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A Differential Equation for Lines of Curvature on Surfaces Immersed in IR^4

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Uma equação diferencial para as linhas de curvatura em superfícies imersas no $I\!R^4$.

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Resumo

Se estabelece a equação diferencial das linhas de curvatura para superfícies imersas no R^4. Se mostra que -do ponto de vista das linhas de curvatura- a equação diferencial apresentada leva consigo uma informação bastante completa da superfície imersa.

A Differential Equation For Lines Of Curvature On Surfaces Immersed In $I\!R^4$

by

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Abstract

It is stablished the differential equation of the *lines of curvature* for immersions of surfaces into $\mathbb{R}^4$. It is shown that -from the point of view of principal lines of curvature- the considered differential equations carries a fairly complete information of the immersed surface

1 Introduction

The notion of principal direction for a smooth immersion $f : M \to \mathbb{R}^4$ which we introduce and use is due to J.A. Little [Lit]; it is an extension of the classical three–dimensional concept (see [R-S] for another possible extension). A *principal direction at* p is a line in T_pM generated by a unitary vector which makes the length of $\alpha(X,X)$ take a extreme value, where α is the second fundamental form of the immersion f at p and X varies on the unitary circle in T_pM. The set $\mathcal{E}$ of values of $\alpha(X,X)$ called *ellipse of curvature* is an ellipse that may degenerate into a line segment, a circle or a point. Also, it is easily seen that as X goes once around the circle, $\alpha(X,X)$ goes twice around $\mathcal{E}$. Therefore, when $\mathcal{E}$ is either an ellipse or a line segment, there are four principal directions at p; when $\mathcal{E}$ is either a circle or a point we say p is a *umbilic point of* f. The principal lines of curvature of α are those curves in M that are disjoint from the umbilic points and tangent to principal lines.

In this note we clarify some points of the original proof of Theorem 1.1 (see [GGTG, Theorem 2.1]), which we consider very important in the study of lines of curvature of surfaces inmersed in $\mathbb{R}^4$. Also, we show some quartic differencials that are invariant by isothermal coordinates. The holomorphic quartic differentials live naturally in Riemman surfaces and may be as interesting as quadratic differentials. The main result of this note is:

Theorem 1.1 *Let $f : M \to I\!R^4$ be a smooth immersion of a surface M. In isothermic coordinates $(u, v) : (M, p) \to (I\!R^2, 0)$, the differential equation of the lines of curvature of f is given by*

$$(1) \qquad 4a(u,v)(du^2 - dv^2)dudv + b(u,v)(du^4 - 6du^2dv^2 + dv^4) = 0$$

where $a = a(u,v)$ and $b = b(u,v)$ are smooth real-valued functions. Moreover, p is a umbilic point if and only if $a(0,0) = b(0,0) = 0$. $a(0,0) = b(0,0) = 0$.

Conversely, for any given analytic functions $a, b : U \to I\!R$ defined on an open neighborhood $U \subset I\!R^2$ of a point p, there exists an immersion $f : V \to I\!R^4$ where $V \subset U$ is some small open neighborhood of p such that the differential equation of the lines of curvature of f is given by (6) and the coordinates (u, v) are isothermic.

2 Differential Equation of Lines of Curvature

Let $f : M \to I\!R^4$ be a smooth immersion of a surface M. Let $U \subset M$ be an open neighborhood with isothermic coordinates (u, v). Let $z = u + iv$, and let $\lambda = |\partial_u| = |\partial_v|$, where $\partial_u = \dfrac{\partial}{\partial u}$ and $\partial_v = \dfrac{\partial}{\partial v}$.
We introduce the two Wirtingen operators

$$(2) \qquad \partial_z = \frac{1}{\sqrt{2}}(\partial_u - i\partial_v) \qquad \text{and} \qquad \partial_{\bar{z}} = \frac{1}{\sqrt{2}}(\partial_u + i\partial_v),$$

and the corresponding dual forms

$$(3) \qquad dz = \frac{1}{\sqrt{2}}(du + idv) \qquad \text{and} \qquad d\bar{z} = \frac{1}{\sqrt{2}}(du - idv),$$

We denote

$$(4) \qquad \sigma = \alpha(\partial_z, \partial_z) \qquad \text{and} \qquad \tau = \alpha(\partial_z, \partial_{\bar{z}})\,.$$

Here $\langle , \rangle$ is a bilinear complex extension of the inner product of $T^\perp M$ to $T^\perp M \otimes \mathbb{C}$, where $T^\perp M$ denotes the normal bundle.

Proposition 2.1 *Let $f : M \to \mathbb{R}^4$ be a smooth immersion of a surface M. In isothermic coordinates $(u, v) : (M, p) \to (\mathbb{R}^2, 0)$, the differential equation of the lines of curvature of f is given by*

$$Im(\langle \sigma, \sigma \rangle dz^4) = 0 \tag{5}$$

or equivalently

$$4a(u,v)(du^2 - dv^2)dudv + b(u,v)(du^4 - 6du^2dv^2 + dv^4) = 0 \tag{6}$$

where

$$a = a(u,v) = Re(\langle \sigma, \sigma \rangle) \qquad \textit{and} \qquad b = b(u,v) = Im(\langle \sigma, \sigma \rangle)$$

are smooth real–valued functions.

Moreover, p is a umbilic point if and only if $a(0,0) = b(0,0) = 0$.

The proof of Proposition 2.1 follows the proof of Lemma 2.3.

Under the conditions of the latter proposition, $\epsilon_1 = \lambda^{-1}\partial_u$, $\epsilon_2 = \lambda^{-1}\partial_v$ is an orthonormal frame.

Lemma 2.2 *Let $\alpha_{ij} = \alpha(\epsilon_i, \epsilon_j)$. Then ϵ_1 is tangent to a principal line at $p \in M$ if and only if $Im(\langle \sigma, \sigma \rangle dz^4)(\epsilon_1) = 0$ at p.*

Proof:

Let $\nu : [0, 2\pi] \to TM_p$ be given by

$$\nu(\theta) = \cos\theta\, \epsilon_1 + \sin\theta\, \epsilon_2.$$

Since the image of ν is an ellipse centered at the mean curvature vector $H = 1/2(\alpha_{11} + \alpha_{22})$ (see Little [Lit]), ϵ_1 is a principal direction at p if and only if the function

$$f(\theta) = ||\alpha(\nu(\theta), \nu(\theta)) - H||^2$$

has a critical value at $\theta = 0$. Now

$$0 = f'(0) = 4\left\langle \alpha(\nu'(0), \nu(0)), \alpha(\nu(0), \nu(0))\right\rangle - H$$

if and only if

$$0 = \langle \alpha_{12}, \alpha_{11} - H\rangle = \langle \alpha_{12}, \alpha_{11} - \alpha_{22}\rangle\,,$$

or equivalently

$$0 = Im(||\alpha_{11} - \alpha_{22}||^2 - 4||\alpha_{12}||^2 - 4i\langle \alpha_{11} - \alpha_{22}, \alpha_{12}\rangle) = Im(\langle \sigma, \sigma\rangle dz^4(\epsilon_1))$$

because $dz(\epsilon_1)$ is a real number. The proof is complete. ∎

Lemma 2.3 *The quartic differential form*

$$\langle \sigma, \sigma\rangle dz^4$$

does not depend on the chosen isotermic coordinates.

Proof:

In fact, let $(u'.v')$ be another system of isotermic coordinates. Let

$$\partial_{z'} = \frac{1}{\sqrt{2}}(\partial_{u'} - i\partial_{v'}) \qquad \text{and} \qquad \partial_{\overline{z'}} = \frac{1}{\sqrt{2}}(\partial_{u'} + i\partial_{v'}) \tag{7}$$

be their corresponding dual forms

$$dz' = \frac{1}{\sqrt{2}}(du' + idv') \qquad \text{and} \qquad d\overline{z'} = \frac{1}{\sqrt{2}}(du' - idv'). \tag{8}$$

Hence, at a given point $p \in M$ belonging to the domain of definition of both (u, v) and (u', v'), there exist $0 \leq \theta < 2\pi$ and $K > 0$ such that

$$\partial_{z'} = K e^{i\theta} \partial_z.$$

Therefore, using

$$\begin{aligned} dz(\partial_z) = d\overline{z}(\partial_{\overline{z}}) &= 1, & dz(\partial_{\overline{z}}) = d\overline{z}(\partial_z) &= 0, \\ dz'(\partial_{z'}) = d\overline{z'}(\partial_{\overline{z'}}) &= 1, & dz'(\partial_{\overline{z'}}) = d\overline{z'}(\partial_{z'}) &= 0, \end{aligned}$$

we conclude that

$$dz' = K^{-1} e^{-i\theta} dz.$$

Summarizing, at the given point p, we have

$$\langle \alpha(\partial_z, \partial_z), \alpha(\partial_z, \partial_z) \rangle (dz)^4 = \langle \alpha(\partial_{z'}, \partial_{z'}), \alpha(\partial_{z'}, \partial_{z'}) \rangle (dz')^4$$

and the proof is complete. ■

Proof of Proposition 2.1

The proof follows directly from the previous two lemmas. ■

Proposition 2.4 *For any given analytic functions $a, b : U \to \mathbb{R}$ defined on an open neighborhood $U \subset \mathbb{R}^2$ of a point $p = (u_0, v_0)$, there exists an immersion $f : V \to \mathbb{R}^4$ where $V \subset U$ is some small open neighborhood of p such that the differential equation of the lines of curvature of f is given by (6) and the coordinates (u, v) are isothermic.*

To prove the Proposition 2.4 we need the next results.

Lemma 2.5 *Suppose the assumptions above. Suppose that, for some $\epsilon > 0$, $R = \{(u, v) : |u - u_0| < \epsilon, |v - v_0| < \epsilon\}$ is covered by our coordinate systems. Then there* *exists a normal frame $\{e_3, e_4\}$ defined on R and a smooth function $\eta = \eta(u, v)$ such that*

$$\begin{aligned} \nabla^{\perp}_{\partial_u} e_3 &= \eta e_4 \quad , \quad \nabla^{\perp}_{\partial_u} e_4 = -\eta e_3 \\ \nabla^{\perp}_{\partial_v} e_3 &= 0 \quad , \quad \nabla^{\perp}_{\partial_v} e_4 = 0. \end{aligned} \tag{9}$$

Proof:

Take a normal unitary vector field e_3 along $\{v = v_0\}$. With respect to the riemannian connection $\nabla^{\perp}$, by parallel translation along the curves $\{u = \text{constant}\}$, extend e_3 to R. This implies that $\nabla^{\perp}_{\partial_v} e_3 = 0$. Let e_4 be a unitary vector field normal to both M and e_3; certainly, $\nabla^{\perp}_{\partial_v} e_4 = 0$. Since

$$\begin{aligned} 0 = \frac{\partial}{\partial u}\langle e_3, e_4\rangle &= \langle \nabla^{\perp}_{\partial_u} e_3, e_4\rangle + \langle \nabla^{\perp}_{\partial_u} e_4, e_3\rangle \\ 0 = \frac{\partial}{\partial u}\langle e_i, e_i\rangle &= 2\langle \nabla^{\perp}_{\partial_u} e_i, e_i\rangle, \qquad \text{for } i = 3, 4, \end{aligned}$$

equations (9) are satisfied ■

The proof of the following lemma can also be obtained by a direct use of the main result of [Jac].

Lemma 2.6 *Let $\{e_3, e_4\}$ be a normal frame and let $\eta = \eta(u, v)$ be a smooth function which verify (9).*
If we denote

$$\sigma_{\beta 1} = Re(\langle \sigma, e_\beta\rangle), \quad \sigma_{\beta 2} = Im(\langle \sigma, e_\beta\rangle), \quad \tau_\beta = \langle \tau, e_\beta\rangle, \quad \beta = 3, 4, \tag{10}$$

then the Gauss, Ricci and Codazzi equations may be written, respectively, as

$$\lambda_{vv} = \frac{1}{\lambda}(-\sigma_{31}^2 - \sigma_{32}^2 - \sigma_{41}^2 - \sigma_{42}^2 + \tau_3^2 + \tau_4^2 + \lambda_u^2 + \lambda_v^2 - \lambda\lambda_{uu}) \tag{11}$$

$$(\eta)_v = \frac{2}{\lambda^2}(\sigma_{41}\sigma_{32} - \sigma_{31}\sigma_{42}) \tag{12}$$

$$
(13) \qquad \begin{aligned}
(\sigma_{32})_v &= (\sigma_{31})_u - (\tau_3)_u - \eta\sigma_{41} + \eta\tau_4 &&+ \frac{2}{\lambda}\lambda_u\tau_3 \\
(\sigma_{31}+\tau_3)_v &= -(\sigma_{32})_u + \eta\sigma_{42} &&+ \frac{2}{\lambda}\lambda_v\tau_3 \\
(\sigma_{42})_v &= (\sigma_{41})_u - (\tau_4)_u + \eta\sigma_{31} - \eta\tau_3 &&+ \frac{2}{\lambda}\lambda_u\tau_4 \\
(\sigma_{41}+\tau_4)_v &= -(\sigma_{42})_u - \eta\sigma_{32} &&+ \frac{2}{\lambda}\lambda_v\tau_4
\end{aligned}
$$

Proof:

We use notations (1) and (2).

Let us first consider the Gauss equation

$$
(14) \qquad \langle R(\partial_z, \partial_{\bar{z}})\partial_z, \partial_{\bar{z}}\rangle = \langle \alpha(\partial_z, \partial_z), \alpha(\partial_{\bar{z}}, \partial_{\bar{z}})\rangle - |\alpha(\partial_z, \partial_{\bar{z}})|^2.
$$

Note that if, for $i = 3, 4$, $\sigma_i = \langle \sigma, e_i\rangle$, then,

$$
\begin{aligned}
R(\partial_z, \partial_{\bar{z}})\partial_z &= \nabla_{\partial_z}\nabla_{\partial_{\bar{z}}}\partial_z - \nabla_{\partial_{\bar{z}}}\nabla_{\partial_z}\partial_z \\
&= -\nabla_{\partial_{\bar{z}}}\Big(\frac{2}{\lambda}\frac{\partial\lambda}{\partial z}\Big)\partial_z \\
&= -\nabla_{\partial_{\bar{z}}}\Big(2\frac{\partial \log\lambda}{\partial z}\Big)\partial_z \\
&= -\Delta \log\lambda \partial_z
\end{aligned}
$$

which implies

$$
\langle R(\partial_z, \partial_{\bar{z}})\partial_z, \partial_{\bar{z}}\rangle = -\lambda^2\Delta\log\lambda.
$$

Hence the Gauss equation has the form

$$
-\lambda^2\Delta\log\lambda = |\sigma|^2 - |\tau|^2
$$

and may be rewritten as (11).

We now consider the Ricci equation

$$R^{\perp}(\partial_z, \partial_{\overline{z}})v = \alpha(\sigma_v \partial_{\overline{z}}, \partial_z) - \alpha(\sigma_v \partial_z, \partial_{\overline{z}}), \ v \in T^{\perp}M. \tag{15}$$

Note that

$$\begin{aligned} R^{\perp}(\partial_z, \partial_{\overline{z}})e_3 &= \alpha(\sigma_{e_3}\partial_{\overline{z}}, \partial_z) - \alpha(\sigma_{e_3}\partial_z, \partial_{\overline{z}}) \\ &= \frac{1}{\lambda^2}\Big[\alpha(\overline{\sigma}_3\partial_z + \tau_3\partial_{\overline{z}}, \partial_z) + \alpha(\sigma_3\partial_{\overline{z}} + \tau_3\partial_z, \partial_z)\Big] \\ &= \frac{1}{\lambda^2}(\overline{\sigma}_3\sigma - \sigma_3\overline{\sigma}) \\ &= 2\frac{i}{\lambda^2}Im(\overline{\sigma}_3\sigma), \end{aligned}$$

hence

$$\langle R^{\perp}(\partial_z, \partial_{\overline{z}})e_3, e_4\rangle = 2\frac{i}{\lambda^2}Im(\overline{\sigma}_3\sigma_4).$$

We also obtain that

$$\begin{aligned} R^{\perp}(\partial_z, \partial_{\overline{z}})e_3 &= \nabla^{\perp}_{\partial_z}\nabla^{\perp}_{\partial_{\overline{z}}}e_3 - \nabla^{\perp}_{\partial_{\overline{z}}}\nabla^{\perp}_{\partial_z}e_3 \\ &= \nabla^{\perp}_{\partial_z}\Big(\frac{\eta}{\sqrt{2}}e_4\Big) - \nabla^{\perp}_{\partial_{\overline{z}}}\Big(\frac{\eta}{\sqrt{2}}e_4\Big) \\ &= \frac{1}{\sqrt{2}}\Big(\frac{\partial\eta}{\partial z} - \frac{\partial\eta}{\partial\overline{z}}\Big)e_4 \\ &= -i\frac{\partial\eta}{\partial v}\ e_4\ , \end{aligned}$$

and thus

$$\langle R^{\perp}(\partial_z, \partial_{\overline{z}})e_3, e_4\rangle = -i\frac{\partial\eta}{\partial v}.$$

This implies that the Ricci equation has the form

$$\frac{\partial\eta}{\partial v} = -\frac{2}{\lambda^2}Im(\overline{\sigma}_3\sigma_4)$$

which may be rewritten as (12).

Finally, we consider the Codazzi equation

$$(\nabla^{\perp}_{\partial_z}\alpha)(\partial_{\bar{z}},\partial_z) = (\nabla^{\perp}_{\partial_{\bar{z}}}\alpha)(\partial_z,\partial_z). \tag{16}$$

We have

$$\nabla^{\perp}_{\partial_{\bar{z}}}\sigma = \nabla^{\perp}_{\partial_z}\tau - 2\frac{\partial \log \lambda}{\partial z}\tau.$$

Also, we find that

$$\nabla^{\perp}_{\partial_{\bar{z}}}\sigma = \left(\frac{\partial \sigma_3}{\partial \bar{z}} - \sigma_4\eta\right)e_3 + \left(\frac{\partial \sigma_4}{\partial \bar{z}} + \sigma_3\eta\right)e_4$$

and that

$$\nabla^{\perp}_{\partial_z}\tau = \left(\frac{\partial \tau_3}{\partial z} - \tau_4\eta\right)e_3 + \left(\frac{\partial \tau_4}{\partial z} + \tau_3\eta\right)e_4.$$

Hence

$$\begin{aligned}
\frac{\partial \sigma_3}{\partial \bar{z}} - \sigma_4\eta &= \frac{\partial \tau_3}{\partial z} - \tau_4\eta - 2\frac{\partial \log \lambda}{\partial z}\tau_3 \\
\frac{\partial \sigma_4}{\partial \bar{z}} + \sigma_3\eta &= \frac{\partial \tau_4}{\partial z} + \tau_3\eta - 2\frac{\partial \log \lambda}{\partial z}\tau_4
\end{aligned} \tag{17}$$

which may be rewritten as (13) ∎

Proof of Proposition 2.4:

We need to prove that, for any given local analytic functions $a, b : U \to I\!R$ as in the assumptions, there exists a local analytic immersion f such that the differential equation of the lines of curvature of f is given by (6) and that the coordinates (u, v) are isothermic.

If we find a solution $\lambda > 0,\ \eta > 0,\ \sigma_{31},\ \sigma_{32},\ (C_4 - \tau_4),\ \sigma_{42}, \tau_3, \tau_4$ of system (11) – (13) of Lemma 2.6 such that each one of these functions is defined in an open neighborhood $V \subset U$ of p, then the theorem of existence and unicity of immersions [Jac] guarantees the existence of a local immersion $f : V \to I\!R^4$ which has $\lambda^2 = E = G$ and $F = 0$ as coefficients of its first fundamental form. On the other hand, if this solution satisfies the system

$$(18) \qquad a = \sigma_{31}^2 - \sigma_{32}^2 + \sigma_{41}^2 - \sigma_{42}^2 \quad , \quad b = 2(\sigma_{31}\sigma_{32} + (C_4 - \tau_4)\sigma_{42}) \ ,$$

then the differential equation of the principal lines of curvature is given by (6) and thus the proof of the theorem will follow. For this we first define $\Lambda_i = \Lambda_i(u, v, a, b, \sigma_{32}, \sigma_{42}),\ i = 3, 4$, by

$$(19) \qquad \begin{aligned} \Lambda_3 &= \frac{b\,\sigma_{32} + \sigma_{42}\sqrt{4(\sigma_{32}^2 + \sigma_{42}^2)(a + \sigma_{32}^2 + \sigma_{42}^2) - b^2}}{2(\sigma_{32}^2 + \sigma_{42}^2)} , \\ \Lambda_4 &= \frac{b\,\sigma_{42} - \sigma_{32}\sqrt{4(\sigma_{32}^2 + \sigma_{42}^2)(a + \sigma_{32}^2 + \sigma_{42}^2) - b^2}}{2(\sigma_{32}^2 + \sigma_{42}^2)} . \end{aligned}$$

Now, for $i = 3, 4,$ we define the funcion

$$\Omega_i = \Omega_i(u, v, a, a_u, b, b_u, \sigma_{32}, (\sigma_{32})_u, \sigma_{42}, (\sigma_{42})_u)$$

obtained by formally taking the partial derivative of $\Lambda_i(u, v, a, b, \sigma_{32}, \sigma_{42})$, with respect to the variable u, by assuming that $a = a(u, v)$, $b = b(u, v)$, $\sigma_{32} = \sigma_{32}(u, v)$, $\sigma_{42} = \sigma_{42}(u, v)$.

We next introduce the following system of linear PDE's:

$$\frac{\partial U_1}{\partial v} = U_2$$

$$
(20)\quad
\begin{aligned}
\frac{\partial U_3}{\partial v} &= \frac{\partial U_2}{\partial u}\\
\frac{\partial U_2}{\partial v} &= \frac{1}{U_1}\left(C_3^2 - 2\Lambda_3 C_3 - 2\Lambda_4 C_4 + C_4^2 - \sigma_{32}^2 - \sigma_{42}^2 + U_2^2 - U_3^2 - U_1\frac{\partial U_3}{\partial u}\right)\\
\frac{\partial \eta}{\partial v} &= \frac{2}{U_1^2}(\Lambda_4\sigma_{32} - \Lambda_3\sigma_{42})\\
\frac{\partial \sigma_{32}}{\partial v} &= 2\Omega_3 - \frac{\partial C_3}{\partial u} - 2\eta\Lambda_4 + \eta C_4 + \frac{2}{U_1}U_3(C_3 - \Lambda_3)\\
\frac{\partial C_3}{\partial v} &= -\frac{\partial \sigma_{32}}{\partial u} + \eta\sigma_{42} + \frac{2}{U_1}U_2(C_3 - \Lambda_3)\\
\frac{\partial \sigma_{42}}{\partial v} &= 2\Omega_4 - \frac{\partial C_4}{\partial u} + 2\eta\Lambda_3 - \eta C_3 + \frac{2}{U_1}U_3(C_4 - \Lambda_4)\\
\frac{\partial C_4}{\partial v} &= -\frac{\partial \sigma_{42}}{\partial u} - \eta\sigma_{32} + \frac{2}{U_1}U_2(C_4 - \Lambda_4) \quad ,
\end{aligned}
$$

with initial conditions

$$
\begin{aligned}
U_1(u,0) &\equiv 1 \quad , & U_2(u,0) &\equiv 0\\
U_3(u,0) &\equiv 0 \quad , & \eta(u,0) &\equiv 1\\
C_3(u,0) &\equiv 0 \quad , & C_4(u,0) &\equiv 0\\
\sigma_{32}(u,0) &\equiv 0 \quad ,\\
\sigma_{42}(u,0) &\equiv \tfrac{\sqrt{2}}{2}\sqrt{-a(u,0) + \sqrt{(a(u,0))^2 + (b(u,0))^2 + 1}}\ .
\end{aligned}
$$

(21)

Then the Cauchy-Kowalewsky theorem [Spi] implies the existence of an analytic solution around $(0,0)$ for the entire system (20). Note that the chosen initial conditions guarantee that $U_1 > 0$ and that the expression

$$
4(\sigma_{32}^2 + \sigma_{42}^2)(a + \sigma_{32}^2 + \sigma_{42}^2) - b^2
$$

inside the square root of the definition $\Lambda_i = \Lambda_i(u,v)$, $i = 3,4$, is positive and analytic in a small neighborhood of $(0,0)$. If we define $\lambda = U_1$, then the first two equations of (20) together with the chosen initial conditions imply that $U_2 = \lambda_v$ and that $U_3 = \lambda_u$. Also, if we rewrite system (20) by making, for $i = 3,4$ the following substitutions

$$
(22) \qquad \begin{array}{rcl} \sigma_{i1}(u,v) & := & \Lambda_i(u,v,\sigma_{32}(u,v),\sigma_{42}(u,v)) \\ (\sigma_{i1})_u(u,v) & := & \Omega_i(u,v,a,a_u,b,b_u,\sigma_{32},(\sigma_{32})_u,\sigma_{42},(\sigma_{42})_u) \\ \tau_i & := & C_i - \sigma_{i1} \ , \end{array}
$$

then system (20) implies that the structural equations (11) – (13) are satisfied. Moreover by (19) and (22), we have that (18) is satisfied. ■

Proof of Theorem 1.1

It follows directly from Proposition 2.1 and 2.4 ■

Remark 2.7 *Let $\{e_3, e_4\}$ be a normal frame and let $\eta = \eta(u,v)$ be a smooth function which verify (9).*
If the first and second fundamental forms are denoted as follows,

$$
\begin{array}{rcl} \alpha(\partial_u,\partial_u) & = & e\, e_3 + \bar{e}\, e_4 \\ \alpha(\partial_u,\partial_v) & = & f\, e_3 + \bar{f}\, e_4 \\ \alpha(\partial_u,\partial_u) & = & g\, e_3 + \bar{g}\, e_4 \\ E & = & \lambda^2 \end{array}
$$

then, by using Lemma 2.6, we obtain that the Gauss, Ricci and Codazzi equations can be written, respectively, as

$$
(23) \qquad eg - f^2 + \bar{e}\bar{g} - \bar{f}^2 = \frac{1}{2E}(E_u^2 + E_v^2 - E(E_{uu} + E_{vv})),
$$

$$
(24) \qquad \eta_v = \frac{(e-g)\bar{f} - f(\bar{e} - \bar{g}),}{E}
$$

and

$$
e_v - f_u = -\bar{f}\eta + E_v\frac{e+g}{2E},
$$

$$
(25)\qquad
\begin{aligned}
\bar{e}_v - \bar{f}_u &= f\eta + E_v\frac{\bar{e}+\bar{g}}{2E}, \\
f_v - g_u &= -\bar{g}\eta - E_u\frac{e+g}{2E}, \\
\bar{f}_v - \bar{g}_u &= g\eta - E_u\frac{\bar{e}+\bar{g}}{2E}.
\end{aligned}
$$

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