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Rigid Immersions of Vector Bundles of Rank 1 on a Surface into $I\!R^3$

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Imersões rígidas no $I\!\!R^3$ de fibrados vetoriais, de posto um, sobre uma superfície

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Resumo

Considerar um fibrado vetorial Riemanniano, de posto um, definido por uma seção normal ν sobre uma superfície bidimensional M de $I\!\!R^4$. Seja α a segunda forma fundamental de M com respeito a ν. Neste artigo damos condições em α para obter uma imersão isométrica no $I\!\!R^3$, do referido fibrado vetorial, cuja segunda forma fundamental é precisamente α. Estas condições determinam restrições geométricas fortes na superfície M.

Rigid immersions of vector bundles of rank 1 on a surface into $\mathbb{R}^3$

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Abstract

Consider a Riemannian vector bundle of rank 1 defined by a normal section ν on a surface M in $\mathbb{R}^4$. Let α be the second fundamental form with respect to ν which determines a configuration of lines of curvature. In this article we obtain conditions on α to immerse isometrically the vector bundle with α as a second fundamental form into $\mathbb{R}^3$. Geometric restrictions on the surface are determined by these conditions.

Key words: ν-Principal configuration, second fundamental form, fiber bundle immersions, structural equations.
MSC 1995: 53A05, 57R25, 57R30

1 Introduction

For any surface in $\mathbb{R}^3$, the eigenspaces of the second fundamental form define two orthogonal line fields (principal directions) whose singularities are exactly the umbilics.

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In a more general context, given a surface immersed in $I\!R^4$ with a normal unitary vector field ν, we can consider the shape operator and the second fundamental form with respect to this vector field. This gives rise to two orthogonal line fields, the ν-principal directions, whose singularities are known as ν-umbilics. The integral lines of these fields are called ν-lines of curvature. The generic topological behaviour of the ν-lines of curvature, near isolated ν-umbilics is the same as that of the lines of curvature near umbilics of surfaces in $I\!R^3$. This topological type is robust and depends only on the number of umbilical separatrices which might be one, two or three ([9]).

Nevertheless, there are some interesting differences between these two contexts of definition of lines of curvature which suggest to analize more carefully their relationships. For instance, to each isolated umbilic of a surface in $I\!R^3$ we can attach the index of either one of the two fields. This index has to be of the form $n/2$, with $n \in Z\!\!\!Z$. Examples of umbilics of index j are known for all $j \leq 1$. The classical (local) Loewner's Conjecture states that every umbilic of a smooth surface immersed in $I\!R^3$ must have an index less than or equal to one. This conjecture has been asserted to be true for analytic surfaces by several authors among whom are H. Hamburger [3], G. Bol [1], T. Klotz [5], C. J. Titus [11] and H. Scherbel [10].

On the other hand, in [2] it was proved that, given $n \in Z\!\!\!Z$, there exists an analytic surface M immersed in $I\!R^4$ and an analytic unitary vector field ν normal to M having an isolated ν-umbilic of index $n/2$. This fact contrast with the Loewner's conjecture stated above.

In order to study the conditions to immerse in $I\!R^3$ an arbitrary ν-umbilic or more generally an arbirtrary set of local ν-lines of curvature on a surface immersed in $I\!R^4$ we stablish the following considerations. Since the differential equation of ν-lines of curvature depends only on the first and the second fundamental forms, it is enough to fix these fundamental forms on the vector bundle of rank one on M, whose fiber at $p \in M$ is the normal line in the direction of $\nu(p)$. In section 2 we present the basic tools used along the sequel. In section 3 we obtain from the structural equations of Gauss and Codazzi, two conditions on the second fundamental form which are equivalent to the fact that the curvature tensor of the vector bundle vanishes (Theorem 3.3). In particular this is equivalent to the fact that locally the bundle is immersible in $I\!R^3$ (Corollary 3.4). Besides, it is interesting to remark that Condition 1 of Theorem 3.3 implies that M must have a global foliation of asymptotic lines whose associated binormal is $\nu^{\perp}$ ([6]). These conditions impose some geometric properties of the surface to immerse it in $I\!R^4$. For instance, M has to be locally convex. Therefore minimal surfaces in $I\!R^4$ never admit isometric immersions in $I\!R^3$ for any of their normal fields (Proposition 3.8). On the other hand, closed locally convex generic surfaces in $I\!R^4$ do not admit global isometric immersions in $I\!R^3$ for any of their normal fields (Corollary 3.11).

Finally, we would like to thank Francesco Mercuri for his comments and support at the begining of this work.

2 Vector bundles of rank one on M

Let M be a smooth oriented surface immersed in $I\!R^4$ with the Riemannian metric induced by the standard Riemannian metric of $I\!R^4$. For each $p \in M$ consider the decomposition $T_pI\!R^4 = T_pM \bigoplus (T_pM)^{\perp}$, where $(T_pM)^{\perp}$ is the orthogonal complement of T_pM in $I\!R^4$. Let $\chi(M)$ and $\chi(M)^{\perp}$ be the space of smooth vector fields on M and the space of smooth vector fields normal to M, respectively.

Let $\bar{\nabla}$ be the Riemannian connection of $I\!R^4$. Given vector fields X, Y in $\chi(M)$, let $\bar{X}$, $\bar{Y}$ be some local extensions to $I\!R^4$. The Riemannian connection on M is well defined by the tangent component of the Riemannian connection of $I\!R^4$: $\nabla_X Y = (\bar{\nabla}_{\bar{X}}\bar{Y})^{\top}$. On the other hand, given a normal vector field $\xi \in \chi^{\perp}(M)$ let $\nabla^{\perp}_X \xi = (\bar{\nabla}_{\bar{X}}\bar{\xi})^{\perp}$ be the normal component of $\bar{\nabla}_{\bar{X}}\bar{\xi}$, this way we have a compatible connection in $TM^{\perp}$.

Consider *the second fundamental map*,

$$\alpha : \chi(M) \times \chi(M) \to (\chi(M))^{\perp}, \ \alpha(X,Y) = \bar{\nabla}_{\bar{X}}\bar{Y} - \nabla_X Y,$$

this map is well defined, symmetric and bilinear.

If $p \in M$ and $\nu \in (T_pM)^{\perp}$, $\nu \neq 0$, define the function

$$l_\nu : T_pM \times T_pM \to I\!R, \ l_\nu(X,Y) = < \alpha(X,Y), \nu > .$$

Then, this function as well is symmetric and bilinear. *The second fundamental form* of M at p is the associated quadratic form,

$$II_\nu : T_pM \to I\!R, \ II_\nu(X) = l_\nu(X,X).$$

Recall *the shape operator*

$$S_\nu : T_pM \to T_pM,\ S_\nu(X) = -(\bar{\nabla}_{\bar{X}}\bar{\nu})^\top,$$

where $\bar{\nu}$ is a local extension to $\mathbb{R}^4$ of the normal vector field ν at p and $\top$ means the tangent component. This operator is bilinear, self-adjoint and for any $X,\ Y \in T_pM$ satisfies the following equation: $< S_\nu(X), Y >= l_\nu(X,Y)$ [2]. Therefore, the second fundamental form can be expressed by $II_\nu(X) =< S_\nu(X), X >$.

Assume that $\{\nu, \nu^\perp\}$ is an orthonormal basis of the normal vector bundle $(TM)^\perp$. Consider the vector bundle of rank one $\pi : \tilde{\nu} \to M$, where:

- $\tilde{\nu}$ is the normal vector bundle on M whose fiber at $p \in M$ is the normal line in the direction $\nu(p)$.
- π is the natural projection.

Endow this vector bundle with the connection $\tilde{\nabla}$ defined as the *projection on* $\tilde{\nu}$ *of the normal connection* $\nabla^\perp$ restricted to $\tilde{\nu}$, namely:

$$\tilde{\nabla}_X\eta = \langle \nabla^\perp_X\eta, \nu\rangle\nu, \tag{1}$$

for $\eta \in \tilde{\nu}$ and $X \in TM$.

Therefore $\tilde{\nu}$ is a Riemannian vector bundle with the metric induced by the standard metric on $\mathbb{R}^4$. The connection $\tilde{\nabla}$ is compatible with the metric.

Let $\tilde{\alpha}$ be the symmetric section of the vector bundle

$$\rho : Hom\,(TM \times TM, \tilde{\nu}) \to M,$$

whose fiber at $p \in M$ is $Hom(T_pM \times T_pM, \tilde{\nu}_p)$, defined by $\tilde{\alpha}(X,Y) = l(X,Y)\nu$, where $\alpha(X,Y) = l(X,Y)\nu + l^\perp(X,Y)\nu^\perp,\ \nu \in \tilde{\nu}$.

Consider the *Withney sum* of vector bundles: $E_\nu = TM \oplus_W \tilde{\nu}$, where the metric on E_ν is the orthogonal sum of the metrics on TM and $\tilde{\nu}$. This Riemannian vector bundle E_ν can be endow with a connection ∇', compatible with its metric, defined by:

$$\begin{aligned} \nabla'_X Y &= \nabla_X Y + \tilde{\alpha}(X,Y), X, Y \in TM \\ \nabla'_X \xi &= -S_\xi X + \tilde{\nabla}_X \xi;\ X \in TM, \xi \in \tilde{\nu}. \end{aligned}$$

This way *curvature tensor* of E_ν is:

$$R_\nu(X,Y)\eta = \nabla'_X \nabla'_Y \eta - \nabla'_Y \nabla'_X \eta - \nabla'_{[X,Y]}\eta,$$

$\eta \in TM \oplus_W \tilde{\nu}$.

Since the curvature of $I\!R^4$ vanishes, the Gauss and Codazzi structural equations satisfied by the immersion of M can be written respectively as [4]

$$\begin{aligned} \langle R(X,Y), W\rangle &= \langle \alpha(X,W), \alpha(Y,Z)\rangle - \langle \alpha(X,Z), \alpha(Y,W)\rangle, \\ \left(\nabla^\perp_X \alpha\right)(Y,Z) &= \nabla^\perp_Y \alpha(X,Z), \textit{ or equivalently}, \\ (\nabla_X S)(Y,\xi) &= (\nabla_Y S)(X,\xi). \end{aligned} \tag{2}$$

3 Conditions on the second fundamental form determined by the rigid immersions

We begin obtanining some equivalences which let us express Codazzi structural equation in a convenient way used in the proof of Theorem 3.3.

Lemma 3.1 *The following conditions are equivalent,*
1. $l^\perp(X,Z)\langle \nabla^\perp_Y \nu^\perp, \nu\rangle + l^\perp(Y,Z)\langle \nabla^\perp_X \nu, \nu^\perp\rangle = 0$, *for all* $X, Y, Z \in TM$.
2. $S_{\nu^\perp}\left(\langle \nabla^\perp_Y \nu^\perp, \nu\rangle X + \langle \nabla^\perp_X \nu, \nu^\perp\rangle Y\right) = 0$, for all $X, Y \in TM$.

Proof The following equations imply the result,

$$\begin{aligned} l^\perp(X,Z)\langle \nabla^\perp_Y \nu^\perp, \nu\rangle + l^\perp(Y,Z)\langle \nabla^\perp_X \nu, \nu^\perp\rangle &= \\ \langle \nabla^\perp_Y \nu^\perp, \nu\rangle\langle \alpha(X,Z), \nu^\perp\rangle + \langle \nabla^\perp_X \nu, \nu^\perp\rangle\langle \alpha(Y,Z), \nu^\perp\rangle &= \\ \langle \langle \nabla^\perp_Y \nu^\perp, \nu\rangle S_{\nu^\perp} X + \langle \nabla^\perp_X \nu, \nu^\perp\rangle S_{\nu^\perp} Y, Z\rangle. \ \blacksquare \end{aligned}$$

Lemma 3.2 *Condition 2 of the previous lemma holds if and only if*

$$l(X, \nabla_Y Z) - l(Y, \nabla_X Z) - l([X,Y], Z) + X\left(l(Y,Z)\right) - Y\left(l(X,Z)\right) = 0.$$

Proof Consider Codazzi equation in $I\!\!R^4$ (2).

$$\left(\nabla^{\perp}_{X}\alpha\right)(Y,Z) = \left(\nabla^{\perp}_{Y}\alpha\right)(X,Z),$$

where,

$$\left(\nabla^{\perp}_{X}\alpha\right)(Y,Z) = \nabla^{\perp}_{X}\alpha(Y,Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z),$$

and

$$\left(\nabla^{\perp}_{Y}\alpha\right)(X,Z) = \nabla^{\perp}_{Y}\alpha(X,Z) - \alpha(\nabla_Y X, Z) - \alpha(X, \nabla_Y Z).$$

Substituting the values of the image of α in the normal basis, Codazzi condition has the following expression,

$$\begin{aligned}
\nabla^{\perp}_{X}\left(l(Y,Z)\nu + l^{\perp}(Y,Z)\nu^{\perp}\right) - \left(l(\nabla_X Y,Z)\nu + l^{\perp}(\nabla_X Y,Z)\nu^{\perp}\right) &\; - \\
\left(l(Y,\nabla_X Z)\nu + l^{\perp}(Y,\nabla_X Z)\nu^{\perp}\right) &\; = \\
\nabla^{\perp}_{Y}\left(l(X,Z)\nu + l^{\perp}(X,Z)\nu^{\perp}\right) - \left(l(\nabla_Y X,Z)\nu + l^{\perp}(\nabla_Y X,Z)\nu^{\perp}\right) &\; - \\
\left(l(X,\nabla_Y Z)\nu + l^{\perp}(X,\nabla_Y Z)\nu^{\perp}\right). &
\end{aligned}$$

The component of this vector field in the direction ν is,

$$\begin{aligned}
\langle\nabla^{\perp}_{X}\left(l(Y,Z)\nu + l^{\perp}(Y,Z)\nu^{\perp}\right),\nu\rangle - l(\nabla_X Y,Z) - l(Y,\nabla_X Z) &\; = \\
\langle\nabla^{\perp}_{Y}\left(l(X,Z)\nu + l^{\perp}(X,Z)\nu^{\perp}\right),\nu\rangle - l(\nabla_Y X,Z) - l(X,\nabla_Y Z). &
\end{aligned} \tag{3}$$

Since $\langle\nabla'_X\nu,\nu\rangle = 0$, $\nabla^{\perp}_{X}\nu$ is orthogonal to ν. Considering that $\nabla^{\perp}_{X}\left(l(Y,Z)\nu\right) = X\left(l(Y,Z)\right)\nu + l(Y,Z)\nabla^{\perp}_{X}\nu$ we obtain,

$$\langle\nabla^{\perp}_{X}\left(l(Y,Z)\nu + l^{\perp}(Y,Z)\nu^{\perp}\right),\nu\rangle = X\left(l(Y,Z)\right) + l^{\perp}(Y,Z)\langle\nabla^{\perp}_{X}\nu^{\perp},\nu\rangle.$$

Thus,

$$\begin{aligned}
X\left(l(Y,Z)\right) + l^{\perp}(Y,Z)\langle\nabla^{\perp}_{X}\nu^{\perp},\nu\rangle - l(\nabla_X Y,Z) - l(Y,\nabla_X Z) &\; = \\
Y\left(l(X,Z)\right) + l^{\perp}(X,Z)\langle\nabla^{\perp}_{Y}\nu^{\perp},\nu\rangle - l(\nabla_Y X,Z) - l(X,\nabla_Y Z), &
\end{aligned} \tag{4}$$

$$X\left(l(Y,Z)\right)-Y\left(l(X,Z)\right)+l^{\perp}(Y,Z)\langle\nabla^{\perp}_{X}\nu^{\perp},\nu\rangle-l^{\perp}(X,Z)\langle\nabla^{\perp}_{Y}\nu^{\perp},\nu\rangle = l(\nabla_X Y,Z)-l(\nabla_Y X,Z)+l(Y,\nabla_X Z)-l(X,\nabla_Y Z),$$

$$X\left(l(Y,Z)\right)-Y\left(l(X,Z)\right)+l^{\perp}(Y,Z)\langle\nabla^{\perp}_{X}\nu^{\perp},\nu\rangle-l^{\perp}(X,Z)\langle\nabla^{\perp}_{Y}\nu^{\perp},\nu\rangle = l([X,Y],Z)+l(Y,\nabla_X Z)-l(X,\nabla_Y Z).$$

Finally since the normal connection is compatible with the metric, condition 2 implies,

$$\begin{aligned} l^{\perp}(X,Z)\langle\nabla^{\perp}_{Y}\nu^{\perp},\nu\rangle+l^{\perp}(Y,Z)\langle\nabla^{\perp}_{X}\nu,\nu^{\perp}\rangle &= \\ l^{\perp}(X,Z)\langle\nabla^{\perp}_{Y}\nu^{\perp},\nu\rangle-l^{\perp}(Y,Z)\langle\nu,\nabla^{\perp}_{X}\nu^{\perp}\rangle &= 0, \end{aligned}$$

therefore equation (4)

$$l(X,\nabla_Y Z)-l(Y,\nabla_X Z)-l([X,Y],Z)+X\left(l(Y,Z)\right)-Y\left(l(X,Z)\right)=0,$$

holds if and only if condition 2 holds. ■

Theorem 3.3 *Let M be an oriented surface immersed in $I\!R^4$. Let ν be an smooth unitary vector field normal to M. Consider the vector bundle defined by $\pi : E_\nu \rightarrow M$, with Riemannian connection ∇'. Then,*
1. $l^{\perp}$ is degenerated, namely for any $X, Y, Z, X \in T_pM$

$$l^{\perp}(X,W)l^{\perp}(Y,Z)-l^{\perp}(X,Z)l^{\perp}(Y,W)=0, \quad \textit{and}$$

2. $w=\langle\nabla^{\perp}_{Y}\nu^{\perp},\nu\rangle X+\langle\nabla^{\perp}_{X}\nu,\nu^{\perp}\rangle Y \in \ker S_{\nu^{\perp}}$, for every $X, Y \in TM$,
if and only if,
The vector bundle $\pi : E_\nu \rightarrow M$ has constant curvature 0.

Proof In the first part we analyze the curvature tensor for $\xi \in \tilde{\nu}$.
By definition of ∇'

$$
\begin{aligned}
\nabla'_X \nabla'_Y \xi &= -\nabla'_X (S_\xi Y) + \nabla'_X \left(\tilde{\nabla}_Y \xi\right) \\
&= -\nabla_X (S_\xi Y) - l(X, S_\xi Y)\nu - S_{\tilde{\nabla}_Y \xi} X + \tilde{\nabla}_X \left(\tilde{\nabla}_Y \xi\right),
\end{aligned}
$$

and

$$
\nabla'_{[X,Y]}\xi = -S_\xi [X,Y] + \tilde{\nabla}_{[X,Y]}\xi,
$$

we obtain,

$$
\begin{aligned}
R_\nu(X,Y)\xi &= -\nabla_X (S_\xi Y) - l(X, S_\xi Y)\nu - S_{\tilde{\nabla}_Y \xi} X + \tilde{\nabla}_X \left(\tilde{\nabla}_Y \xi\right) + \nabla_Y (S_\xi X) \\
&+ l(Y, S_\xi X)\nu + S_{\tilde{\nabla}_X \xi} Y - \tilde{\nabla}_Y \left(\tilde{\nabla}_X \xi\right) + S_\xi [X,Y] - \tilde{\nabla}_{[X,Y]}\xi \\
&= (l(Y, S_\xi X) - l(X, S_\xi Y))\nu + \tilde{\nabla}_X \left(\tilde{\nabla}_Y \xi\right) - \tilde{\nabla}_Y \left(\tilde{\nabla}_X \xi\right) - \tilde{\nabla}_{[X,Y]}\xi \\
&+ (\nabla_Y (S_\xi X) - \nabla_X (S_\xi Y)) + S_\xi [X,Y] + \left(S_{\tilde{\nabla}_X \xi} Y - S_{\tilde{\nabla}_Y \xi} X\right) \\
&= (l(Y, S_\xi X) - l(X, S_\xi Y))\nu + \tilde{\nabla}_X \left(\tilde{\nabla}_Y \xi\right) - \tilde{\nabla}_Y \left(\tilde{\nabla}_X \xi\right) - \tilde{\nabla}_{[X,Y]}\xi \\
&+ (\nabla_Y (S_\xi X) - \nabla_X (S_\xi Y)) + S_\xi [X,Y] + \left(S_{\nabla^\perp_X \xi} Y - S_{\nabla^\perp_Y \xi} X\right) \\
&+ S_{\langle \nabla^\perp_Y \xi, \nu^\perp\rangle \nu^\perp} X - S_{\langle \nabla^\perp_X \xi, \nu^\perp\rangle \nu^\perp} Y. \qquad (5)
\end{aligned}
$$

Now let us pick up the terms of this expression separately in order to determine when they vanish. First consider,

$$
\begin{aligned}
l(Y, S_\xi X) - l(X, S_\xi Y) &= l(Y, S_\xi X) - \langle l(X, S_\xi Y)\nu, \nu\rangle \\
&= l(Y, S_\xi X) - \langle l(X, S_\xi Y)\nu + l^\perp(X, S_\xi Y)\nu^\perp, \nu\rangle \\
&= l(Y, S_\xi X) - \langle \alpha(X, S_\xi Y), \nu\rangle \\
&= l(Y, S_\xi X) - \langle \alpha(S_\xi Y, X), \nu\rangle \\
&= l(Y, S_\xi X) - \langle S_\nu S_\xi Y, X\rangle = l(Y, S_\xi X) - \langle S_\xi S_\nu X, Y\rangle \\
&= l(Y, S_\xi X) - \langle S_\nu S_\xi X, Y\rangle \\
&= l(Y, S_\xi X) - \langle \alpha(Y, S_\xi X), \nu\rangle \\
&= l(Y, S_\xi X) - \langle \alpha(S_\xi X, Y), \nu\rangle \\
&= l(Y, S_\xi X) - l(S_\xi X, Y) = 0. \qquad (6)
\end{aligned}
$$

Let us compute now, $\tilde{\nabla}_X \left(\tilde{\nabla}_Y \xi\right) - \tilde{\nabla}_Y \left(\tilde{\nabla}_X \xi\right) - \tilde{\nabla}_{[X,Y]}\xi$.

Write $\xi = h\nu$, where h is a positive function.

Considering that ∇' is compatible with the metric and $\langle \nu, \nu \rangle = 1$ we have $\langle \nabla'_X \nu, \nu \rangle = \langle \nabla'_Y \nu, \nu \rangle = 0$, thus,

$$\begin{aligned}
\langle \tilde{\nabla}_X \tilde{\nabla}_Y \xi - \tilde{\nabla}_Y \tilde{\nabla}_X \xi - \tilde{\nabla}_{[X,Y]} \xi, \nu \rangle &= \langle \nabla'_X \tilde{\nabla}_Y \xi, \nu \rangle - \langle \nabla'_Y \tilde{\nabla}_X \xi, \nu \rangle - \langle \nabla'_{[X,Y]} \xi, \nu \rangle \\
&= \langle \nabla'_X \left(\langle \nabla'_Y \xi, \nu \rangle \nu \right), \nu \rangle \\
&- \langle \nabla'_Y \left(\langle \nabla'_X \xi, \nu \rangle \nu \right), \nu \rangle - \langle \langle \nabla'_{[X,Y]} \xi, \nu \rangle \nu, \nu \rangle \\
&= \langle \nabla'_X \left(\langle \nabla'_Y \xi, \nu \rangle \nu \right) - \nabla'_Y \left(\langle \nabla'_X \xi, \nu \rangle \nu \right) \\
&- \langle \nabla'_{[X,Y]} \xi, \nu \rangle \nu, \nu \rangle.
\end{aligned}$$

Since $\xi = h\nu$,

$$\begin{aligned}
\langle \nabla'_Y \xi, \nu \rangle &= \langle (Yh)\, \nu + h \nabla'_Y \nu, \nu \rangle = Yh \\
\langle \nabla'_X \xi, \nu \rangle &= Xh; \ , \langle \nabla'_{[X,Y]} \xi, \nu \rangle = [X,Y]h.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle \tilde{\nabla}_X \tilde{\nabla}_Y \xi, \nu \rangle - \langle \tilde{\nabla}_Y \tilde{\nabla}_X \xi, \nu \rangle - \langle \tilde{\nabla}_{[X,Y]} \xi, \nu \rangle &= \langle \nabla'_X \left((Yh)\, \nu \right) - \nabla'_Y \left((Xh)\, \nu \right) \\
&- \langle \left([X,Y]h \right) \nu, \nu \rangle \\
&= XYh - YXh - [X,Y]h \\
&= 0. \qquad (7)
\end{aligned}$$

Therefore substituting (6) and (7) in (5) we obtain,

$$\begin{aligned}
R_\nu(X,Y)\xi &= \left(\nabla_Y \left(S_\xi X \right) - \nabla_X \left(S_\xi Y \right) \right) + S_\xi [X,Y] + \left(S_{\nabla^\perp_X \xi} Y - S_{\nabla^\perp_Y \xi} X \right) + \\
&+ S_{\langle \nabla^\perp_Y \xi, \nu^\perp \rangle \nu^\perp} X - S_{\langle \nabla^\perp_X \xi, \nu^\perp \rangle \nu^\perp} Y
\end{aligned}$$

Since M is immersed in $I\!R^4$ satisfies Codazzi equation (2),

$$\left(\nabla_Y S \right)(X, \xi) = \left(\nabla_X S \right)(Y, \xi),$$

namely,

$$\nabla_Y \left(S_\xi X \right) - S_\xi \left(\nabla_Y X \right) - S_{\nabla^\perp_Y \xi} X = \nabla_X \left(S_\xi Y \right) - S_\xi \left(\nabla_X Y \right) - S_{\nabla^\perp_X \xi} Y,$$

thus,

$$\nabla_Y \left(S_\xi X \right) - \nabla_X \left(S_\xi Y \right) = S_\xi \left(\nabla_Y X \right) + S_{\nabla^\perp_Y \xi} X - S_\xi \left(\nabla_X Y \right) - S_{\nabla^\perp_X \xi} Y. \ (8)$$

then substituting in the previous expression we have,

$$\begin{aligned} R_\nu(X,Y)\xi &= -S_\xi\left(\nabla_X Y\right) + S_\xi\left(\nabla_Y X\right) + S_\xi[X,Y] \\ &+ S_{\langle\nabla_Y^\perp\xi,\nu^\perp\rangle\nu^\perp}X - S_{\langle\nabla_X^\perp\xi,\nu^\perp\rangle\nu^\perp}Y \\ &= S_\xi\left(-\nabla_X Y + \nabla_Y X + [X,Y]\right) + S_{\langle\nabla_Y^\perp\xi,\nu^\perp\rangle\nu^\perp}X - S_{\langle\nabla_X^\perp\xi,\nu^\perp\rangle\nu^\perp}Y \\ &= S_{\langle\nabla_Y^\perp\xi,\nu^\perp\rangle\nu^\perp}X - S_{\langle\nabla_X^\perp\xi,\nu^\perp\rangle\nu^\perp}Y = \langle\nabla_Y^\perp\xi,\nu^\perp\rangle S_{\nu^\perp}X - \langle\nabla_X^\perp\xi,\nu^\perp\rangle S_{\nu^\perp}Y \\ &= \langle h\nabla_Y^\perp\nu,\nu^\perp\rangle S_{\nu^\perp}X - \langle h\nabla_X^\perp\nu,\nu^\perp\rangle S_{\nu^\perp}Y \\ &= \langle\nabla_Y^\perp\nu,\nu^\perp\rangle S_{\nu^\perp}X - \langle\nabla_X^\perp\nu,\nu^\perp\rangle S_{\nu^\perp}Y \\ &= \langle\nabla_Y^\perp\nu^\perp,\nu\rangle S_{\nu^\perp}X - \langle\nabla_X^\perp\nu,\nu^\perp\rangle S_{\nu^\perp}Y = 0. \end{aligned}$$

Therefore, $R_\nu(X,Y)\xi$ vanishes if and only if condition 2 holds.

In the second part of the proof we analyze the curvature tensor for the tangent directions. To finish the proof we will show that for $Z \in TM$:

$$R_\nu(X,Y)Z = \nabla'_X\nabla'_Y Z - \nabla'_Y\nabla'_X Z - \nabla'_{[X,Y]}Z = 0.$$

$\nabla'_Y Z = \nabla_Y Z + l(Y,Z)\nu$, thus

$$\begin{gathered} \nabla'_X\nabla'_Y Z = \nabla'_X\left(\nabla_Y Z\right) + \nabla'_X\left(l(Y,Z)\nu\right) = \\ \nabla_X\nabla_Y Z + l(X,\nabla_Y Z)\nu - S_{l(Y,Z)\nu}X + \tilde{\nabla}_X l(Y,Z)\nu, \end{gathered}$$

so the expression for the curvature tensor is

$$\begin{aligned} R_\nu(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z + l(X,\nabla_Y Z)\nu - l(Y,\nabla_X Z)\nu + S_{l(X,Z)\nu}Y \\ &- S_{l(Y,Z)\nu}X + \tilde{\nabla}_X\left(l(Y,Z)\nu\right) - \tilde{\nabla}_Y\left(l(X,Z)\nu\right) - \nabla_{[X,Y]}Z - l([X,Y],Z)\nu \\ &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z + S_{l(X,Z)\nu}Y \\ &- S_{l(Y,Z)\nu}X + \left(l(X,\nabla_Y Z) - l(Y,\nabla_X Z) - l([X,Y],Z)\right. \\ &+ \left. X\left(l(Y,Z)\right) - Y\left(l(X,Z)\right)\right)\nu. \end{aligned}$$

Therefore lemma 3.2 implies,

$$R_\nu(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z + S_{l(X,Z)\nu}Y - S_{l(Y,Z)\nu}X,$$

if and only if condition 2 holds. Now, since M satisfies Gauss equation in $I\!R^4$, the following equations hold,

$$
\begin{aligned}
\langle R(X,Y)Z,W\rangle &= \langle \alpha(X,W),\alpha(Y,Z)\rangle - \langle \alpha(X,Z),\alpha(Y,W)\rangle \\
&= \langle l(X,W)\nu + l^{\perp}(X,W)\nu^{\perp}\rangle\langle l(Y,Z)\nu + l^{\perp}(Y,Z)\nu^{\perp}\rangle \\
&- \langle l(X,Z)\nu + l^{\perp}(X,Z)\nu^{\perp}\rangle\langle l(Y,W)\nu + l^{\perp}(Y,W)\nu^{\perp}\rangle \\
&= l(X,W)l(Y,Z) + l^{\perp}(X,W)l^{\perp}(Y,Z) - l(X,Z)l(Y,W) \\
&+ l^{\perp}(X,Z)l^{\perp}(Y,W).
\end{aligned}
$$

On the other hand write

$$
\begin{aligned}
\langle S_{l(X,Z)\nu}Y,W\rangle - \langle S_{l(Y,Z)\nu}X,W\rangle &= \langle \alpha(Y,W), l(X,Z)\nu\rangle - \langle \alpha(X,W), l(Y,Z)\nu\rangle \\
&= l(Y,W)l(X,Z) - l(X,W)l(Y,Z).
\end{aligned}
$$

Thus,

$$
\langle R(X,Y)Z,W\rangle = l^{\perp}(X,W)l^{\perp}(Y,Z) - l^{\perp}(X,Z)l^{\perp}(Y,W).
$$

So this expression vanishes if and only if condition 1 holds. ■

Conditions 1 and 2 imply that $\tilde{\alpha}$ and $\tilde{\nabla}$ satisfy Gauss, Codazzi equations for the case of constant sectional curvature 0. Then the Fundamental Theorem of Immersions of Submanifolds implies that there exists locally, an isometric immersion $\psi : M \to \mathbb{R}^3$ and a vector bundle isomorphism $\tilde{\psi} : E_\nu \to TM^{\perp}$, such that for every X, $Y \in TM$ and local sections η, ν of E_ν

$$
\begin{aligned}
\langle \tilde{\psi}(\eta), \tilde{\psi}(\nu)\rangle &= \langle \eta, \nu\rangle \\
\tilde{\psi}\tilde{\alpha}(X,Y) &= \hat{\alpha}(X,Y) \\
\tilde{\psi}\tilde{\nabla}_X\eta &= \nabla^{\perp}\tilde{\psi}(\eta),
\end{aligned}
$$

where $\hat{\alpha}$ and $\nabla^{\perp}$ are the second fundamental form and the normal connection of the immersion, respectively. The existence of such immersion clearly implies conditions 1,2 because they are satisfied trivially considering $M \subset \mathbb{R}^3 \subset \mathbb{R}^4$.

Therefore we obtain the following corollary expressing the second fundamental form (condition 1) in local coordinates.

Suppose that $U \subset M$ is an open neighborhood with local coordinates (u,v). The coefficients of the second fundamental form with respect to ν are

$$
\begin{aligned}
e_\nu &= II_\nu(\partial_u) = - < \alpha(\partial_u,\partial_u), \nu >, \\
f_\nu &= - < \alpha(\partial_u,\partial_v), \nu > = - < \alpha(\partial_v,\partial_u), \nu >, \\
g_\nu &= II_\nu(\partial_v) = - < \alpha(\partial_v,\partial_v), \nu >,
\end{aligned}
$$

where $\partial_u = \frac{\partial}{\partial u}$ and $\partial_v = \frac{\partial}{\partial v}$.

Corollary 3.4 *Let M be an oriented surface immersed in $I\!R^4$ and $U \subset M$ an open simply connected neigborhood parametrized by a coordinate chart $\phi : V \to U \in I\!R^4$. Assume that ν and $\nu^\perp$ are unitary smooth orthogonal vector fields normal to $\phi(V) \subset I\!R^4$.*
Then, there exists an isometric immersion $\psi : V \subset I\!R^2 \to I\!R^3$, whose second fundamental form is defined by e_ν, f_ν, g_ν if and only if the following conditionds hold:
1. $e_{\nu^\perp} g_{\nu^\perp} - f^2_{\nu^\perp} = 0$.
2. $w = \langle \nabla^\perp_Y \nu^\perp, \nu \rangle X + \langle \nabla^\perp_X \nu, \nu^\perp \rangle Y \in \ker S_{\nu^\perp}$ for all X, $Y \in TM$.

Remark 3.5 *Since $e_{\nu^\perp} g_{\nu^\perp} - f^2_{\nu^\perp}$ is the determinant of $S_{\nu^\perp}$, condition 1 implies that on the region where $S_{\nu^\perp}$ does not vanishes, there is a global foliation defined by an asymptotic direction field whose binormal field is $\nu^\perp$. On the other hand, the points where $S_{\nu^\perp}$ vanishes are $\nu^\perp$-umbilic with zero constant curvature.*

Lemma 3.6 *Suppose that $\nu^\perp$ is a non (locally) constant binormal vector field with corresponding asymptotic field X. Then the condition 2 above is equivalent to $\nabla^\perp_X \nu^\perp = 0$*

Proof Suppose that Y is any other tangent field linearly independent to X at any point. Since $X \in KerS_{\nu^\perp}$ we have that

$$< \nabla^\perp_Y \nu^\perp, \nu > X + < \nabla^\perp_X \nu, \nu^\perp > Y \in KerS_{\nu^\perp}$$

if and only if $< \nabla^\perp_X \nu, \nu^\perp > Y = 0$
except perhaps at the isolated inflection points of M (at which $Y \in KerS_{\nu^\perp}$). But this amounts to say that $< \nabla^\perp_X \nu, \nu^\perp >= 0$ except perhaps at isolated points, and from the continuity of these functions we have that $< \nabla^\perp_X \nu, \nu^\perp >= 0$ at every point. On the other hand since $\nu^\perp$ is a unitary normal field, we have that $< \nabla^\perp_X \nu^\perp, \nu^\perp >= 0$ and hence the expression above is equivalent to $\nabla^\perp_X \nu^\perp = 0$. ■

Remark 3.7 *a) Since $\nu^\perp$ is a binormal field with asymptotic direction X, we have that $S_{\nu^\perp}(X) = 0$ and thus $\nabla_X \nu^\perp = 0$. Therefore the above expression becomes equivalent to $\bar{\nabla}_X \nu^\perp = 0$.*
b) It also follows from the above lemma that $\nabla^\perp_X \nu = 0$ (for $< \nu, \nu^\perp >= 0$ and hence $< \nabla^\perp_X \nu, \nu^\perp > + < \nabla^\perp_X \nu^\perp, \nu >= 0$)

In general a surface immersed in $\mathbb{R}^4$ does not need to have globally defined binormal fields. A surface M immersed in $\mathbb{R}^4$ is said to be locally convex provided it admits a local support hyperplane at each one of its points. Surfaces contained in the hypersphere S^3 give us examples of locally convex surfaces. It was shown in [7], that a necessary and sufficient condition for the local convexity of a generic surface in $\mathbb{R}^4$ is the global existence of two binormal fields on it. On the other hand minimal surfaces in $\mathbb{R}^4$ (i.e., those having vanishing mean curvaure vector at every point) give us examples of surfaces without binormal directions at any one of their points ([8]). In view of these facts we can state:

Proposition 3.8 *A necessary condition for the existence of some normal field ν on a generic surface Msuch that E_ν admits an isometric immersion into $\mathbb{R}^3$ is that M be locally convex. In particular, minimal surfaces of $\mathbb{R}^4$ do not admit isometric immersions into $\mathbb{R}^3$ for any one of their normal fields.*

Remark 3.9 *Since surfaces in $\mathbb{R}^4$ have at most two binormal fields, there are at most two normal with possible isometric immersions in $\mathbb{R}^3$. In fact, we have: Let M be a locally convex surface in $\mathbb{R}^4$ and let ν be a normal field such that $\nu^\perp$ is one of the binormal fields on M. Then E_ν can be isometrically immersed in $\mathbb{R}^3$ if and only if ν is parallel along the asymptotic lines associated to $\nu^\perp$.*

Proposition 3.10 *Let ν be the normal field on M such that $\nu^\perp$ is not constant, nor totally umbilic and suppose that $\phi : E_\nu \to \mathbb{R}^3$ is an isometric immersion. Then ϕ takes the asymptotic lines of M, corresponding to the binormal $\nu^\perp$ into the asymptotic lines of its image $\phi(M)$ as a surface in $\mathbb{R}^3$.*

Proof As we have seen, $\nabla^\perp_X \nu = 0$. But since ϕ takes the field ν into the normal field N of $\phi(M)$ in $\mathbb{R}^3$ and ϕ is an isometric immersion we get that $\nabla^\perp_{\phi_*(X)} N = 0$ and thus $\phi_*(X)$, must be an asymptotic direction in $\phi(M)$. ∎

Corollary 3.11 *Closed (i.e., compact without boundary) locally convex generic surfaces in $\mathbb{R}^4$ do not admit global isometric immersions in $\mathbb{R}^3$ for any of their normal fields.*

Proof Otherwise we would be able to find some closed surface in $\mathbb{R}^3$ with globally defined asymptotic lines which is impossible. ∎

Let us consider now the two remining possibilities:
a) $\nu^{\perp}$ is constant.
b) $\nu^{\perp}$ is totally umbilic with vanishing curvature.

It is not difficult to show that both cases are equivalent and correspond to flat surfaces in the sense that they lie in some hyperplane. In this case the natural isometric immersion is given by the identity. These being the only cases in which it is possible to find a (trivial) globally defined isometric immersion

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