y_\epsilon$. Define on $Z_\epsilon = X_0\times Y_\epsilon$ the following norm:
$$||(u,v)||_\epsilon := \max\{\,|u|_{X_0}, |v|_{X_\epsilon}\,\} \text{ for } (u,v)\in Z_\epsilon.$$
We will denote by $\Gamma_\epsilon$ the metric on $Z_\epsilon$ induced by the norm $||\cdot||_\epsilon$. For each $\epsilon\in]0,\epsilon_0]$, define $\theta_\epsilon := 0$.

Let $\Psi_\epsilon \colon X_\epsilon \to Z_\epsilon$ be the linear map defined by

$$\Psi_\epsilon(w) := (Q_\epsilon w, (I - Q_\epsilon)w) \text{ for } w \in X_\epsilon.$$

It follows that $\Psi_\epsilon$ is a bijective linear map and its inverse map is given by

$${\Psi_\epsilon}^{-1}((u,v)) = u + v \text{ for } (u,v) \in Z_\epsilon.$$

Moreover both $\Psi_\epsilon$ and ${\Psi_\epsilon}^{-1}$ are continuous maps. This fact is a consequence of the following inequalities:

$$||\Psi_\epsilon(w)||_\epsilon \leq \max\{1, C\}|w|_{X_\epsilon} \text{ for } w \in X_\epsilon, \tag{12}$$

$$|{\Psi_\epsilon}^{-1}((u,v))|_{X_\epsilon} \leq (1 + C^2)^{1/2}||(u,v)||_\epsilon \text{ for } (u,v) \in Z_\epsilon, \tag{13}$$

where the positive constant $C$ was defined in hypothesis (Spec).

Given $(u,v) \in Z_\epsilon$ and $t \in [0, \infty[$ define

$$(u,v)\tilde{\pi}_\epsilon t := \Psi_\epsilon({\Psi_\epsilon}^{-1}((u,v))\pi_\epsilon t).$$

Hence $\tilde{\pi}_\epsilon$ is a local semiflow on $Z_\epsilon$ which is conjugated to $\pi_\epsilon$. Theorem 5.5 and inequalities (12) and (13) immediately imply the following

**Corollary 5.6.** *Under the above hypotheses the family $(\tilde{\pi}_\epsilon)_{\epsilon \in ]0,\epsilon_0]}$ converges singularly to $\pi_0$.* □

Singular admissibility is implied by the following hypothesis on the families of nonlinearities in damped wave equations.

**Definition 5.7.** *We say that the family $(f_\epsilon)_{\epsilon \in [0,\epsilon_0]}$ satisfies hypothesis* (Comp) *if the following hypotheses are satisfied:*

1. *for every $\epsilon \in [0, \epsilon_0]$ the map $f_\epsilon \colon H_1^\epsilon \to H^\epsilon$ is compact, i.e. maps bounded sets in $(H_1^\epsilon, |\cdot|_{H_1^\epsilon})$ into compact sets in $(H^\epsilon, |\cdot|_{H^\epsilon})$;*
2. *whenever $(\epsilon_n)_{n \in \mathbb{N}}$ is a sequence of positive numbers converging to zero as $n \to \infty$, $\tilde{C} \in [0, \infty[$ and $(\xi_n)_{n \in \mathbb{N}}$ is a sequence in $H_1$ such that*

$$\sup_{n \in \mathbb{N}} |\xi_n|_{H_1^{\epsilon_n}} \leq \tilde{C},$$

*then there is a sequence $(n_k)_{k \in \mathbb{N}}$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that*

$$|f_{\epsilon_{n_k}}(\xi_{n_k}) - v|_{H^{\epsilon_n}} \to 0 \text{ as } k \to \infty$$

*for some $v \in H^0$.*

The following statement is easily proved:

**Example 5.8.** *Let $\epsilon_0 > 0$ be arbitrary and the families $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon \in [0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon \in [0,\epsilon_0]}$ be as in Example 5.2. Let $G \colon \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be as in Example 5.4. If $n := M + N > 2$ then assume also that $\delta < (2^*/2) - 1$, where $2^* := 2n/(n-2)$ (subcritical growth). Then, the family $(f_\epsilon)_{\epsilon \in [0,\epsilon_0]}$ satisfies hypothesis* (Comp).

The admissibility result proved in [2] reads as follows:

**Theorem 5.9.** *Let* $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ *and* $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *satisfy condition* (Spec). *Suppose the family of maps* $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *satisfies hypotheses* (Conv) *and* (Comp).

*Then for every* $\epsilon \in [0, \epsilon_0]$ *every closed bounded subset of* $X_\epsilon$ *is strongly* $\pi_\epsilon$*-admissible.*

*Suppose* $N \subset X_0$ *is a closed and bounded set and* $(\epsilon_n)_{n\in\mathbb{N}}$ *is a sequence of positive numbers converging to zero as* $n \to \infty$. *Let* $\eta > 0$ *be arbitrary and define*

$$N_n = N_{n,\eta} := \{\, u \in X_{\epsilon_n} \mid Q_{\epsilon_n} u \in N \text{ and } |(I - Q_{\epsilon_n})u|_{X_{\epsilon_n}} \leq \eta \,\}.$$

*Suppose* $(t_n)_{n\in\mathbb{N}}$ *is a sequence of positive numbers such that* $t_n \to \infty$ *as* $n \to \infty$ *and* $(u_n)_{n\in\mathbb{N}}$ *is a sequence in* $H_1 \times H$ *such that*

$$u_n \pi_{\epsilon_n} [0, t_n] \subset N_n \text{ for all } n \in \mathbb{N}.$$

*Then there exist a* $v \in N$ *and a sequence* $(n_k)_{k\in\mathbb{N}}$ *in* $\mathbb{N}$ *with* $n_k \to \infty$ *as* $k \to \infty$ *such that*

$$|u_{n_k} \pi_{\epsilon_{n_k}} t_{n_k} - v|_{X_{\epsilon_{n_k}}} \to 0 \text{ as } k \to \infty. \quad \square$$

Let $N$ be a closed and bounded subset of $X_0$. Recall that, given $\epsilon \in \,]0, \epsilon_0]$ and $\eta > 0$,

$$\begin{aligned}[N]_{\epsilon,\eta} &= \{\, (u,v) \in Z_\epsilon \mid u \in N \text{ and } v \in \mathrm{cl}_\epsilon\, B_\epsilon(0,\eta) \,\} \\ &= \{\, (u,v) \in Z_\epsilon \mid u \in N \text{ and } \|v\|_\epsilon \leq \eta \,\}.\end{aligned}$$

Theorem 5.9 and inequalities (12) and (13) easily imply the following

**Corollary 5.10.** *Let* $N$ *be a closed and bounded subset of* $X_0$. *Then for every* $\eta > 0$ *the set* $N$ *is singularly strongly admissible with respect to* $\eta$ *and the family* $(\tilde{\pi}_\epsilon)_{\epsilon\in[0,\epsilon_0]}$, *where* $\tilde{\pi}_0 = \pi_0$. $\square$

We can finally state the following Conley index continuation principle for 'singular' families of abstract damped wave equations:

**Theorem 5.11.** *Let* $N$ *be a closed and bounded isolating neighborhood for* $\pi_0$. *Then for every* $\eta > 0$, *there exists an* $\epsilon^c = \epsilon^c(\eta) > 0$ *such that for every* $\epsilon \in \,]0, \epsilon^c]$ *the set* $N_{\epsilon,\eta}$ *is a strongly admissible isolating neighborhood relative to* $\pi_\epsilon$ *and*

$$h(\pi_\epsilon, N_{\epsilon,\eta}) = h(\pi_0, N).$$

*Furthermore, for every* $\eta > 0$, *the family* $(K_{\epsilon,\eta})_{\epsilon\in[0,\epsilon^c(\eta)]}$ *of invariant sets, where* $K_{\epsilon,\eta} := A_{\pi_\epsilon}(N_{\epsilon,\eta})$, $\epsilon \in \,]0, \epsilon^c(\eta)]$, *and* $K_{0,\eta} = K_0 := A_{\pi_0}(N)$ *is upper semicontinuous at* $\epsilon = 0$ *with respect to the family* $|\cdot|_{X_\epsilon}$ *of norms.*

*Proof.* The isomorphism $\Psi_\epsilon$ conjugates the local semiflow $\pi_\epsilon$ to the local semiflow $\tilde{\pi}_\epsilon$. Thus whenever $S$ is a strongly admissible isolating neighborhood with respect to $\pi_\epsilon$, then $\Psi_\epsilon(S)$ is a strongly admissible isolating neighborhood with respect to $\tilde{\pi}_\epsilon$ and

$$h(\pi_\epsilon, S) = h(\tilde{\pi}_\epsilon, \Psi_\epsilon(S)).$$

Corollaries 5.6 and 5.10 imply that the family of semiflows $(\tilde{\pi}_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ and the set $N$ satisfy the hypotheses of Theorem 4.1. Notice also that any closed ball in $Y_\epsilon$ is contractible. Hence Theorem 4.1 and Corollary 4.10 completes the proof. $\square$

The last result, together with Examples 5.2, 5.4 and 5.8 yield a singular Conley index continuation principle for damped wave equations on thin domains. The precise formulation is left to the reader. In particular, we obtain Theorem D stated in the Introduction.

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