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Contracting Sets and Dissipation

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Abstract

In this work we prove bounded dissipativeness for some parabolic problems in fractional power spaces which are embedded in L^∞. We do this under very few hypotheses on the nonlinearity. These hypotheses are natural and easy to verify in many applications. The bounded dissipativeness provides the existence of global attractors. The tolls employed are an enhanced theory of invariant regions for systems of parabolic partial differential equations, the notion of contracting sets and the variation of constants formula. Several examples are considered to emphasize the applicability of these techniques.

1. Introduction

Consider the parabolic problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= D\Delta u + f(u) \quad \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} &= 0 \qquad\qquad\quad\; \text{in} \quad \partial\Omega \end{aligned} \tag{1}$$

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where Ω is a bounded smooth open set in $\mathbb{R}^n$, $u \in \mathbb{R}^N$, D is a diagonal matrix with non-negative entries and $f : \mathbb{R}^N \to \mathbb{R}^N$ is a locally Lipschitz function.

Let $X = L^2(\Omega, \mathbb{R}^N)$ and define the fractional power spaces X^α associated to the Laplacian with homogeneous Neumann boundary conditions.

Let A be the linear operator defined by

$$
\begin{aligned}
&D(A) = \{\phi \in H^2(\Omega, \mathbb{R}^N) : \frac{\partial u}{\partial n} = 0 \text{ in } \partial\Omega\} \\
&A\phi = -\Delta\phi, \quad \forall \phi \in D(A)
\end{aligned}
\tag{2}
$$

Then, the linear semigroup e^{-At} generated by $-A$ satisfies the following estimates

$$
\begin{aligned}
&\|e^{-At}u_0\|_{X^\alpha} \le M\|u_0\|_{X^\alpha}, \quad t \ge 0, \\
&\|e^{-At}u_0\|_{X^\alpha} \le Mt^{-\alpha}\|u_0\|_X, \quad t > 0,
\end{aligned}
\tag{3}
$$

for some $M \ge 1$.

In recent years there have been many authors dealing with the existence of a global attractor for such problems. Some chose to work in the natural energy spaces $H^1(\Omega, \mathbb{R}^N)$ (see, for example, Marion [1987], Hale [1989], Hale and Raugel [1991], Carvalho and Pereira [1991] and Carvalho and Oliveira [1992] and Carvalho [1992]). That choice lead to restrictions in the class of nonlinearities in consideration since the problem had to be locally well posed. These restrictions are limitations on the growth of the nonlinearity which are related to Sobolev embeddings. On the other hand, these energy spaces enjoy a natural concept of dissipation, which would make the proof of existence of a global attractor much simpler.

Some authors worked in fractional powers spaces which are embedded in L^∞ (see, for example, Hale [1986], Fusco [1987], Hale and Rocha [1987a,b], Hale and Sakamoto [1989] and Carvalho and Hale [1991]). These spaces have the advantage that no growth condition is required for local existence of solutions of (1). However, there were no concept of global dissipation available in such spaces. In Hale [1986], a local concept of dissipation would come from large diffusivity.

In Marion [1989] the question of existence of global attractors in $H^1(\Omega, \mathbb{R}^N)$ is considered for nonlinearities with polynomial growth (see also, Hale [1989], Hale and Raugel [1991], Carvalho and Pereira [1991] and Carvalho and Oliveira [1992] and Carvalho [1992]) . For the case when the system has an invariant region, Marion [1989] considers the existence of local attractors; that is, only those functions whose image is contained in the invariant region are taken into account.

In Hale [1986] (see also Hale and Rocha [1987a,b] and Hale and Sakamoto [1989]), the existence of local attractors in X^α, $\alpha > \frac{3}{4}$, is proved regardless of the existence of an invariant region but assuming that the problem (6) has a global attractor and only for large diffusion coefficients.

In Carvalho and Ruas-Filho [1993], we described a global concept of dissipation for such fractional power spaces. The question of global dissipation in X^α spaces is addressed for the case when f satisfies the dissipativeness condition

$$\limsup_{|u_i|\to\infty} \frac{f_i(u_1,\cdots,u_N)}{u_i} \leq -\delta < 0, \quad 1 \leq i \leq N \tag{4}$$

and the symmetry condition

$$\frac{\partial f_i}{\partial u_j}(u) = \frac{\partial f_j}{\partial u_i}(u), \quad \forall u \in \mathbb{R}^N. \tag{5}$$

The condition (4) seems quite reasonable but the condition (5) does not seem at all necessary.

Indeed, such condition, rules out important problems such as the Fitzhugh-Nagumo equations and the Hodgkin-Huxley Equations.

Therefore, we have only been able to prove the existence of global attractors for systems with gradient structure. We use the invariance theory, keeping a link with the energy spaces.

Our aim in this paper is to eliminate that link and prove the existence of global attractors for systems of reaction diffusion equations which are not gradient such as the Hodgkin-Huxley equations, the Fitzhugh-Nagumo equations and many other models of applied sciences.

We now proceed to describe the results in more detail. Consider the ordinary differential equation

$$\dot{u} = f(u) \tag{6}$$

where $u = (u_1, \cdots, u_N) \in \mathbb{R}^N$ and $f : \mathbb{R}^N \to \mathbb{R}^N$ is locally Lipschitz continuous function such that there are positive constants $M_1, \cdots, M_N$ with

$$u_i f_i(u) < 0, \quad |u_i| > M_i, \quad 1 \leq i \leq N \tag{7}$$

Under these hypothesis, the problem (6) has a global attractor in $\mathbb{R}^N$; that is, there is a compact invariant connected set $\mathcal{A}_0$ which attracts bounded subsets of $\mathbb{R}^N$.

We assume throughout this paper that $\alpha > \max\{\frac{1}{2}, \frac{n}{4}\}$ if $n = 2, 3$ and $\alpha \geq \frac{1}{2}$ if $n = 1$. In this case, $X^\alpha \subset L^\infty(\Omega, \mathbb{R}^N)$ (see, for example, Henry[1981]).

If the diffusion is taken into account in (6) we expect that it still has a global attractor; that is (1) has a global attractor $\mathcal{A}_D$ if f satisfies (7). Actually, we expect that $\mathcal{A}_D$ strongly resembles $\mathcal{A}_0$ is some sence. We would like to say that whenever $u \in \mathcal{A}_D$, we have that

$$u(x) \in \mathcal{A}_0 \quad \forall x \in \Omega. \tag{8}$$

Even though we have not been able to prove this assertion in this general context we prove this result in some cases and conjecture that it is true in general.

The condition (7) is not the most general situation that we can consider but it has many applications and it is more suitable to introduce the results.

The condition (5) is imposed by Carvalho and Ruas-Filho [1993] due to the fact that they employ some energy function and the invariance principle to obtain point dissipativeness, mantainig a link with the energy spaces through the energy functionals.

In this paper, we work directly with the equations to obtain point dissipativeness and do not need the symmetry condition (5), eliminating completely the link with the energy spaces.

In Section 2 we consider the question of local existence for some parabolic problems that are posed in a complete metric subspace of a fractional power space. In Section 3 we prove that under the above hypothesis, the problem (1) has a global attractor. In Section 4 we consider several applications to mathematical models in chemistry, physics and biology. In Section 5 we introduce the notion of a diffusion independent attractors and image independent attractors giving conditions under which the problem (1) has diffusion independent attractor or an image independent attractor.

2. Local Existence for Parabolic Problems in Complete Metric Spaces

Suppose that $\mathcal{M}$ is a closed rectangle in $\mathbb{R}^N$, $N \geq 1$; that is, $\mathcal{M} = \Pi_{j=1}^{N} I_j$, where $I_j \subset \mathbb{R}$ is a closed interval (bounded or not). Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be a locally lipschitzian function such that $f(u) \cdot \mathbf{n}(u) \leq 0$ for every $u \in \partial\mathcal{M}$.

Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $n \leq 3$ and $D = \mathrm{diag}(d_1, \cdots, d_N)$, $d_i > 0$, $1 \leq i \leq N$. Consider the following problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= D\Delta u + f(u) \quad x \in \Omega \\ \frac{\partial u}{\partial n} &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{9}$$

where n is the outward normal to $\partial\Omega$ at x.

This problem defines a local semigroup in X^α. Let $Y = \{\phi \in X^\alpha : \phi(x) \in \mathcal{M}, \ \forall x \in \Omega\}$ with the metric inherited from X^α. Then, Y is a complete metric space. Our goal is to prove that any solution of (9) with initial data in Y remains in Y for as long as it exists. To prove this result we need a few auxiliary lemmas.

Lemma 1 *Given $\epsilon > 0$, there exists a locally Lipschitz function f_ϵ such that*

$$\sup_{u \in \mathbb{R}^N} |f_\epsilon(u) - f(u)|_{\mathbb{R}^N} < \epsilon \tag{10}$$

and $f_\epsilon(u) \cdot \mathbf{n}(u) < 0$ for all $u \in \partial\mathcal{M}$, where $\mathbf{n}(u)$ is the outward normal to $\mathcal{M}$ at u.

Proof: The construction of f_ϵ can be carried on as follows. Consider now a set of N locally Lipschitz functions $g_j : \mathbb{R}^N \to \mathbb{R}$, $1 \leq j \leq N$ such that $g_j(u) < 0$ for u in the j^{th} right face of

$\mathcal{M}$, if this face is defined, and $g_j(u) > 0$ for u in the j^{th} left face of $\mathcal{M}$, if this face is defined. Next, choose $\eta > 0$ such that $\eta|g(u)| < \epsilon$ for any $u \in \mathbb{R}^N$ where $g(u) = (g_1(u), \cdots, g_N(u))$. Define $f_\epsilon := f(u) + \eta g(u)$ and the result will follow.

Let $f_\epsilon^e : X^\alpha \to L^2(\Omega, \mathbb{R}^N)$ be the function defined by

$$f_\epsilon^e(\phi)(x) = f_\epsilon(\phi(x)), \ \forall x \in \Omega, \ \forall \phi \in X^\alpha. \tag{11}$$

Then, if $f_0 = f$ and $f_0^e = f^e$, the following result holds

Lemma 2 *Given $\epsilon > 0$, there exists $\delta > 0$, such that*

$$\|f_s^e(\phi) - f^e(\phi)\|_{L^2(\Omega,\mathbb{R}^N)} < \epsilon \tag{12}$$

for every $s < \delta$.

Proof: Since $X^\alpha \subset L^\infty(\Omega, \mathbb{R}^N)$ we have that,

$$|f_s^e(\phi)(x) - f^e(\phi)(x)| < \delta \tag{13}$$

and

$$\|f_s^e(\phi) - f^e(\phi)\|_{L^2(\Omega,\mathbb{R}^N)} \leq \delta|\Omega|^{\frac{1}{2}} < \epsilon \tag{14}$$

whenever $s < \delta < \epsilon\frac{1}{|\Omega|^{\frac{1}{2}}}$.

Corollary 1 *There exists a sequence of functions $f_j^e : X^\alpha \to L^2(\Omega, \mathbb{R}^N)$ which are Lipschitz continuous in bounded subsets of X^α and satisfy*

$$\sup_{u \in X^\alpha} \|f_j^e(\phi) - f^e(\phi)\|_{L^2(\Omega,\mathbb{R}^N)} \to 0 \tag{15}$$

as $j \to \infty$.

Next we consider the problem

$$\begin{aligned} \frac{\partial u_j}{\partial t} &= D\Delta u_j + f_j(u_j) \quad x \in \Omega \\ \frac{\partial u_j}{\partial n} &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{16}$$

where n is the outward normal to $\partial\Omega$ at x.

Denote by $u_j(t, \phi)$ the local solution of (16) with initial data $\phi \in X^\alpha$. If $\phi \in Y$, it follows from Smoller [1983] that $u_j(t, \phi)(x) \in \mathcal{M}$, $\forall x \in \Omega$ and whenever it exists.

We are now in position to state and prove the main result of this section.

Theorem 2 *Suppose that $\phi \in Y$ and suppose that the solution $u(t, \phi)$, of (9) with initial data ϕ, is defined on $[0, T]$ for some $T > 0$. Then, the solution $u_j(t, \phi)$ of (16) with initial data ϕ is defined on $[0, T]$ for j large enough. Furthermore, $u(t, \phi)(x) \in \mathcal{M}$, for all $x \in \Omega$ and for all $t \in [0, T]$.*

Proof: The first part of this theorem is a standard result in theory of semilinear parabolic problems and can be found in Henry [1981], for example. We only need to prove that $u(t,\phi)(x) \in \mathcal{M}$, $\forall x \in \Omega$ and $\forall t \in [0,T]$. To do that we use the variation of constants formula and the fact that X^α is embedded in $L^\infty(\Omega, \mathbb{R}^N)$. Note that

$$u(t,\phi) - u_j(t,\phi) = \int_0^t e^{-A(t-s)} \left[f^\epsilon(u(s,\phi)) - f_j^\epsilon(u_j(s,\phi))\right] ds. \tag{17}$$

Thus,

$$\|u(t,\phi) - u_j(t,\phi)\|_{X^\alpha} = \int_0^t M(t-s)^{-\alpha} \|f^\epsilon(u(s,\phi)) - f_j^\epsilon(u_j(s,\phi))\|_{L^2(\Omega,\mathbb{R}^N)} ds \tag{18}$$

and for j large enough, we have that

$$\|u(t,\phi) - u_j(t,\phi)\|_{L^\infty(\Omega,\mathbb{R}^N)} \leq \epsilon K M \int_0^T (t-s)^{-\alpha} ds =: \epsilon\beta, \tag{19}$$

where K is the embedding constant for $X^\alpha \subset L^\infty(\Omega, \mathbb{R}^N)$.

This implies that $u(t,\phi)(x)$ is in a $\epsilon\beta$ neighborhood of $\mathcal{M}$ for any $\epsilon > 0$ and therefore, since $\mathcal{M}$ is closed, it follows that $u(t,\phi)(x) \in \mathcal{M}$ for any $x \in \Omega$. This concludes the proof.

3. The Existence of Global Attractors

We are interested in obtaining that the solution operator $\{T(t),\ t \geq 0\}$ for the parabolic system (9) is bounded dissipative, where N, D and $\mathcal{M}$ are as in Section 2 and $f : \mathcal{M} \to \mathbb{R}$ is a locally Lipschitz continuous function satisfying (7).

Suppose that $\mathrm{R} \subset \mathcal{M}$ is a rectangle which contains every bounded face of $\mathcal{M}$, $x_0 \in \mathrm{R}^o$ and $\mathrm{R}_t = (x_0 + t(\mathrm{R} - x_0)) \cap \mathcal{M}$, $t \geq 0$, then $\{\mathrm{R}_t,\ t \geq 0\}$ covers $\mathcal{M}$. Let $p_\mathrm{R} : \mathcal{M} \to \mathbb{R}^+$ defined by

$$p_\mathrm{R}(u) = \inf\{t \geq 0 : u \in \mathrm{R}_{t+1}\} \tag{20}$$

and define $F_\mathrm{R} : Y \to \mathbb{R}^+$ by

$$F_\mathrm{R}(w) = \sup\{p_\mathrm{R}(w(x)) :\ x \in \Omega\}. \tag{21}$$

Definition 1 *A bounded rectangle* $\mathrm{R} \subset \mathcal{M}$ *is contracting for the vector field* $f(u)$ *in* $\mathcal{M}$, *if for every point* u *which belongs to a (closed) face of* R *that is not contained in* $\partial\mathcal{M}$, *we have that* $f(u).\mathbf{n}(u) < 0$, *where* $\mathbf{n}(u)$ *is the outward normal to* R *at* u.

Theorem 3 *Let* $f(u)$ *be a vector field in* $\mathcal{M} \subset \mathbb{R}^N$ *which vanishes in* $\partial\mathcal{M}$, *and* R *be a rectangle in* $\mathcal{M}$ *that contains every bounded face of* $\mathcal{M}$. *Suppose that* $u(\cdot,t) \in X^\alpha$ *is a solution of (9). If* R *is such that, for some* $\tau > 0$, R_τ *is a contracting rectangle for the vector field* $f(u)$; *then, for any* $T > 0$ *for which* $F_\mathrm{R}(u(T)) = \tau$, *there exists* $\eta > 0$ *such that*

$$\limsup_{h \to 0} \frac{F_\mathrm{R}(u(T+h)) - F_\mathrm{R}(u(T))}{h} \leq -\eta. \tag{22}$$

Proof: Let $u = (u_1, \cdots, u_N)$, $f = (f_1, \cdots, f_N)$. If $F_R(u(T)) = \tau$ and $R_\tau = \Pi_{j=1}^N [l_i, r_i]$, we have that $l_i \leq u_i(x,T) \leq r_i$, $1 \leq i \leq N$. If $u(\bar{x},T) \in \partial R_\tau$, then $u(\bar{x},T)$ is in one of the faces of R which is not contained in $\partial \mathcal{M}$, say the right-hand j^{th} face $\partial(R_\tau)_j^r$ of R_τ. Since R_τ is a contracting rectangle for f, there is an $\eta > 0$ such that for all $u \in \partial(R_\tau)_j^r$,

$$f(u(\bar{x},T)) \cdot \mathbf{n}(u(\bar{x},T)) < -\eta,$$

where $\mathbf{n}(\bar{x},T)$ is the outward-pointing normal at $u(\bar{x},T)$. Since $u(\bar{x},T)$ is in the right-hand j^{th} face of R_τ, we have that $u_j(x,T) \leq r_j$, $\forall x$ with $u(\bar{x},T) = r_j$. Thus, $d_j \Delta u_j \leq 0$, since $F_R(u(\cdot,T) = \tau$. Consequently,

$$\frac{\partial u_j}{\partial t} = d_j \Delta u_j + f_j(u) < -\eta,$$

so that $u_j(\bar{x}, T+h) < r_j - \eta$ for small h. By continuity, this holds for all x in a neighborhood of $\bar{x}$.

If $K = \{x : u(x,T) \in \overline{\partial R_\tau \backslash \partial \mathcal{M}}\}$, then K is compact, and by what we have just shown, there is an open set $U \supset K$ such that if $x \in U$, and h is small,

$$u(x, T+h) \in R_{(\tau - \eta h)}.$$

If $x \notin U$, then $u(x,T)$ is interior to R_τ, so there is a compact set $Q \subset R_\tau^\circ$, and an $h_0 > 0$ such that $u(x, T+h) \in Q$ if $|h| < h_0$ for all x. Thus, for sufficiently small h, $u(x, T+h)$ is in $R_{(\tau-\eta h)}$ for all x, and so

$$F_R(u(x, T+h)) \leq \tau - 2\eta h.$$

Therefore, for all x

$$\frac{F_R(u(x,T+h)) - F_R(u(x,T))}{h} \leq -2\eta.$$

and the result is proved.

To prove that there exists a bounded absorbing set for $T(t)$ in X^α we use the variation of constants formula

$$T(t)u_0 = e^{-A(t-t_0)} T(t_0) u_0 + \int_{t_0}^t e^{-A(t-s)} f(T(s)u_0) ds, \quad t \geq t_0 \geq 0 \tag{23}$$

and the fact that X^α is embedded in $L^\infty(\Omega, \mathbb{R}^N)$.

Let $R_0 = \Pi_{i=1}^N (-M_i, M_i) \cap \mathcal{M}$ and $R_\tau = (1+\tau) R_0 \cap \mathcal{M}$ for $\tau \geq 0$ and assume that the M_i's are chosen such that R_0 contains every bounded face of $\mathcal{M}$. Under the hypothesis (7), it is easy to see that $\{R_\tau : \tau \geq 0\}$ is an increasing family of contracting rectangles for the problem (1) and a covering for $\mathcal{M}$.

Let $u \in X^\alpha$; then, there exists $\tau_0 > 0$ such that $u(x) \in R_{\tau_0}$ for all $x \in \Omega$. Since R_{τ_0} is a contracting rectangle we have that the solution $u(t, \cdot, u_0)$ of (1) with inital data u_0, is such that $u(t, x, u_0) \in R_{\tau_0}$ for all $x \in \Omega$, for as long as it exists. Thus, there exists $L > 0$ such that $\|T(t)u_0\|_\infty = \|u(t, \cdot, u_0)\|_\infty \leq L$ and

$$\|T(t)u_0\|_{X^\alpha} \leq M\,L\,t^{-\alpha}|\Omega|^{\frac{1}{2}} + M\,P\,|\Omega|^{\frac{1}{2}}\int_0^t (t-s)^{-\alpha}ds, \tag{24}$$

where $P = \sup_{s\in \mathrm{R}_{\tau_0}} |f(s)|$. Since $\|T(t)u_0\|_{X^\alpha}$ does not blow up in finite time, it must exist for all $t \geq 0$.

To prove that $T(t)$ is point dissipative, we observe that for any $\epsilon > 0$ there exists $t_\epsilon > 0$ such that $T(t)u_0(x) \in \mathrm{R}_\epsilon$, $\forall x \in \Omega$, $\forall t \geq t_\epsilon$. This follows from the fact that for any $\tau > 0$, $\dot{F}_{\mathrm{R}_\tau}(T(t)u_0) \leq -2\eta$ for all $t > 0$ and for some $\eta > 0$.

Thus, for $t \geq t_1 + 1$

$$\|T(t)u_0\|_{X^\alpha} \leq M|\Omega|^{\frac{1}{2}}\|T(t-1)u_0\|_\infty + \frac{M\,P_1\,|\Omega|^{\frac{1}{2}}}{1-\alpha} \leq M\,L_1|\Omega|^{\frac{1}{2}} + \frac{M\,P_1\,|\Omega|^{\frac{1}{2}}}{1-\alpha} \tag{25}$$

where $L_1 = \sup_{s\in \mathrm{R}_1} |s|$ and $P_1 = \sup_{s\in R_1} |f(s)|$. Since L_1 and P_1 do not depend on u_0 we have that $T(t)$ is point dissipative.

Observe that if $B \subset Y$ is a bounded set then there exists $\tau_B > 0$ such that $u(x) \in \mathrm{R}_{\tau_B}$ $\forall x \in \Omega$ and $\forall u \in B$. If $L_B = \sup_{s\in \mathrm{R}_{\tau_B}} |s|$ and $P_B = \sup_{s\in R_{\tau_B}} |f(s)|$, we have

$$\|T(t)u\|_{X^\alpha} \leq M\,L_B|\Omega|^{\frac{1}{2}} + \frac{M\,P_B\,|\Omega|^{\frac{1}{2}}}{1-\alpha}, \tag{26}$$

and orbits of bounded subsets of Y are bounded, proving that

Theorem 4 *Suppose that $f : \mathbb{R}^N \to \mathbb{R}^N$ is a locally lipschitz continuous function satisfying (7). Then, the problem (1) has a global attractor $\mathcal{A}_D$ in X^α. Furthermore,*

$$u(x) \in \mathrm{R}_0,\ \forall x \in \Omega\ \forall u \in \mathcal{A}_D \text{ and } \forall D \geq 0. \tag{27}$$

4. Applications of Theorem 4

4.1. The Hodgkin-Huxley Equations

Consider the system of weakly coupled parabolic partial differential equations

$$\begin{aligned}
c\frac{\partial u}{\partial t} &= R^{-1}\Delta u + g(u,v,w,z)\\
\frac{\partial v}{\partial t} &= \epsilon_1\Delta v + g_1(u)(h_1(u)-v)\\
\frac{\partial w}{\partial t} &= \epsilon_2\Delta w + g_2(u)(h_2(u)-w)\\
\frac{\partial z}{\partial t} &= \epsilon_2\Delta z + g_3(u)(h_3(u)-z)\\
\frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = \frac{\partial z}{\partial n} = 0
\end{aligned} \tag{28}$$

where c, R and ϵ_i, $i = 1,2,3$, are positive constants, and g is defined by

$$\begin{aligned} g(u,v,w,z) = \; & k_1 v^3 w(c_1 - u) + k_2 z^4 (c_2 - u) + k_3(c_3 - u), \\ & c_1 > c_3 > 0 > c_2. \end{aligned} \tag{29}$$

Furthermore, $g_i > 0$, $1 > h_i > 0$, $i = 1,2,3$. In this model the variables v, w, z represent chemical concentrations, and are thus nonegative, while u denotes eletric potential. The equations are a mathematical model for physiological phenomenon of signal tranmission across axons.

Therefore, the above system is locally well posed in $Y = \{(u_1,u_2,u_3,u_4) \in X^\alpha : u_i(x) \geq 0,\, 2 \leq i \leq 4,\, u_1 \in \mathbb{R}\ \forall x \in \Omega\}$. In what follows we prove that the problem (28) has a complete set of contracting rectangles and therefore, by the results in Section 3, it has a global attractor in Y.

Under these hypothesis it is easy to see that the (7) is satisfied for the vector field $F(u,v,w,z) = (g(u,v,w,z), g_1(u)(h_1(u) - v), g_2(u)(h_2(u) - w), g_3(u)(h_3(u) - z))^{\mathsf{T}}$ in $\mathbf{E} = \{(u_1,u_2,u_3,u_4) \in \mathbb{R}^4 : u_i \geq 0,\, 2 \leq i \leq 4,\, u_1 \in \mathbb{R}\}$. The constants M_i, $1 \leq i \leq 4$, can be chosen as follows,

$$M_i \geq 1,\ i = 2,3,4 \quad \text{and} \quad M_1 \geq \max\{c_1, |c_2|\}$$

It follows from the results in Section 3 that (28) has a global attractor in Y.

4.2. The Fitzhugh-Nagumo Equations

Consider the equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + f(u) - v \\ \frac{\partial v}{\partial t} &= \epsilon \Delta v + \sigma u - \gamma v \\ \frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0. \end{aligned} \tag{30}$$

where σ, γ and ϵ are positive constants. The function f has the qualitative form of a cubic polynomial, for example $f(u) = -u(u - \beta)(u - 1)$, where $0 < \beta < \frac{1}{2}$.

Therefore, the system (30) defines a local dynamical system in X^α. If we chose the rectangles such that they have the shape described in the picture, we can easily verify that they constitute a complete family of contracting rectangles.

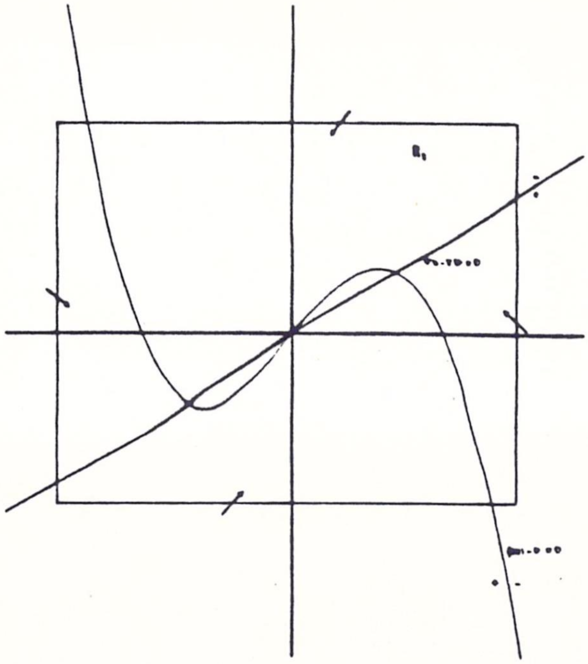


figure 1

Therefore, (7) is satisfied and the results in Section 3 imply the existence of a global attractor for the problem (30) in X^α.

4.3. A Problem In Superconductivity of Liquids

Consider the following system of parabolic partial differential equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= D\Delta u + (1-|u|^2)u, \\ \frac{\partial u}{\partial n} &= 0, \end{aligned} \tag{31}$$

where D is as before. Then, for $M_i > 1$, $1 \leq i \leq$ N, the condition (7) is satisfied and the results of Section 2 imply the existence of a global attractor $\mathcal{A}_D$ for the problem (31).

5. Diffusion Independent Attractors

It is well known that attractors for systems o parabolic equations may vary significantly by changing the diffusion coefficients (see, for example, Chafee and Infante [1974]). Even though there are many possible changes in the attractor there is something that remains unchanged. For example, consider the Chafee and Infante problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \epsilon u_{xx} + f(u), \quad x \in (0,1) \\ u_x(0) &= u_x(1) = 0. \end{aligned} \tag{32}$$

If $\mathcal{A}_\epsilon$ is the global attractor for (32), $f(u) = u - u^3$, it was shown in Carvalho and Ruas-Filho [1993] that,

$$u(x) \in [-1, 1] \tag{33}$$

for every $u \in \mathcal{A}_\epsilon$, whatever $\epsilon \geq 0$ may be. That is, the attractor for (32) equals the attractor for (6), in this sence.

In this section, we prove that this is a general fact that is true for a wide class of parabolic systems.

Definition 2 *We sat that $\mathcal{B} \subset \mathbb{R}^N$ is a* diffusion independent invariant set *if for any $D \geq 0$*

$$\mathcal{B} \supset \{u(x) : x \in \Omega, \ u \in \mathcal{A}_D\}. \tag{34}$$

We say that $\mathcal{B} \subset \mathbb{R}^N$ is a diffusion independent attractor *if it is minimal with the property (34). We say that $\mathcal{B}$ is an* image invariant attractor *for (1), $D \geq 0$, if the set $\{u(x) : x \in \Omega, \ u \in \mathcal{A}_D\}$ does not depend upon D for $D \geq 0$.*

Note that the set $\mathcal{B}$ defined by (34) coincides with the attractor of (6) for the problem (32). Therefore it is an image invariant attractor for (32). We may not expect to obtain image invariant attractors of other type of boundary condition. Thus, for any boundary condition other then Neumann, we study only the existence of diffusion invariant attractors.

Next we introduce the concept of contracting region for the vector field $f(u)$.

Definition 3 *A region C is said a* contracting region *for the vector field $f(u)$, defined in a subdomain of $\mathbb{R}^N$ containing C, if*

$$f(u) \cdot \mathbf{n}(u) < 0, \tag{35}$$

for all $u \in \partial C$, where $\mathbf{n}(u)$ is the outward normal vector to ∂C at u.

Assume $f : \mathbb{R}^N \to \mathbb{R}^N$ is locally Lispschitz continuous and that Ω is a bounded smooth subset of $\mathbb{R}^n$. Assume also that the problem (6) has a global attractor $\mathcal{A}_0$. In what follows we prove that under these and some additional hypotheses the problem (1) has a family of invariant regions.

Definition 4 *We say that $\{\mathcal{O}_s, \ s > 0\}$ is a complete family of contracting sets for the vector field f, if:*

- *For each $s > 0$ the set $\mathcal{O}_s$ is a contracting set for the vector field f;*
- *It is an open covering of $\mathbb{R}^N$;*
- *$\mathcal{O}_{s_1} \subset \mathcal{O}_{s_2}$ whenever $s_1 \leq s_2$;*
- *$\mathcal{O}_s / \cup_{0<\theta<s} \mathcal{O}_\theta = \emptyset$, for all $s > 0$, and*

Theorem 5 *Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be a locally Lipschitz continuous function. If the problem (6) has a global attractor; then, there exists a complete family of contracting sets for $f(u)$.*

Proof. Since the problem (6) has a global attractor $\mathcal{A}_0$ there exists a C^1 function $V : \mathbb{R}^N \to \mathbb{R}^N$ such that

- $V(v) = 0$ if $v \in \mathcal{A}_0$;
- $a(d(v, \mathcal{A}_0)) \leq V(v) \leq b(d(v, \mathcal{A}_0))$ where $a(r)$ is continuous, nondecreasing, $a(r) > 0$ if $r > 0$, $a(r) \to \infty$ as $r \to \infty$ and $b(r)$ is continuous with $b(0) = 0$;
- $\dot{V}_6(v) \leq -V(v)$, for all $v \in \mathbb{R}^N$.

For any $\tau > 0$ define $V_\tau = \{v \in \mathbb{R}^N : V(v) < \tau\}$. We will prove that $\{V_\tau : \tau > 0\}$ is a complete family of contracting sets for the vector field f.

The only property in the definition of a complete family of contracting sets that is not obvious is the fact that V_τ is a contracting set itself, for each $\tau > 0$. To prove this property, note that, if $v_0 \in \partial V_\tau$, then $\nabla V(v_0) := (\frac{\partial V}{\partial v_1}(v_0), \cdots, \frac{\partial V}{\partial v_N}(v_0))$ is a normal vector to the ∂V_τ. Thus,

$$f(v) \cdot \frac{\nabla V(v_0)}{|\nabla V(v_0)|} \leq -\frac{\tau}{|\nabla V(v_0)|} < 0. \tag{36}$$

It follows that $\{V_\tau : \tau > 0\}$ is a complete family of contracting sets and the result is proved.

Definition 5 *A smooth function $V : \mathbb{R}^N \to \mathbb{R}$ is called an eventually convex function if there exists $M > 0$ such that $d^2V(u)(\eta, \eta) \geq 0$, for every u, $\eta \in \mathbb{R}^N$, $\|u\| \geq M$.*

Theorem 6 *Let $V : \mathbb{R}^N \to \mathbb{R}$ be an eventually convex function sucn that $V(u) \to \infty$ when $\|u\| \to \infty$ and define $V_\tau = \{v \in \mathbb{R}^N : V(v) < \tau\}$. Assume that $b = \max\{V(u) : \|u\| = M$, $\{V_\tau : \tau > b\}$ is a complete family of contracting sets for the vector field f and $\nabla V(u) \neq 0$ for $u \notin V_b$. Then, there exists a global attractor $\mathcal{A}_D$ for (1) in X^α. Furthermore, $u(x) \in \mathcal{B} = V_b$ for every $x \in \Omega$ and for every $u \in \mathcal{A}_D$ and (1) has a diffusion independent attractor.*

Proof: Let $u(\cdot, t) \in X^\alpha$ be a solution of (1). Let $t > 0$ fixed and τ_0 such that $u(x,t) \in V_\tau$ $\forall\, x \in \Omega$, $\forall\, \tau > \tau_0$ and τ_0 is the smallest number with this property. Then

$$\frac{d}{dt}V(u(x,t)) = \nabla V(u(x,t)) \cdot \frac{\partial}{\partial t}u(x,t) = \nabla V(u(x,t)) \cdot D\Delta u(x,t) + \nabla V(u(x,t)) \cdot f(u(x,t)) \tag{37}$$

Note that, if $u(\bar{x}, t) \in \partial V_{\tau_0}$ then $\nabla V(u(\bar{x}, t))$ is orthogonal to ∂V_{τ_0} at $u(\bar{x}, t)$, since ∂V_{τ_0} is a level surface for V. Thus, whenever $u(\bar{x}, t) \in \partial V_{\tau_0}$ we have that there exists a η depending only on τ_0 such that

$$\nabla V(u(\bar{x}, t)) \cdot f(u(\bar{x}, t)) < -\eta \tag{38}$$

Also

$$0 \geq D\Delta V(u(\bar{x},t)) = Dd^2V(u(\bar{x},t))(\nabla u(\bar{x},t), \nabla u(\bar{x},t)) + \nabla V(u(\bar{x},t)) \cdot D\Delta u(\bar{x},t) \tag{39}$$

It follows that

$$\frac{d}{dt}V(u(\bar{x},t)) \leq \nabla V(u(\bar{x},t)) \cdot f(u(\bar{x},t)) < -\eta, \tag{40}$$

for every $\bar{x} \in \Omega$ such that $u(\bar{x},t) \in \partial V_{\tau_0}$.

Consequently, $V(u(\bar{x},t+h) < \tau_0 - \eta h$. By continuity, this holds for all x in a neighborhood of $\bar{x}$. If $K = \{x : u(x,t) \in \partial V_{\tau_0}\}$ then K is compact, and by what we have just shown, there is an open set $U \supset K$ such that $x \in U$ and h small, implies

$$V(u(x,t+h)) \leq V(u(x,t)) - \eta h \tag{41}$$

If $x \notin U$, then $u(x,t)$ is interior to V_{τ_0}, so there is a compact set $Q \subset V_{\tau_0}$, and $h_0 > 0$ such that $u(x,t+h) \in Q$ if $|h| < h_0$ for all x. Thus, for sufficiently small h,

$$V(u(x,t+h)) \leq \tau_0 - \eta h, \quad \forall x \in \Omega \tag{42}$$

Therefore, if $F_{V_{\tau_0}} = \sup\{V_{\tau_0}(u(x,t)) : x \in \Omega\}$, we have that

$$\limsup_{h \to 0} \frac{F_{V_{\tau_0}}(u(\cdot,t+h) - F_{V_{\tau_0}}(u(\cdot,t)}{h} \leq -\eta \tag{43}$$

Proceeding as in Section 2, we have that (1) has a global attractor. It remains to prove that $\mathcal{B}$ is a diffusion independent invariant set for (1).

To prove that $\mathcal{B}$ is a diffusion independent invariant set for (1), suppose that $u_0 \in \mathcal{A}_D$, then if $\tau_0 > b$ is as before, we have that for all $x \in \Omega$ and $u_0 \in \mathcal{A}_D$, $\dot{V}(u(x,t)) \leq -\eta$. Therefore $u(h,x,u_0) \in V_\tau$ for some $\tau < \tau_0$, for all $x \in \Omega$ and for all $u_0 \in \mathcal{A}_D$. This contradicts the the definition of τ_0. Therefore $\mathcal{B}$ is a diffusion invariant set for (1) and the proof is completed.

Note that the set $\cap_{\tau>0} V_\tau$ contains the range of every function in $\mathcal{A}_D$ independently of D. An estimate of this type has been obtained for systems satisfying (5) in Carvalho and Ruas-Filho [1993].

Corollary 7 *If the hypothesis of Theorem (6) are satisfied and $\mathcal{A}_0 = \cap_{\{\tau>0\}} V_\tau$. Then, $\mathcal{A}_0$ is a image invariant attractor for the problem (1).*

Corollary 8 *Let $V : \mathbb{R} \to \mathbb{R}$ be the function*

$$V(u) = -\frac{u^2}{2} + \frac{u^4}{4}. \tag{44}$$

Then, V is a convex function for $|u| > 3^{-\frac{1}{2}}$ and $\{V_\tau : \tau > 0\}$ is a complete family of contracting sets for (32). Furthermore, $\mathcal{A}_0 = [-1,1]$ is a image invariant attractor for the problem (32), $\epsilon \geq 0$.

5.1. A Problem Arising from Thin Domains Problems

Consider the following system of parabolic partial differential equations

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = D\Delta u - \tilde{A}u + f(u), \qquad \text{in} \ \ \Omega, \\ &\frac{\partial u}{\partial n} = 0, \qquad \text{in} \ \ \partial\Omega. \end{aligned}\right\} \tag{45}$$

where $u = (u_1, u_2, \cdots, u_N)^T$, $N \geq 1$, $D = \mathrm{diag}(d_1, \cdots, d_N)$, $d_i > 0$, $1 \leq i \leq N$ and the nonlinearity $f := (g(u_1), \cdots, g(u_N))^T : \mathbb{R}^N \to \mathbb{R}^N$, where $g : \mathbb{R} \to \mathbb{R}$ is assumed to be a C^1 function satisfying

$$sg(s) < 0 \qquad \forall s \in \mathbb{R}, \quad s \notin [\bar{\xi}, \xi], \quad 1 \leq i \leq N. \tag{46}$$

The matrix $\tilde{A}$ is taken as $\tilde{A} = M^{-1}B$ where $M = \mathrm{diag}(L_1, \cdots, L_N)$, with $0 \leq L_i \leq 1$ for $1 \leq i \leq N$, $\sum_{i=1}^{N} L_i = 1$ and B be is the tridiagonal symmetric matrix

$$B = \begin{bmatrix} m_1 & r_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ r_1 & m_2 & r_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & r_2 & m_3 & r_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & r_3 & m_4 & r_4 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & & & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & r_{N-3} & m_{N-2} & r_{N-2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & r_{N-2} & m_{N-1} & r_{N-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & r_{N-1} & m_N \end{bmatrix} \tag{47}$$

where

$$\begin{aligned} m_1 &= \tfrac{a_2}{2l_2} + \tfrac{a_1\rho}{\rho l_1 + a_1(1-\rho)}, \\ m_N &= \tfrac{a_N}{2l_N} + \tfrac{a_{N+1}\sigma}{\sigma l_{N+1} + a_{N+1}(1-\sigma)}, \\ m_k &= \tfrac{a_k}{2l_k} + \tfrac{a_{k+1}}{2l_{k+1}}, \quad k = 2, \cdots, N-1, \\ r_k &= \tfrac{-a_{k+1}}{2l_{k+1}}, \quad k = 1, \cdots, N-1. \end{aligned}$$

and $a_k > 0$, $l_k > 0$ for $1 \leq k \leq N+1$. This problem arises as a limiting problem for reaction diffusion equations in thin domains around a point (see Hale and Raugel[1992], Carvalho[1992] and Carvalho and Oliveira[1992], for more details on how these systems appear and Carvalho and Ruas-Filho [1993] for existence of attractors in X^α and uniform estimates on the attractor).

We prove the existence of attractor and the existence of a diffusion independent attractor for the problem (45) applying the results of Section 5.

Let

$$V(u) = \frac{1}{2}\langle Bu, u\rangle - \sum_{i=1}^{N} L_i \int_0^{u_1} g(s)ds. \tag{48}$$

Then, the following holds

Corollary 9 *If the function g satisfy (7) and if there is a bounded set B such that $\tilde{A} - diag(g'(u_1), \cdots, g'(u_N))$ is positive definite for $u \notin B$ and $\nabla V(u_1, \cdots, u_N) \neq 0$ for $u \notin B$. Then, V is convex outside B and $\{V_\tau : \tau > b\}$ is a complete family of contracting sets, where $b = \inf\{\tau > 0 : V_\tau \supset B\}$. Furthermore, V_b contains a diffusion independent attractor for (45), $D \geq 0$.*

Next we consider a particular case of the above problem for which we can better describe the results. Let p and q be to positive parameters and consider the parabolic problem

$$\left.\begin{aligned} \frac{\partial u_1}{\partial t} &= d_1 \Delta u_1 - p(u_1 - u_2) + u_1 - u_1^3 \\ \frac{\partial u_2}{\partial t} &= d_2 \Delta u_2 + q(u_1 - u_2) + u_2 - u_2^3. \end{aligned}\right\} \tag{49}$$

and the associated system of ordinary differential equations

$$\left.\begin{aligned} \dot{u}_1 &= -p(u_1 - u_2) + u_1 - u_1^3 \\ \dot{u}_2 &= q(u_1 - u_2) + u_2 - u_2^3. \end{aligned}\right\} \tag{50}$$

It is easy to see that if $p \neq q$, the system (50) is not gradient. However, the function

$$V(u) = \frac{1}{2}\langle Bu, u\rangle - \frac{1}{2p}u_1^2 + \frac{1}{4p}u_1^4 - \frac{1}{2q}u_2^2 + \frac{1}{4q}u_2^4. \tag{51}$$

where

$$B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{52}$$

is such that

$$\begin{aligned} \frac{\partial V}{\partial u_1}(u_1, u_2) &= -\tfrac{1}{p}(-p(u_1 - u_2) + u_1 - u_1^3) \\ \frac{\partial V}{\partial u_2}(u_1, u_2) &= -\tfrac{1}{q}(q(u_1 - u_2) + u_2 - u_2^3) \end{aligned} \tag{53}$$

and

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \nabla V(u_1, u_2) \tag{54}$$

This shows that the problem (50) has gradient structure an since all the equilibrium points are hyperbolic, its attractor must consist of the union of the unstable manifolds of the equilibrium points. Thus any bifurcation that may occur can only be a static bifurcation; that is, a bifurcation of equilibrium points.

Next, we proceed to describe the static bifurcations of this problem. For this purpose we must consider the solutions of the following equations

$$0 = -p(u_1 - u_2) + u_1 - u_1^3 \tag{55}$$

$$0 = q(u_1 - u_2) + u_2 - u_2^3. \tag{56}$$

Subtracting these two equations, we obtain

$$(u_1 - u_2)(1 - (p+q) - (u_1^2 + u_1 u_2 + u_2^2)) = 0 \tag{57}$$

and if $u_1 \neq u_2$ it becomes

$$1 - (p+q) = u_1^2 + u_1 u_2 + u_2^2 \geq 0. \tag{58}$$

Thus, if $p + q \geq 1$, the only equilibrium points of the problem (50) must satisfy $u_1 = u_2$ and therefore, they are

$$(-1,-1), \quad (1,1), \quad (0,0). \tag{59}$$

On the other hand, if $0 \leq p + q < 1$ and $p \neq 0$, we have from (55) that

$$u_2 = -\frac{u_1(1 - p - u_1^2)}{p} \tag{60}$$

and substituting this value into (58), we obtain

$$1 - (p+q) = u_1^2 - \frac{u_1^2(1-p-u_1^2)}{p} + \frac{u_1^2(1-p-u_1^2)^2}{p^2} \tag{61}$$

by making $u_1^2 = x$, we above equation reduces to

$$1 - (p+q) = x - \frac{x(1-p-x)}{p} + \frac{x(1-p-x)^2}{p^2}, \tag{62}$$

which is a cubic polynomial and, therefore, its roots can be found explicitly as functions of p and q.

We proceed to solve the above equation. Firstly, let's make a change of coordinates that simplify equation (62). If $1 - p - x = t$, it becomes

$$t^3 - t^2 + pt - p^2 q = 0 \tag{63}$$

By making $y = t - \frac{1}{3}$ we obtain that

$$y^3 + 3ly + 2m = 0 \tag{64}$$

where $l = \frac{1}{3}(p - \frac{1}{3})$ and $m = \frac{1}{2}(\frac{p}{3} - p^2 q - \frac{2}{27})$.

One root of this equation is given by

$$y_1 = (-m + \sqrt{l^3 + m^2})^{\frac{1}{3}} + (-m - \sqrt{l^3 + m^2})^{\frac{1}{3}}. \tag{65}$$

Dividing the equation (64) by $(y - y_1)$ we obtain

$$y^2 + y_1 y + (y_1^2 + 3l) = 0, \tag{66}$$

and solving equation (66) we obtain the remaining roots of (64). These roots are given by

$$\begin{aligned} y_{2,3} = & \frac{-(-m+\sqrt{l^3+m^2})^{\frac{1}{3}} - (-m-\sqrt{l^3+m^2})^{\frac{1}{3}}}{2} \\ & \pm \frac{\sqrt{3}\sqrt{-[(-m+\sqrt{l^3+m^2})^{\frac{1}{3}} - (-m-\sqrt{l^3+m^2})^{\frac{1}{3}}]^2}}{2} \end{aligned} \tag{67}$$

Observe that y_1 is real whatever be the sign of $l^3 + m^2$. And $y_{2,3}$ are real only when $l^3 + m^2 \leq 0$.

Note yet that $x = \frac{2}{3} - p - y$ and that we only have real roots for (55) and (56) if $x \geq 0$, since $u_1^2 = x$.

We now proceed to analyse the situations when this occur. Suppose that, $l^3 + m^2 \leq 0$. Then,

$$\begin{aligned} y_1 &= (-m + i\sqrt{-l^3 - m^2})^{\frac{1}{3}} + (-m - i\sqrt{-l^3 - m^2})^{\frac{1}{3}} \\ &= 2\sqrt{-l}\cos\theta \end{aligned} \tag{68}$$

and

$$y_{2,3} = -\sqrt{-l}\left(\cos\theta \pm \sqrt{3}\,sen\,\theta\right) \tag{69}$$

where $\theta = \frac{1}{3}\arccos\frac{-m}{(-l)^{\frac{3}{2}}}$, $\theta \in [-\frac{\pi}{3}, \frac{\pi}{3}]$.

Thus,

$$\begin{aligned} x_1 &= \frac{2}{3} - p - 2\sqrt{\frac{1}{3}(\frac{1}{3} - p)}\cos\theta \\ x_{2,3} &= \frac{2}{3} - p + \sqrt{\frac{1}{3}(\frac{1}{3} - p)}(\cos\theta \pm \sqrt{3}sen\,\theta) \end{aligned} \tag{70}$$

and x_1, x_2, x_3 are nonnegative, since $\cos\theta \pm \sqrt{3}sen\,\theta \leq 2$, for all θ and for $p \leq \frac{1}{3}$, we have that

$$-2\sqrt{\frac{1}{3}(\frac{1}{3} - p)} + \frac{2}{3} - p \geq 0. \tag{71}$$

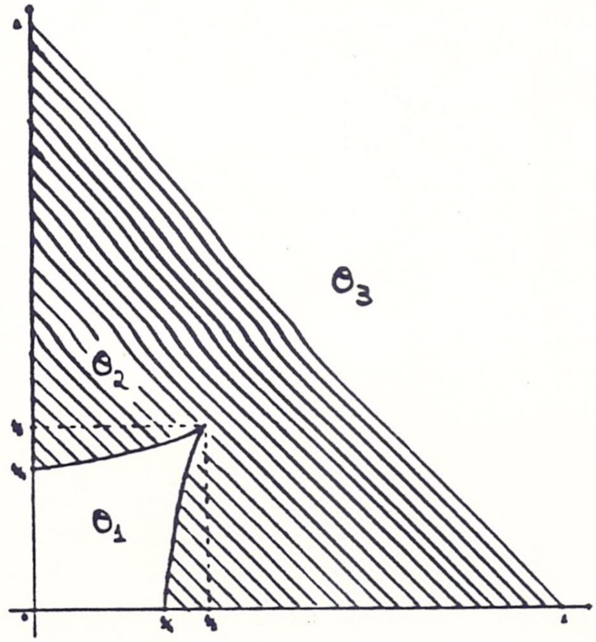


figure 2

This implies that the problem (50) has nine equilibrium points in the region given by $l^3 + m^2 \leq 0$. That is,

$$p^2(-\frac{1}{108} + \frac{p+q}{27} - \frac{pq}{6} + \frac{p^2q^2}{4}) \leq 0. \tag{72}$$

This region in the first quadrant of the plane is shown in O_1, figure 2.

Next we consider the case when $l^3 + m^2 > 0$. In this case, it is clear that $y_1 \in \mathbb{R}$, whereas $y_{2,3} \notin \mathbb{R}$. Furthermore,

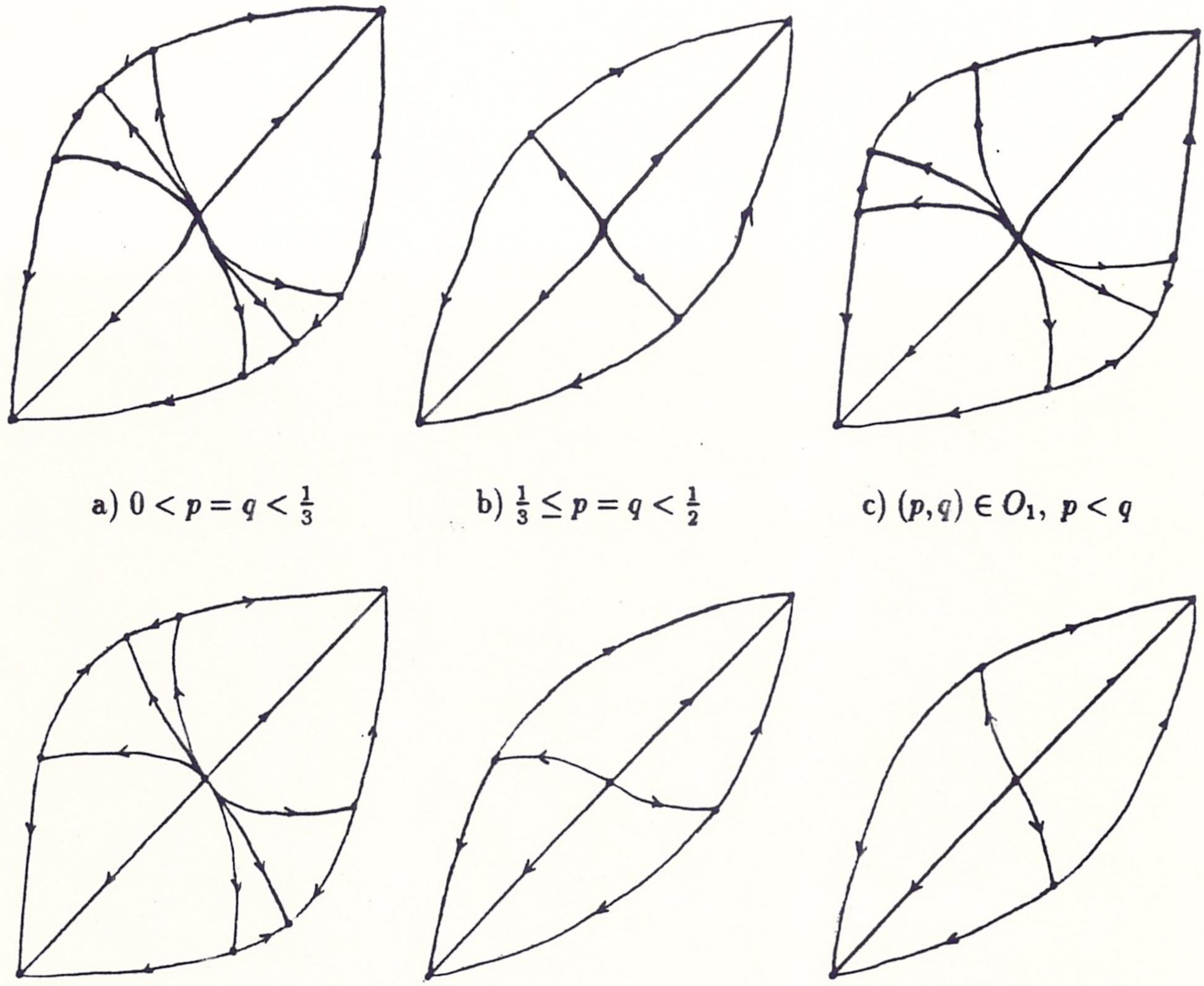
$$x_1 = \frac{2}{3} - p - (-m + \sqrt{l^3 + m^2})^{\frac{1}{3}} - (-m - \sqrt{l^3 + m^2})^{\frac{1}{3}} \geq 0 \tag{73}$$

is equivalent to $p + q \leq 1$ and the region where $l^3 + m^2 > 0$ and $x_1 \geq 0$ is shown in O_2, figure 2.

This region corresponds to the region in space of parameters where the problem (50) has five equilibrium points.

If $p + q \geq 1$, as remarked before, the system (50) has only three equilibrium points; namely, $(0,0)$, $(-1,-1)$ and $(1,1)$.

We conlude that there are two bifurcation curves in the space of parameters dividing the first quadrant into three regions, O_1, O_2 and O_3, where no biffurcation occurs. In figure 3 we see, for each of these regions, the pictures corresponding to the attractor for (50).



a) $0 < p = q < \frac{1}{3}$ b) $\frac{1}{3} \le p = q < \frac{1}{2}$ c) $(p,q) \in O_1,\ p < q$

d) $(p,q) \in O_1,\ p > q$ e) $(p,q) \in O_2,\ p > q$ f) $(p,q) \in O_2,\ p < q$

figure 3

We need to decide when $V : \mathbb{R}^2 \to \mathbb{R}$ is a convex function and when it determines a complete family of contracting sets. For simplicity, we assume that $p = q = \alpha$. Let us compute the second derivative of V. Note that

$$\nabla V(u_1, u_2) = \begin{pmatrix} (\alpha - 1)u_1 - \alpha u_2 + u_1^3 \\ (\alpha - 1)u_2 - \alpha u_1 + u_2^3 \end{pmatrix} \tag{74}$$

and

$$d^2 V(u_1, u_2) = \begin{pmatrix} \alpha - 1 + 3u_1^2 & -\alpha \\ -\alpha & \alpha - 1 + 3u_2^2 \end{pmatrix}. \tag{75}$$

Thus, the eigenvalues of $d^2V(u_1,u_2)$ are given by

$$\lambda_{1,2} = \frac{2(\alpha-1)+(l+m)\pm\sqrt{(l-m)^2+4\alpha^2}}{2}, \tag{76}$$

where $l = 3u_1^2$ and $m = 3u_2^2$.

Assume that $\alpha > 1$ then, $\lambda_{1,2}$ are nonnegative whenever $|(u_1,u_2)| \geq \sqrt{\frac{2\alpha-1}{3(\alpha-1)}}$. Thus, under this condition, $d^2V(u_1,u_2)$ is positive definite and V is convex. Observe that if $\alpha \leq 1$ V is never convex in the outside of a bounded set.

Note that (45) can be rewritten as

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u - \operatorname{diag}(L_1^{-1},\cdots,L_N^{-1})\nabla V(u), \quad \text{in } \Omega, \\ &\frac{\partial u}{\partial n} = 0, \quad \text{in } \partial\Omega. \end{aligned}\right\} \tag{77}$$

In the case of (50) the system is gradient and, therefore, $V_\tau, \tau > 0$ is always a contracting set for (50). It is easy to see that $\{V_\tau, \tau > 0\}$ is a complete family of contracting sets. These contracting sets are convex whenever V_τ contains the ball of radius $r = \sqrt{\frac{2\alpha-1}{3(\alpha-1)}}$ around $(0,0)$.

Corollary 10 *The set*

$$\mathcal{A} = \cap\{V_\tau : \tau > 0,\ V_\tau \supset B_r(0,0)\} \ \cap\ [-1,1]^2$$

is a diffusion independent attractor for (50).

We observe that for $\alpha \geq 1/2$ the only possible equilibrium points must satisfy $u_1 = u_2$ which implies that the only equilibrium points are (0,0), (1,1) and (-1,-1). Note that as the parameter α varies from zero to $1/2$ we observe a bifurcation at $\alpha = 1/3$ where four roots cease to exist and a bifurcation as $\alpha = 1/2$ where two roots cease to exist and there is no bifurcation for $\alpha > 1/2$. All the bifurcations are pitchfork bifurcations.

If $p \neq q$ the same bifurcation patern is observed when going from region O_1 to O_2 and finally to O_3, except that all the bifurcations now are saddle node bifurcations.

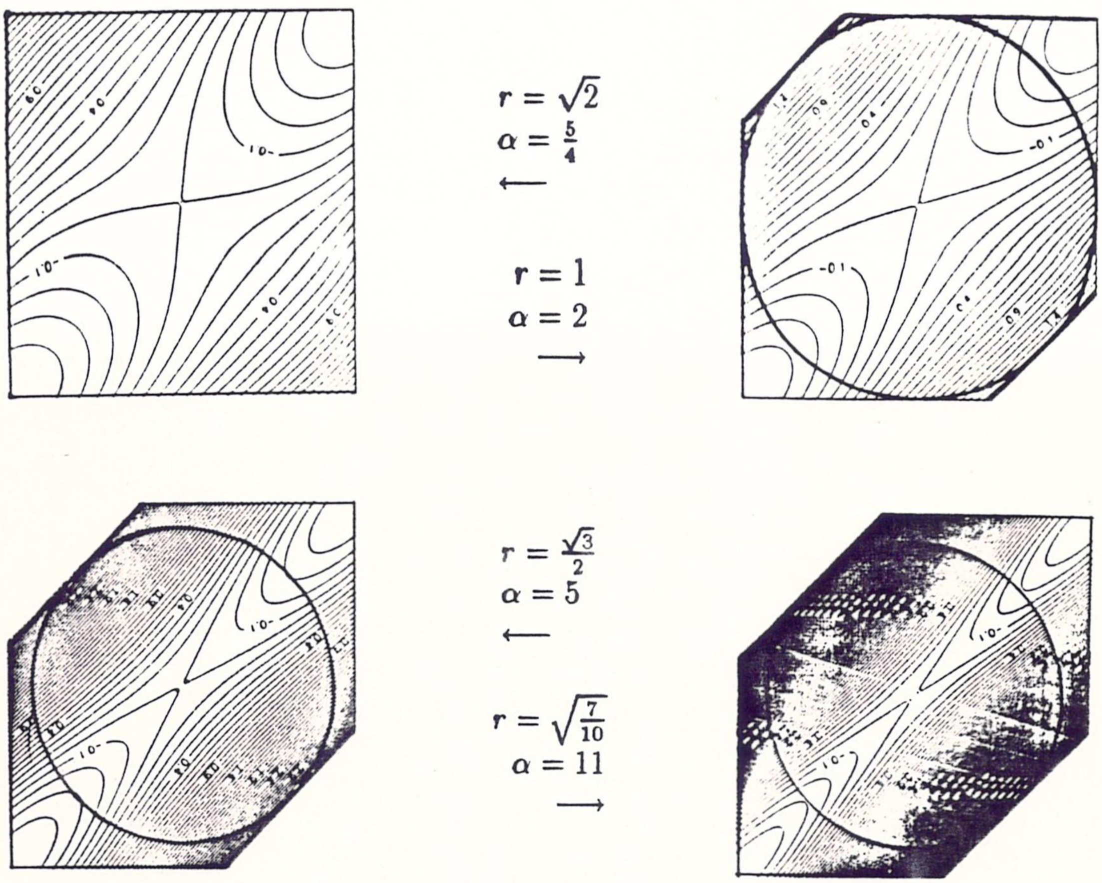


figure 4

Consider the problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \epsilon\Delta u + g(u) \quad x \in \Omega. \\ \frac{\partial u}{\partial n} &= 0 \quad x \in \Omega. \end{aligned} \tag{78}$$

Note that, for any α, there are many possible equilibrium points for (50). Among them, there are those that satisfy $u_1(x) = u_2(x)$ for all $x \in \Omega$. Thus, any such equilibrium point is such that u_1 is an equilibrium point for (78) with $\epsilon = d_1$ and u_2 is an equilibrim point for (78) with $\epsilon = d_2$. If we make $d_1 = d_2 = \epsilon$ and $n = 1$ we know that if ϵ is viewed as a parameter, there is a sequence of bifurcations of the equilibrium point $u = 0$ when the parameter approachs zero. These bifurcations are pichfork bifurcation and all the equilibrium points that bifurcate from zero are unstable (see, Chafee and Infante [1974]). Also, the attractor for (32) is embedded into the attractor for (50, the image invariant attractor for (32) is $[-1,1]$, and its embedding into the diffusion independent attractor for (50) with $d_1 = d_2 = \epsilon$ is the set $\{(s,s) : s \in [-1,1]\}$. Thus the dynamics for (50) with $d_1 = d_2 = \epsilon$ is at least as complicated as the dynamics for (32). We would like to be able to say that the dynamics in the attractor of (50) and the dynamics in the attractor of (32) coincide when $\epsilon = d_1 = d_2$ for $\alpha \geq \frac{1}{2}$. This could be accomplished by proving that (50) with $d_1 = d_2 = \epsilon$ has an image invariant attractor which coincides with $\{(s,s) : s \in [-1,1]\}$. So far, we have not been able to prove that, not even for the case $n = 1$.

Note that for $0 \leq \alpha \leq \frac{5}{4}$ we have that the diffusion independent attractor for (50), $d_1, d_2 \geq 0$ is contained in $[-1,1]^2$. If $\frac{5}{4} \geq \alpha$, the problem (50) has a diffusion independent attractor contained in the intersection of the set contained in the level curve of V that is tangent to the circle of radius $\sqrt{\frac{2\alpha-1}{3(\alpha-1)}}$ and the square $[-1,1]^2$. A few visualizations of these regions can be seen in figure 4.

5.2. Examples

5.2.1. Image Independent Attractors in Superconductivity of Liquids

Consider following problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= D\Delta u + (1 - |u|^2)f(u) \\ \frac{\partial u}{\partial n} &= 0, \end{aligned} \tag{79}$$

where $f := (f_1, \cdots, f_N)$ is a locally Lipschitz function satisfying and $f_i(u)u_i > 0$, $u_i \neq 0$, $1 \leq i \leq N$ and $D = dI$, $d > 0$. Define $V(u) = |u|^2 - 1$. Then, V is a convex function and it is easy to see that $V_\tau = \{v \in \mathbb{R}^N : V(v) < \tau\}$ is a contracting set for every $\tau > 0$. Furthermore, $\{V_\tau, \tau > 0\}$ is a complete family of contracting sets.

It follows from the previous results that $\mathcal{A}_0 = \{u \in \mathbb{R}^N : |u| \leq 1\}$ is a diffusion independent attractor for (79).

Note results obtained for this example can not be obtained by using the results in Carvalho and Ruas-Filho [1993] (not even the existence of a global attractor) since the above nonlinearity does not, necessarily, satisfy (5).

5.2.2. Image Independent Attractors in Lotka-Volterra Models for Two Competing Species

Consider the following Lotka-Volterra Model in $X^\alpha \times X^\alpha$ for two species competing for the same food source, with each species having logistic growth in the absence of the other and taking into account the diffusion.

$$\frac{\partial n_1}{\partial t} = d_1 \Delta n_1 + r_1 n_1 (1 - k_{11} n_1 - k_{12} n_2)$$

$$\frac{\partial n_2}{\partial t} = d_2 \Delta n_2 + r_2 n_2 (1 - k_{21} n_1 - k_{22} n_2) \tag{80}$$

$$\frac{\partial n_1}{\partial n} = \frac{\partial n_2}{\partial n} = 0$$

where n_1, n_2 represent the populations concentrations, d_1, d_2 are the diffusion coefficients, r_1, r_2 are the linear birth rates and k_{ij} are positive coefficients which are inversely proportional to the carrying capacity of species j, k_{ij} also takes into account competition effects between the species. These parameters are all positive and the variables u, v are nonnegative.

The ordinary differential equation bellow

$$\begin{aligned} \dot{u} &= u(1 - u - a\,v) = f_1(u,v) \\ \dot{v} &= \rho v(1 - v - b\,u) = f_2(u,v), \end{aligned} \tag{81}$$

is associated with the restriction of the above problem to the subspace of constant functions in the phase space $X^\alpha \times X^\alpha$ by the following change of variables $u = k_{11}n_1$, $v = k_{22}n_2$, by renaming the constants and reescaling time. If we also reescale the space variable, the problem (80) becomes

$$\frac{\partial u}{\partial t} = d_1 \Delta u + u\,(1 - u - a\,v)$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + \rho\,v\,(1 - v - b\,u) \tag{82}$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0$$

The equilibrium points associated to (81) are

$$\begin{aligned} &u = 0,\ v = 0, \quad u = 1,\ v = 0, \quad u = 0,\ v = 1, \\ &u = \tfrac{1-a}{1-a\,b},\ v = \tfrac{1-b}{1-a\,b} \end{aligned} \tag{83}$$

The last of these equilibrium points is only of relevance if each of the coordinates are nonnegative.

There are four distinct possibilities as a and b vary, figure 5 bellow shows the dynamics in each of these cases.

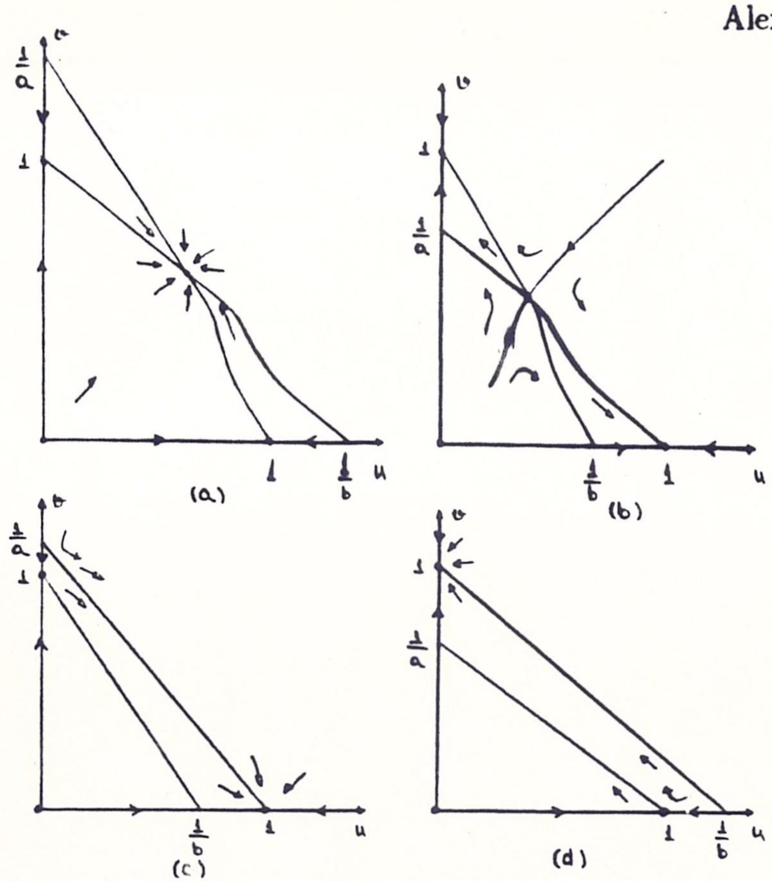


figure 5

Case (a) corresponds to $a < 1$ and $b < 1$ and in this case there is only one stable equilibrium point, namely

$$(\frac{1-a}{1-a\,b}, \frac{1-b}{1-a\,b}),$$

which represents coexistence of the two species. Case (b) corresponds to $a > 1$ and $b > 1$ and in this case there are two stable equilibrium points, $(1,0)$, $(0,1)$ which means that either species may survive depending on which region of attraction the initial data is (the equilibrium point that represents coexistence is unstable having a one dimensional stable manifold, which is a set of measure zero in the phase space). Case (c) corresponds to $a < 1$ and $b > 1$ and in this case thare is only one stable equilibrium point, namely (1.0) which means that only species 1 will survive. Finally, case (d) corresponds to $a > 1$ and $b < 1$ and in this case the only stable equilibrium point is $(0,1)$, meaning that only species 2 will survive. For a more detailed analysis of these cases see Murray [1989].

In what follows we construct a complete family of convex contracting set for the problem (80). First note that

$$f_i(u,v)\,u_i < 0,\ \forall |u_i| > 1.$$

Therefore, the problem (82) has a global attractor $\mathcal{A}$ with the following property

$$u(x) \in [0,1] \times [0,1],\ \forall u \in \mathcal{A}, \forall x \in \Omega.$$

In some particular cases we can obtain better information than the provided above. Consider the particular case of (a) when $\rho = 1$. Let

$$\begin{aligned} \Sigma_\tau^1 &= \left\{(u,v) \in (\mathbb{R}^+)^2 \ : v + b\,v < \tau + 1,\ \tau > c_1\right\} \\ \Sigma_\tau^2 &= \left\{(u,v) \in (\mathbb{R}^+)^2 \ : u + a\,v < \tau + 1,\ \tau > c_2\right\} \end{aligned} \tag{84}$$

where $c_1 = \frac{b(1-a)^2}{4-2a\,b-b-a\,b^2}$ and $c_2 = \frac{a(1-b)^2}{4-2a\,b-a-a^2\,b}$.

Next we prove that, for $i = 1,2$, the family $\{\Sigma_\tau^i : \tau > c_i\}$, is a complete family of convex contracting sets. It only remains to prove that $(f_1, f_2)\cdot\vec{n} < 0$ on $\partial\Sigma_\tau^i$ for $\tau > c_i$ ($\vec{n}$ is the outward normal to Σ_τ^i). In fact, on $\{(u,v) \in (\mathbb{R}^+)^2 : \tau + 1 - v - b\,u = 0\}$, we have that $\vec{n} = (b,1)$ and

$$(u(1-u-a\,v), v(1-v-b\,v))\cdot(b,1) \geq -(\tau+1)\left(\tau\left(1 - \frac{b(1-a)^2}{4(1-ab)}\right) + \frac{b(1-a)^2}{4(1-ab)}\right) < 0$$

if $\tau > c_1$. Similarly, on $\{(u,v) \in (\mathbb{R}^+)^2 : \tau + 1 - u - a\,v = 0\}$, we have that

$$(f_1(u,v), f_2(u,v))\cdot\vec{n} = (u(1-u-a\,v), v(1-v-b\,v))\cdot(1,a) < 0$$

if $\tau > c_2$.

It follows from the results in Sections 2 and 3 that the (80) has a global attractor $\mathcal{A}$ in X^α and that it satisfies

$$u(x) \in \Sigma_{c_1}^1 \cap \Sigma_{c_2}^2,\ \forall u \in \mathcal{A},\ \forall x \in \Omega. \tag{85}$$

This region is seen in figure 6.

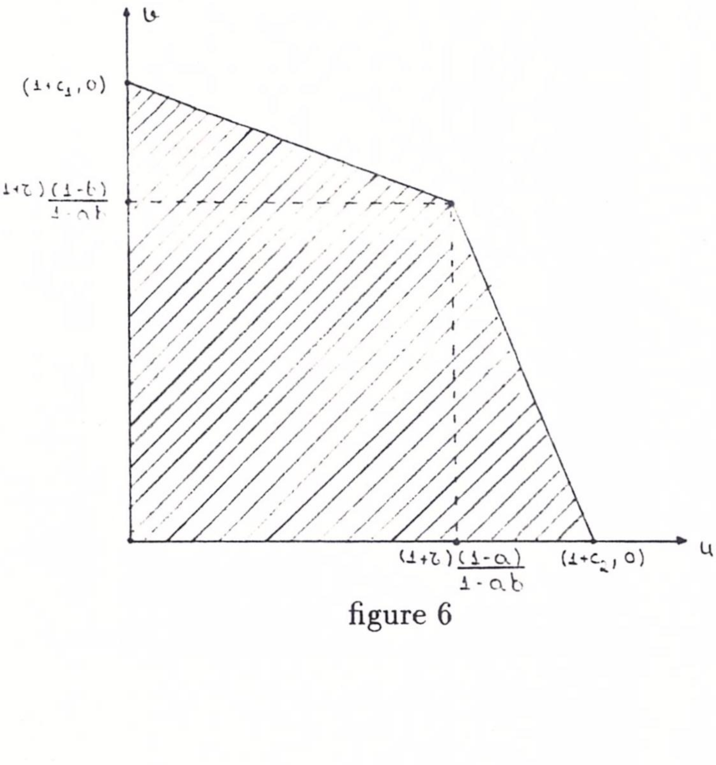


figure 6

5.2.3. Predation-Mediated Coexistence

We now consider a mathematical model for the coexistence of two competing species mediated by the presence of a predator. The model we consider is a reaction-diffusion equation with Lotka-Volterra interaction (see, for example, Ikeda and Mimura [1993]); that is,

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= d_1 \Delta u_1 + f_1(u_1, u_2, v) u_1 \\
\frac{\partial u_2}{\partial t} &= d_2 \Delta u_2 + f_2(u_1, u_2, v) u_2 \\
\frac{\partial v}{\partial t} &= D \Delta v + g(u_1, u_2, v) v \\
\frac{\partial u_1}{\partial n} &= \frac{\partial u_2}{\partial n} = \frac{\partial v}{\partial n} = 0.
\end{aligned} \tag{86}$$

with

$$\begin{aligned}
f_1(u_1, u_2, v) &= a_1 - b_1 u_1 - c_1 u_2 - k_1 v \\
f_2(u_1, u_2, v) &= a_2 - c_2 u_1 - b_2 u_2 - k_2 v \\
g(u_1, u_2, v) &= -r + \alpha_1 k_1 u_1 + \alpha_2 k_2 u_2,
\end{aligned} \tag{87}$$

where u_1, u_2 and v are respectively the spatial average densities of two competing species and their predator. These variables are, therefore, nonegative. The constants $a_i, b_i, c_i, k_i, \alpha_i, d_i$, $i = 1, 2$, D and r are all positive constants.

It is claimed in Ikeda and Mimura [1993], that if d_1, d_2 and D are large then the dynamics of (86) can be told by the dynamics of the ordinary differential equation

$$\begin{aligned}
\frac{du_1}{dt} &= f_1(u_1, u_2, v) u_1 \\
\frac{du_2}{dt} &= f_2(u_1, u_2, v) u_2 \\
\frac{dv}{dt} &= g(u_1, u_2, v) v.
\end{aligned} \tag{88}$$

This can be obtained using the results in Carvalho [1992], if we obtain an apriori estimate on the size of the attractor for (86), independently of the size of D, d_1, d_2. To do that we must first guarantee the existence of global attractors for (86) and then obtain such apriori estimates. This will be done by employing the results of this Section.

Once some apriori bounds are known, the results of Carvalho [1992] say that the asymptotic dynamics of (86) is the same as the dynamics of (88); that is, the attractors for (86) and (88) coincide.

Our first task is to prove that the problem (86) defines a local dynamical system in $Y := \{(\phi_1, \phi_2, \psi) \in X^\alpha : (\phi_1(x), \phi_2(x), \psi(x)) \in (\mathbb{R}^+)^3, \forall x \in \Omega\}$. This follows from Theorem 2 in Section 2. and from the fact that Y is embedded in $L^\infty(\Omega, (\mathbb{R}^+)^3)$.

Let's proceed to find a complete family of convex contracting sets for the problem (86).

Theorem 11 *Let $V : (\mathbb{R}^+)^3 \to \mathbb{R}$ be the function defined by $V(u_1, u_2, v) = \frac{u_1^2}{2} + \frac{u_2^2}{2} + \delta v$, where*

$$\delta = \frac{2r}{\alpha_1^2 k_1 + \alpha_2^2 k_2};$$

then, V is convex. Furthermore, if $V_\tau := \{(u_1, u_2, v) \in (\mathbb{R}^+)^3 : V(u_1, u_2, v) < \tau\}$, we have that $\{V_\tau, \tau \geq T\}$ is a complete family of contracting sets, where T is a large positive number.

Proof. It is clear that V is a convex function and we only have to prove that $\{V_\tau, \tau \geq T\}$ is a complete family of contracting sets.

Let

$$F(u_1, u_2, v) := \begin{pmatrix} (a_1 - b_1 u_1, -c_1 u_2 - k_1 v) u_1 \\ (a_2 - c_2 u_1, -b_2 u_2 - k_2 v) u_2 \\ (-r + \alpha_1 k_1 u_1 + \alpha_2 k_2 u_2) v \end{pmatrix}. \tag{89}$$

Thus, for $(u_1, u_2, v) \in (\mathbb{R}^+)^3$, we have that

$$\nabla V(u_1, u_2, v) \cdot F(u_1, u_2, v) \leq a_1 u_1^2 - b_1 u_1^3 + a_2 u_2^2 - b_2 u_2^3 + h(u_1. u_2) v, \tag{90}$$

where

$$h(u_1, u_2) := \alpha_1 k_1 \delta u_1 + \alpha_2 k_2 \delta u_2 - k_1 u_1^2 - k_2 u_2^2 - \delta r. \tag{91}$$

It is easy to see that h is bounded from above and that its maximum is attained at $(u_1, u_2) = (\frac{\alpha_1 \delta}{2}, \frac{\alpha_2 \delta}{2})$. Furthermore, its maximum value is $-\frac{r\delta}{2}$.

Thus, we have that

$$\nabla V(u_1, u_2, v) \cdot F(u_1, u_2, v) \leq a_1 u_1^2 - b_1 u_1^3 + a_2 u_2^2 - b_2 u_2^3 - \frac{\delta r}{2} v \tag{92}$$

Next, it is also easy to see that

$$a_1 u_1^2 - b_1 u_1^3 + a_2 u_2^2 - b_2 u_2^3 \leq -\frac{1}{3}\left(b_1 u_1^3 + b_2 u_2^3\right) + C, \tag{93}$$

where

$$C = \frac{1}{3}\left(\frac{a_1^3}{b_1^2} + \frac{a_2^3}{b_2^2}\right).$$

This implies that

$$\nabla V(u_1, u_2, v) \cdot F(u_1, u_2, v) \leq -\frac{1}{3}\left(b_1 u_1^3 + b_2 u_2^3\right) - \frac{\delta r}{2} v + C \tag{94}$$

and $\nabla V \cdot F < 0$ whenever $\max\{u_1, u_2, v\} > \max\{\left(\frac{3C}{b_1}\right)^{\frac{1}{3}}, \left(\frac{3C}{b_2}\right)^{\frac{1}{3}}, \frac{2C}{\delta r}\}$.

And the proof is completed by choosing T such that V_τ contains the ball of radius

$$R = 2\max\{\left(\frac{3C}{b_1}\right)^{\frac{1}{3}}, \left(\frac{3C}{b_2}\right)^{\frac{1}{3}}, \frac{2C}{\delta r}\}. \tag{95}$$

for any $\tau \geq T$.

The results in this section now imply that the problem (86) has a global attractor $\mathcal{A}$ in X^α which satisfy

$$u(x) \in \cap_{\tau \geq T} V_\tau, \quad \forall x \in \Omega, \quad \forall u \in \mathcal{A} \tag{96}$$

Conjecture 1 *Suppose (6) has global attractor $\mathcal{A}_0$. Assume that $\mathcal{A}_0$ is a convex set, then the hypotheses of Theorem (6) are satisfied and therefore, the problem (1) has a global attractor in X^α.*

Conjecture 2 *Suppose (6) has a global attractor $\mathcal{A}_0$ then the problem (1) has a global attractor in X^α.*

6. Further Comments

Let us now consider the following problem. Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be a C^1 vector field. Then the Phase Space for the ordinary differential equation

$$\dot{u} = f(u) \tag{97}$$

is $\mathbb{R}^N$ and we may consider the phase portrait of the flow.

Let $S_N = \{(x_1, \cdots, x_{N+1}) \in \mathbb{R}^{N+1} : x_1^2 + \cdots + x_N^2 + x_{N+1}^2 = 1\}$. And consider the map $T : \mathbb{R}^N \to S_N \backslash \{(0, \cdots, 0, 1)\}$ defined by

$$(Tu)_i = x_i = \frac{2u_i}{u_1^2 + \cdots + u_N^2 + 1}, \; 1 \leq i \leq \mathrm{N}, \; x_{N+1} = \frac{u_1^2 + \cdots + u_N^2 - 1}{u_1^2 + \cdots + u_N^2 + 1}. \tag{98}$$

Then, T is invertible and its inverse is given by

$$u_i = \frac{x_i}{1 - x_{N+1}}, \; 1 \leq i \leq \mathrm{N} + 1. \tag{99}$$

To the phase portrait of (97) we associate its image in $S_N \backslash \{(0, \cdots, 0, 1)\}$.

Next we consider the problem of defining a vector field in $S_N \backslash \{(0, \cdots, 0, 1)\}$ which reflects the vector field f. We do this in the following way. Consider $u(t, u_0)$ a solution of (97) with initial data $u_0 \in \mathbb{R}^N$. Then $f(u_0) = \frac{d}{dt} u(t, u_0)\big|_{t=0}$ and we can associate to $f(u_0)$ the vector $DT(u_0)f(u_0)$ in the tangent bundle of $S_N \backslash \{(0, \cdots, 0, 1)\}$, which corresponds to

$$DT(u_0)f(u_0) = \frac{d}{dt} T(u(t, u_0))\big|_{t=0} \tag{100}$$

Observe that if the $N+1^{th}$ coordinate of $DT(u_0)f(u_0)$ is negative in a neighborhood of the north pole minus the north pole. Then, there is an eventually convex function $V : \mathbb{R}^N \to \mathbb{R}$ satisfying the conditions of Theorem 6. Of course, it may happens that the conditions of Theorem 6 are satisfied and above condition is not. This condition can be writen as

$$(u_1, \cdots, u_N) \cdot f(u_1, \cdots, u_N) < 0 \tag{101}$$

outside a large ball around the origin.

It is easy to see that there are many examples for which this condition can be verified.

REFERENCES

R. Adams, Sobolev Spaces, Academic Press, [1971].

H. Brézis, Analyse Fonctionnelle, Masson , Paris, [1983].

A. N. Carvalho, Infinite Dimensional Dynamics Described by Ordinary Differential Equations, CDSNS - Report series # 97, Georgia Tech, [1992].

A. N. Carvalho and J. K. Hale, Large Diffusion with Dispersion, Nonlinear Analysis - Theory, Methods and Applications, 1991.

A. N. Carvalho and A. L. Pereira , A Scalar Parabolic Equation Whose Asymptotic Behavior is Dictated by a System of Ordinary Differential Equations, J. Diff. Equations (to appear), [1991].

A. N. Carvalho and J. G. Ruas-Filho, Global Attractors for Parabolic Problems in Fractional Power Spaces, ICMSC-USP Report Series # 128, [1993] (submited).

N. Chafee and E. F. Infante, A Bifurcation Problem for a Nonlinear Partial Differential Equation of Parabolic Type, Aplicable Analysis, Vol. 4, pp. 17-37, [1974].

K. Chueh, C. Conley and J. Smoller, Positively Invariant Regions for Systems of Nonlinear Diffusion Equations, Indiana Univ. Math. J., **26**, 373-392, [1977].

E. Conway, D. Hoff, and J. Smoller, Large time behavior of solutions of systems of nonlinear reaction-diffusion equations, SIAM J. Appl. Math., **35**, 1-16, [1978].

R. Courant and D. Hilbert , Methods of Mathematical Physics, **I**, John Wiley & Sons, [1989].

A. Friedman, Partial Differential Equations, Krieger, [1983].

G. Fusco, On the Explicit Construction of an ODE Which Has the Same Dynamics as a Scalar Parabolic PDE, Journal of Differential Equations, **69**, 85-110, [1987].

J. K. Hale, Asymptotic Behavior and Dynamics in Infinite Dimensions, Research Notes in Mathematics, Pitman, **132**,1-42, [1985].

J. K. Hale, Large Diffusivity and Asymptotic Behavior in Parabolic Systems, Journal of Mathematical Analysis and Applications, **118**, No. 2, 455-466, [1986].

J. K. Hale, Asymptotic Behavior of Dissipative Systems, Mathematical Surveys and Monographs **25**, AMS, [1988].

J. K.. Hale and G.Raugel, Attractors for Dissipative Evolutionary Equations, CDSNS - Report series # 72, Georgia Tech, [1992].

J. K. Hale and C. Rocha, Varying Boundary Conditions with Large Diffusivity, J. Mat. Pures et Appl., **66**, 139-158, [1987].

J. K. Hale and C. Rocha, Interaction of Diffusion and Boundary Conditions, Nonlinear Analysis, Theory, Methods and Applications, **11**, No. 5, 633-649, [1987a].

J. K. Hale and K. Sakamoto, Shadow Systems and Attractors in Reaction-Diffusion Equations, Applicable Analysis, **32**, 287-303, [1989].

D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lectures Notes in Mathematics, **840**, Springer-Verlag, [1981].

T. Ikeda, M. Mimura, An Interfacial Approach to Regional Segregation of Two Competing Species Mediated by a Predator, J. Math. Biol., **31**, pp. 215-240, [1993].

K. Kishimoto and H. F. Weinberger, The Spatial Homogeneity of Stable Equilibria of Some Reaction-Diffusion Systems in Convex Domains, Journal of Differential Equations, **58**, 15-21, [1985].

M. Marion, Finite-Dimensional Attractors Associated with Partly Dissipative Reaction-Diffusion Systems, SIAM J. Math. Anal., Vol. 20, No. 4, pp. 816-844, [1989].

J. Moser, A Sharp Form of an Inequality by N. Trudinger, Indiana Univ. Math. J., **20**, 1077-1092, [1971].

J. D. Murray, Mathematical Biology, Biomathematics Texts, Springer-Verlag, [1989].

A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, **44**, Springer-Verlag, [1983].

M. F. Protter and H. F. Weinberger, Maximum Principles in Differential Equations, Prentice-Hall, [1967].

J. Smoller, Shock Waves and Reaction-Diffusion Equations, Grundlehren Math. Wiss., **258**, Springer-Verlag, [1983].

R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Applied Mathematical Sciences **68**, Springer-Verlag, [1988].

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