UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação
ISSN 0103-2577

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Nº 107

NOTAS

Série Matemática

São Carlos – SP
Out./2000

Pontos umbílicos simples de superfícies imersas no $I\!R^4$.

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Outubro 2000

Resumo

Estudamos problemas locais, ao redor de pontos umbílicos simples, em superfícies bidimensionais de $I\!R^4$ tais como determinação finita e desdobramento versal.

Simple Umbilic Points on Surfaces Immersed in $I\!R^4$

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Abstract

We study local problems around simple umbilic points of surfaces immersed in $I\!R^4$ such as finite determinacy and versal unfolding.

1 Introduction

We proceed with the work started in [GGTG], which is devoted to the study of the possible configurations of the lines of principal curvature around umbilic points on

*Part of this work was supported by the CNPq and by PRONEX/FINEP/MCT Grant # 376.97.1080.00. Teoria Qualitativa das Equações Diferenciais Ordinárias.

†Part of this work was supported by FONDECYT Grant # 1980778 and by IMPA/CNPq Grant

surfaces that are immersed in $I\!R^4$, namely, the locally topologically stable umbilic points are classified and it is shown that they appear generically.

The notion of principal direction for a smooth immersion $f : M \to I\!R^4$ which we introduce and use is due to J.A. Little [Lit]; it is an extension of the classical three–dimensional concept (see [R-S] for another possible extension). A *principal direction at* p is a line in T_pM generated by a unitary vector which makes the length of $\alpha(X,X)$ take a extreme value, where α is the second fundamental form of the immersion f at p and X varies on the unitary circle in T_pM. The set $\mathcal{E}$ of values of $\alpha(X,X)$ called *ellipse of curvature* is an ellipse that may degenerate into a line segment, a circle or a point. Also, it is easily seen that as X goes once around the circle, $\alpha(X,X)$ goes twice around $\mathcal{E}$. Therefore, when $\mathcal{E}$ is either an ellipse or a line segment, there are four principal directions at p; when $\mathcal{E}$ is either a circle or a point we say p is a *umbilic point of* f. The principal lines of curvature of α are those curves in M that are disjoint from the umbilic points and tangent to principal lines.

This article is organized as follows:

In Sections 2-4, we summarize the results obtained in [GGTG].

Section 5 is devoted to the proof of Theorem 1.1, which we state next.

Theorem 1.1 *Under generic conditions, if $f : M \to I\!R^4$ is a smooth immersion of a surface M and $p \in M$ is a umbilic point of f, then there are isothermal coordinates $(u,v) : (M,p) \to (I\!R^2, 0)$ such that the differential equation of the principal lines of f in these coordinates is of the form*

$$4(Au + Bv + S(u,v))(du^2 - dv^2)dudv + (v + R(u,v))(du^4 - 6du^2dv^2 + dv^4) = 0$$

where $A \neq 0$ and B are real numbers, and $S(u,v)$ and $R(u,v)$ are real–valued functions that satisfy

$$S(0,0) = R(0,0) = \frac{\partial S}{\partial u}(0,0) = \frac{\partial S}{\partial v}(0,0) = \frac{\partial R}{\partial u}(0,0) = \frac{\partial R}{\partial v}(0,0) = 0.$$

Moreover, under each of the conditions (a)–(c), the umbilic point p is locally topologically stable. Finally, under each of the conditions (a)–(e), the corresponding phase portrait is obtained making into one, by means of a rigid translation, the pair of pictures (i.e., nets) in the figure that is indicated below.

(a) Condition H_3 (Fig. 1): $\Delta < 0$.

(b) Condition H_4 (Fig. 2): $\Delta > 0$, $A < 0$ *and* $A \neq -1/4$.

(c) *Condition* H_5 *(Fig. 3):* $\Delta > 0,\ A > 0$.

(d) *Condition* H_{34} *(Fig. 4):* $\Delta > 0,\ A = -1/4$ *and* $B \neq 0$.

(e) *Condition* $\tilde{H}_3$ *(Fig. 5):* $\Delta > 0,\ A = -1/4$ *and* $B = 0$,

where

$$\begin{aligned} \Delta \;=\; & 16[4(1+B^2)^3 + 24(1+B^2)^2 A + 8(5-B^2)(1+B^2)A^2 + \\ & 4(9+B^2)A^3 + (17+B^2)A^4 + 4A^5] \,. \end{aligned}$$

Remark 1.2

(a) The statements on the umbilics H_3, H_4, H_5 are proved in [GGTG]; in this article we prove those on H_{34} and $\tilde{H}_3$.

(b) $A \neq 0$ is a transversality (generic) condition that characterizes a simple umbilic point.

(c) The principal lines around the umbilic point p of Theorem 1.1 make up two pairwise transversal nets $\mathcal{F}_1$ and $\mathcal{F}_2$. We say p is *locally topologically stable* if $\mathcal{F}_1$ and $\mathcal{F}_2$ are locally topologically stable around p. This definition for nets is similar to the one for the case of principal lines of surfaces immersed in $I\!R^3$ around an isolated umbilic point (see [G-S]). The umbilics H_3, H_4 and H_5 are locally topologically stable, whereas $\tilde{H}_3$ and H_{34} are not.

(d) In the coordinates under consideration, the slope – at the origin – of a principal line approaching the origin (i.e., the umbilic point) must be a real root of the polynomial $g(s) = -sQ(s)$, where

$$Q(s) = s^4 - 4Bs^3 - 2(3+2A)s^2 + 4Bs + 1 + 4A\,.$$

The discriminant of this polynomial is Δ. Conversely, given a real root λ of $g(s)$, there is a principal line approaching the umbilic point with slope λ.

(e) Under condition H_3, the polynomial $g(s)$ has three simple real roots and two complex ones; each net $\mathcal{F}_i$ has three separatrices.

(f) Under condition H_4 (resp. H_5), the polynomial $g(s)$ has five simple real roots; each net $\mathcal{F}_i$ has four (resp. five) separatrices.

(g) Under condition $\tilde{H}_3$, zero is a triple root of $g(s)$ and the other two roots are real and simple; each net $\mathcal{F}_i$ has three separatrices.

(h) Under condition $\tilde{H}_{34}$, zero is a double root of $g(s)$ and the other three roots are real and simple; each net $\mathcal{F}_i$ has four separatrices.

Finally in Section 6, we obtain the versal unfolding of the umbilic points H_{34} and $\tilde{H}_3$ thus proving that the former are of codimension one and while the latter of codimension two.

2 Differential Equation of Lines of Curvature

We shall need the following result of [GGTG, Theorem 2.1.] (See also [GGGT]), i.e.,

Theorem 2.1 *Let $f : M \to I\!R^4$ be a smooth immersion of a surface M. In isothermic coordinates $(u,v) : (M,p) \to (I\!R^2, 0)$, the differential equation of the lines of curvature of f is given by*

$$(1) \qquad 4a(u,v)(du^2 - dv^2)dudv + b(u,v)(du^4 - 6du^2dv^2 + dv^4) = 0$$

where $a = a(u,v)$ and $b = b(u,v)$ are smooth real-valued functions. Moreover, p is a umbilic point if and only if $a(0,0) = b(0,0) = 0$. $a(0,0) = b(0,0) = 0$.

Conversely, for any given analytic functions $a, b : U \to I\!R$ defined on an open neighborhood $U \subset I\!R^2$ of a point p, there exists an immersion $f : V \to I\!R^4$ where $V \subset U$ is some small open neighborhood of p such that the differential equation of the lines of curvature of f is given by (??) and the coordinates (u,v) are isothermic.

3 Simple Umbilic Points

Let $f : M \to I\!R^4$ be a smooth immersion of a surface M, and let $p \in M$ be a umbilic point of f. The point p is called a *simple umbilic point* of f if there are isothermic coordinates $(u,v) : (M,p) \to (I\!R^2, 0)$ such that the differential equation of the principal lines of f in these coordinates has the form of equation (??), and $\{a = 0\}$ and $\{b = 0\}$ are regular curves meeting to each other transversally at the origin.

Lemma 3.1 *Let $f : M \to I\!R^4$ be a smooth immersion of a surface M, and let $p \in M$ be a simple umbilic point of f. There are isothermic coordinates $(u,v) : (M,p) \to (I\!R^2, 0)$ such that the differential equation of the principal lines of f in these coordinates is of the form*

$$(2) \qquad 4(Au + Bv + S)(du^2 - dv^2)dudv + (v + R)(du^4 - 6du^2dv^2 + dv^4) = 0$$

where $A \neq 0$ and B are real numbers, $S = S(u,v)$ and $R = R(u,v)$ are real-valued functions satisfying

$$S(0,0) = R(0,0) = \frac{\partial S}{\partial u}(0,0) = \frac{\partial S}{\partial v}(0,0) = \frac{\partial R}{\partial u}(0,0) = \frac{\partial R}{\partial v}(0,0) = 0.$$

Let M be a smooth, compact and oriented surface. We let $Imm(M, I\!R^4)$ denote the space of smooth immersions of M into $I\!R^4$ endowed with the smooth topology. The following two propositions are needed in the next section.

Proposition 3.2 *The set of immersions $f \in Imm(M, I\!R^4)$ whose umbilic points are simple is dense in $Imm(M, I\!R^4)$.*

Also, since simple umbilic points are defined by transversality conditions, they are persistent under perturbations of the immersion. We make this result more precise in the next proposition.

Proposition 3.3 *Let p_0 be a simple umbilic point of a smooth immersion $f : M \to I\!R^4$ of a surface M. Then there exist a neighborhood U of p in M, a neighborhood $\mathcal{V}$ of f in $Imm(M, I\!R^4)$, and a smooth map $p : \mathcal{V} \to U$ which, to each $g \in \mathcal{V}$, associates the unique umbilic point of $g|_U$ and this umbilic is simple.*

Proof. By [Ga-S, Proposition 2.1], the equation of the lines of curvature of a smooth immersion $g : M \to I\!R^4$ relative to an arbitrary chart (u,v) is given by

$$(3) \qquad \begin{aligned} (A_{40}a_0 + A_{41}a_1)dv^4 + (A_{30}a_0 + A_{31}a_1)dv^3du + (A_{20}a_0 + A_{21}a_1)dv^2du^2 \\ + a_1dvdu^3 + a_0du^4 = 0\,, \end{aligned}$$

where $a_0 = a_0(u,v), a_1 = a_1(u,v)$ and $A_{ij} = A_{ij}(u,v)$, with $i = 2,3,4$ and $j = 0,1$ are smooth functions. Moreover, the umbilic points of g are given by the equations $a_0 = a_1 = 0$.

We now consider isothermic coordinates $(u,v) : (M,p) \to (I\!R^2, 0)$ such that the differential equation of the principal lines of f is of form (2). Therefore $A \neq 0$.

For g in a neighborhood $\tilde{\mathcal{V}}$ of f in $Imm(M, I\!R^4)$, the equation of the lines of curvature, in the same coordinates, is the form (3) where $a_0(f)(u,v) = v + R(u,v)$ and $a_1(f)(u,v) = 4(Au + Bv + S(u,v))$.

Next consider the smooth map $F : \tilde{\mathcal{V}} \times I\!R^2 \to I\!R^2$ defined by

$$F(g,(u,v)) = (a_0(g)(u,v), a_1(g)(u,v)) .$$

Since $F(f,(0,0)) = (0,0)$ and the matrix

$$D_2F(f,(0,0)) = \begin{pmatrix} 4A & 4B \\ 0 & 1 \end{pmatrix}$$

is non-singular, there exist a neighborhood $\tilde{U}$ of $(0,0)$ in $I\!R^2$, a neighborhood $\mathcal{V} \subset \tilde{\mathcal{V}}$ of f in $Imm(M, I\!R^4)$, and a smooth map $q : \mathcal{V} \to \tilde{U}$ such that $q(f) = (0,0)$ and $F(g, q(g)) = (0,0)$ for all $g \in \mathcal{V}$, and the proof now follows. ■

4 The Manifold LM and the Semi-local Vector Field $\mathcal{L}'$

We consider the projective line bundle PM over M, which is defined by the tangent bundle with the zero section 0 removed ($TM \setminus O$) modulo the equivalent relation: two elements (p_1, v_1) and (p_2, v_2) are equivalent if their first components coincide and the second ones are collinear. We let P denote the projection of PM onto M. In terms of the chart (u,v) with domain U in M, the charts $(u, v, t = du/dv)$ and $(u, v, s = dv/du)$ are defined on $P^{-1}(U)$ and their domains cover this open set.

Consider the surface LM in PM defined by the solutions of equation (1) of Theorem 2.1

$$\omega = 4a(u,v)(du^2 - dv^2)dudv + b(u,v)(du^4 - 6du^2dv^2 + dv^4) = 0 .$$

In the chart $(u, v, s = dv/du)$, the surface LM is written as

$$\mathcal{L}(u, v, s) = 4a(u,v)(1 - s^2)s + b(u,v)(1 - 6s^2 + s^4) = 0$$

and in the chart $(u, v, t = du/dv)$, it is expressed as

$$\mathcal{L}(u, v, t) = 4a(u,v)(t^2 - 1)t + b(u,v)(t^4 - 6t^2 + 1) = 0 .$$

It is clear that the surface LM is determined by the principal directions and does not depend on the particular chart used.

Let Sm be the set of umbilic points of the immersion $f : M \to I\!R^4$. Outside $P^{-1}(Sm)$, we have that LM is a regular submanifold of $P^{-1}(M)$ where it is a 4-fold regular covering of $M \setminus Sm$. In local (u, v) coordinates around $p \in Sm$, the set Sm corresponds to the set $a^{-1}(0) \cap b^{-1}(0)$ (see Theorem 2.1).

Lemma 4.1 *Let $p \in Sm$. The point p is simple if and only if LM is regular around $P^{-1}(p)$.*

In local (u, v, s) coordinates (i.e., $t \neq 0$) for a point of PM, we consider the vector field

$$\mathcal{L}' = u'\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v} + s'\frac{\partial}{\partial s}$$

whose components are given by:

$$\begin{aligned} u' &= \tilde{u}(u, v, s) = 4a(u, v)(1 - 3s^2) + 4b(u, v)s(-3 + s^2)\,, \\ v' &= s\tilde{u}(u, v, s)\,, \\ s' &= -[\,\mathcal{L}_u(u, v; s) + s\mathcal{L}_v(u, v; s)]. \end{aligned}$$

A simple calculation shows that $\mathcal{L}'$ is tangent to LM. In what follows, we only deal with its restriction to LM, maintaining the same notation $\mathcal{L}'$ notwithstanding. Its projection $P_*\mathcal{L}'$ only vanishes at the umbilic points. In the complement of Sm, the vector field $\mathcal{L}'$ generates the principal line fields of M; that is, for each non-umbilic (u, v), the four P-preimages (u, v, r_1), (u, v, r_2), (u, v, r_3) and (u, v, r_4) verify that $P_*\mathcal{L}'(u, v, r_i)$ generates the principal line with direction r_i.

If (u, v) are the coordinates of Lemma 3.1, then $(0, 0)$ is umbilic and the singularities of $\mathcal{L}'$ are the zeros of s' on the s-axis given by the equation

$$g(s) = -sQ(s) = 0$$

where

$$Q(s) = s^4 - 4Bs^3 - 2(3 + 2A)s^2 + 4Bs + 1 + 4A\,.$$

The main results of [GGTG] are summarized in the next theorem.

Theorem 4.2 *The set of smooth immersions $f : M \to I\!R^4$ such that every umbilic point is locally topologically stable, is open and dense in $Imm(M, I\!R^4)$.*

5 Phase Portrait

In order to determine the phase portrait of the nets formed by the lines of curvature around a simple umbilic point p, we consider isothermic coordinates $(u, v) : (M, p) \to (I\!R^2, 0)$ such that the differential equation of the principal lines of f is of form (2). Under these conditions, the polynomial

$$g(s) = -s\,Q(s)\,, \tag{4}$$

where

$$Q(s) \;=\; s^4 - 4Bs^3 - 2(3+2A)s^2 + 4Bs + 1 + 4A\,,$$

is called the *separatrix polynomial* of (2). The name comes from the fact that the real roots of this polynomial correspond exactly to all of the possible directions of asymptotic convergence to the point p by the leaves of both principal nets.

In relation to these polynomials we have the next result.

Lemma 5.1 *The discriminant of the polynomial of degree four $Q(s)$ is*

$$\begin{aligned} \Delta \;&=\; 16[4(1+B^2)^3 + 24(1+B^2)^2A + 8(5-B^2)(1+B^2)A^2 + \\ &\quad\; 4(9+B^2)A^3 + (17+B^2)A^4 + 4A^5]\,. \end{aligned} \tag{5}$$

Moreover,

(a) $\Delta < 0$ *implies that $g(s)$ has three simple real roots and two conjugate complex roots.*

(b) $\Delta > 0$ *and $A_{10} \neq -1/4$ imply that $g(s)$ has five simple real roots.*

(c) The multiplicity of a real root s_0 of $g(s) = -s\,Q(s)$ is less than or equal to three. Moreover, if s_0 has multiplicity two or three, then the other roots of $g(s)$ are real and simple.

Proof. Statements a) and b) are proved in [GGGT, Lemma 4.2]. For statement c), note that, with an appropriated change of coordinates (rigid rotation), we may suppose that $s_0 = 0$. Hence $A = -\dfrac{1}{4}$ and

$$g(s) = -s^2(s^3 - 4Bs^2 - 5s + 4B)\,.$$

If $B = 0$, then the multiplicity of $s_0 = 0$ is three,

$$g(s) = -s^3(s^2 - 5)\,,$$

and the other roots are $s = \pm\sqrt{5}$.
If $B \neq 0$, then the multiplicity of $s_0 = 0$ is two and the remaining three roots of $g(s)$ are real and simple, since

$$g(1) = 4\,, \quad g(-1) = -4\,, \quad \lim_{s\to+\infty} g(s) = -\infty\,, \lim_{s\to-\infty} g(s) = +\infty\,. \quad \blacksquare$$

Definition 5.2 *Let p be a simple umbilic point of a smooth immersion $f : M \to \mathbb{R}^4$ of a surface M. Consider isothermic coordinates $(u, v) : (M, p) \to (\mathbb{R}^2, 0)$ such that the differential equation of the principal lines of f is of form (2). Then the point p is called:*

a) A hyperbolic umbilic point if the separatrix polynomial (4) has only simple roots.

b) A H_{34} umbilic point if the separatrix polynomial (4) has a root of multiplicity two.

c) A $\tilde{H}_3$ umbilic point if the separatrix polynomial (4) has a root of multiplicity three.

In [GGGT, Theorem 1.1], it was proved that the hyperbolic umbilic points are locally topologically stable and that there are three different topological types. To describe them, consider isothermical coordinates $(u, v) : (M, p) \to (\mathbb{R}^2, 0)$ such that the equation of the lines of curvature is given by (2). Thus the discriminant Δ of the corresponding polynomial $Q(s)$ is given by (5). Then:

(H_3) if $\Delta < 0$, then the configurations of the principal nets are homeomorphic to the ones shown in Figure 1;

(H_4) if $\Delta > 0$ $A < 0$ and $A \neq -\frac{1}{4}$, then the configurations of the principal nets are homeomorphic to those shown in Figure 2;

(H_5) if $\Delta > 0$ and $A > 0$, then the configurations of the principal nets are homeomorphic to the ones shown in Figure 3.

Figure 1

Figure 2

Figure 3

Theorem 5.3 *Let $f : M \to I\!R^4$ be a smooth immersion of a surface M, and let $p \in M$ be a umbilic point of f of type H_{34} (resp. of type $\tilde{H}_3$). The phase portrait of the principal nets around p are homeomorphic to the ones shown in Figure 4 (resp. Figure 5).*

Figure 4

Figure 5

Proof. For the proof we use arguments analogous to those of the proof of [GGGT, Lemma 5.1]. Indeed, the surface LM defined in the projective line bundle PM over M by the solutions of equation (2) is regular in $P^{-1}(p)$, where P denotes the projection of PM onto M. In local $(u, v, s = \frac{dv}{du})$ coordinates, we consider the vector field (which is tangent to LM)

$$u'\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v} + s'\frac{\partial}{\partial s}$$

whose components are given by

$$\begin{aligned} u' &= \tilde{u}(u, v, s) = 4a(u, v)(1 - 3s^2) + 4b(u, v)s(-3 + s^2)\,, \\ v' &= s\tilde{u}(u, v, s)\,, \\ s' &= -[\,\mathcal{L}_u(u, v; s) + s\mathcal{L}_v(u, v; s)], \end{aligned}$$

with $a(u,v) = Au + Bv + S(u,v)$ and $b(u,v) = v + R(u,v)$, where S and R are as in (2).
Again, we use the notation $\mathcal{L}'$ for the restriction of this vector field to LM.
If (u,v) are the coordinates used in (2), then $(0,0)$ is umbilic and the singularities of $\mathcal{L}'$ are the zeros of s' on the s-axis given by the equation

$$g(s) = -sQ(s) = 0$$

where

$$Q(s) = s^4 - 4Bs^3 - 2(3+2A)s^2 + 4Bs + 1 + 4A\,.$$

The projection W of our vector field $\mathcal{L}'$ onto the plane u', s' is given by

$$\begin{aligned} u' &= \tilde{u}(u, v(u,s), s) = 4a(u, v(u,s))(1-3s^2) + 4b(u, v(u,s))s(-3+s^2) \\ &= uh(s) + U(u,s)\,, \\ s' &= -[\,\mathcal{L}_u(u, v(u,s); s) + s\mathcal{L}_v(u, v(u,s); s)] \\ &= g(s) + P(u,s)\,, \end{aligned}$$

with $U(0,s) = \frac{\partial U}{\partial u}(0,s) = 0\,,\ P(0,s) = \frac{\partial P}{\partial u}(0,s) = 0$ and

$$h(s) = \frac{4A(1+s^2)^3}{s^4 - 4Bs^3 - 6s^2 + 4Bs + 1}\,.$$

For the case p a umbilic point of type H_{34}, we may assume that $s_0 = 0$ is a double root of $g(s)$. Then $A = -\dfrac{1}{4}$ and $B \neq 0$; the other roots s_1, s_2 and s_3 verify: $s_1 < -1$, $s_2 > 1$ and, if $B < 0$ (resp. $B > 0$), then $0 < s_3 < 1$ (resp. $-1 < s_3 < 0$). Thus the singular points $(0,0,s_1)$, $(0,0,s_2)$ and $(0,0,s_3)$ are hyperbolic saddles, and $(0,0,0)$ is a semi-hyperbolic singular point, which is a saddle node over its central manifold.
For the case p a umbilic point of type $\tilde{H}_3$, we may assume that $s_0 = 0$ is a root of multiplicity three of $g(s)$. Then the other roots of $g(s)$ are $s_1 < -1$ and $s_2 > 1$. Also, the singular points $(0,0,s_1)$ and $(0,0,s_2)$ are hyperbolic saddles, and $(0,0,0)$ is a semi-hyperbolic singular point, which is a repellor over its central manifold.
The proof of Theorem 4.3 now follows. ■

6 Versal Unfolding

In this section, we obtain the versal unfolding of the umbilic points H_{34} and $\tilde{H}_3$. They are presented as a family of surfaces in the Monge form $f_\lambda : \mathbb{R}^2 \to \mathbb{R}^4$

$$f_\lambda(x,y) = (x, y, R_\lambda(x,y), S_\lambda(x,y))\,.$$

The notion of equivalence of families of immersions that we use is the following.

Definition 6.1 *Consider two smooth families (f_μ) and (g_μ) in $Imm(M, I\!R^4)$ with (the same) parameter $\mu \in I\!R^k$. Let $\mathcal{F}_1(f_\mu)$ and $\mathcal{F}_2(f_\mu)$ (resp. $\mathcal{F}_1(g_\mu)$ and $\mathcal{F}_2(g_\mu)$) be the nets associated to f_μ (resp. g_μ). The families (f_μ) and (g_μ) are called principal C^0-equivalent (over the identity) if, for every $\mu \in I\!R^k$, the nets $\mathcal{F}_i(f_\mu)$ and $\mathcal{F}_i(g_\mu)$ are topologically equivalent, with $i = 1, 2$.*

In [Ga-S, Proposition 3.1], it was proved that if p is a simple umbilic point of $f \in Imm(M, I\!R^4)$, then there exists a Monge chart

$$z = R(u,v), \quad w = S(u,v)$$

and a homotety in $I\!R^4$ such that the equation of the lines of curvature is given by

$$\begin{array}{ll} (6) & \begin{array}{c} 4(Au+Bv)(du^2-dv^2)dudv + v(du^4 - 6du^2dv^2 + dv^4) + \\ A_4dv^4 + A_3dv^3du + A_2dv^2du^2 + A_1dvdu^3 + A_0du^4 = 0 \end{array} \end{array}$$

where $A \neq 0$ and $A_i(0,0) = \dfrac{\partial A_i}{\partial u}(0,0) = \dfrac{\partial A_i}{\partial v}(0,0) = 0$, for $i = 0,1,2,3,4$.

Since the linear parts of equations (6) and (2) coincide, Definition 5.2 also applies to equation (6) with the same parameters A and B.

In Lemma 6.3 above we prove that the normal form (6) also hold for families in $Imm(M, I\!R^4)$ passing through an immersion which has a simple umbilic point. First we need the next result.

Lemma 6.2 *Let (f_μ) be an arbitrary smooth family in $Imm(M, I\!R^4)$ with parameter $\mu \in I\!R^k$ such that f_0 has a simple umbilic point p_0. For f_0, consider a Monge chart*

$$z = R(u,v), \quad w = S(u,v)$$

such that the differential equation of the principal lines of f_0 in these coordinates is of form (6). Then there exists a change of coordinates of the form $(u,v,\mu) = H(s,t,\mu)$ such that for all μ, with $|\mu|$ small, the origin is a simple umbilic point of $f_\mu(s,t)$.

Proof. For μ in a neighborhood V of the origin in $I\!R^k$, let $p(\mu)$ be the smooth continuation of the simple umbilic point $p_0 = p(0)$ in U. Then it suffices to consider the change of coordinates

$$(u,v,\mu) = (s,t,\mu) + ((u,v)(p(\mu)), \bar{0})$$

where $\bar{0}$ is the origin in $I\!R^k$. ■

Lemma 6.3 *Let (f_μ) be an arbitrary smooth family in $Imm(M, I\!R^4)$, with parameter $\mu \in I\!R^k$, such that f_0 has a simple umbilic point p_0. Then there exits a Monge chart*

$$z = R(u, v, \mu), \quad w = S(u, v, \mu)$$

such that the equation of the lines of curvature is given by

$$\begin{aligned} &4(A(\mu)u + B(\mu)v)(du^2 - dv^2)dudv + v(du^4 - 6du^2dv^2 + dv^4) + \\ &A_4(\mu)dv^4 + A_3(\mu)dv^3du + A_2(\mu)dv^2du^2 + A_1(\mu)dvdu^3 + A_0(\mu)du^4 = 0\,, \end{aligned} \tag{7}$$

with $A_i(\mu)(0,0) = \dfrac{\partial A_i}{\partial u}(\mu)(0,0) = \dfrac{\partial A_i}{\partial v}(\mu)(0,0) = 0,$ *for* $i = 0, 1, 2, 3, 4.$

Proof. For μ in a neighborhood V of the origin in $I\!R^k$, we consider a Monge chart

$$z = R(s, t, \mu), \quad w = S(s, t, \mu)$$

such that the equation of the lines of curvature is given by

$$\begin{aligned} &4(\tilde{A}(\mu)s + \tilde{B}(\mu)t)(ds^2 - dt^2)dsdt + (\tilde{C}(\mu)s + \tilde{D}(\mu)t)(ds^4 - 6ds^2dt^2 + dt^4) + \\ &\tilde{A}_4(\mu)dt^4 + \tilde{A}_3(\mu)dt^3ds + \tilde{A}_2(\mu)dt^2ds^2 + \tilde{A}_1(\mu)dtds^3 + \tilde{A}_0(\mu)ds^4 = 0 \end{aligned}$$

where $A = \tilde{A}(0) \neq 0$, $\tilde{C}(0) = 0$, $\tilde{D}(0) = 1$ and $\tilde{A}_i(\mu)(0,0) = \dfrac{\partial \tilde{A}_i}{\partial s}(\mu)(0,0) = \dfrac{\partial \tilde{A}_i}{\partial t}(\mu)(0,0) = 0,$ for $i = 0, 1, 2, 3, 4.$

Let $L_{(\alpha,\beta,\mu)} : I\!R^2 \to I\!R^2$ be a family of linear isomorphisms, with parameter $(\alpha, \beta, \mu) \in I\!R^2 \times V$, such that the inverse of $L = L_{(\alpha,\beta,\mu)}$ is given by

$$L^{-1}(s, t, \mu) = ((1+\alpha)s - \beta t, \beta s + (1+\alpha)t, \mu)\,.$$

Observe that for all $(\alpha, \beta, \mu) \in I\!R^2 \times V$, $L_{(\alpha,\beta,\mu)}$ is a linear rotation. This implies that, in coordinates

$$(s, t, \mu) = ((1+\alpha)u - \beta v, \beta u + (1+\alpha)v, \mu)\,,$$

the equation of the lines of curvature correspond to a Monge chart and is given by

$$\begin{aligned} &4(A(\mu)u + B(\mu)v)(du^2 - dv^2)dudv + (C(\mu)u + D(\mu)v)(du^4 - 6du^2dv^2 + dv^4) + \\ &A_4(\mu)dv^4 + A_3(\mu)dv^3du + A_2(\mu)dv^2du^2 + A_1(\mu)dvdu^3 + A_0(\mu)du^4 = 0\,, \end{aligned}$$

with $A_i(\mu)(0,0) = \dfrac{\partial A_i}{\partial u}(\mu)(0,0) = \dfrac{\partial A_i}{\partial v}(\mu)(0,0) = 0$, for $i = 0,1,2,3,4$.
To finish the proof of the lemma, it is enough to show that for $\mu \in V$ close enough to 0, there exists $(\alpha,\beta) = (\alpha(\mu),\beta(\mu))$ such that $(C(\mu), D(\mu)) \equiv (0,1)$. In fact

$$\begin{aligned}
C(\mu) &= 4(1+\alpha)^4\tilde{A}(\mu)\beta + 4(1+\alpha)^3\tilde{B}(\mu)\beta^2 - 4(1+\alpha)^2\tilde{A}(\mu)\beta^3 - \\
&\quad 4(1+\alpha)\tilde{B}(\mu)\beta^4 + (1+\alpha)^4\beta\tilde{D}(\mu) - 6(1+\alpha)^2\beta^3\tilde{D}(\mu) + \beta^5\tilde{D}(\mu) + \\
&\quad (1+\alpha)^5\tilde{C}(\mu) - 6(1+\alpha)^3\beta^2\tilde{C}(\mu) + (1+\alpha)\beta^4\tilde{C}(\mu) \qquad \text{and} \\
D(\mu) &= 4(1+\alpha)^4\tilde{B}(\mu)\beta - 4(1+\alpha)^3\tilde{A}(\mu)\beta^2 - 4(1+\alpha)^2\tilde{B}(\mu)\beta^3 + 4(1+ \\
&\quad \alpha)\tilde{A}(\mu)\beta^4 + (1+\alpha)^5\tilde{D}(\mu) - 6(1+\alpha)^3\beta^2\tilde{D}(\mu) + (1+\alpha)\beta^4\tilde{D}(\mu) - \\
&\quad (1+\alpha)^4\beta\tilde{C}(\mu) + 6(1+\alpha)^2\beta^3\tilde{C}(\mu) - \beta^5\tilde{C}(\mu).
\end{aligned}$$

If $\tilde{C}(\mu) = 0$, then $\tilde{D}(\mu) \neq 0$; therefore, we may set $\beta = 0$ and $1+\alpha = \frac{1}{\tilde{D}(\mu)^{\frac{1}{5}}}$ and so $C(\mu) = 0$ and $D(\mu) = 1$.
If $\tilde{C}(\mu) \neq 0$, we set $1+\alpha = m\beta$, with m a real root of the equation

$$\begin{aligned}
&\tilde{C}(\mu)x^5 + 2(2\tilde{B}(\mu) - 3\tilde{C}(\mu))x^4 + 2(2\tilde{B}(\mu) - 3\tilde{C}(\mu))x^3 - \\
&\quad 2(2\tilde{A}(\mu) + 3\tilde{D}(\mu))x^2 + (\tilde{C}(\mu) - 4\tilde{B}(\mu))x + \tilde{D}(\mu) = 0\,.
\end{aligned}$$

Then $C(\mu) = 0$ and we are under the condition of the first case. ■

To obtain a versal unfolding for a simple umbilic point we shall need the following lemma.

Lemma 6.4 *Let (f_μ) be an arbitrary smooth family in $Imm(M, I\!R^4)$, with parameter $\mu \in I\!R^k$, such that f_0 has a simple umbilic point p_0. Consider a Monge chart*

$$z = R(u,v,\mu), \quad w = S(u,v,\mu)$$

such that the equation of the lines of curvature is given by

$$\begin{aligned}
&4(A(\mu)u + B(\mu)v)(du^2 - dv^2)dudv + v(du^4 - 6du^2dv^2 + dv^4) + \\
&A_4(\mu)dv^4 + A_3(\mu)dv^3du + A_2(\mu)dv^2du^2 + A_1(\mu)dvdu^3 + A_0(\mu)du^4 = 0\,,
\end{aligned} \tag{8}$$

with $A_i(\mu)(0,0) = \dfrac{\partial A_i}{\partial u}(\mu)(0,0) = \dfrac{\partial A_i}{\partial v}(\mu)(0,0) = 0$, for $i = 0,1,2,3,4$.
Then, for $\mid \mu \mid$ small, the family (f_μ) is equivalent to the family of surfaces in Monge form

$$z = \tilde{R}(u,v,\mu), \quad w = \tilde{S}(u,v,\mu)\,,$$

with

$$\begin{aligned}\tilde{R}(u,v,\mu) &= v^2 - \frac{1}{24}A(\mu)u^3 + +\frac{1}{24}B(\mu)v^3 \qquad \text{and} \\ \tilde{S}(u,v,\mu) &= u^2 + uv + v^2 + \frac{1}{24}v^3 .\end{aligned}$$

Proof. Consider the Monge chart of the previous lemma

$$z = R(u,v,\mu), \quad w = S(u,v,\mu) . \tag{9}$$

Using [Ga-S, Section 3], if

$$\begin{aligned}R(u,v,\mu) &= \frac{r_{20}(\mu)}{2}u^2 + r_{11}(\mu)uv + \frac{r_{02}(\mu)}{2}v^2 + \frac{r_{30}(\mu)}{6}u^3 + \frac{r_{21}(\mu)}{2}u^2v + \\ &\quad \frac{r_{12}(\mu)}{2}uv^2 + \frac{r_{03}(\mu)}{6}v^3 + h.o.t. \qquad \text{and} \\ S(u,v,\mu) &= \frac{s_{20}(\mu)}{2}u^2 + s_{11}(\mu)uv + \frac{s_{02}(\mu)}{2}v^2 + \frac{s_{30}(\mu)}{6}u^3 + \frac{s_{21}(\mu)}{2}u^2v + \\ &\quad \frac{s_{12}(\mu)}{2}uv^2 + \frac{s_{03}(\mu)}{6}v^3 + h.o.t. ,\end{aligned}$$

then $(0,0)$ is an umbilic point if and only if

$$r_{11}(\mu)(r_{20}(\mu) - r_{02}(\mu)) + s_{11}(\mu)(s_{20}(\mu) - s_{02}(\mu)) = 0$$

and

$$4(r_{11}(\mu)^2 + s_{11}(\mu)^2) - (r_{20}(\mu) - r_{02}(\mu))^2 - (s_{20}(\mu) - s_{02}(\mu))^2 = 0 .$$

Note that these conditions are equivalent to

$$2r_{11}(\mu) = \pm(s_{20}(\mu) - s_{02}(\mu)) \quad \text{and} \quad 2s_{11}(\mu) = \mp(r_{20}(\mu) - r_{02}(\mu)) . \tag{10}$$

Under these circuntances, observing (8) at $(0,0)$ we get

$$\begin{aligned}0 &= r_{21}(\mu)(r_{02}(\mu) - r_{20}(\mu)) + (r_{12}(\mu) - r_{30}(\mu))r_{11}(\mu) + \\ &\quad (s_{12}(\mu) - s_{30}(\mu))s_{11}(\mu) + s_{21}(\mu)(s_{02}(\mu) - s_{20}(\mu)) ,\end{aligned}$$

$$1 = 4r_{12}(\mu)(r_{02}(\mu) - r_{20}(\mu)) + 4(r_{03}(\mu) - r_{21}(\mu))r_{11}(\mu) +$$

$$4s_{12}(\mu)(s_{02}(\mu) - s_{20}(\mu)) + 4(s_{03}(\mu) - s_{21}(\mu))s_{11}(\mu)\,,$$

$$\begin{aligned} A(\mu) &= 2(r_{12}(\mu) - r_{30}(\mu))(r_{02}(\mu) - r_{20}(\mu)) - 8r_{21}(\mu)r_{11}(\mu) + \\ &\quad 2(s_{12}(\mu) - s_{30}(\mu))(s_{02}(\mu) - s_{20}(\mu)) - 8s_{21}(\mu)s_{11}(\mu)) \qquad \text{and} \end{aligned}$$

$$\begin{aligned} B(\mu) &= 2(r_{03}(\mu) - r_{21}(\mu))(r_{02}(\mu) - r_{20}(\mu)) - 8r_{12}(\mu)r_{11}(\mu) + \\ &\quad 2(s_{03}(\mu) - s_{21}(\mu)(s_{02}(\mu) - s_{20}(\mu)) - 8s_{12}(\mu)s_{11}(\mu)\,. \end{aligned}$$

We shall proceed considering, in (10), the case

$$2r_{11}(\mu) = (s_{20}(\mu) - s_{02}(\mu)) \quad \text{and} \quad 2s_{11}(\mu) = -(r_{20}(\mu) - r_{02}(\mu))\,.$$

Using the notations

$$\begin{aligned} \alpha(\mu) &= (r_{12} - r_{30} - 2s_{21})(\mu)\,, \\ \beta(\mu) &= (s_{12} - s_{30} + 2r_{21})(\mu)\,, \\ \gamma(\mu) &= (r_{03} - r_{21} - 2s_{12})(\mu)\,, \\ \delta(\mu) &= (s_{03} - s_{21} + 2r_{12})(\mu)\,, \end{aligned}$$

we have the relations

$$(11) \qquad \begin{cases} (\alpha r_{11} + \beta s_{11})(\mu) &= 0 \\ (\gamma r_{11} + \delta s_{11})(\mu) &= \frac{1}{4} \\ (-\beta r_{11} + \alpha s_{11})(\mu) &= \frac{1}{4}A(\mu) \\ (-\delta r_{11} + \gamma s_{11})(\mu) &= \frac{1}{4}B(\mu)\,. \end{cases}$$

Consider the family of surfaces in Monge form

$$(12) \qquad z = \tilde{R}(u, v, \mu), \quad w = \tilde{S}(u, v, \mu)\,,$$

with

$$\begin{aligned} \tilde{R}(u, v, \mu) &= v^2 - \frac{1}{24}A(\mu)u^3 + + \frac{1}{24}B(\mu)v^3 \qquad \text{and} \\ \tilde{S}(u, v, \mu) &= u^2 + uv + v^2 + \frac{1}{24}v^3\,. \end{aligned}$$

For this family, we obtain

$$2\tilde{r}_{11}(\mu) = (s_{20} - s_{02})(\mu) = 0\,, \quad 2s_{11}(\mu) = -(r_{20} - r_{02})(\mu) = 2\,,$$

and

$$\tilde{\alpha}(\mu) = \frac{1}{4}A(\mu)\,, \quad \tilde{\beta}(\mu) = 0\,, \quad \tilde{\gamma}(\mu) = \frac{1}{4}B(\mu)\,, \quad \tilde{\delta}(\mu) = \frac{1}{4}\,,$$

and relations (11) hold.

Therefore the families (9) and (12) have respective equation of lines of principal curvature that share the same linear part, namely,

$$4(A(\mu)u + B(\mu)v)(du^2 - dv^2)dudv + v(du^4 - 6du^2dv^2 + dv^4) = 0\,.$$

Hence the families are equivalent. ∎

Now we shall present a versal unfolding for umbilic points of type H_{34}.

Lemma 6.5 *Let $f : M \to I\!R^4$ be a smooth immersion of a surface M, and let $p \in M$ be a H_{34}–umbilic point of f. Then there exists a Monge chart*

$$z = R(u,v), \quad w = S(u,v)$$

such that the differential equation of the principal lines of f in these coordinates is of form (6) with $A \neq \frac{1}{4}$.

Proof. Consider a Monge chart

$$z = R(u,v), \quad w = S(u,v)$$

such that the differential equation of the principal lines of f in these coordinates is of form (6). Then $A = \frac{1}{4}$ if and only if the root of multiplicity two of the polynomial $g(s)$ is $s = 0$. Now if $A = \frac{1}{4}$ and s_0 is a simple root of $g(s)$, then we preserve (6) as the differential equation of the principal lines of f by making a rotation that sends s_0 over $s = 0$. This says that $A \neq \frac{1}{4}$ and the proof is complete. ∎

Theorem 6.6 *A versal unfolding of a umbilic point of type H_{34} is the family of surfaces in Monge form $f_\lambda : I\!R^2 \to I\!R^4$ defined by*

$$f_\lambda(x,y) \;=\; (x, y, R_\lambda(x,y), S_\lambda(x,y))\,,$$

with

$$R_\lambda(x,y) \;=\; y^2 - \frac{1}{24}\left(\lambda - \frac{125}{32}\right)x^3 + \frac{1}{24}\frac{51}{32}y^3 \qquad \text{and}$$

$$S_\lambda(x,y) \;=\; x^2 + xy + y^2 + \frac{1}{24}\,y^3\,.$$

Proof. Let $(g_\mu : M \to I\!R^4)$ be an arbitrary family of smooth immersions of a surface M, with parameter $\mu \in I\!R^k$, and let $p \in M$ be a H_{34}–umbilic point of g_0. Using Lemmas 6.4 and 6.5 we may suppose that the family (g_μ) is in Monge form

$$g_\mu(x,y) = (x, y, R_\mu(x,y), S_\mu(x,y))\,,$$

with

$$\begin{aligned} R_\mu(x,y) &= y^2 - \frac{1}{24}A(\mu)x^3 + +\frac{1}{24}B(\mu)y^3 \qquad \text{and} \\ S_\mu(x,y) &= x^2 + xy + y^2 + \frac{1}{24}y^3\,, \end{aligned}$$

and that $A(0) \neq \frac{1}{4}$.

Consider the real–valued function ψ defined in a neighborhood of the origin of $I\!R^k$ which is given by

$$\begin{aligned} \psi(\mu) &= 16[4(1+B(\mu)^2)^3 + 24(1+B(\mu)^2)^2A(\mu) + 8(5-B(\mu)^2)(1+B(\mu)^2)A(\mu)^2 \\ &\quad + 4(9+B(\mu)^2)A(\mu)^3 + (17+B(\mu)^2)A(\mu)^4 + 4A(\mu)^5]\,. \end{aligned}$$

Then the unfolding induced by ψ from the family of surfaces $(f_\lambda)_{\lambda\in I\!R}$ is

$$\tilde f_\mu(x,y) = (x, y, \tilde R_\mu(x,y), \tilde S_\mu(x,y))\,,$$

with

$$\begin{aligned} \tilde R_\mu(x,y) &= y^2 - \frac{1}{24}\left(\psi(\mu) - \frac{125}{32}\right)x^3 + \frac{1}{24}\frac{51}{32}y^3 \qquad \text{and} \\ \tilde S_\mu(x,y) &= x^2 + xy + y^2 + \frac{1}{24}\,y^3\,. \end{aligned}$$

Both families are equivalent for $|\,\mu\,|$ small, since the discriminant associated to the family (g_μ) is $\Lambda(\mu) = \psi(\mu)$ and the one to the family $(\tilde f_\mu)$ is

$$\begin{aligned} \tilde\Lambda(\mu) &= \tfrac{\psi(\mu)}{8192}\,\Big(6640625 - 48348750\psi(\mu) + 30426304\psi(\mu)^2 \\ &\qquad - 6680064\psi(\mu)^3 + 524288\psi(\mu)^4\Big)\,. \end{aligned}$$

The proof is now complete. ■

Now we consider the umbilic points of type $\tilde H_3$.

Theorem 6.7 *A versal unfolding of a umbilic point of type $\tilde{H}_3$ is the family of surfaces in Monge form $f_\lambda : \mathbb{R}^2 \to \mathbb{R}^4$, with $\lambda = (\lambda_1, \lambda_2)$, defined by*

$$f_\lambda(x,y) = (x, y, R_\lambda(x,y), S_\lambda(x,y))$$

where

$$\begin{aligned} R_\lambda(x,y) &= y^2 - \frac{1}{24}\left(\lambda_1 - \frac{1}{4}\right)x^3 + \frac{1}{24}\lambda_2 y^3 \qquad \text{and} \\ S_\lambda(x,y) &= x^2 + xy + y^2 + \frac{1}{24}y^3 . \end{aligned}$$

Proof. Let $(g_\mu : M \to \mathbb{R}^4)$ be an arbitrary family of smooth immersions of a surface M, with parameter $\mu \in \mathbb{R}^k$, and let $p \in M$ be a $\tilde{H}_3$–umbilic point of g_0. Using Lemma 6.4 we may suppose that the family (g_μ) is in Monge form

$$g_\mu(x,y) = (x, y, R_\mu(x,y), S_\mu(x,y)),$$

with

$$\begin{aligned} R_\mu(x,y) &= y^2 - \frac{1}{24}A(\mu)x^3 + +\frac{1}{24}B(\mu)y^3 \qquad \text{and} \\ S_\mu(x,y) &= x^2 + xy + y^2 + \frac{1}{24}y^3 . \end{aligned}$$

Let us consider the real bi–valued function ψ defined in a neighborhood of the origin of $\mathbb{R}^k$ by

$$\psi(\mu) = (A(\mu), B(\mu)) .$$

Then the unfolding induced by ψ from the family of surfaces $(f_\lambda)_{\lambda \in \mathbb{R}}$ is

$$\tilde{f}_\mu(x,y) = (x, y, \tilde{R}_\mu(x,y), \tilde{S}_\mu(x,y)),$$

with

$$\begin{aligned} \tilde{R}_\mu(x,y) &= y^2 - \frac{1}{24}\left(A(\mu) - \frac{1}{4}\right)x^3 + \frac{1}{24}B(\mu)y^3 \qquad \text{and} \\ \tilde{S}_\mu(x,y) &= x^2 + xy + y^2 + \frac{1}{24}y^3 . \end{aligned}$$

Finally, since the discriminant associated to the family (g_μ) is equal to the one of the family $(\tilde{f}_\mu)$, for every μ, both families are equivalent. This completes the proof. ■

The next two theorems give us the bifurcation diagrams of these types of umbilic points.

Theorem 6.8 *Consider the one–parameter family of immersions $f_\lambda : I\!R^2 \to I\!R^4$ defined by*

$$f_\lambda(x,y) = (x, y, R(x,y), S(x,y)),$$

with

$$R(x,y) = \frac{1}{2}x^2 + \frac{1}{24}\left(1 - \left(\lambda - \frac{125}{32}\right)\right)x^3 + \frac{1}{8}xy^2 + \frac{1}{48}\frac{51}{32}y^3 \quad \text{and}$$
$$S(x,y) = \frac{1}{2}xy.$$

Then the origin is a umbilic point for all values of λ. Moreover, for $|\lambda|$ small, the origin is of type H_3 for $\lambda < 0$, of type H_{34} for $\lambda = 0$, and of type H_4 for $\lambda > 0$.

Proof. The corresponding values for A, B and Λ are

$$A = \lambda - \frac{125}{32}, \quad B = \frac{51}{32},$$

and

$$\Lambda = \frac{\lambda}{8192}(6640625 - 48348750\lambda + 30426304\lambda^2 - 6680064\lambda^3 + 524288\lambda^4),$$

and the proof now follows. ∎

Theorem 6.9 *Consider the two–parameter family of immersions $f_\lambda : I\!R^2 \to I\!R^4$ defined by*

$$f_\lambda(x,y) = (x, y, R(x,y), S(x,y)),$$

with

$$R(x,y) = \frac{1}{2}x^2 + \frac{1}{24}\left(1 - \left(\lambda_1 - \frac{1}{4}\right)\right)x^3 + \frac{1}{8}xy^2 + \frac{1}{48}\lambda_2 y^3 \quad \text{and}$$
$$S(x,y) = \frac{1}{2}xy.$$

Then the origin is a umbilic point for all values of $\lambda = (\lambda_1, \lambda_2)$. Moreover, for $|\lambda|$ small, we have:

i) The origin is of type H_3 if $\Lambda < 0$.

ii) The origin is of type H_{34} if $\Lambda = 0$ and $\lambda_1 \neq 0$, or if $\lambda_1 = 0$ and $\lambda_2 \neq 0$.

iii) The origin is of type H_4 for $\Lambda > 0$ and $\lambda_1 \neq 0$.

iv) The origin is of type $\tilde{H}_3$ for $\lambda = (\lambda_1, \lambda_2) = (0,0)$.

Here

$$\Lambda = \frac{1}{4}(625\,\lambda_1 + 1200\,\lambda_2 + 1376\,\lambda_1^3 + 768\,\lambda_1^4 + 256\,\lambda_1^5 + 125\,\lambda_2^2 + 2080\,\lambda_1\,\lambda_2^2 + 1952\,\lambda_1^2\,\lambda_2^2 + 256\,\lambda_1^4\,\lambda_2^2 + 352\,\lambda_2^4 + 1792\,\lambda_1\,\lambda_2^4 - 512\,\lambda_1^2\,\lambda_2^4 + 256\,\lambda_2^6)\,.$$

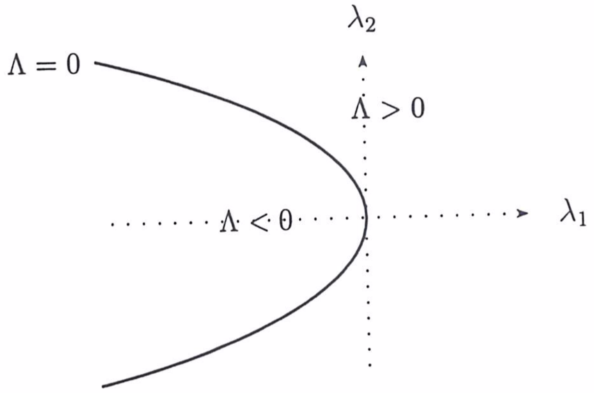

Figure 6

Proof. For the proof, it suffices to observe that the corresponding values of A and B are

$$A = \lambda_1 - \frac{1}{4} \quad \text{and} \quad B = \lambda_2\,. \qquad \blacksquare$$

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