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Resumo

Seja Ω um domínio limitado suave não-vazio em $\mathbb{R}^M \times \mathbb{R}^N$. Dado $\epsilon > 0$ comprima Ω pelo fator ϵ na direção y para obter o domínio comprimido $\Omega_\epsilon := \{ (x, \epsilon y) \mid (x, y) \in \Omega \}$. Sejam γ e ν constantes positivas. Considere a seguinte equação das ondas amortecidas em Ω_ϵ:

$$(\tilde{D}_\epsilon) \qquad \begin{aligned} & u_{tt} = -\nu u_t + \Delta u - \gamma u + f(\epsilon, x, y, u), && t > 0,\ (x, y) \in \Omega_\epsilon, \\ & \partial_{\nu_\epsilon} u = 0, && t > 0,\ (x, y) \in \partial\Omega_\epsilon, \end{aligned}$$

onde ν_ϵ é a normal exterior a $\partial\Omega_\epsilon$ e f é uma não-linearidade de classe C^1 com apropriados crescimentos polinomiais em ϵ, y and u os quais garantem que $(\tilde{D}_\epsilon)$ gere um fluxo (local) $\tilde{\pi}_\epsilon$ em $X := H^1(\Omega_\epsilon) \times L^2(\Omega_\epsilon)$. Mostramos que existe um subespaço fechado X_0 de X e um fluxo $\tilde{\pi}_0$ em X_0 o qual é o fluxo limite da família $\tilde{\pi}_\epsilon$, $\epsilon > 0$. Mostramos que, quando $\epsilon \to 0$, a família $\tilde{\pi}_\epsilon$ converge em um sentido singular para $\tilde{\pi}_0$ e estabelecemos uma propriedade técnica de compacidade assintótica singular. Como um corolário, obtemos um resultado de semicontinuidade superior de atratores globais da família $\tilde{\pi}_\epsilon$, $\epsilon \geq 0$, generalizando resultados anteriores obtidos por Hale e Raugel para domínios os quais são conjuntos ordenados de uma função positiva.

Nossos resultados também implicam um princípio de continuação singular do índice de Conley para equações das ondas amortecidas definidas em domínios finos, o qual permite que se conclua propriedades de conjuntos invariantes dos fluxos $\tilde{\pi}_\epsilon$, $\epsilon > 0$ pequeno, a partir de propriedades análogas de conjuntos invariantes do fluxo limite.

ON CONVERGENCE, ADMISSIBILITY AND ATTRACTORS FOR DAMPED WAVE EQUATIONS ON SQUEEZED DOMAINS

M. C. CARBINATTO AND K. P. RYBAKOWSKI

ABSTRACT. Let Ω be an arbitrary nonempty smooth bounded domain in $\mathbb{R}^M \times \mathbb{R}^N$. Given $\epsilon > 0$ squeeze Ω by the factor ϵ in the y-direction to obtain the squeezed domain $\Omega_\epsilon := \{\,(x, \epsilon y) \mid (x, y) \in \Omega\,\}$. Let γ and ν be positive constants. Consider the following semilinear damped wave equation on Ω_ϵ:

$$(\tilde{D}_\epsilon) \qquad \begin{aligned} &u_{tt} = -\nu u_t + \Delta u - \gamma u + f(\epsilon, x, y, u), && t > 0,\ (x, y) \in \Omega_\epsilon, \\ &\partial_{\nu_\epsilon} u = 0, && t > 0,\ (x, y) \in \partial\Omega_\epsilon, \end{aligned}$$

where ν_ϵ is the exterior normal vector field on $\partial\Omega_\epsilon$ and f is a C^1-nonlinearity of appropriate polynomial growth in ϵ, y and u which ensures that $(\tilde{D}_\epsilon)$ generates a (local) flow $\tilde{\pi}_\epsilon$ on $X := H^1(\Omega_\epsilon) \times L^2(\Omega_\epsilon)$. We show that there is a closed subspace X_0 of X and a flow $\tilde{\pi}_0$ on X_0 which is the limit flow of the family $\tilde{\pi}_\epsilon$, $\epsilon > 0$. We show that, as $\epsilon \to 0$, the family $\tilde{\pi}_\epsilon$ converges in some singular sense to $\tilde{\pi}_0$ and establish a technical singular asymptotic compactness property. As a corollary, we obtain an upper-semicontinuity result for global attractors of the family $\tilde{\pi}_\epsilon$, $\epsilon \geq 0$, generalizing previous results obtained by Hale and Raugel for domains which are ordinate sets of a positive function.

Our results also imply a singular Conley index continuation principle for damped wave equations on thin domains, which allows one to conclude properties of invariant set of the flows $\tilde{\pi}_\epsilon$, $\epsilon > 0$ small, from analogous properties of invariant sets of the limit flow.

1. Introduction

Let M and N be positive integers and Ω be an arbitrary nonempty smooth bounded domain in $\mathbb{R}^M \times \mathbb{R}^N$. Write (x, y) for a generic point of $\mathbb{R}^M \times \mathbb{R}^N$. Given $\epsilon > 0$ squeeze Ω by the factor ϵ in the y-direction to obtain the squeezed domain Ω_ϵ. More precisely, let $T_\epsilon \colon \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}^M \times \mathbb{R}^N$, $(x, y) \mapsto (x, \epsilon y)$, be the *squeezing operator* and define $\Omega_\epsilon := \{\,(x, \epsilon y) \mid (x, y) \in \Omega\,\}$. Let γ and ν be positive constants. Consider the following semilinear damped wave equation on Ω_ϵ:

$$(\tilde{D}_\epsilon) \qquad \begin{aligned} &u_{tt} = -\nu u_t + \Delta u - \gamma u + f(\epsilon, x, y, u), && t > 0,\ (x, y) \in \Omega_\epsilon, \\ &\partial_{\nu_\epsilon} u = 0, && t > 0,\ (x, y) \in \partial\Omega_\epsilon, \end{aligned}$$

where ν_ϵ is the exterior normal vector field on $\partial\Omega_\epsilon$ and f is a C^1-nonlinearity of appropriate polynomial growth in ϵ, y and u which ensures that $(\tilde{D}_\epsilon)$ generates a (local) flow $\tilde{\pi}_\epsilon$ on $H^1(\Omega_\epsilon) \times L^2(\Omega_\epsilon)$.

As $\epsilon \to 0$ the thin domain Ω_ϵ degenerates to an M-dimensional open set. One may ask what happens in the limit to the family $(\tilde{\pi}_\epsilon)_{\epsilon > 0}$ of flows.

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More specifically, suppose that $f = f(u)$ has subcritical growth and is *dissipative* in the sense that

$$\limsup_{|s|\to\infty} f(s)/s \le 0.$$

Then for $\epsilon > 0$ the flow $\tilde{\pi}_\epsilon$ possesses a global attractor $\tilde{\mathcal{A}}_\epsilon$. If there is a limit flow $\tilde{\pi}_0$, does it also have a global attractor $\tilde{\mathcal{A}}_0$? Does the family $(\tilde{\mathcal{A}}_\epsilon)_{\epsilon>0}$ 'converge' to $\tilde{\mathcal{A}}_0$ in some sense?

This latter problem was first considered by Hale and Raugel in [7] for the case when $N = 1$, U is a bounded domain in $\mathbb{R}^M$ and the domain Ω is the *ordinate set* of a smooth positive function g defined on $\operatorname{cl} U$, i.e.

$$\Omega = \{\, (x,y) \mid x \in U \text{ and } 0 < y < g(x) \,\}.$$

The authors prove that, in this case, there exists a limit flow $\tilde{\pi}_0$, which is defined by the Neumann boundary value problem

$$(H_0) \qquad \begin{aligned} &u_{tt} = -\nu u_t + (1/g)\sum_{i=1}^{M}(g u_{x_i})_{x_i} - \gamma u + f(u), && t > 0,\ x \in U\\ &\partial_\nu u = 0, && t > 0,\ x \in \partial U. \end{aligned}$$

The flow $\tilde{\pi}_0$ has a global attractor $\tilde{\mathcal{A}}_0$ and, in some sense, the family $(\tilde{\mathcal{A}}_\epsilon)_{\epsilon\ge 0}$ is upper-semicontinuous at $\epsilon = 0$.

If the domain Ω is not the ordinate set of some function (e.g. if Ω has holes or different horizontal branches) then an equation of the form (H_0) cannot be a limiting equation for the family $(\tilde{D}_\epsilon)$, $\epsilon > 0$. The purpose of this paper is to show that in this general case a limit equation and the corresponding limit flow still exist. We also show that the flows $\tilde{\pi}_\epsilon$ converge in some singular sense to the limit flow and that they satisfy a singular admissibility (i.e. asymptotic compactness) property as $\epsilon \to 0$. If f is dissipative and has subcritical growth then these results imply that the limit flow has a global attractor and that the upper semicontinuity result of Hale and Raugel continues to hold on arbitrary squeezed domains.

Our results also imply a singular Conley index continuation principle for damped wave equations on thin domains, which allows one to conclude properties of invariant set of the flows $\tilde{\pi}_\epsilon$, $\epsilon > 0$ small, from analogous properties of invariant sets of the limit flow.

This paper extends to damped wave equations some earlier results obtained by Hale and Raugel [6] for reaction-diffusion equations on thin domains of ordinate set type and by Prizzi and Rybakowski [10] as well as by the present authors in [1] for reaction-diffusion equations on general squeezed domains. It is instructive to give a brief description of the latter results. Consider the following Neumann boundary value problem:

$$(\tilde{E}_\epsilon) \qquad \begin{aligned} &u_t = \Delta u - \gamma u + f(\epsilon, x, y, u), && t > 0,\ (x,y) \in \tilde{\Omega}_\epsilon,\\ &\partial_{\nu_\epsilon} u = 0, && t > 0,\ (x,y) \in \partial\tilde{\Omega}_\epsilon, \end{aligned}$$

on the squeezed domain $\tilde{\Omega}_\epsilon$.

In order to describe some of the results obtained in [10] and [1] we first use the transformation $u \mapsto u \circ T_\epsilon$ to transfer problem $(\tilde{E}_\epsilon)$ to the following equivalent problem on the fixed domain Ω:

$$(E_\epsilon) \qquad \begin{array}{ll} u_t = \Delta_x u + (1/\epsilon^2)\Delta_y u - \gamma u + f(\epsilon, x, \epsilon y, u), & t > 0,\ (x,y) \in \Omega, \\ \nu_1 \cdot \nabla_x u + (1/\epsilon^2)\nu_2 \cdot \nabla_y u = 0, & t > 0,\ (x,y) \in \partial\Omega. \end{array}$$

Here, $\nu = (\nu_1, \nu_2)$ is the exterior normal vector field on $\partial\Omega$.

If the nonlinearity f is a C^1-function of its arguments and has appropriate polynomial growth in u, ϵ and y, then we can define, for every $\epsilon > 0$, the Nemitski operator

$$f_\epsilon \colon H^1(\Omega) \to L^2(\Omega)$$

$$f_\epsilon(u)(x,y) := f(\epsilon, x, \epsilon y, u(x,y)).$$

Then equation (E_ϵ) can be written in the abstract form

$$\dot{u} + A_\epsilon u = f_\epsilon(u),$$

where A_ϵ is the linear operator defined by

$$A_\epsilon u = -\Delta_x u - (1/\epsilon^2)\Delta_y u + \gamma u \in L^2(\Omega)$$

for $u \in H^2(\Omega)$ with $\nu_1 \cdot \nabla_x u + (1/\epsilon^2)\nu_2 \cdot \nabla_y u = 0$ on $\partial\Omega$. Equation (E_ϵ) defines a (local) semiflow Π_ϵ on $H^1(\Omega)$. Note that the operator A_ϵ is generated, in the usual way, by the bilinear form

$$a_\epsilon \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$$

given by

$$a_\epsilon(u,v) := \int_\Omega \left(\gamma uv + \nabla_x u \cdot \nabla_x v + (1/\epsilon^2)\nabla_y u \cdot \nabla_y v\right) \, dx\, dy.$$

(Cf. formula (F) below). On $H^1(\Omega)$ we define the norm $|\cdot|_\epsilon$ by

$$|u|_\epsilon^2 := a_\epsilon(u,u).$$

Now define the space

$$H^1_s(\Omega) := \{\, u \in H^1(\Omega) \mid \nabla_y u = 0 \,\}.$$

This is a closed linear subspace of $H^1(\Omega)$. The bilinear form given by the formula:

$$a_0(u,v) := \int_\Omega (\gamma uv + \nabla_x u \cdot \nabla_x v) \, dx\, dy, \qquad u, v \in H^1_s(\Omega)$$

uniquely determines a densely defined selfadjoint linear operator $A_0 \colon D(A_0) \subset H^1_s(\Omega) \to L^2_s(\Omega)$ by the usual formula

$$(F) \qquad a_0(u,v) = \langle A_0 u, v\rangle_{L^2(\Omega)}, \qquad \text{for } u \in D(A_0) \text{ and } v \in H^1_s(\Omega).$$

Here, $L^2_s(\Omega)$ is the closure of $H^1_s(\Omega)$ in the L^2-norm, so $L^2_s(\Omega)$ is a closed linear subspace of $L^2(\Omega)$. It is proved that the Nemitski operator f_0 defined by

$$f_0(u)(x,y) := f(0, x, 0, u(x,y)),$$

maps the space $H^1_s(\Omega)$ into $L^2_s(\Omega)$. Consequently the abstract parabolic equation

$$(E_0) \qquad \dot{u} = -A_0 u + f_0(u)$$

defines a semiflow Π_0 on the space $H^1_s(\Omega)$. This is the (singular) limit semiflow of the family Π_ϵ.

In fact, the family $(\Pi_\epsilon)_{\epsilon>0}$ converges to Π_0 in the following restricted sense(see [1]): whenever $\epsilon_n \to 0^+$, $t_n \to t_0$ and $|u_n - u_0|_{\epsilon_n} \to 0$, where t_n, $t_0 \in [0, \infty[$, $u_n \in H^1(\Omega)$, $n \geq 1$, and $u_0 \in H^1_s(\Omega)$ then $|\Pi_{\epsilon_n}(t_n, u_n) - \Pi_0(t_0, u_0)|_{\epsilon_n} \to 0$. If $t_0 > 0$ we need only assume $|u_n - u_0|_{L^2} \to 0$ (see [10]). This latter fact follows since A_ϵ has compact resolvent.

If $f = f(u)$ and f is *dissipative*, then each semiflow Π_ϵ, $\epsilon \geq 0$, possesses a global attractor $\mathcal{A}_\epsilon$. Using the second part of the convergence result just quoted it is proved that the family of attractors $(\mathcal{A}_\epsilon)_{\epsilon \geq 0}$ is upper-semicontinuous at $\epsilon = 0$ with respect to the family $|\cdot|_\epsilon$ of norms, i.e.

$$\lim_{\epsilon \to 0^+} \sup_{u \in \mathcal{A}_\epsilon} \inf_{v \in \mathcal{A}_0} |u - v|_\epsilon = 0.$$

(See [10, Theorem 5.10], which generalizes previous results of [6]).

Notice that each attractor $\mathcal{A}_\epsilon$ for $\epsilon \geq 0$ is a compact isolated Π_ϵ-invariant set with Conley index Σ^0. Therefore the above theorem implies that the invariant set $\mathcal{A}_0$ can be continued to a family of invariant sets with the same Conley index.

It was proved in our previous paper [1] that the latter statement holds true for an arbitrary isolated invariant set of the limit semiflow:

Theorem C (Singular Conley index continuation principle). *Let $K_0 \subset H^1_s(\Omega)$ be a compact isolated invariant set of the limit semiflow Π_0. Then there is an $\epsilon_0 > 0$ and for each $\epsilon \in]0, \epsilon_0]$ there is a compact isolated invariant set $K_\epsilon \subset H^1(\Omega)$ of the semiflow Π_ϵ such that the Conley indices of K_ϵ and K_0 are the same. Moreover, the family $(K_\epsilon)_{\epsilon \in [0, \epsilon_0]}$ is upper-semicontinuous at $\epsilon = 0$ with respect to the family $|\cdot|_\epsilon$ of norms, i.e.*

$$\lim_{\epsilon \to 0^+} \sup_{u \in K_\epsilon} \inf_{v \in K_0} |u - v|_\epsilon = 0.$$

For the precise assumptions on f and for various applications of this result the reader is referred to the paper [1]. In the special case of cylindrical domains, i.e. for Ω of the form $\Omega = U \times]0, 1[$ an analogue of Theorem C was independently proved by Huang [8]. His proof method, unlike ours, cannot be generalized to the case of damped wave equations.

In order to extend the above results to damped wave equations $(\tilde{D}_\epsilon)$, let us proceed as before and transfer problem $(\tilde{D}_\epsilon)$ to the following equivalent equation on the fixed domain Ω:

$$(D_\epsilon) \qquad \begin{array}{ll} u_{tt} = -\nu u_t + \Delta_x u + (1/\epsilon^2)\Delta_y u - \gamma u + f(\epsilon, x, \epsilon y, u), & t > 0,\ (x, y) \in \Omega, \\ \nu_1 \cdot \nabla_x u + (1/\epsilon^2)\nu_2 \cdot \nabla_y u = 0, & t > 0,\ (x, y) \in \partial\Omega. \end{array}$$

Set $X_\epsilon \equiv X := H^1(\Omega) \times L^2(\Omega)$ for $\epsilon > 0$ and $X_0 := H^1_s(\Omega) \times L^2_s(\Omega)$. Given $u \in X_\epsilon$ we will write u_0 (resp. u_1) for the first (resp. second) component of u, i.e. $u = (u_0, u_1)$. For $\epsilon > 0$ endow X_ϵ with the norm $|\cdot|_{X_\epsilon}$ defined by

$$|(u_0, u_1)|^2_{X_\epsilon} := |u_0|^2_\epsilon + |u_1|^2_{L^2(\Omega)}.$$

For $\epsilon \geq 0$ define $D(B_\epsilon) := D(A_\epsilon) \times H^1(\Omega) \subset X_\epsilon$ and let $B_\epsilon \colon D(B_\epsilon) \to X_\epsilon$ be the operator

$$B_\epsilon(u_0, u_1) := (-u_1, \nu u_1 + A_\epsilon u_0) \text{ for } (u_0, u_1) \in D(B_\epsilon).$$

Moreover, for all $\epsilon \geq 0$ define the map $g_\epsilon \colon X_\epsilon \to X_\epsilon$ by

$$g_\epsilon(u_0, u_1) = (0, f_\epsilon(u_0)) \text{ for } (u_0, u_1) \in X_\epsilon.$$

Then, for all $\epsilon > 0$, equation (D_ϵ) can be written in the equivalent abstract form

$$\dot{u} = -B_\epsilon u + g_\epsilon(u).$$

This equation defines a (local) semiflow (actually, a flow) π_ϵ on X. The limit flow of the family $(\pi_\epsilon)_{\epsilon\in]0,1]}$ is given by the flow π_0 generated by the equation

$$\dot{u} = -B_0 u + g_0(u).$$

We prove (see Theorem 3.5 below) that the family $(\pi_\epsilon)_{\epsilon>0}$ converges to π_0 in a restricted sense, analogously to the first part of the convergence result in the parabolic case.

On the other hand, since the linear operators B_ϵ do not have compact resolvents, the arguments in [10] (resp. in [1]) used to prove the upper semicontinuity of attractors (resp. the singular Conley index continuation principle) cannot be used in the present case. To remedy this, we establish in section 4 a singular asymptotic compactness property for the family $(\pi_\epsilon)_{\epsilon\geq 0}$ (Theorems 4.3 and 4.5). This property enables us to prove, in the dissipative case, the existence of a global attractor for the limit flow and an upper semicontinuity result for attractors (see Theorem 5.6).

Our convergence and asymptotic compactness property, together with an abstract result established in [2] also imply a version of the singular Conley index continuation result for damped wave equations on squeezed domains (see Theorem 5.8 below). The complete proof of this latter result is given in [2]. For the precise assumptions on the nonlinearity f, see Examples 3.3 and 4.2 below. Let us only mention, that unlike in Theorem 3.5, we assume, in Theorems 5.6 and 5.8, subcritical growth of f with respect to u. If f has critical growth, then, under some additional conditions, the same results obtain. The proofs are more difficult and are given in [3].

We keep the presentation of our results on an abstract level. In fact we only assume certain spectral convergence properties on the family $(A_\epsilon)_{\epsilon\geq 0}$ of linear operators (see hypothesis (Spec) in Section 2). We also make abstract convergence and admissibility assumptions (conditions (Conv) in Section 3 and (Comp) in Section 4) on the family $(f_\epsilon)_{\epsilon\geq 0}$ of nonlinear operators. For the existence and upper semicontinuity of attractors we also need an assumption (condition (Liap) in section 5) guaranteeing the existence of Liapunov functions for the flows π_ϵ satisfying appropriate estimates.

As a consequence of our abstract approach all our results can also be applied to more general situations, e.g. to damped wave equations on curved thin domains introduced in the recent paper [13]. Moreover, we can replace the Neumann boundary conditions by, say, mixed boundary conditions. In addition, the convergence and admissibility results remain valid for appropriate *systems* of damped wave equations on thin domains.

2. Singular families of linear damped wave operators

In this section we introduce the abstract framework for the study of damped wave equations on thin domains. We introduce a spectral convergence condition on a family $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ of linear operators, which will later enable us to obtain a basic singular convergence theorem for linear damped wave equations.

We start by recalling the definition of a semiflow and a few related concepts.

Let X be a topological space, let D be an open subset of $[0,\infty[\times X$ and $\pi\colon D\to X$ be a continuous map. We write $x\pi t := \pi(t,x)$ for $(t,x)\in D$. The map π is called a *local semiflow on* X if the following properties are satisfied:

1. For every $x \in X$ there is a number $\omega_x = \omega_x^\pi \in]0, \infty]$ such that $(t, x) \in D$ if and only if $0 \leq t < \omega_x$.
2. $x\pi 0 = x$ for all $x \in X$.
3. If $(t, x) \in D$ and $(s, x\pi t) \in D$ then $(t + s, x) \in D$ and
$$x\pi(t + s) = (x\pi t)\pi s.$$

If $\omega_x = \infty$ for every $x \in X$, then π is called a *global semiflow on* X.

Let N be an arbitrary subset of X. We say that the local semiflow π *does not explode* in N, if whenever $x\pi t \in N$ for all $t \in [0, \omega_x[$ then $\omega_x = \infty$.

Let J be an arbitrary interval in $\mathbb{R}$. A map $\sigma \colon J \to X$ is called a *solution of* π if for all $t \in J$ and $s \in [0, \infty[$ for which $t + s \in J$, it follows that $\sigma(t)\pi s$ is defined and $\sigma(t)\pi s = \sigma(t + s)$. If $0 \in J$ and $\sigma(0) = x$, we say that σ is a *solution through* x. If $J = \mathbb{R}$ (respectively $J =]-\infty, 0]$), then σ is called a *full solution* (respectively *full left solution*) relative to π.

Suppose H is a linear space which is complete with respect to the scalar product $\langle \cdot, \cdot \rangle_H$ and let $A \colon D(A) \subset H \to H$ be a (densely defined) positive self-adjoint operator on $(H, \langle \cdot, \cdot \rangle_H)$ with $A^{-1} \colon H \to H$ compact. Let $\lambda_1 := \lambda_1(A) > 0$ be the first eigenvalue of A and $H_1(A)$ be the linear space $D(A^{1/2})$. Endow $H_1(A)$ with the (complete) scalar product
$$\langle u, v \rangle_{H_1(A)} := \langle A^{1/2}u, A^{1/2}v \rangle_H.$$
Set $X(A) := H_1(A) \times H$. Given $u \in X(A)$ we will write u_0 (resp. u_1) for the first (resp. second) component of u, i.e. $u = (u_0, u_1)$.

Endow $X(A)$ with the (complete) scalar product
$$\langle (u_0, u_1), (v_0, v_1) \rangle_{X(A)} := \langle u_0, v_0 \rangle_{H_1(A)} + \langle u_1, v_1 \rangle_H.$$
Write $|\cdot|_H$, $|\cdot|_{H_1(A)}$ and $|\cdot|_{X(A)}$ for the corresponding Hilbert space norms.

Let $\nu > 0$. Define $D(B_{\nu,A}) := D(A) \times H_1(A) \subset X(A)$ and let $B_{\nu,A} \colon D(B_{\nu,A}) \to X(A)$ be the operator
$$B_{\nu,A}(u_0, u_1) := (-u_1, \nu u_1 + Au_0) \text{ for } (u_0, u_1) \in D(B_{\nu,A}).$$

The following result is well-known (cf [9, 16]):

Proposition 2.1. *Under the above hypotheses the operator $-B := -B_{\nu,A}$ is the generator of a C^0-group e^{-tB}, $t \in \mathbb{R}$, of operators on $X(A)$. Let $\alpha \in \mathbb{R}$ be an arbitrary number satisfying the inequalities*
$$0 < \alpha \leq \frac{\nu}{2} \text{ and } 1 - \frac{\alpha(\nu - \alpha)}{\lambda_1} > 0.$$
Set $M := \mu_1^2/m$, where
$$\mu_1 := \max\{\sqrt{1 + \alpha}, \sqrt{1 + \alpha\lambda_1^{-1} + \alpha^2\lambda_1^{-1}}\} \text{ and } m := \sqrt{1 - \frac{\alpha(\nu - \alpha)}{\lambda_1}}.$$
Then the following estimate holds:
$$|e^{-tB}u|_{X(A)} \leq Me^{-\alpha t}|u|_{X(A)} \text{ for } t \geq 0 \text{ and } u \in X(A).$$

Now suppose that $f\colon H_1(A) \to H$ is a map which is Lipschitzian on bounded subsets of $H_1(A)$. Define the map $g = g_{A,f}\colon X(A) \to X(A)$ by

$$g_{A,f}(u_0, u_1) = (0, f(u_0)) \text{ for } (u_0, u_1) \in X(A).$$

It follows that $g_{A,f}$ is Lipschitzian on bounded subsets of $X(A)$.

Consequently (cf [15]) there exists a well-defined local semiflow (actually a local flow) $\pi_{\nu,A,f}$ generated by the mild solutions of the following abstract damped wave equation

$$\dot{u} = -B_{\nu,A}u + g_{A,f}(u).$$

Moreover, $\pi_{\nu,A,f}$ does not explode in bounded subsets of $X(A)$. This follows, e.g. from Theorem 1 in [15] and its proof.

Let us now introduce the following basic hypothesis:

Definition 2.2. *Given a number $\epsilon_0 > 0$ we say that the families $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$* satisfy condition (Spec) *if the following properties are satisfied:*

1. *for every $\epsilon \in [0, \epsilon_0]$, $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})$ is a Hilbert space and $A_\epsilon\colon D(A_\epsilon) \subset H^\epsilon \to H^\epsilon$ is a densely defined positive self-adjoint operator on $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})$ with ${A_\epsilon}^{-1}\colon H^\epsilon \to H^\epsilon$ compact. Write $H_1^\epsilon := H_1(A_\epsilon)$ and $X_\epsilon := X(A_\epsilon)$;*
2. *the linear spaces $H := H^\epsilon$ and $H_1 := H_1^\epsilon$ are independent of ϵ, for $\epsilon \in]0, \epsilon_0]$, H^0 is a linear subspace of H and H_1^0 is a linear subspace of H_1;*
3. *there exists a constant $C \in]0, \infty[$ such that*
$$|u|_{X_\epsilon} \le C|u|_{X_0} \text{ and } |u|_{X_0} \le C|u|_{X_\epsilon}$$
for all $u \in X_0$ and all $\epsilon \in]0, \epsilon_0]$;
4. *for every $\epsilon \in]0, \epsilon_0]$ let $(\lambda_{\epsilon,j})_{j\in\mathbb{N}}$ be the repeated sequence of eigenvalues of A_ϵ and $(w_{\epsilon,j})_{j\in\mathbb{N}}$ be a corresponding H^ϵ-orthonormal sequence of eigenfunctions. Furthermore, let $(\lambda_{0,j})_{j\in\mathbb{N}}$ be the repeated sequence of eigenvalues of A_0.*
   Whenever $(\epsilon_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers converging to zero then
   (a) *for every $j \in \mathbb{N}$,*
   $$\lambda_{\epsilon_n,j} \to \lambda_{0,j} \text{ as } n \to \infty;$$
   (b) *there is a subsequence of $(\epsilon_n)_{n\in\mathbb{N}}$ again denoted by $(\epsilon_n)_{n\in\mathbb{N}}$ and there is an H^0-orthonormal sequence of eigenfunctions $(w_{0,j})_{j\in\mathbb{N}}$ corresponding to $(\lambda_{0,j})_{j\in\mathbb{N}}$ such that*
   $$|w_{\epsilon_n,j} - w_{0,j}|_{H_1^{\epsilon_n}} \to 0 \text{ as } n \to \infty$$
   for all $j \in \mathbb{N}$;
   (c) *for every $u \in H_1^0$,*
   $$\langle u, w_{\epsilon_n,j}\rangle_{H_1^{\epsilon_n}} \to \langle u, w_{0,j}\rangle_{H_1^0} \text{ as } n \to \infty$$
   for all $j \in \mathbb{N}$.

The following example shows that the thin domain problems for reaction-diffusion equations and damped wave equations described in the Introduction provide families of Hilbert spaces and positive self-adjoint operators satisfying condition (Spec).

Example 2.3. *Let Ω be an arbitrary nonempty bounded domain in $\mathbb{R}^M \times \mathbb{R}^N$ with Lipschitz boundary and let $\epsilon > 0$ be arbitrary. As in the Introduction, write (x, y) for a generic point of $\mathbb{R}^M \times \mathbb{R}^N$. Let $\gamma > 0$ be a fixed constant. We will now formally describe the family of differential operators on Ω arising from the Laplacian on Ω_ϵ with Neumann boundary* *conditions. Here, as in the Introduction, Ω_ϵ is the domain obtained by the squeezing operator T_ϵ. Define the symmetric bilinear form*

$$a_\epsilon\colon H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$$

by

$$a_\epsilon(u,v) := \int_\Omega \left(\gamma uv + \nabla_x u\cdot\nabla_x v + (1/\epsilon^2)\nabla_y u\cdot\nabla_y v\right)\,\mathrm{d}x\,\mathrm{d}y.$$

The pair $(a_\epsilon,\langle\cdot,\cdot\rangle_{L^2(\Omega)})$ generates a linear operator $A_\epsilon\colon D(A_\epsilon)\subset H^1(\Omega)\to L^2(\Omega)$ such that $u\in D(A_\epsilon)$ if and only if there is a $w_u\in L^2(\Omega)$ for which

$$a_\epsilon(u,v) = \langle w_u, v\rangle_{L^2(\Omega)} \text{ for all } v\in H^1(\Omega).$$

The element $w_u\in L^2(\Omega)$ is then necessarily unique and we write $A_\epsilon(u) := w_u$.

Now define the "limit" space $H^1_s(\Omega)$ by

$$H^1_s(\Omega) = \{\, u\in H^1(\Omega) \mid \nabla_y u = 0\,\}.$$

Note that $H^1_s(\Omega)$ is a closed linear subspace of $H^1(\Omega)$. Let us also define the space $L^2_s(\Omega)$ to be the closure of the set $H^1_s(\Omega)$ in $L^2(\Omega)$. It follows that $L^2_s(\Omega)$ is a Hilbert space under the scalar product of $L^2(\Omega)$. Now let $a_0\colon H^1_s(\Omega)\times H^1_s(\Omega)\to\mathbb{R}$ be the "limit" bilinear form defined by

$$a_0(u,v) := \int_\Omega (\gamma uv + \nabla u\cdot\nabla v)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega (\gamma uv + \nabla_x u\cdot\nabla_x v)\,\mathrm{d}x\,\mathrm{d}y.$$

Denote by A_0 the operator defined, as above, by the pair $(a_0,\langle\cdot,\cdot\rangle_{L^2_s(\Omega)})$.

For $\epsilon>0$ write $H^\epsilon := L^2(\Omega)$ and $\langle\cdot,\cdot\rangle_{H^\epsilon} := \langle\cdot,\cdot\rangle_{L^2(\Omega)}$. Moreover, define $H^0 := L^2_s(\Omega)$ and $\langle\cdot,\cdot\rangle_{H^0} := \langle\cdot,\cdot\rangle_{L^2_s(\Omega)}$. With these notations it follows from the results of [10] *that for every $\epsilon_0>0$ the families $(H^\epsilon,\langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfy condition* (Spec).

Remark 2.4. *There are other examples of families $(H^\epsilon,\langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfying condition* (Spec).

First, instead of Neumann boundary conditions, one may consider, say, mixed boundary conditions on Ω_ϵ. This merely leads to an appropriate modification in the definition of the limit spaces $H^1_s(\Omega)$ and $L^2_s(\Omega)$. The proofs of the results of [10] *carry over with obvious changes.*

Next, instead of squeezing the domain Ω in a 'flat' fashion, we may use a 'curved' squeezing in the normal directions of some M-dimensional submanifold of $\mathbb{R}^{M+N}$. As it has recently been established (see [13]*) the Laplacian on the curved squeezed domain $\hat\Omega_\epsilon$ again gives rise to families $(H^\epsilon,\langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfying condition* (Spec).

Lastly, we may, of course, consider systems *of differential operators, like $u\mapsto D\Delta u$ on Ω_ϵ (or $\hat\Omega_\epsilon$), where now u is a vector-valued function taking its values in $\mathbb{R}^p$ and D is a diagonal $p\times p$-matrix with positive coefficients. The resulting families $(H^\epsilon,\langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ again satisfy condition* (Spec).

For the rest of the paper, let $\nu>0$ be fixed arbitrarily. If the families $(H^\epsilon,\langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfy condition (Spec) then we shall write $B_\epsilon := B_{\nu,A_\epsilon}$, $\epsilon\in[0,\epsilon_0]$.

Remark 2.5. *Let $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfy condition* (Spec). *Then for every $u \in H_1^\epsilon$*

$$|u|^2_{H_1^\epsilon} = \sum_{j=1}^{\infty} \lambda_{\epsilon,j} |\langle u, w_{\epsilon,j}\rangle_{H^\epsilon}|^2 \geq \lambda_{\epsilon,1}|u|^2_{H^\epsilon}.$$

Since $\lim_{\epsilon\to 0^+} \lambda_{\epsilon,1} = \lambda_{0,1} > 0$, there is an $\epsilon_0' \in]0,\epsilon_0]$ and a $C' \in]0,\infty[$ such that

$$\lambda_{\epsilon,1}^{1/2} \geq C' \text{ for all } \epsilon \in [0,\epsilon_0'] \tag{1}$$

and so

$$|u|_{H_1^\epsilon} \geq C'|u|_{H^\epsilon}, \text{ for all } \epsilon \in [0,\epsilon_0'] \text{ and all } u \in H_1^\epsilon. \tag{2}$$

Moreover, using Proposition 2.1 we may also assume that there are constants M, $\alpha \in]0,\infty[$ such that

$$|e^{-tB_\epsilon}u|_{X_\epsilon} \leq Me^{-\alpha t}|u|_{X_\epsilon} \text{ for } \epsilon \in [0,\epsilon_0'],\ t \geq 0 \text{ and } u \in X_\epsilon. \tag{3}$$

The next two propositions are consequences of Definition 2.2.

Proposition 2.6. *Let $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfy property* (Spec). *Then for every $\epsilon \in]0,\epsilon_0]$, the subspace $(X_0, |\cdot|_{X_0})$ is closed in $(X_\epsilon, |\cdot|_{X_\epsilon})$.*

Proof. Let $\epsilon \in]0,\epsilon_0]$ and suppose $(u_n)_{n\in\mathbb{N}}$ is a sequence in X_0 with $|u_n - u|_{X_\epsilon} \to 0$ as $n\to\infty$ for some $u \in X_\epsilon$. Condition (Spec)3 implies that

$$|u_n - u_m|_{X_0} \leq C|u_n - u_m|_{X_\epsilon},$$

it follows that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X_0, |\cdot|_{X_0})$ which is a complete space. Therefore $(u_n)_{n\in\mathbb{N}}$ converges in X_0 to some v in X_0. But condition (Spec)3 implies that

$$|u_n - v|_{X_\epsilon} \leq C|u_n - v|_{X_0}.$$

Hence $u = v$ and thus $u \in X_0$. This proves the proposition. □

Proposition 2.7. *Let $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfy property* (Spec). *Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to zero. For every $u \in H^0$,*

$$\langle u, w_{\epsilon_n,j}\rangle_{H^{\epsilon_n}} \to \langle u, w_{0,j}\rangle_{H^0} \text{ as } n\to\infty$$

for all $j \in \mathbb{N}$.

Proof. First we will prove the result for $u \in H_1^0$. In this case, condition (Spec)4(c) implies

$$\langle u, w_{\epsilon_n,j}\rangle_{H_1^{\epsilon_n}} \to \langle u, w_{0,j}\rangle_{H_1^0} \text{ as } n\to\infty$$

for all $j \in \mathbb{N}$. Therefore for each $j \in \mathbb{N}$ we have

$$\lambda_{\epsilon_n,j}\langle u, w_{\epsilon_n,j}\rangle_{H^{\epsilon_n}} \to \lambda_{0,j}\langle u, w_{0,j}\rangle_{H^0} \text{ as } n\to\infty.$$

Since $\lambda_{\epsilon_n,j} \to \lambda_{0,j}$ as $n\to\infty$, $\lambda_{\epsilon_n,j} > 0$ and $\lambda_{0,j} > 0$ for all j, $n \in \mathbb{N}$, it follows that

$$\langle u, w_{\epsilon_n,j}\rangle_{H^{\epsilon_n}} \to \langle u, w_{0,j}\rangle_{H^0} \text{ as } n\to\infty.$$

Now, let $u \in H^0$ and $\delta > 0$. Since H_1^0 is dense in $(H^0, |\cdot|_{H^0})$ there exists a $u' \in H_1^0$ such that $|u' - u|_{H^0} < \delta$. By what we have proved so far, there is an $n_0 \in \mathbb{N}$ such that

$$|\langle u', w_{\epsilon_n,j}\rangle_{H^{\epsilon_n}} - \langle u', w_{0,j}\rangle_{H^0}| < \delta \text{ for all } n \geq n_0.$$

Using the Cauchy-Schwarz inequality and hypothesis (Spec)3 we thus obtain for all $n \geq n_0$ that

$$
\begin{aligned}
|\langle u, w_{\epsilon_n,j}\rangle_{H^{\epsilon_n}} - \langle u, w_{0,j}\rangle_{H^0}| &< |u-u'|_{H^{\epsilon_n}} + \delta + |u-u'|_{H^0} \\
&= |(0,u)-(0,u')|_{X_{\epsilon_n}} + \delta + |u-u'|_{H^0} \leq C|(0,u)-(0,u')|_{X_0} + \delta + |u-u'|_{H^0} \\
&= C|u-u'|_{H^0} + \delta + |u-u'|_{H^0} < (C+2)\delta.
\end{aligned}
$$

Since $\delta > 0$ is arbitrary, the proof is complete. □

We can now state our first convergence result:

Theorem 2.8. *Let $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfy condition* (Spec). *Suppose $(\epsilon_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers converging to zero as $n \to \infty$, $(u_n)_{n\in\mathbb{N}}$ is a sequence in $H_1 \times H$ and $u_\infty \in H_1^0 \times H^0$ with*

$$|u_n - u_\infty|_{X_{\epsilon_n}} \to 0 \text{ as } n \to \infty.$$

If $(t_n)_{n\in\mathbb{N}}$ is a sequence in $[0,\infty[$, $t_\infty \in [0,\infty[$ and $t_n \to t_\infty$ as $n \to \infty$, then

$$|e^{-t_nB_{\epsilon_n}}u_n - e^{-t_\infty B_0}u_\infty|_{X_{\epsilon_n}} \to 0 \text{ as } n \to \infty.$$

Proof. Note that, for each $n \in \mathbb{N}$, the vector u_n has the form $u_n = (u_{n,0}, u_{n,1})$, where $u_{n,0} \in H_1$ and $u_{n,1} \in H$. Let $(w_{0,j})_{j\in\mathbb{N}}$ be as in hypothesis (Spec)4. For every positive integer N and $\epsilon \in [0,\epsilon_0]$ let $P_{\epsilon,N}\colon X_\epsilon \to X_\epsilon$ be the X_ϵ-orthogonal projection of X_ϵ onto the subspace generated by the vectors $(w_{\epsilon,j},0)$, $(0,w_{\epsilon,j})$, $j = 1,\dots,N$. Then

$$
\begin{aligned}
|e^{-t_nB_{\epsilon_n}}u_n - e^{-t_\infty B_0}u_\infty|_{X_{\epsilon_n}} &\leq |e^{-t_nB_{\epsilon_n}}(P_{\epsilon_n,N}u_n) - e^{-t_\infty B_0}(P_{0,N}u_\infty)|_{X_{\epsilon_n}} \\
&\quad + |e^{-t_nB_{\epsilon_n}}(I-P_{\epsilon_n,N})u_n|_{X_{\epsilon_n}} + |e^{-t_\infty B_0}(I-P_{0,N})u_\infty|_{X_{\epsilon_n}} \\
&=: T^1_{n,N} + T^2_{n,N} + T^3_{n,N}.
\end{aligned}
$$

Moreover note that for fixed $\epsilon \in]0,\epsilon_0]$ the sequence $(\lambda_{\epsilon,j}{}^{-1/2}w_{\epsilon,j})_{j\in\mathbb{N}}$ is an orthonormal basis of H_1 with respect to the scalar product $\langle\cdot,\cdot\rangle_{H_1^\epsilon}$. Thus for $v = (v_0,v_1) \in X_\epsilon = H_1 \times H$ and $N \in \mathbb{N}$ we have

$$
\begin{aligned}
P_{\epsilon,N}v &= \Big(\sum_{j=1}^N \langle v_0, \lambda_{\epsilon,j}{}^{-1/2}w_{\epsilon,j}\rangle_{H_1^\epsilon}\lambda_{\epsilon,j}{}^{-1/2}w_{\epsilon,j}, \sum_{j=1}^N \langle v_1, w_{\epsilon,j}\rangle_{H^\epsilon}w_{\epsilon,j}\Big) \\
&= \sum_{j=1}^N \langle v_0, w_{\epsilon,j}\rangle_{H^\epsilon}(w_{\epsilon,j},0) + \sum_{j=1}^N \langle v_1, w_{\epsilon,j}\rangle_{H^\epsilon}(0,w_{\epsilon,j}).
\end{aligned}
$$

Similarly for $v = (v_0,v_1) \in H_1^0 \times H^0$ and $N \in \mathbb{N}$ we have

$$P_{0,N}v = \sum_{j=1}^N \langle v_0, w_{0,j}\rangle_{H^0}(w_{0,j},0) + \sum_{j=1}^N \langle v_1, w_{0,j}\rangle_{H^0}(0,w_{0,j}).$$

For $v = (v_0,v_1) \in H_1 \times H$ and $\epsilon \in]0,\epsilon_0']$

$$|v|^2_{X_\epsilon} = |v_0|^2_{H_1^\epsilon} + |v_1|^2_{H^\epsilon} \geq C'^2|v_0|^2_{H^\epsilon} + |v_1|^2_{H^\epsilon}, \tag{4}$$

where ϵ_0' and the positive constant C' were defined in Remark 2.5. The fact that $|u_n - u_\infty|_{X_{\epsilon_n}} \to 0$ as $n \to \infty$ and inequality (4) imply that

$$|u_{n,0} - u_{\infty,0}|_{H^{\epsilon_n}} \to 0 \text{ and } |u_{n,1} - u_{\infty,1}|_{H^{\epsilon_n}} \to 0 \text{ as } n \to \infty$$

Using Proposition 2.7 we thus obtain for every $j \in \mathbb{N}$

$$\begin{aligned}|\langle u_{n,0}, w_{\epsilon_n,j}\rangle_{H^{\epsilon_n}} - \langle u_{\infty,0}, w_{0,j}\rangle_{H^0}| \le |\langle u_{n,0}, w_{\epsilon_n,j}\rangle_{H^{\epsilon_n}} - \langle u_{\infty,0}, w_{\epsilon_n,j}\rangle_{H^{\epsilon_n}}| \\ + |\langle u_{\infty,0}, w_{\epsilon_n,j}\rangle_{H^{\epsilon_n}} - \langle u_{\infty,0}, w_{0,j}\rangle_{H^0}| \le |u_{n,0} - u_{\infty,0}|_{H^{\epsilon_n}} \\ + |\langle u_{\infty,0}, w_{\epsilon_n,j}\rangle_{H^{\epsilon_n}} - \langle u_{\infty,0}, w_{0,j}\rangle_{H^0}| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus

$$\langle u_{n,0}, w_{\epsilon_n,j}\rangle_{H^{\epsilon_n}} \to \langle u_{\infty,0}, w_{0,j}\rangle_{H^0} \text{ as } n \to \infty. \tag{5}$$

Similarly,

$$\langle u_{n,1}, w_{\epsilon_n,j}\rangle_{H^{\epsilon_n}} \to \langle u_{\infty,1}, w_{0,j}\rangle_{H^0} \text{ as } n \to \infty. \tag{6}$$

Since the condition (Spec)4(b) implies that $|w_{\epsilon_n,j} - w_{0,j}|_{H_1^{\epsilon_n}} \to 0$, it follows from inequality (2) that $|w_{\epsilon_n,j} - w_{0,j}|_{H^{\epsilon_n}} \to 0$ as $n \to \infty$. Formulas (5) and (6) thus imply that

$$|P_{\epsilon_n,N}u_n - P_{0,N}u_\infty|_{X_{\epsilon_n}} \to 0 \text{ as } n \to \infty. \tag{7}$$

To complete the proof we need the following lemmas:

Lemma 2.9. *For every $N \in \mathbb{N}$ the sequence $(T^1_{n,N})_{n\in\mathbb{N}}$ converges to zero as $n \to \infty$.*

Proof. For every $\lambda, a, b \in \mathbb{R}$ define the function $x(\lambda, a, b)\colon \mathbb{R} \to \mathbb{R}$ as the solution of the initial value problem

$$x''(t) + \nu x'(t) + \lambda x(t) = 0, x(0) = a, x'(0) = b.$$

Clearly the functions $(\lambda, a, b, t) \mapsto x(\lambda, a, b)(t)$ and $(\lambda, a, b, t) \mapsto x(\lambda, a, b)'(t)$ are continuous. Now note that

$$\begin{aligned}e^{-t_n B_{\epsilon_n}} P_{\epsilon_n,N}u_n &= \sum_{j=1}^{N} e^{-t_n B_{\epsilon_n}}(a_{n,j}w_{\epsilon_n,j}, b_{n,j}w_{\epsilon_n,j}) \\ &= \sum_{j=1}^{N}(x(\lambda_{\epsilon_n,j}, a_{n,j}, b_{n,j})(t_n)w_{\epsilon_n,j}, x(\lambda_{\epsilon_n,j}, a_{n,j}, b_{n,j})'(t_n)w_{\epsilon_n,j}),\end{aligned}$$

where $a_{n,j} = \langle u_{n,0}, w_{\epsilon_n,j}\rangle_{H^{\epsilon_n}}$ and $b_{n,j} = \langle u_{n,1}, w_{\epsilon_n,j}\rangle_{H^{\epsilon_n}}$. Similarly

$$e^{-t_\infty B_0} P_{0,N}u_\infty = \sum_{j=1}^{N}(x(\lambda_{0,j}, a_{0,j}, b_{0,j})(t_\infty)w_{0,j}, x(\lambda_{0,j}, a_{0,j}, b_{0,j})'(t_\infty)w_{0,j}),$$

where $a_{0,j} = \langle u_{\infty,0}, w_{0,j}\rangle_{H^0}$ and $b_{0,j} = \langle u_{\infty,1}, w_{0,j}\rangle_{H^0}$. This fact together with formulas (5) and (6) imply the lemma. □

Lemma 2.10. *For every $\delta > 0$ there exist an $N_0 = N_0(\delta) \in \mathbb{N}$ and $n_0 = n_0(\delta) \in \mathbb{N}$ such that for all $n \ge n_0$*

$$T^2_{n,N_0} < \delta \text{ and } T^3_{n,N_0} < \delta.$$

Proof. From our hypotheses and formula (7) we obtain that

$$|(I - P_{\epsilon_n,N})u_n - (I - P_{0,N})u_\infty|_{X_{\epsilon_n}} \to 0 \text{ as } n \to \infty. \tag{8}$$

Now $|(I - P_{0,N})u_\infty|_{X_{\epsilon_n}} \le C|(I - P_{0,N})u_\infty|_{X_0}$. Notice that the constant C is independent of n and N. Given $\delta > 0$ choose $N_0 \in \mathbb{N}$ such that

$$|(I - P_{0,N_0})u_\infty|_{X_0} < \delta.$$

This is possible since $|P_{0,N}u_\infty - u_\infty|_{X_0} \to 0$ as $n \to \infty$. Formula (8) implies that there exists an $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$|(I - P_{\epsilon_n,N_0})u_n - (I - P_{0,N_0})u_\infty|_{X_{\epsilon_n}} < \delta.$$

Then for all $n \geq n_0$

$$|(I - P_{\epsilon_n,N_0})u_n|_{X_{\epsilon_n}} < \delta + |(I - P_{0,N_0})u_\infty|_{X_0} < \delta + C\delta.$$

Let M be as in Remark 2.5. It follows that

$$T^2_{n,N_0} \leq M|(I - P_{\epsilon_n,N_0})u_n|_{X_{\epsilon_n}} \leq M\delta + MC\delta$$

and

$$T^3_{n,N_0} \leq CM|(I - P_{0,N_0})u_\infty|_{X_0} \leq CM\delta.$$

This completes the proof of the lemma. □

We can now complete the proof of the theorem: Recall that

$$|e^{-t_n B_{\epsilon_n}}u_n - e^{-t_\infty B_0}u_\infty|_{X_{\epsilon_n}} \leq T^1_{n,N} + T^2_{n,N} + T^3_{n,N}.$$

For $\delta > 0$ arbitrary choose $N_0 = N_0(\delta) \in \mathbb{N}$ and $n_0 = n_0(\delta) \in \mathbb{N}$ as in Lemma 2.10 such that for all $n \geq n_0$

$$T^2_{n,N_0} < \delta \text{ and } T^3_{n,N_0} < \delta.$$

By Lemma 2.9 we can choose $n_1 \geq n_0$ such that

$$T^1_{n,N_0} < \delta \text{ for all } n \geq n_1.$$

Hence for $n \geq n_1$

$$T^1_{n,N_0} + T^2_{n,N_0} + T^3_{n,N_0} < 3\delta.$$

This proves the theorem. □

3. Singular Convergence of nonlinear damped wave equations

We will now introduce a natural condition on the nonlinearities f_ϵ, $\epsilon \in [0, \epsilon_0]$, which implies a general 'singular convergence' theorem of the flows $\pi_{\nu,A_\epsilon,f_\epsilon}$ to the limit flow π_{ν,A_0,f_0}.

Definition 3.1. *(cf Definition 2.5 in [1]) Let $\epsilon_0 > 0$ be arbitrary and $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ be families satisfying condition* (Spec). *We say that the family $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ of maps satisfies hypothesis* (Conv) *if the following properties are satisfied:*

1. *$f_\epsilon\colon H^\epsilon_1 \to H^\epsilon$ for every $\epsilon \in [0, \epsilon_0]$.*
2. *$\lim_{\epsilon\to 0^+} |f_\epsilon(u) - f_0(u)|_{H^\epsilon} = 0$ for every $u \in H^0_1$.*
3. *For every $M \in [0, \infty[$ there is an $L = L_M \in [0, \infty[$ such that*

$$|f_\epsilon(u) - f_\epsilon(v)|_{H^\epsilon} \leq L|u - v|_{H^\epsilon_1}$$

for all $\epsilon \in [0, \epsilon_0]$ and u, $v \in H^\epsilon_1$ satisfying $|u|_{H^\epsilon_1}$, $|v|_{H^\epsilon_1} \leq M$.

Remark 3.2. *Let $(\tilde{f}_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ be a family of maps which satisfies hypothesis* (Conv) *and $(h_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ be a family with $h_\epsilon \in H^\epsilon$ for $\epsilon \in [0, \epsilon_0]$ and such that $|h_\epsilon - h_0|_{H^\epsilon} \to 0$ as $\epsilon \to 0^+$. Define $f_\epsilon(u) := \tilde{f}_\epsilon(u) + h_\epsilon$ for $\epsilon \in [0, \epsilon_0]$ and $u \in H^\epsilon$. Then the family $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ also satisfies hypothesis* (Conv). *This remark will be used in Section 5.*

Families of maps satisfying hypothesis (Conv) can be obtained as follows:

Example 3.3. *Let $\epsilon_0 > 0$ be arbitrary and the families $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ be as in Example 2.3. Let $G\colon \mathbb{R}\times\mathbb{R}^M\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$, $(\epsilon,x,y,\xi)\mapsto G(\epsilon,x,y,\xi)$, be a C^1-function for which there are constants β, δ and $C\in[0,\infty[$ such that for all $(\epsilon,x,y,\xi)\in\mathbb{R}\times\mathbb{R}^M\times\mathbb{R}^N\times\mathbb{R}$ the following estimates are satisfied:*

1. $|\partial_\epsilon G(\epsilon,x,y,\xi)|\le C(1+|\xi|^\beta)$;
2. $|\nabla_y G(\epsilon,x,y,\xi)|\le C(1+|\xi|^\beta)$;
3. $|\partial_\xi G(\epsilon,x,y,\xi)|\le C(1+|\xi|^\delta)$.

If $n := M+N > 2$ then we also assume that $\beta \le 2^*/2$ and $\delta\le(2^*/2)-1$, where $2^* := 2n/(n-2)$.

For $\epsilon>0$ define the map f_ϵ by

$$f_\epsilon(u)(x,y) := G(\epsilon,x,\epsilon y,u(x,y)),\ \text{for } u\in H^1(\Omega) \text{ and } (x,y)\in\Omega.$$

Furthermore, for $u\in H^1_s(\Omega)$ define $f_0(u)\colon\Omega\to\mathbb{R}$ by

$$f_0(u)(x,y) := G(0,x,0,u(x,y)) \text{ for } (x,y)\in\Omega.$$

It is easy to prove (cf. [1]) that the family $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfies hypothesis (Conv).

Remark 3.4. *Example 3.3 can be generalized to functions G taking values in $\mathbb{R}^p$ and having the argument ξ lying in $\mathbb{R}^p$. This allows applications to* systems *of damped wave equations.*

In the sequel, if the family $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ of maps satisfies hypothesis (Conv) then we will write $g_\epsilon := g_{A_\epsilon,f_\epsilon}$ and $\pi_\epsilon := \pi_{\nu,A_\epsilon,f_\epsilon}$ for every $\epsilon\in[0,\epsilon_0]$.

We can now state our main convergence result:

Theorem 3.5. *Let $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfy condition* (Spec), *and suppose that $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfies hypothesis* (Conv). *Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence in $]0,\epsilon_0]$ with $\epsilon_n\to0$ as $n\to\infty$ and $(t_n)_{n\in\mathbb{N}}$ be a sequence in $[0,\infty[$ with $t_n\to t_\infty$ as $n\to\infty$, for some $t_\infty\in[0,\infty[$. Moreover, let $(u_n)_{u\in\mathbb{N}}$ be a sequence in $H_1\times H$ such that*

$$|u_n-u_\infty|_{X_{\epsilon_n}}\to 0 \text{ as } n\to\infty$$

for some $u_\infty\in H^0_1\times H^0$. Under these assumptions if $u_\infty\pi_0t_\infty$ is defined then, for all n large enough $u_n\pi_{\epsilon_n}t_n$ is defined and

$$|u_n\pi_{\epsilon_n}t_n-u_\infty\pi_0t_\infty|_{X_{\epsilon_n}}\to 0 \text{ as } n\to\infty.$$

The proof of this theorem requires the following

Lemma 3.6. *For every $\overline{u}\in X_0$ there exist a $\delta>0$ and a $\tau>0$ such that for every $a_\infty\in X_0$ with $|a_\infty-\overline{u}|_{X_0}\le\delta$, $a_\infty\pi_0\tau$ is defined and whenever $(\epsilon_n)_{n\in\mathbb{N}}$ is a sequence in $]0,\epsilon_0]$ with $\epsilon_n\to0$ and $(a_n)_{n\in\mathbb{N}}$ is a sequence in $H_1\times H$ with*

$$|a_n-a_\infty|_{X_{\epsilon_n}}\to 0 \text{ as } n\to\infty,$$

then there is an $n_0\in\mathbb{N}$ such that $a_n\pi_{\epsilon_n}\tau$ is defined for $n\ge n_0$ and

$$\sup_{t\in[0,\tau]}|a_n\pi_{\epsilon_n}t-a_\infty\pi_0t|_{X_{\epsilon_n}}\to0$$

as $n\to\infty$.

Proof. Let $C_1 \in]0, \infty[$ be such that $|\overline{u}|_{X_0} \le C_1$. Without loss of generality we can assume that the constant C in condition (Spec)3 is such that

$$C > 1. \tag{9}$$

Set

$$M' := 2C_1 + 3CC_1 \tag{10}$$

and let $L := L_{M'}$ be as in Definition 3.1 with M replaced by M'. Since $|g_\epsilon(\overline{u}) - g_0(\overline{u})|_{X_\epsilon} \to 0$ as $\epsilon \to 0^+$ we may assume that

$$|g_0(\overline{u})|_{X_0} \le C_2$$

and

$$\sup_{\epsilon \in]0, \epsilon'_0]} |g_\epsilon(\overline{u})|_{X_\epsilon} \le C_2 < \infty$$

for some positive constant C_2.

Now choose $\tau > 0$ and $\delta > 0$ such that

$$ML\tau \le 1/2, \tag{11}$$

$$M\tau(2LC_1 + C_2) \le C_1/4, \tag{12}$$

$$(CM + C)\delta \le C_1/4 \tag{13}$$

and

$$C|e^{-tB_0}\overline{u} - \overline{u}|_{X_0} \le C_1/4 \text{ for } t \in [0, \tau], \tag{14}$$

where the positive constant M is as in Remark 2.5.

For every $\epsilon \in [0, \epsilon'_0]$ and $a \in X_\epsilon$ with

$$|a - \overline{u}|_{X_\epsilon} \le C_1, \tag{15}$$

define

$$S_{\epsilon,a} := \{\, u \mid u\colon [0, \tau] \to X_\epsilon \text{ is continuous and } |u(t) - a|_{X_\epsilon} \le C_1 \text{ for all } t \in [0, \tau] \,\}.$$

For $u \in S_{\epsilon,a}$ define the map $T_{\epsilon,a}(u)\colon [0, \tau] \to X_\epsilon$ by

$$T_{\epsilon,a}(u)(t) := e^{-tB_\epsilon} a + \int_0^t e^{-(t-s)B_\epsilon} g_\epsilon(u(s))\, \mathrm{d}s.$$

Note that whenever $u \in S_{\epsilon,a}$, then for all $t \in [0, \tau]$

$$|u(t)|_{X_\epsilon} \le C_1 + |a|_{X_\epsilon} \le C_1 + C_1 + |\overline{u}|_{X_\epsilon} \le 2C_1 + C_1C \le M',$$

where the last inequality follows from (10). Thus for all $u, v \in S_{\epsilon,a}$ arbitrary and for all $t \in [0, \tau]$, we have

$$\begin{aligned} |T_{\epsilon,a}(u)(t) - T_{\epsilon,a}(v)(t)|_{X_\epsilon} &= \Big|\int_0^t e^{-(t-s)B_\epsilon}(g_\epsilon(u(s)) - g_\epsilon(v(s)))\, \mathrm{d}s\Big|_{X_\epsilon} \\ &\le M\tau L \sup_{s \in [0,\tau]} |u(s) - v(s)|_{X_\epsilon} \le 1/2 \sup_{s \in [0,\tau]} |u(s) - v(s)|_{X_\epsilon}. \end{aligned} \tag{16}$$

Notice that the last inequality follows from (11). Moreover for all $t \in [0, \tau]$

$$|T_{\epsilon,a}(u)(t) - a|_{X_\epsilon} \le |e^{-tB_\epsilon} a - a|_{X_\epsilon} + \Big|\int_0^t e^{-(t-s)B_\epsilon} g_\epsilon(u(s))\, \mathrm{d}s\Big|_{X_\epsilon} =: K^1_\epsilon + K^2_\epsilon.$$

Hence

$$K_\epsilon^2 \le M\tau \sup_{s\in[0,\tau]} |g_\epsilon(u(s))|_{X_\epsilon}.$$

Now for $s \in [0,\tau]$ we have

$$\begin{aligned}|g_\epsilon(u(s))|_{X_\epsilon} &\le |g_\epsilon(u(s)) - g_\epsilon(a)|_{X_\epsilon} + |g_\epsilon(a)|_{X_\epsilon}\\ &\le L|u(s)-a|_{X_\epsilon} + |g_\epsilon(a) - g_\epsilon(\overline{u})|_{X_\epsilon} + |g_\epsilon(\overline{u})|_{X_\epsilon}\\ &\le LC_1 + LC_1 + C_2 = 2LC_1 + C_2.\end{aligned}$$

Thus, it follows from (12) that $K_\epsilon^2 \le M\tau(2LC_1+C_2) \le C_1/4$. In the previous computation we used the fact that

$$|a|_{X_\epsilon} \le C_1 + |\overline{u}|_{X_\epsilon} \le C_1 + CC_1 \le M'$$

and $|\overline{u}|_{X_\epsilon} \le M'$.

If $a_0 \in X_0$ satisfies $|a_0 - \overline{u}|_{X_0} \le \delta$, then

$$\begin{aligned}K_\epsilon^1 &\le |e^{-tB_\epsilon}a - e^{-tB_0}a_0|_{X_\epsilon} + |e^{-tB_0}a_0 - e^{-tB_0}\overline{u}|_{X_\epsilon}\\ &\quad + |e^{-tB_0}\overline{u} - \overline{u}|_{X_\epsilon} + |\overline{u} - a_0|_{X_\epsilon} + |a - a_0|_{X_\epsilon}\\ &\le |e^{-tB_\epsilon}a - e^{-tB_0}a_0|_{X_\epsilon} + |a-a_0|_{X_\epsilon} + CM\delta + C|e^{-tB_0}\overline{u} - \overline{u}|_{X_0} + C\delta\\ &\le |e^{-tB_\epsilon}a - e^{-tB_0}a_0|_{X_\epsilon} + |a-a_0|_{X_\epsilon} + C_1/2,\end{aligned}$$

where the last inequality follows from (13) and (14). Thus putting things together, we obtain for all $\epsilon \in [0,\epsilon_0']$, all $a \in X_\epsilon$ satisfying (15) and all $a_0 \in X_0$ with $|a_0 - \overline{u}|_{X_0} \le \delta$ that

$$|T_{\epsilon,a}(u)(t) - a|_{X_\epsilon} \le 3C_1/4 + |e^{-tB_\epsilon}a - e^{-tB_0}a_0|_{X_\epsilon} + |a - a_0|_{X_\epsilon}. \tag{17}$$

In particular, for all $a \in X_0$ satisfying $|a - \overline{u}|_{X_0} \le \delta$ we have that

$$|T_{0,a}(u)(t) - a|_{X_0} \le 3C_1/4 \le C_1 \text{ for all } t \in [0,\tau].$$

Hence we conclude that $T_{0,a}(S_{0,a}) \subset S_{0,a}$ and so, by Banach Fixed Point Theorem, there is a unique fixed point of $T_{0,a}$ in $S_{0,a}$. In particular $a\pi_0\tau$ is defined.

Now let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence in $]0,\epsilon_0]$ with $\epsilon_n \to 0$ as $n \to \infty$. Suppose $a_\infty \in X_0$ satisfies $|a_\infty - \overline{u}|_{X_0} \le \delta$ and $(a_n)_{u\in\mathbb{N}}$ is a sequence in $H_1 \times H$ with $|a_n - a_\infty|_{X_{\epsilon_n}} \to 0$ as $n \to \infty$. By what we just proved, it follows that $a_\infty\pi_0\tau$ is defined. Theorem 2.8 implies that there is an $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$

$$\sup_{t\in[0,\tau]} |e^{-tB_{\epsilon_n}}a_n - e^{-tB_0}a_\infty|_{X_{\epsilon_n}} + |a_n - a_\infty|_{X_{\epsilon_n}} \le C_1/4.$$

Thus, it follows from (17) that $T_{\epsilon_n,a_n}(S_{\epsilon_n,a_n}) \subset S_{\epsilon_n,a_n}$ for all $n \ge n_0$. So T_{ϵ_n,a_n} has a fixed point in S_{ϵ_n,a_n}. In particular $a_n\pi_{\epsilon_n}\tau$ is defined for all $n \ge n_0$. Thus for all $n \ge n_0$ we have

$$\begin{aligned}|a_n\pi_{\epsilon_n}t - a_\infty\pi_0 t|_{X_{\epsilon_n}} &\le |e^{-tB_{\epsilon_n}}a_n - e^{-tB_0}a_\infty|_{X_{\epsilon_n}}\\ &\quad + \left|\int_0^t (e^{-(t-s)B_{\epsilon_n}}g_{\epsilon_n}(a_n\pi_{\epsilon_n}s) - e^{-(t-s)B_0}g_0(a_\infty\pi_0 s))\,ds\right|_{X_{\epsilon_n}}\end{aligned}$$

Now for $s \in [0,\tau]$ we have

$$\begin{aligned}|a_\infty\pi_0 s|_{X_{\epsilon_n}} &\le C|a_\infty\pi_0 s|_{X_0} \le C(|a_\infty\pi_0 s - a_\infty|_{X_0} + |a_\infty - \overline{u}|_{X_0} + |\overline{u}|_{X_0})\\ &\le 3CC_1 \le M'.\end{aligned}$$

Therefore we have that

$$(18)\quad |a_n\pi_{\epsilon_n}t - a_\infty\pi_0 t|_{X_{\epsilon_n}} \le |e^{-tB_{\epsilon_n}}a_n - e^{-tB_0}a_\infty|_{X_{\epsilon_n}} + M\tau L \sup_{s\in[0,\tau]} |a_n\pi_{\epsilon_n}s - a_\infty\pi_0 s|_{X_{\epsilon_n}}$$
$$+ |\int_0^t (e^{-(t-s)B_{\epsilon_n}}g_{\epsilon_n}(a_\infty\pi_0 s) - e^{-(t-s)B_0}g_0(a_\infty\pi_0 s))\,\mathrm{d}s|_{X_{\epsilon_n}}.$$

Now

$$|\int_0^t (e^{-(t-s)B_{\epsilon_n}}g_{\epsilon_n}(a_\infty\pi_0 s) - e^{-(t-s)B_0}g_0(a_\infty\pi_0 s))\,\mathrm{d}s|_{X_{\epsilon_n}}$$
$$\le \tau \sup_{t\in[0,\tau]}\sup_{s\in[0,t]} |e^{-(t-s)B_{\epsilon_n}}g_{\epsilon_n}(a_\infty\pi_0 s) - e^{-(t-s)B_0}g_0(a_\infty\pi_0 s)|_{X_{\epsilon_n}}$$

We claim that

$$(19)\qquad \sup_{t\in[0,\tau]}\sup_{s\in[0,t]} |e^{-(t-s)B_{\epsilon_n}}g_{\epsilon_n}(a_\infty\pi_0 s) - e^{-(t-s)B_0}g_0(a_\infty\pi_0 s)|_{X_{\epsilon_n}} \to 0 \text{ as } n\to\infty.$$

Otherwise there is a sequence $(t_n)_{n\in\mathbb{N}}$ in $[0,\tau]$ and there is a sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n\in[0,t_n]$ for all $n\in\mathbb{N}$ such that

$$|e^{-(t_n-s_n)B_{\epsilon_n}}g_{\epsilon_n}(a_\infty\pi_0 s_n) - e^{-(t_n-s_n)B_0}g_0(a_\infty\pi_0 s_n)|_{X_{\epsilon_n}} \ge \delta_0$$

for some $\delta_0>0$ and all $n\in\mathbb{N}$. Without loss of generality we can assume that $t_n\to t$ and $s_n\to s$ as $n\to\infty$ with $s\le t$. Notice that

$$|g_{\epsilon_n}(a_\infty\pi_0 s_n) - g_0(a_\infty\pi_0 s)|_{X_{\epsilon_n}} \le |g_{\epsilon_n}(a_\infty\pi_0 s_n) - g_{\epsilon_n}(a_\infty\pi_0 s)|_{X_{\epsilon_n}}$$
$$+ |g_{\epsilon_n}(a_\infty\pi_0 s) - g_0(a_\infty\pi_0 s)|_{X_{\epsilon_n}}$$
$$\le LC|a_\infty\pi_0 s_n - a_\infty\pi_0 s|_{X_0} + |g_{\epsilon_n}(a_\infty\pi_0 s) - g_0(a_\infty\pi_0 s)|_{X_{\epsilon_n}}.$$

Hence $|g_{\epsilon_n}(a_\infty\pi_0 s_n) - g_0(a_\infty\pi_0 s)|_{X_{\epsilon_n}} \to 0$ as $n\to\infty$. Thus Theorem 2.8 implies that

$$(20)\qquad |e^{-(t_n-s_n)B_{\epsilon_n}}g_{\epsilon_n}(a_\infty\pi_0 s_n) - e^{-(t-s)B_0}g_0(a_\infty\pi_0 s)|_{X_{\epsilon_n}} \to 0 \text{ as } n\to\infty.$$

Notice that $|g_0(a_\infty\pi_0 s_n) - g_0(a_\infty\pi_0 s)|_{X_0} \to 0$ as $n\to\infty$ and

$$|e^{-(t_n-s_n)B_0}g_0(a_\infty\pi_0 s_n) - e^{-(t-s)B_0}g_0(a_\infty\pi_0 s)|_{X_{\epsilon_n}}$$
$$\le C|e^{-(t_n-s_n)B_0}g_0(a_\infty\pi_0 s_n) - e^{-(t-s)B_0}g_0(a_\infty\pi_0 s)|_{X_0}.$$

Therefore $|e^{-(t_n-s_n)B_0}g_0(a_\infty\pi_0 s_n) - e^{-(t-s)B_0}g_0(a_\infty\pi_0 s)|_{X_{\epsilon_n}} \to 0$ $n\to\infty$. This fact together with (20) yields a contradiction which proves (19).

Putting (18) and (19) together, using Theorem 2.8 and the fact that $M\tau L\le 1/2$ we see that

$$\sup_{t\in[0,\tau]} |a_n\pi_{\epsilon_n}t - a_\infty\pi_0 t|_{X_{\epsilon_n}} \to 0$$

as $n\to\infty$. The lemma is proved. □

We can now give a

Proof of Theorem 3.5. Since $u_\infty\pi_0 t_\infty$ is defined, there is a $b > t_\infty$, $b \in \,]0,\infty[$, such that $u_\infty\pi_0 t$ is defined for all $t \in [0, b[$. Define

$$I := \{\, t \in [0,b[\mid \text{there exists an } n_0 \in \mathbb{N} \text{ such that } u_n\pi_{\epsilon_n} t \text{ is defined for } n \geq n_0 \text{ and } \sup_{s\in[0,t]} |u_n\pi_{\epsilon_n} s - u_\infty\pi_0 s|_{X_{\epsilon_n}} \to 0 \text{ as } n \to \infty \,\}.$$

It is clear that $0 \in I$. Furthermore if $0 \leq t' < t$ and $t \in I$, then $t' \in I$. Let

$$\bar{t} := \sup I.$$

It follows that $\bar{t} \leq b$ and so $[0, \bar{t}[\,\subset I$.

An application of Lemma 3.6 with $\overline{u} := u_\infty$ shows that $\bar{t} > 0$. We claim that $\bar{t} = b$. Suppose, on the contrary, that $\bar{t} < b$. It follows that $\overline{u} := u_\infty\pi_0\bar{t}$ is defined. Let $\delta > 0$ and $\tau > 0$ be as in Lemma 3.6 with respect to this choice of $\overline{u}$.

Choose $t \in \mathbb{R}$ with $0 < t < \bar{t} < t + \tau$ and $|u_\infty\pi_0 t - u_\infty\pi_0\bar{t}|_{X_0} < \delta$. We have that $t \in I$ so there exists an $n_0 \in \mathbb{N}$ such that $u_n\pi_{\epsilon_n} t$ is defined for all $n \geq n_0$ and

$$\sup_{s\in[0,t]} |u_n\pi_{\epsilon_n} s - u_\infty\pi_0 s|_{X_{\epsilon_n}} \to 0 \text{ as } n \to \infty. \tag{21}$$

Set $\tilde{u}_n := u_n\pi_{\epsilon_n} t$ and $\tilde{u} := u_\infty\pi_0 t$. Applying Lemma 3.6 with a_n replaced by $\tilde{u}_n$ and a_∞ replaced by $\tilde{u}$ we thus have that $\tilde{u}\pi_0\tau$ is defined and we obtain the existence of an $n_1 \geq n_0$ such that $\tilde{u}_n\pi_{\epsilon_n}\tau$ is defined for all $n \geq n_1$ and

$$\sup_{s\in[0,\tau]} |\tilde{u}_n\pi_{\epsilon_n} s - \tilde{u}\pi_0 s|_{X_{\epsilon_n}} \to 0 \text{ as } n \to \infty. \tag{22}$$

Formulas (21) and (22) imply that $u_\infty\pi_0(t+\tau)$ is defined, $u_n\pi_{\epsilon_n}(t+\tau)$ is also defined for all $n \geq n_1$ and

$$\sup_{s\in[0,t+\tau]} |u_n\pi_{\epsilon_n} s - u_\infty\pi_0 s|_{X_{\epsilon_n}} \to 0 \text{ as } n \to \infty.$$

Thus $t + \tau \in I$, but $t + \tau > \bar{t}$, a contradiction, which proves that $\bar{t} = b$.

Since $t_\infty \in [0, b[$, it follows that there is a $t \in [0, b[$ with $t_\infty < t$ and $t_n < t$ for all n large enough. In particular $u_\infty\pi_0 t_\infty$ is defined and $u_\infty\pi_0 t_n$, $u_n\pi_{\epsilon_n} t_n$ are defined for all n large enough and

$$|u_n\pi_{\epsilon_n} t_n - u_\infty\pi_0 t_n|_{X_{\epsilon_n}} \to 0 \text{ as } n \to \infty.$$

Since, by condition (Spec)3,

$$|u_\infty\pi_0 t_n - u_\infty\pi_0 t_\infty|_{X_{\epsilon_n}} \leq C|u_\infty\pi_0 t_n - u_\infty\pi_0 t_\infty|_{X_0}$$

and $|u_\infty\pi_0 t_n - u_\infty\pi_0 t_\infty|_{X_0} \to 0$ as $n \to \infty$, the theorem follows. □

4. Admissibility

In this section we shall prove that under a certain abstract compactness hypothesis on the family $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ the corresponding family of flows satisfies a singular asymptotic compactness property. This property, together with the singular convergence result obtained in the previous section, is all that we need to establish the singular Conley index continuation principle for damped wave equations on thin domains.

We begin with a definition:

Definition 4.1. *We say that the family* $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *satisfies hypothesis* (Comp) *if the following hypotheses are satisfied:*

1. *For every* $\epsilon \in [0,\epsilon_0]$ *the map* $f_\epsilon\colon H_1^\epsilon \to H^\epsilon$ *is compact, i.e. maps bounded sets in* $(H_1^\epsilon, |\cdot|_{H_1^\epsilon})$ *into compact sets in* $(H^\epsilon, |\cdot|_{H^\epsilon})$.
2. *Whenever* $(\epsilon_n)_{n\in\mathbb{N}}$ *is a sequence of positive numbers converging to zero as* $n\to\infty$, $\tilde{C}\in[0,\infty[$ *and* $(\xi_n)_{n\in\mathbb{N}}$ *is a sequence in* H_1 *such that*
$$\sup_{n\in\mathbb{N}} |\xi_n|_{H_1^{\epsilon_n}} \le \tilde{C},$$
then there is a sequence $(n_k)_{k\in\mathbb{N}}$ *in* $\mathbb{N}$ *with* $n_k\to\infty$ *as* $k\to\infty$ *such that*
$$|f_{\epsilon_{n_k}}(\xi_{n_k}) - v|_{H^{\epsilon_n}} \to 0 \text{ as } k\to\infty$$
for some $v\in H^0$.

Example 4.2. *Let* $\epsilon_0>0$ *be arbitrary and the families* $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ *and* $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *be as in Example 2.3. Let* $G\colon \mathbb{R}\times\mathbb{R}^M\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ *and the family* $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *be as in Example 3.3. If* $n := M+N > 2$ *then assume also that* $\delta < (2^*/2)-1$, *where* $2^* := 2n/(n-2)$ *(subcritical growth). Then* $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *satisfies hypothesis* (Comp).

In the sequel, given a subset S of a linear space E we denote by $\operatorname{co} S$ the convex hull of S.

We now obtain the first admissibility result:

Theorem 4.3. *(Cf Theorem III 4.4 in* [14]*) Let* $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ *and* $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *satisfy condition* (Spec). *Suppose the family of maps* $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *satisfies hypotheses* (Conv) *and* (Comp). *Let* $\epsilon\in[0,\epsilon_0]$ *be arbitrary. Suppose* N *is a closed and bounded set in* X_ϵ, $(t_n)_{n\in\mathbb{N}}$ *is a sequence of positive numbers such that* $t_n\to\infty$ *as* $n\to\infty$ *and* $(a_n)_{n\in\mathbb{N}}$ *is a sequence in* X_ϵ *such that*
$$a_n\pi_\epsilon[0,t_n]\subset N \text{ for all } n\in\mathbb{N}.$$
Then there is a sequence $(n_k)_{k\in\mathbb{N}}$ *in* $\mathbb{N}$ *with* $n_k\to\infty$ *as* $k\to\infty$ *such that*
$$|a_{n_k}\pi_\epsilon t_{n_k} - v|_{X_\epsilon} \to 0 \text{ as } k\to\infty,$$
for some $v\in X_\epsilon$.

Remark 4.4. *In the terminology of* [14], *the set* N *is* π_ϵ-admissible.

Proof. Let $\beta := \beta_\epsilon$ be the Kuratowski measure of noncompactness on X_ϵ (cf [4]). Let $\delta>0$ be arbitrary. Then there is an $r := r(\delta)\in\,]0,\infty[$ such that
$$Me^{-\alpha r} < \delta,$$
where the constants M and α are as in Remark 2.5. Since $t_n\to\infty$ as $n\to\infty$, there is an $n_0\in\mathbb{N}$ such that $t_n\ge r$ for all $n\ge n_0$. Therefore for every $n\ge n_0$ we have
$$a_n\pi_\epsilon t_n = e^{-rB_\epsilon}a_n\pi_\epsilon(t_n-r) + \int_0^r e^{-(r-s)B_\epsilon}g_\epsilon(a_n\pi_\epsilon(s+t_n-r))\,ds.$$
By hypothesis (Comp) there is a compact set K_1 in X_ϵ such that
$$g_\epsilon(a_n\pi_\epsilon(s+t_n-r))\in K_1 \text{ for all } n\in\mathbb{N} \text{ and } s\in[0,r].$$

Let $K_2 := \{ e^{-(r-s)B_\epsilon} v \mid v \in K_1 \text{ and } s \in [0,r] \}$. Of course K_2 is compact in X_ϵ. Thus

$$\int_0^r e^{-(r-s)B_\epsilon} g_\epsilon(a_n\pi_\epsilon(s+t_n-r))\, ds \in K_3 \text{ for all } n \in \mathbb{N},$$

where $K_3 = r \operatorname{cl} \operatorname{co} K_2$. Notice that the set K_3 is compact by Mazur's Theorem and so we obtain

$$\begin{aligned}\beta(\{ a_n\pi_\epsilon t_n \mid n \in \mathbb{N}\}) &= \beta(\{ a_n\pi_\epsilon t_n \mid n \geq n_0 \}) \\ &\leq \beta(\{ e^{-rB_\epsilon} a_n\pi_\epsilon(t-r) \mid n \geq n_0 \}) + \beta(K_3) \\ &\leq 2MC_N e^{-\alpha r} \leq 2C_N\delta,\end{aligned}$$

where $C_N := \sup_{u\in N} |u|_{X_\epsilon} < \infty$. Since $\delta > 0$ is arbitrary, it follows that

$$\beta(\{ a_n\pi_\epsilon t_n \mid n \in \mathbb{N}\}) = 0$$

so $(a_n\pi_\epsilon t_n)_{n\in\mathbb{N}}$ has a subsequence converging in X_ϵ. This proves the theorem. □

Recall that Proposition 2.6 implies that for every $\epsilon \in]0, \epsilon_0]$, the set $X_0 = H^0_1 \times H^0$ is a closed subspace of X_ϵ. For each $\epsilon \in]0, \epsilon_0]$, let $Q_\epsilon \colon X_\epsilon \to X_\epsilon$ be the X_ϵ-orthogonal projection of X_ϵ onto X_0.

We can now state our second admissibility result:

Theorem 4.5. *Let $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfy condition* (Spec). *Suppose the family of maps $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfies hypotheses* (Conv) *and* (Comp). *Suppose $(\epsilon_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers converging to zero as $n \to \infty$, $(t_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers such that $t_n \to \infty$ as $n \to \infty$ and $(a_n)_{n\in\mathbb{N}}$ is a sequence in $H_1 \times H$. Assume that there exists a $C'' \in]0,\infty[$ such that*

$$|a_n\pi_{\epsilon_n} s|_{X_{\epsilon_n}} \leq C'' \text{ for all } n \in \mathbb{N} \text{ and for all } s \in [0, t_n].$$

Then there exist a $v \in X_0$ and a sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$|a_{n_k}\pi_{\epsilon_{n_k}} t_{n_k} - v|_{X_{\epsilon_{n_k}}} \to 0 \text{ as } k \to \infty.$$

Proof. Let $\beta := \beta_0$ be the Kuratowski measure of noncompactness on X_0. Let $\delta > 0$ be arbitrary. Then there is an $r := r(\delta) \in]0,\infty[$ such that

$$Me^{-\alpha r} < \delta,$$

where the constants M and α are as in Remark 2.5. Since $t_n \to \infty$ as $n \to \infty$, there is an $n_0 \in \mathbb{N}$ such that $t_n \geq r$ for all $n \geq n_0$. Therefore for every $n \geq n_0$ we have

$$Q_{\epsilon_n}(a_n\pi_{\epsilon_n}t_n) = Q_{\epsilon_n}(e^{-rB_{\epsilon_n}} a_n\pi_{\epsilon_n}(t_n-r)) + \int_0^r Q_{\epsilon_n} e^{-(r-s)B_{\epsilon_n}} g_{\epsilon_n}(a_n\pi_{\epsilon_n}(s+t_n-r))\, ds.$$

Using condition (Spec)3 we have

$$\begin{aligned}|Q_{\epsilon_n}(e^{-rB_{\epsilon_n}} a_n\pi_{\epsilon_n}(t_n-r))|_{X_0} &\leq C|Q_{\epsilon_n}(e^{-rB_{\epsilon_n}} a_n\pi_{\epsilon_n}(t_n-r))|_{X_{\epsilon_n}} \\ &\leq CMe^{-\alpha r}|a_n\pi_{\epsilon_n}(t_n-r)|_{X_{\epsilon_n}} \leq CMe^{-\alpha r}C'',\end{aligned}$$

Thus

$$\beta(\{ Q_{\epsilon_n}(e^{-rB_{\epsilon_n}} a_n\pi_{\epsilon_n}(t_n-r)) \mid n \geq n_0 \}) \leq 2MCC''\delta. \tag{23}$$

Let

$$K := \{ Q_{\epsilon_n} e^{-(r-s)B_{\epsilon_n}} g_{\epsilon_n}(a_n\pi_{\epsilon_n}(s+t_n-r)) \mid n \in \mathbb{N} \text{ and } s \in [0,r] \}.$$

We claim that K is relatively compact in X_0. In fact, let $(n_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ and $(s_k)_{k\in\mathbb{N}}$ be a sequence in $[0,r]$. We have to show that

$$\left(Q_{\epsilon_{n_k}} e^{-(r-s_k)B_{\epsilon_{n_k}}} g_{\epsilon_{n_k}}(a_{n_k}\pi_{\epsilon_{n_k}}(s_k+t_{n_k}-r))\right)_{k\in\mathbb{N}}$$

has a subsequence converging to some element in X_0. We may assume that there exists a $s\in[0,r]$ such that $s_k\to s$ as $k\to\infty$. We will consider two cases.

First suppose $(n_k)_{k\in\mathbb{N}}$ is a bounded sequence in $\mathbb{N}$. Therefore we may assume that $n_k=m$ for some $m\in\mathbb{N}$. Since $|a_m\pi_{\epsilon_m}(s_k+t_m-r)|_{X_{\epsilon_m}}\le C''$, hypothesis (Comp)1 implies that we may assume that

$$|g_{\epsilon_m}(a_m\pi_{\epsilon_m}(s_k+t_m-r))-v|_{X_{\epsilon_m}}\to 0 \text{ as } k\to\infty$$

for some $v\in X_{\epsilon_m}$. This implies that

$$|Q_{\epsilon_m}e^{-(r-s_k)B_{\epsilon_m}}g_{\epsilon_m}(a_m\pi_{\epsilon_m}(s_k+t_m-r))-Q_{\epsilon_m}e^{-(r-s_k)B_{\epsilon_m}}v|_{X_{\epsilon_m}}\to 0 \text{ as } k\to\infty.$$

Now we consider the case for which $(n_k)_{k\in\mathbb{N}}$ is an unbounded sequence in $\mathbb{N}$. Then without loss of generality we suppose $n_k\to\infty$ as $k\to\infty$ and so $(\epsilon_{n_k})_{k\in\mathbb{N}}$ is a sequence of positive numbers converging to zero as $k\to\infty$. Using hypothesis (Comp)2 we may also assume that

$$|g_{\epsilon_{n_k}}(a_{n_k}\pi_{\epsilon_{n_k}}(s_k+t_{n_k}-r))-v|_{X_{\epsilon_{n_k}}}\to 0 \text{ as } k\to\infty$$

for some $v\in X_0$. Since

$$\begin{aligned}|Q_{\epsilon_{n_k}}&e^{-(r-s_k)B_{\epsilon_{n_k}}}g_{\epsilon_{n_k}}(a_{n_k}\pi_{\epsilon_{n_k}}(s_k+t_{n_k}-r))-e^{-(r-s)B_0}v|_{X_{\epsilon_{n_k}}}\\ &=|Q_{\epsilon_{n_k}}e^{-(r-s_k)B_{\epsilon_{n_k}}}g_{\epsilon_{n_k}}(a_{n_k}\pi_{\epsilon_{n_k}}(s_k+t_{n_k}-r))-Q_{\epsilon_{n_k}}e^{-(r-s)B_0}v|_{X_{\epsilon_{n_k}}}\\ &\qquad\le|e^{-(r-s_k)B_{\epsilon_{n_k}}}g_{\epsilon_{n_k}}(a_{n_k}\pi_{\epsilon_{n_k}}(s_k+t_{n_k}-r))-e^{-(r-s)B_0}v|_{X_{\epsilon_{n_k}}},\end{aligned}$$

Theorem 2.8 implies

$$|Q_{\epsilon_{n_k}}e^{-(r-s_k)B_{\epsilon_{n_k}}}g_{\epsilon_{n_k}}(a_{n_k}\pi_{\epsilon_{n_k}}(s_k+t_{n_k}-r))-e^{-(r-s)B_0}v|_{X_{\epsilon_{n_k}}}\to 0 \text{ as } k\to\infty.$$

The claim follows.

Thus, as before, we have

$$\begin{aligned}(24)\quad \beta(\{\,Q_{\epsilon_n}(a_n\pi_\epsilon t_n)\mid n\in\mathbb{N}\})=\beta(\{\,Q_{\epsilon_n}(e^{-rB_{\epsilon_n}}a_n\pi_{\epsilon_n}(t_n-r))\mid n\ge n_0\,\})\\ +\beta(r\,\mathrm{cl}\,\mathrm{co}\,\mathrm{cl}\,K)\le 2MCC''\delta,\end{aligned}$$

where the last inequality follows from (23). Since $\delta>0$ is arbitrary, the relation in (24) implies that there exist a sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ with $n_k\to\infty$ as $k\to\infty$ and $v\in X_0$ such that

$$|Q_{\epsilon_{n_k}}(a_{n_k}\pi_{\epsilon_{n_k}}t_{n_k})-v|_{X_0}\to 0 \text{ as } k\to\infty.$$

Since condition (Spec)3 implies

$$|Q_{\epsilon_{n_k}}(a_{n_k}\pi_{\epsilon_{n_k}}t_{n_k})-v|_{X_{\epsilon_{n_k}}}\le C|Q_{\epsilon_{n_k}}(a_{n_k}\pi_{\epsilon_{n_k}}t_{n_k})-v|_{X_0},$$

it follows that

$$|Q_{\epsilon_{n_k}}(a_{n_k}\pi_{\epsilon_{n_k}}t_{n_k})-v|_{X_{\epsilon_{n_k}}}\to 0 \text{ as } k\to\infty. \tag{25}$$

Therefore to prove the theorem it is sufficient to show that

$$|(I - Q_{\epsilon_n})a_n\pi_{\epsilon_n}t_n|_{X_{\epsilon_n}} \to 0 \text{ as } n \to \infty. \tag{26}$$

Let $\delta > 0$ be arbitrary and choose $r > 0$ and $n_0 \in \mathbb{N}$ as before. For all $n \geq n_0$ we obtain

$$\begin{aligned} |(I - Q_{\epsilon_n})a_n\pi_{\epsilon_n}t_n|_{X_{\epsilon_n}} &\leq |(I - Q_{\epsilon_n})e^{-rB_{\epsilon_n}}a_n\pi_{\epsilon_n}(t_n - r)|_{X_{\epsilon_n}} \\ &\quad + |\int_0^r (I - Q_{\epsilon_n})e^{-(r-s)B_{\epsilon_n}}g_{\epsilon_n}(a_n\pi_{\epsilon_n}(s + t_n - r))\, ds|_{X_{\epsilon_n}} \\ &\leq Me^{-\alpha r}C'' + r\delta_n \leq \delta C'' + r\delta_n, \end{aligned}$$

where $\delta_n := \sup_{s\in[0,r]} |(I - Q_{\epsilon_n})e^{-(r-s)B_{\epsilon_n}}g_{\epsilon_n}(a_n\pi_{\epsilon_n}(s + t_n - r))\, ds|_{X_{\epsilon_n}}$.

We claim that $\delta_n \to 0$ as $n \to \infty$. Suppose this is not true. Then for some subsequence $(\delta_{n_k})_{k\in\mathbb{N}}$ of the sequence $(\delta_n)_{n\in\mathbb{N}}$ and for some $\delta' > 0$ we have

$$\delta_{n_k} > \delta' \text{ for all } k \in \mathbb{N}.$$

Thus there is a sequence $(s_k)_{k\in\mathbb{N}}$ in $[0, r]$ with

$$|(I - Q_{\epsilon_{n_k}})e^{-(r-s_k)B_{\epsilon_{n_k}}}g_{\epsilon_{n_k}}(a_{n_k}\pi_{\epsilon_{n_k}}(s_k + t_{n_k} - r))|_{X_{\epsilon_{n_k}}} \geq \delta' > 0 \text{ for all } k \in \mathbb{N}. \tag{27}$$

We may assume that there exists a $s \in [0, r]$ such that $s_k \to s$, as $k \to \infty$. Since

$$|a_{n_k}\pi_{\epsilon_{n_k}}(s_k + t_{n_k} - r)|_{X_{\epsilon_{n_k}}} \leq C'' \text{ for all } k \in \mathbb{N},$$

we may also assume, using hypothesis (Comp), that

$$|g_{\epsilon_{n_k}}(a_{n_k}\pi_{\epsilon_{n_k}}(s_k + t_{n_k} - r)) - w|_{X_{\epsilon_{n_k}}} \to 0 \text{ as } k \to \infty$$

for some $w \in X_0$. Therefore Theorem 2.8 implies

$$|e^{-(r-s_k)B_{\epsilon_{n_k}}}g_{\epsilon_{n_k}}(a_{n_k}\pi_{\epsilon_{n_k}}(s_k + t_{n_k} - r)) - e^{-(r-s)B_0}w|_{X_{\epsilon_{n_k}}} \to 0 \text{ as } k \to \infty.$$

Since $(I - Q_{\epsilon_{n_k}})e^{-(r-s)B_0}w = 0$, we obtain

$$\begin{aligned} &|(I - Q_{\epsilon_{n_k}})e^{-(r-s_k)B_{\epsilon_{n_k}}}g_{\epsilon_{n_k}}(a_{n_k}\pi_{\epsilon_{n_k}}(s_k + t_{n_k} - r))|_{X_{\epsilon_{n_k}}} \\ &\quad = |(I - Q_{\epsilon_{n_k}})e^{-(r-s_k)B_{\epsilon_{n_k}}}g_{\epsilon_{n_k}}(a_{n_k}\pi_{\epsilon_{n_k}}(s_k + t_{n_k} - r)) - (I - Q_{\epsilon_{n_k}})e^{-(r-s)B_0}w|_{X_{\epsilon_{n_k}}} \\ &\quad \leq |e^{-(r-s_k)B_{\epsilon_{n_k}}}g_{\epsilon_{n_k}}(a_{n_k}\pi_{\epsilon_{n_k}}(s_k + t_{n_k} - r)) - e^{-(r-s)B_0}w|_{X_{\epsilon_{n_k}}}. \end{aligned}$$

But this contradicts (27). The theorem is proved. □

Theorem 4.5 has the following corollaries:

Corollary 4.6. *Let $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfy condition* (Spec). *Suppose the family of maps $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfies hypotheses* (Conv) *and* (Comp). *Suppose $N \subset X_0$ is a closed and bounded set and $(\epsilon_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers converging to zero as $n \to \infty$. Let $\eta > 0$ be arbitrary and define*

$$N_n = N_{n,\eta} := \{\, u \in X_{\epsilon_n} \mid Q_{\epsilon_n}u \in N \text{ and } |(I - Q_{\epsilon_n})u|_{X_{\epsilon_n}} \leq \eta \,\}.$$

Suppose $(t_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers such that $t_n \to \infty$ as $n \to \infty$ and $(a_n)_{n\in\mathbb{N}}$ is a sequence in $H_1 \times H$ such that

$$a_n\pi_{\epsilon_n}[0, t_n] \subset N_n \text{ for all } n \in \mathbb{N}.$$

Then there exist a $v \in N$ and a sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$|a_{n_k}\pi_{\epsilon_{n_k}}t_{n_k} - v|_{X_{\epsilon_{n_k}}} \to 0 \text{ as } k \to \infty.$$

Corollary 4.7. *Let $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfy condition* (Spec). *Suppose the family of maps $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfies hypotheses* (Conv) *and* (Comp). *Suppose $(\epsilon_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers converging to zero as $n \to \infty$. For every $n \in \mathbb{N}$, let $\sigma_n \colon J_n \to X_{\epsilon_n}$ be a solution of π_{ϵ_n}. Assume there is a constant $C'' \in [0,\infty[$ such that*

$$|\sigma_n(t)|_{X_{\epsilon_n}} \leq C'' \text{ for all } n \in \mathbb{N} \text{ and for all } t \in J_n.$$

If $J_n \equiv \mathbb{R}$ for every $n \in \mathbb{N}$ (resp. if $J_n = [-t_n, 0]$, $n \in \mathbb{N}$, where $t_n \to \infty$), then there is a subsequence of $(\sigma_n)_{n\in\mathbb{N}}$, again denoted by $(\sigma_n)_{n\in\mathbb{N}}$ and there is a solution $\sigma \colon \mathbb{R} \to X_0$ (resp. $\sigma\colon\ ]-\infty,0] \to X_0$) of π_0 such that for every $t \in \mathbb{R}$ (resp. every $t \in\ ]-\infty,0]$)

$$|\sigma_n(t) - \sigma(t)|_{X_{\epsilon_n}} \to 0 \text{ as } n \to \infty.$$

5. Upper semicontinuity of attractors and Conley index continuation

In this final section we prove that, under the conditions (Spec), (Conv), (Comp) introduced in the previous section and the additional condition (Liap), all flows π_ϵ, $\epsilon \in [0, \epsilon_0]$, have a global attractor and the family of attractor is upper semicontinuous at $\epsilon = 0$.

We start with the following essentially known result:

Proposition 5.1. *Let X be a Banach space and $B \colon D(B) \to X$ be a linear operator such that $-B$ is a generator of a C^0-semigroup on X. Let $g \colon X \to X$ be a map which is Lipschitzian on every bounded subset of X. Let π be the local semiflow on X generated by the mild solutions of the equation*

$$\dot{u} + Bu = g(u).$$

Suppose π is gradient-like with respect to a continuous function $V \colon X \to \mathbb{R}$, i.e. for every nonconstant solution $\sigma \colon J \to X$ of π the function $t \mapsto V(\sigma(t))$ is non-increasing and nonconstant. Moreover, assume that there is a $\kappa \in [0,\infty[$ such that

$$|u|_X \leq \kappa(V(u)+1) \text{ for every } u \in X.$$

Then π is a global semiflow, i.e. $u\pi t$ is defined for every $u \in X$ and for every $t \in [0,\infty[$.

Proof. In fact, given $u \in X$, let $\omega \in\ ]0,\infty]$ be such that $u\pi t$ is defined if and only if $t \in [0,\omega[$. Then our assumptions imply that

$$|u\pi t|_X \leq \kappa(V(u\pi t)+1) \leq \kappa(V(u)+1) < \infty \text{ for every } t \in [0,\omega[.$$

It was remarked earlier in this paper that π does not explode in bounded sets. Consequently $\omega = \infty$. The proposition is proved. □

The following result is well-known (cf e.g. [5], [9] or [16]):

Theorem 5.2. *Assume the hypotheses of Proposition 5.1. In addition, assume that for every bounded subset B of X*

$$\sup_{u\in B} V(u) < \infty.$$

Finally assume π is asymptotically smooth and that the set E of equilibria of π is bounded in X. Let $\mathcal{A}_\pi$ be the set of all full bounded orbits of π, i.e. the set of all $u \in X$ for which there exists a solution $\sigma \colon \mathbb{R} \to X$ of π with $\sigma(0) = u$ and $\sup_{t\in\mathbb{R}} |\sigma(t)|_X < \infty$.

Under these assumptions the following properties are satisfied:

1. *π is a global semiflow, i.e. $u\pi t$ is defined for every $u \in X$ and for every $t \in [0, \infty[$;*
2. *$\mathcal{A}_\pi$ is nonempty, compact, connected and attracts every bounded subset B of X, i.e. for every $\delta > 0$ there is a $t(B, \delta) \in [0, \infty[$ such that $\inf_{v \in \mathcal{A}_\pi} |u\pi t - v|_X < \delta$ for every $u \in B$ and every $t \geq t(B, \delta)$.*

We call $\mathcal{A}_\pi$ the global attractor *of π.*

Remark 5.3. *Let π be a global semiflow on a Banach space X such that every closed bounded subset of X is π-admissible. Then π is asymptotically smooth. This easily follows from the definitions of admissibility and asymptotic smoothness.*

We require the following

Definition 5.4. *We say that the family $(f_\epsilon)_{\epsilon \in [0, \epsilon_0]}$ of maps satisfies hypothesis* (Liap) *if there are families $(h_\epsilon)_{\epsilon \in [0, \epsilon_0]}$ and $(\mathcal{G}_\epsilon)_{\epsilon \in [0, \epsilon_0]}$ such that:*

1. *For every $\epsilon \in [0, \epsilon_0]$, h_ϵ is a fixed vector in H^ϵ and $C_3 := \sup_{\epsilon \in [0, \epsilon_0]} |h_\epsilon|_{H^\epsilon} < \infty$.*
2. *For every $\epsilon \in [0, \epsilon_0]$, $\mathcal{G}_\epsilon \colon H_1^\epsilon \to \mathbb{R}$ is a Fréchet differentiable function, $\mathcal{G}_\epsilon(0) = 0$ and*
$$D\mathcal{G}_\epsilon(u)v = \langle f_\epsilon(u), v \rangle_{H^\epsilon} - \langle h_\epsilon, v \rangle_{H^\epsilon} \text{ for all } u,\ v \in H_1^\epsilon.$$
3. *There are constants $\nu_1 \in]0, 1/2]$ and $C_1 \in [0, \infty[$ such that*
$$\mathcal{G}_\epsilon(u) \leq (\frac{1}{2} - \nu_1)|u|^2_{H_1^\epsilon} + C_1 \text{ for all } \epsilon \in [0, \epsilon_0] \text{ and } u \in H_1^\epsilon.$$
4. *There are constants $\nu_2 \in]0, 1]$ and $C_2 \in [0, \infty[$ such that*
$$\langle f_\epsilon(u) - h_\epsilon, u \rangle_{H^\epsilon} \leq (1 - \nu_2)|u|^2_{H_1^\epsilon} + C_2 \text{ for all } \epsilon \in [0, \epsilon_0] \text{ and } u \in H_1^\epsilon.$$

Example 5.5. *Let $(H^\epsilon, \langle \cdot, \cdot \rangle_{H^\epsilon})_{\epsilon \in [0, \epsilon_0]}$ and $(A_\epsilon)_{\epsilon \in [0, \epsilon_0]}$ be as in Example 2.3. Moreover, let the function G be as in Example 3.3. For simplicity, assume that G is independent of ϵ, x and y, so $G = G(\xi)$. Assume that the exponent δ satisfies the assumptions of Example 4.2 (subcritical growth) and suppose that G is* dissipative, *i.e.*
$$\limsup_{|\xi| \to \infty} \frac{G(\xi)}{\xi} \leq 0.$$
Define
$$F(\xi) := \int_0^\xi f(s)\,\mathrm{d}s,\ \xi \in \mathbb{R}.$$
and, for $\epsilon \in [0, \epsilon_0]$, let $\mathcal{G}_\epsilon \colon H_1^\epsilon \to \mathbb{R}$ be defined by
$$\mathcal{G}_\epsilon(u) := \int_\Omega F(u(x, y))\,\mathrm{d}x\,\mathrm{d}y.$$
Let $(h_\epsilon)_{\epsilon \in [0, \epsilon_0]}$ be a family such that h_ϵ is a fixed vector in H^ϵ for every $\epsilon \in [0, \epsilon_0]$, $\sup_{\epsilon \in [0, \epsilon_0]} |h_\epsilon|_{H^\epsilon} < \infty$ and $|h_\epsilon - h_0|_{H^\epsilon} \to 0$ as $\epsilon \to 0^+$. One can obtain such a family, e.g., by taking an arbitrary continuous function $h \colon \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}$ and setting $h_\epsilon(x, y) := h(\epsilon, x, \epsilon y)$, $\epsilon \in [0, \epsilon_0]$, $(x, y) \in \Omega$. Define $f_\epsilon(u)(x, y) := G(u(x, y)) + h_\epsilon(x, y)$ for $\epsilon \in [0, \epsilon_0]$, $u \in H_1^\epsilon$ and $(x, y) \in \Omega$. Using Examples 3.3, 4.2 and Remark 3.2 we see that $(f_\epsilon)_{\epsilon \in [0, \epsilon_0]}$ satisfies conditions (Conv) *and* (Comp). *The dissipativeness condition implies that for every $\eta \in]0, \infty[$ there is a $c = c(\eta) \in]0, \infty[$ such that*
$$F(\xi) \leq (1/2)\eta\xi^2 + c$$
$$G(\xi)\xi \leq \eta\xi^2 + c$$

for all $\xi \in \mathbb{R}$. *These relations and the definition of* $\mathcal{G}_\epsilon$ *easily imply condition* (Liap).

We can now prove the following

Theorem 5.6. *Assume that* $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ *and* $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *satisfy condition* (Spec) *and let* $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *be a family of maps satisfying hypotheses* (Conv), (Comp) *and* (Liap). *Then the following properties hold:*

1. *for every* $\epsilon \in [0,\epsilon_0]$, *the set* $\mathcal{A}_\epsilon := \mathcal{A}_{\pi_\epsilon}$ *of all full bounded orbits of* π_ϵ *is compact, connected and attracts every bounded subset of* X_ϵ*;*
2. *the family* $(\mathcal{A}_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ *of attractors is upper-semicontinuous at* $\epsilon = 0$ *with respect to the family of norms* $|\cdot|_{X_\epsilon}$. *In other words,*
$$\lim_{\epsilon\to 0^+} \sup_{u\in\mathcal{A}_\epsilon} \inf_{v\in\mathcal{A}_0} |u-v|_{X_\epsilon} = 0.$$

Proof. For every $\epsilon \in [0,\epsilon_0]$, define the function $V_\epsilon\colon X_\epsilon \to \mathbb{R}$ by
$$V_\epsilon(u_0,u_1) := \frac{1}{2}|u_0|^2_{H_1^\epsilon} + \frac{1}{2}|u_1|^2_{H^\epsilon} - \mathcal{G}_\epsilon(u_0) - \langle h_\epsilon, u_0\rangle_{H^\epsilon}.$$
It follows that for every $\epsilon \in [0,\epsilon_0]$ and for every solution $u = (u_0,u_1)\colon J \to X_\epsilon$ of π_ϵ the function $t \mapsto (V_\epsilon \circ u)(t)$ is differentiable and
$$\frac{d}{dt}(V_\epsilon \circ u)(t) = -\nu|u_1(t)|^2_{H^\epsilon},\ t \in J.$$
Therefore for every $\epsilon \in [0,\epsilon_0]$, the local semiflow π_ϵ is gradient-like with respect to V_ϵ. For every $\epsilon \in [0,\epsilon_0]$, $u \in H_1^\epsilon$ the function $t \mapsto \mathcal{G}_\epsilon(tu)$ is also differentiable and
$$\frac{d}{dt}\mathcal{G}_\epsilon(tu) = \langle f_\epsilon(tu), u\rangle_{H^\epsilon} - \langle h_\epsilon, u\rangle_{H^\epsilon}.$$
Hence
$$\mathcal{G}_\epsilon(u) = \int_0^1 \langle f_\epsilon(tu), u\rangle_{H^\epsilon}\,\mathrm{d}t - \langle h_\epsilon, u\rangle_{H^\epsilon}, \text{ for } u \in H_1^\epsilon.$$
This equality implies that for every $\epsilon \in [0,\epsilon_0]$ and $u = (u_0,u_1) \in X_\epsilon$,
$$\begin{aligned}V_\epsilon(u_0,u_1) &\le \frac{1}{2}|u_0|^2_{H_1^\epsilon} + \frac{1}{2}|u_1|^2_{H^\epsilon} + |\mathcal{G}_\epsilon(u_0)| + |h_\epsilon|_{H^\epsilon}\cdot|u_0|_{H^\epsilon}\\ &\le \frac{1}{2}|u_0|^2_{H_1^\epsilon} + \frac{1}{2}|u_1|^2_{H^\epsilon} + \sup_{t\in[0,1]}|f_\epsilon(tu_0)|_{H^\epsilon}\cdot|u_0|_{H^\epsilon} + |h_\epsilon|_{H^\epsilon}\cdot|u_0|_{H^\epsilon}\\ &\le \frac{1}{2}|u_0|^2_{H_1^\epsilon} + \frac{1}{2}|u_1|^2_{H^\epsilon} + \sup_{t\in[0,1]}|f_\epsilon(tu_0) - f_\epsilon(0)|_{H^\epsilon}\cdot|u_0|_{H^\epsilon}\\ &+ |f_\epsilon(0)|_{H^\epsilon}\cdot|u_0|_{H^\epsilon} + |h_\epsilon|_{H^\epsilon}\cdot|u_0|_{H^\epsilon}.\end{aligned}$$
Using our hypotheses it easily follows that for every $C_4 \in [0,\infty[$,
$$\sup_{u\in X_\epsilon, |u|_{X_\epsilon}\le C_4} V_\epsilon(u) < \infty \text{ for every } \epsilon \in [0,\epsilon_0]. \tag{28}$$
Moreover, if $\epsilon_0' > 0$ is chosen small enough then
$$\sup_{\epsilon\in[0,\epsilon_0']} |f_\epsilon(0)|_{H^\epsilon} < \infty$$

and so for every $C_4 \in [0,\infty[$,

$$C_5(C_4) := \sup_{\epsilon\in[0,\epsilon_0']} \sup_{u\in X_\epsilon, |u|_{X_\epsilon}\le C_4} V_\epsilon(u) < \infty. \tag{29}$$

Furthermore, since $1 \ge \nu_1$ and since $a^2 \ge 2a-1$ for every $a \in [0,\infty[$, we obtain

$$\begin{aligned} V_\epsilon(u_0,u_1) &\ge \frac{1}{2}|u_0|^2_{H_1^\epsilon} + \frac{1}{2}|u_1|^2_{H^\epsilon} - \mathcal{G}_\epsilon(u_0) - |\langle h_\epsilon, u_0\rangle_{H^\epsilon}| \\ &\ge \frac{1}{2}|u_0|^2_{H_1^\epsilon} + \frac{1}{2}|u_1|^2_{H^\epsilon} - (\frac{1}{2}-\nu_1)|u_0|^2_{H_1^\epsilon} - C_1 - {\lambda_{\epsilon,1}}^{-1/2}C_3|u_0|_{H_1^\epsilon} \\ &\ge \frac{\nu_1}{2}|u_0|^2_{H_1^\epsilon} + \frac{1}{2}|u_1|^2_{H^\epsilon} - (2\nu_1\lambda_{\epsilon,1})^{-1}{C_3}^2 - C_1 \ge \nu_1(|(u_0,u_1)|_{X_\epsilon}-1) - (2\nu_1\lambda_{\epsilon,1})^{-1}{C_3}^2 - C_1. \end{aligned}$$

This inequality implies that for every $\epsilon \in [0,\epsilon_0]$ and $(u_0,u_1) \in X_\epsilon$,

$$\nu_1|(u_0,u_1)|_{X_\epsilon} \le V_\epsilon(u_0,u_1) + \nu_1 + (2\nu_1\lambda_{\epsilon,1})^{-1}{C_3}^2 + C_1. \tag{30}$$

Now let $\epsilon \in [0,\epsilon_0]$ and let $u = (u_0,u_1) \in X_\epsilon$ be an equilibrium of $\pi_{\epsilon,f_\epsilon}$. Then $u_1 = 0$, $u_0 \in D(A_\epsilon)$ and $A_\epsilon u_0 = f_\epsilon(u_0)$. Hence

$$\begin{aligned} |u_0|^2_{H_1^\epsilon} &= \langle A_\epsilon u_0, u_0\rangle_{H^\epsilon} = \langle f_\epsilon(u_0) - h_\epsilon, u_0\rangle_{H^\epsilon} + \langle h_\epsilon, u_0\rangle_{H^\epsilon} \\ &\le (1-\nu_2)|u|^2_{H_1^\epsilon} + C_2 + {\lambda_{\epsilon,1}}^{-1/2}C_3|u_0|_{H_1^\epsilon}. \end{aligned}$$

It follows that

$$|u_0|_{H_1^\epsilon} \le (\frac{C_2}{\nu_2} + \frac{C_3{\lambda_{\epsilon,1}}^{-1/2}}{2\nu_2})^{1/2} + \frac{C_3{\lambda_{\epsilon,1}}^{-1/2}}{2\nu_2}. \tag{31}$$

Now a application of Theorem 4.3, Remark 4.4, Remark 5.3 and Theorem 5.2 together with (28), (30) and (31) complete the proof of Property 1 of the theorem.

The estimates (1), (30) and (31) show that there is a constant $C_4 \in [0,\infty[$ such that

$$|u|_{X_\epsilon} \le C_4(V_\epsilon(u)+1) \text{ for every } \epsilon \in [0,\epsilon_0'] \text{ and every } u \in X_\epsilon \tag{32}$$

and

$$|v|_{X_\epsilon} \le C_4 \text{ for all } \epsilon \in [0,\epsilon_0'] \text{ and every equilibrium } v \text{ of } \pi_\epsilon. \tag{33}$$

We claim that

$$\sup_{\epsilon\in[0,\epsilon_0]} \sup_{u\in\mathcal{A}_\epsilon} |u|_{X_\epsilon} \le K := C_4\cdot(C_5(C_4)+1) < \infty. \tag{34}$$

In fact, let $\epsilon \in [0,\epsilon_0']$ be arbitrary and $u \in \mathcal{A}_\epsilon$. Then there is a full bounded solution σ of π_ϵ with $\sigma(0) = u$. The set $\sigma(\mathbb{R})$ lies in a compact subset of X_ϵ so, in particular, the α-limit set of σ contains an element v. It follows that $V_\epsilon(u) \le V_\epsilon(v)$ and v is an equilibrium of π_ϵ. From (32) we thus obtain that

$$|u|_{X_\epsilon} \le C_4(V_\epsilon(u)+1) \le C_4(V_\epsilon(v)+1).$$

Since $|v|_{X_\epsilon} \le C_4$ by (33), the estimate (29) implies that

$$V_\epsilon(v) \le C_5(C_4).$$

This proves our claim.

Suppose Property 2 of the theorem is not true. Then there exist a $\delta > 0$, a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ in $]0,\epsilon_0]$ converging to zero and a sequence of full bounded solutions σ_n of π_{ϵ_n} such that

$$\inf_{v\in\mathcal{A}_0} |u_n - v|_{X_{\epsilon_n}} > \delta \text{ for all } n \in \mathbb{N}, \tag{35}$$

where $u_n := \sigma_n(0)$. The above claim implies

$$|\sigma_n(t)|_{X_{\epsilon_n}} \leq K < \infty \text{ for all } n \in \mathbb{N} \text{ large enough and for all } t \in \mathbb{R}.$$

Corollary 4.7 implies that there is a subsequence of $(\sigma_n)_{n\in\mathbb{N}}$, again denoted by $(\sigma_n)_{n\in\mathbb{N}}$ and there is a solution $\sigma\colon \mathbb{R} \to X_0$ of π_0 such that for every $t \in \mathbb{R}$

$$|\sigma_n(t) - \sigma(t)|_{X_{\epsilon_n}} \to 0 \text{ as } n \to \infty.$$

This limit and condition (Spec)3 imply that

$$|\sigma(t)|_{X_0} \leq CK \text{ for all } t \in \mathbb{R}$$

and so $\sigma(\mathbb{R}) \subset \mathcal{A}_0$. In particular,

$$|u_n - v|_{X_\epsilon} \to 0 \text{ as } n \to \infty,$$

where $v := \sigma(0) \in \mathcal{A}_0$. However this contradicts (35). The theorem is proved. □

We thus obtain the following Corollary, which extends, to general bounded domains, part (i) of Theorem 1.1 in [7] (note that the notation of [7] is different from ours):

Corollary 5.7. *Let $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ be as in Example 2.3 and $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ be as in Example 5.5. Then the following properties hold:*

1. *for every $\epsilon \in [0,\epsilon_0]$, the set $\mathcal{A}_\epsilon := \mathcal{A}_{\pi_\epsilon}$ of all full bounded orbits of π_ϵ is compact, connected and attracts every bounded subset of X_ϵ;*
2. *the family $(\mathcal{A}_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ of attractors is upper-semicontinuous at $\epsilon = 0$ with respect to the family of norms $|\cdot|_{X_\epsilon}$. In other words,*

$$\lim_{\epsilon\to 0^+} \sup_{u\in\mathcal{A}_\epsilon} \inf_{v\in\mathcal{A}_0} |u - v|_{X_\epsilon} = 0.$$

As it was said in the Introduction, each attractor $\mathcal{A}_\epsilon$ for $\epsilon \in [0,\epsilon_0]$ is a compact isolated π_ϵ-invariant set with Conley index Σ^0. Therefore the above theorem implies that the invariant set $\mathcal{A}_0$ can be continued to an upper semicontinuous family of invariant sets all having the same Conley index. The following result shows that the latter statement holds under more general hypotheses for an arbitrary isolated invariant set of the limit semiflow π_0. A proof, given in [2], is an application of Corollary 4.6 and an abstract continuation principle established in [2].

Theorem 5.8. *Assume that $(H^\epsilon, \langle\cdot,\cdot\rangle_{H^\epsilon})_{\epsilon\in[0,\epsilon_0]}$ and $(A_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ satisfy condition* (Spec) *and let $(f_\epsilon)_{\epsilon\in[0,\epsilon_0]}$ be a family of maps satisfying hypotheses* (Conv) *and* (Comp). *Let N be a closed and bounded isolating neighborhood for π_0. For every $\eta > 0$ and $\epsilon \in]0,\epsilon_0]$ let $K_{\epsilon,\eta}$ be the largest π_ϵ-invariant set included in $N_{\epsilon,\eta}$. Moreover, let K_0 be the largest π_0-invariant set included in N.*

Then for every $\eta > 0$, there exists an $\epsilon^c = \epsilon^c(\eta) > 0$ such that for every $\epsilon \in]0,\epsilon^c]$ the set $N_{\epsilon,\eta}$ is a strongly admissible isolating neighborhood of $K_{\epsilon,\eta}$ relative to π_ϵ and the Conley indices of $K_{\epsilon,\eta}$ and K_0 are the same. Furthermore, for every $\eta > 0$, the family

$(K_{\epsilon,\eta})_{\epsilon\in[0,\epsilon^c(\eta)]}$, *where* $K_{0,\eta} := K_0$, *is upper semicontinuous at* $\epsilon = 0$ *with respect to the family* $|\cdot|_{X_\epsilon}$ *of norms, i.e.*

$$\lim_{\epsilon\to 0^+} \sup_{u\in K_{\epsilon,\eta}} \inf_{v\in K_0} |u-v|_{X_\epsilon} = 0.$$

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References

[1] M. C. Carbinatto and K. P. Rybakowski, Conley index continuation and thin domain problems, submitted.

[2] M. C. Carbinatto and K. P. Rybakowski, On a general Conley index continuation principle for singular perturbation problems, submitted.

[3] M. C. Carbinatto and K. P. Rybakowski, On Conley index and attractors for damped wave equations on thin domains: the critical exponent case, in preparation.

[4] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[5] J. Hale, *Asymptotic Behavior of Dissipative Systems*, Math. Surveys and Monographs, no. 25, Amer. Math. Soc., Providence, R. I., 1988.

[6] J. Hale and G. Raugel, Reaction-diffusion equations on thin domains *J. Math. Pures Appl.* **IX Ser.71** (1992), 33 - 95.

[7] J. Hale and G. Raugel, A damped hyperbolic equation on thin domains *Trans. AMS* **329** (1992), 185 - 219.

[8] Q. Huang, The continuation of Conley index for singular perturbations and the Conley indices in gradient-like systems. I: Theory, preprint.

[9] O. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, Cambridge University Press, Cambridge, 1991.

[10] M. Prizzi and K. P. Rybakowski, The effect of domain squeezing upon the dynamics of reaction-diffusion equations, *J. Diff. Equations*, to appear.

[11] M. Prizzi and K. P. Rybakowski, Some recent results on thin domain problems, *Topological Methods in Nonlinear Analysis* **14** (1999), 239 - 255.

[12] M. Prizzi and K. P. Rybakowski, Inertial manifolds on squeezed domains, submitted.

[13] M. Prizzi, M. Rinaldi and K. P. Rybakowski, Parabolic equations on curved squeezed domains, preprint.

[14] K. P. Rybakowski, *The Homotopy Index and Partial Differential Equations*, Universitext, Springer-Verlag, Berlin, 1987.

[15] I. Segal, Non-linear semi-groups, *Annals of Math.* (2) **78** (1963), 339 - 364.

[16] T. Témam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, Berlin, 1988.

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