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Injectivity of C^1 Maps $IR^2 \to IR^2$ at Infinity and Planar Vector Fields

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Injetividade, ao redor do infinito, de aplicações $\mathbb{R}^2 \to \mathbb{R}^2$, de classe C^1, e campos vetoriais planares

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Resumo

Dado $r > 0$, denotemos por $C(r) = \{p \in \mathbb{R}^2 : ||p|| > r\}$.

(1) Sejam $r > 0$ e $X : C(r) \to \mathbb{R}^2$ uma aplicação C^1 tais que, para algum $\epsilon > 0$ e para todo $p \in C(r)$, nenhum dos autovalores de DX(p) pertence a $(-\epsilon, \infty)$. Então, existe $s \geq r$, tal que $X_{|C(s)}$ é injetivo.

(2) Sejam $r > 0$ e $X : C(r) \to \mathbb{R}^2$ uma aplicação C^1 tais que, para algum $\epsilon > 0$ e para todo $p \in C(r)$, nenhum dos autovalores de DX(p) pertence a $(-\epsilon, 0] \cup \{z \in \mathbb{C} : \Re(z) \geq 0\}$. Então, existe $p_0 \in \mathbb{R}^2$ tal que o infinito é ou um atrator ou um repulsor de $x' = X(x) + p_0$.

Injectivity of C^1 maps $\mathbb{R}^2 \to \mathbb{R}^2$ at infinity and planar vector fields

Carlos Gutierrez and Alberto Sarmiento

Abstract

Let $X : \mathbb{R}^2 \setminus \overline{D}_r \to \mathbb{R}^2$ be a C^1 map, where $r > 0$ and $\overline{D}_r = \{p \in \mathbb{R}^2 : ||p|| \leq r\}$.
(i) If for some $\epsilon > 0$ and for all $p \in \mathbb{R}^2 \setminus \overline{D}_r$, no eigenvalue of $DX(p)$ belongs to $(-\epsilon, \infty)$, there exists $s \geq r$, such that $X|_{\mathbb{R}^2 \setminus \overline{D}_s}$ is injective;
(ii) If for some $\epsilon > 0$ and for all $p \in \mathbb{R}^2 \setminus \overline{D}_r$, no eigenvalue of $DX(p)$ belongs to $(-\epsilon, 0] \cup \{z \in \mathbb{C} : \Re(z) \geq 0\}$, there exists $p_0 \in \mathbb{R}^2$ such that the point ∞, of the Riemann sphere $\mathbb{R}^2 \cup \{\infty\}$,is either an attractor or a repellor of $x' = X(x) + p_0$.

1 Introduction

The study of planar vector fields around singularities has somehow motivated the present work. A sample of this study is the work done by C. Chicone, F. Dumortier, J. Sotomayor, R. Roussarie, F. Takens. See for instance [Chi], [DRS], [Rou], [Tak]. Here we study the behaviour of a vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$ around infinity. While a C^1 vector field around a singularity is quite regular, we work under conditions that do not imply, a priori, any regularity of the vector field around infinity. Our main result is the following

Theorem 1. *Let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_r \to \mathbb{R}^2$ be a C^1 map, where $r > 0$ and $\overline{D}_r = \{p \in \mathbb{R}^2 : ||p|| \leq r\}$. The following is satisfied:*

(i) if for some $\epsilon > 0$ and for all $p \in \mathbb{R}^2 \setminus \overline{D}_r$, no eigenvalue of $DX(p)$ belongs to $(-\epsilon, \infty)$, then there exists $s \geq r$, such that $X|_{\mathbb{R}^2 \setminus \overline{D}_s}$ is injective;

(ii) if for some $\epsilon > 0$ and for all $p \in \mathbb{R}^2 \setminus \overline{D}_r$, no eigenvalue of $DX(p)$ belongs to $(-\epsilon, 0] \cup \{z \in \mathbb{C} : \Re(z) \geq 0\}$, then, there exists $p_0 \in \mathbb{R}^2$ such that

the point ∞, of the Riemann sphere $\mathbb{R}^2 \cup \{\infty\}$, is either an attractor or a repellor of $x' = X(x) + p_0$.

To give an idea of the proof of this result, let us introduce the following definition.

Let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a C^1 map as in Theorem 1. Since $f : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}$ is a C^1 submersion, $q \in \mathbb{R}^2 \to \nabla f^\#(q) = (-f_y(q), f_x(q))$, the Hamiltonian vector field of f, has no singularities. Let $g_0(x, y) = xy$ and consider the set

$$B = \{(x, y) \in [0, 2] \times [0, 2] : 0 < x + y \leqslant 2\}.$$

We will say that $\mathcal{A} \subset \mathbb{R}^2$ is a *HRC* (*Half-Reeb Component*) of $\nabla f^\#$ (see figure 1) if there is a homeomorphism $h : B \to \mathcal{A}$ which is a topological equivalence between $\nabla f^\#|_{\mathcal{A}}$ and $\nabla {g_0}^\#|_B$, and such that

(1) $h(\{(x, y) \in B : x + y = 2\})$ (called *the compact edge of $\mathcal{A}$*) is a smooth segment transversal to $\nabla f^\#$ in the complement of $h(1, 1)$, and

(2) both $h(\{(x, y) \in B : x = 0\})$ and $h(\{(x, y) \in B : y = 0\})$ are full half-trajectories of $\nabla f^\#$.

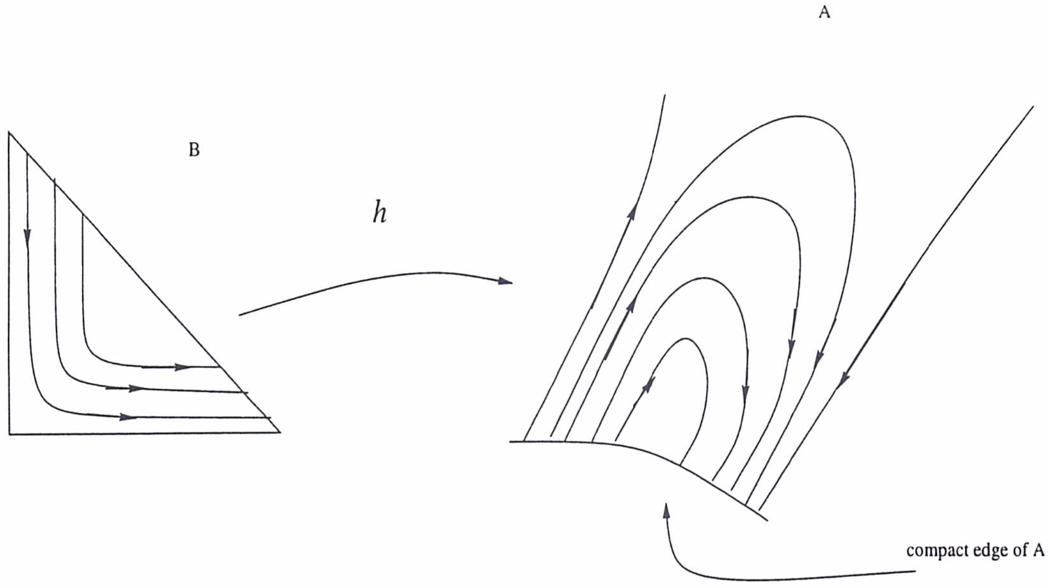

Figure 1: A half Reeb component

See figure 1.

Observe that $\mathcal{A}$ may not be a closed subset of $I\!R^2$.

We shall need the following lemma which is contained in the proof of [Gut, Lemma 2.5]. For $\theta \in I\!R$: let R_θ denote the linear rotation

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Lemma 1. *Let $X = (f, g) : I\!R^2 \setminus \overline{D}_r \to I\!R^2$ be a C^1 map as in Theorem 1. Suppose that $\nabla^{\#} f$ has an HRC which is unbounded (as a subset of $I\!R^2$) but whose projection on the x-axis is a compact interval. Then, there exists $\epsilon > 0$ such that, for all $\theta \in (-\epsilon, 0) \cup (0, \epsilon)$ $\nabla^{\#} f_\theta$ has a HRC whose projection on the x-axis is an interval of infinite length; here $(f_\theta, g_\theta) = R_\theta \circ X \circ R_{-\theta}$.*

The proof of Proposition 1 and Theorem 2 can be found in [CGL] but, as we have already said and for sake of completeness, they are included here.

Proposition 1. *Let $X = (f, g) : I\!R^2 \setminus \overline{D}_r \to I\!R^2$ be a C^1 map as in Theorem 1. Then any HRC of $\nabla^{\#} f$ is a bounded subset of $I\!R^2$.*

Proof: Let $\mathcal{A}$ be a half Reeb component for f. Let $\Pi : I\!R^2 \to I\!R$ be the projection on the first coordinate. By composing with a rotation if necessary, in the way that is stated in Lemma 1, we may suppose that $\Pi(\mathcal{A})$ is an interval of infinite length, say $[b, \infty)$. We may also assume that X is smooth and - By Thom's Transversality Theorem for jets [G-G]- that

(a1) the set
$$T = \{(x, y) \in I\!R^2 : f_y(x, y) = 0\}$$
is made up of regular curves;

(a2) There is a discret subset Δ of T such that if $p \in T \backslash \Delta$ (resp. $p \in \Delta$), $\nabla^{\#} f$ has quadratic contact (resp. cubic contact) with the vertical foliation of $I\!R^2$.

Then, if $a > b$ is large enough,

(b) for any $x \geq a$, the vertical line $\Pi^{-1}(x)$ intersects exactly one trajectory $\alpha_x \subset \mathcal{A}$ of $\nabla f^{\#}|_{\mathcal{A}}$ such that $\Pi(\alpha_x) \cap (x, \infty) = \emptyset$; in other words, x is the maximum for the restriction $\Pi|_{\alpha_x}$.

It follows that

(c) if $x \geq a$ and $p \in \alpha_x \cap \Pi^{-1}(x)$ then $p \in T \cap \mathcal{A} \setminus \Delta$.

Let T_m be the set of $p \in \mathcal{A}$ such that, for some $x \geq a$, $p \in \alpha_x \cap \Pi^{-1}(x)$. Notice that, for every $x \geq a$, $\alpha_x \cap \Pi^{-1}(x)$ is a finite set; nevertheless, by (b), (c) and by using Thom's Transversality Theorem for jets, we may get the following stronger statement:

(d) There is a sequence $F = \{a_1, a_2, \cdots, a_i, \cdots\}$ in $[a, \infty)$, which may be either empty or finite or else countable, such that if $x \in F$ (resp. $x \in [a, \infty) \setminus F$), then $\Pi^{-1}(x) \cap T_m$ is a two-point-set (resp. a one-point-set).

If $x \in [a, \infty) \setminus F$, define $\eta(x) = (x, \eta_2(x)) = \Pi^{-1}(x) \cap T_m$. Certainly $\eta : [a, \infty) \setminus F \to T_m$ is a smooth embedding and f is strictly monotone along η. Therefore, as $f|_{\mathcal{A}}$ is bounded,

$$\int_{a_1}^{\infty} \frac{d}{dx} f(\eta(x))\, dx = \sum_i \int_{a_i}^{a_{i+1}} \nabla f \cdot \dot{\eta} = \sum_i \int_{a_i}^{a_{i+1}} f_x(\eta(x))\, dx$$
$$\leq \lim_{i \to \infty} |f \circ \eta(a_{i+1}) - f \circ \eta(a_i)| < \infty,$$

which implies that, for some sequence $x_n \to \infty$, $\lim_{n \to \infty} f_x(\eta(x_n)) = 0$. This is the required contradiction. $\square$

3 Index of a vector field along a circle

Theorem 2. *Let $X : I\!R^2 \to I\!R^2$ be a C^1 map. Suppose that, for some $\epsilon > 0$,* $\mathrm{spec}(X) \cap (-\epsilon, \epsilon) = \emptyset$. *Then X is injective.*

Proof: By Proposition 1, the Hamiltonian vector fields induced by the coordinate functions of $X = (f, g)$ have no Reeb component. therefore X is injective. □

We shall say that a collar neighborhood U of an embedded circle $C \subset R^2 \backslash \overline{D}_r$ is *interior* (resp. *exterior*) if U is contained in $\overline{D}(C)$ (resp. $I\!R^2 \backslash D(C)$).

Proposition 2. *Let $C \subset R^2 \setminus \overline{D}_r$ be a smooth circle surrounding the origin. Suppose that $X(C)$ is an embedded circle and that there exists an exterior collar neighborhood $U \subset I\!R^2 \setminus \overline{D}_r$ of C such that $X(U)$ is also an exterior collar neighborhood of $X(C)$. Then X is an embedding.*

Proof: By the assumptions, X can be extended to a C^1 map $Y : R^2 \to \dot{R}^2$ which takes $\overline{D}(C)$ difeomorphically onto the $\overline{D}(X(C))$. See [Hir]. Under these conditions we may apply either Theorem 2 or Gutierrez and Fessler Injectivity Theorem [Gut], [Fes] to conclude that Y is an embedding and, a fortiori, that X is an embedding too. □

The theorem below on indexes of singularities of vector fields will be used to prove theorem 1. The proof can be found in [Har, Theorem 9.2]

Let C be a simple closed curve of R^2. A C^1 vector field $Y : R^2 \to R^2$ is said to be *internally (or externally) tangent to* C at $x_0 \in C$, if there exists an $\epsilon > 0$ such that the solution arc $\phi(t)$ of the equation

$$x' = Y(x), \qquad x(0) = x_0,$$

is interior (or exterior) to C for $0 < |t| \leq \epsilon$. We shall denote by $j_Y(C)$ the index of Y along C.

Theorem 3. *Let Y be a C^1 vector field on a connected open set $E \subset R^2$. Let C be a positive oriented Jordan curve of class C^1 in E with the property that $Y(x) \neq 0$ on C and that Y is tangent to C at only a finite number of points $x_1, \cdots, x_n$ of C. Let n^i (resp. n^e be the number of these points x_j where the solution arc $\phi(t)$ of $x' = Y(x)$, $\phi(0) = x_j$ for small $|t|$ is internally (resp. externally) tangent to C at x_j (so that $n^i + n^e \leq n$). Then $2j_Y(C) = 2 + n^i - n^e$.*

Corollary 1. *Let assume the notation and conditions of Theorem 1. In particular let* $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_r \to \mathbb{R}^2$ *be a* C^1 *map. If* $C \subset \mathbb{R}^2 \setminus \overline{D}_r$ *is a smooth circle surrounding the origin, then* $j_{\nabla f^\#}(C) = j_{\nabla f}(C) = 0$.

Proof: If $j_{\nabla f}(C) \neq 0$ there would exist a point $p \in C$ and a real $a > 0$ such that $\nabla f(p) = (a, 0)$. In particular a will be an eigenvalue of $DX(p)$. This contradition with the assumptions of Theorem 1 proves the corollary. □

4 Avoiding internal tangencies

We say that $\nabla f^\#$ is *in general position with* an embedded circle $C \subset \mathbb{R}^2 \setminus \overline{D}_r$ if there exists a subset F of C, at most finite such that (i) $\nabla f^\#$ is transversal to $C \setminus F$, (ii) $\nabla f^\#$ has a quadratic tangency with C at each point of F, and (iii) a trajectory of $\nabla f^\#$ can meet tangentially C at most at one point.

Lemma 2. *Suppose that* $\nabla f^\#$ *is in general position with a smooth circle* $C \subset \mathbb{R}^2 \setminus \overline{D}_r$ *which surrounds the origin. Suppose also that a trajectory* γ *of* $\nabla f^\#$ *meets* C *transversally somewhere and with an external tangency at a point* p. *Then the trajectory* γ *contains a closed subinterval* $[p, r]_f$ *which meets* C *exactly at* $\{p, r\}$ *(doing it transversally at* r*) and the following is satisfied:*

(i) If $[p, r]$ *denotes the closed subinterval of* C *such that* $\Gamma = [p, r] \cup [p, r]_f$ *bounds a compact disc* $\overline{D}(\Gamma)$ *contained in* $\mathbb{R}^2 \setminus D(C)$, *then points of* $\gamma \setminus [p, r]_f$ *nearby* p *do not belong to* $\overline{D}(\Gamma)$;

(ii) Let $(\tilde{p}, \tilde{r})$ *and* $[\tilde{p}, \tilde{r}]$ *be subintervals of* C *satisfying* $[p, r] \subset (\tilde{p}, \tilde{r}) \subset [\tilde{p}, \tilde{r}]$. *If* $\tilde{p}$ *and* $\tilde{r}$ *are close enough to* p *and* r, *respectively, then we may deform* C *into a smooth circle* C_1 *in such a way that the deformation fixes* $C \setminus (\tilde{p}, \tilde{r})$ *and takes* $[\tilde{p}, \tilde{r}] \subset C$ *to a closed interval* $[\tilde{p}, \tilde{r}]_1 \subset C_1$ *which is close to* $[p, r]_f$. *Furthermore,* $\nabla f^\#$ *is in general position with* C_1 *and the number of tangencies of* $\nabla f^\#$ *with* C_1 *is smaller than that of* $\nabla f^\#$ *with* C.

Proof:

Certainly, γ contains a closed subinterval $[p, r]_f \subset \gamma$ which meets C exactly at $\{p, r\}$. As $\nabla f^\#$ is in general position with C, γ meet C transversally at r. Let $[p, r]$ be the closed subinterval of C such that $\Gamma = [p, r] \cup [p, r]_f$ bounds a compact disc $\overline{D}(\Gamma)$ contained in $\mathbb{R}^2 \setminus D(C)$.

If $[p, r]_f$ does not satisfy (i) then the points of $\gamma \setminus [p, r]_f$ nearby p belong to $\overline{D}(\Gamma)$. Hence, as $\nabla f^\#$ has no singularities in $\mathbb{R}^2 \setminus D(C)$, points of $\gamma \setminus [p, r]_f$

nearby p belong to $\overline{D}(\Gamma)$. Therefore, there must be a closed subinterval $[q,p]_f \subset \gamma \cap \overline{D}(\Gamma)$ such that:

(a1) the union Γ_1 of the closed interval $[p,q] \subset [p,r]$ and $[p,q]_f$ bounds a compact disc $\overline{D}(\Gamma_1)$ contained in $(I\!R^2 \setminus D(C)) \cap \overline{D}(\Gamma)$;

(a2) $[q,p]_f$ meets C exactly at $\{p,q\}$, doing it transversally at q (with an external tangency at p); also, points of $\gamma \setminus [q,p]_f$ nearby p do not belong to $\overline{D}(\Gamma_1)$.

Summarizing either $[p,r]_f$ or $[q,p]_f$ satisfies (i). Therefore we may denote by $[p,r]_f$ the arc which satisfies (i).

We claim that (i) implies (ii). In fact, let us choose a small flow box B of $\nabla^{\#} f$ whose interior contains $[p,r]_f$. By the assumptions, we may suppose that $\tilde{p}, \tilde{r} \in B \cap C$. We may see that there exists a closed interval $[\tilde{p}, \tilde{r}]_T \subset B$ transversal to $\nabla f^{\#}$ and such that $C_1 = (C \setminus [\tilde{p}, \tilde{r}]) \cup [\tilde{p}, \tilde{r}]_T$ is a smooth circle, surrounding the origin, contained in $I\!R^2 \setminus D(C)$. Moreover, $\nabla f^{\#}$ meets C_1 with a smaller number of tangencies than it does it with C. The remainning conclusions of (ii) can easily be checked. Therefore (i) implies (ii). $\square$

Remark 1. *Let suppose that $\nabla f^{\#}$ is in general position with a smooth circle $C \subset I\!R^2 \setminus \overline{D}_r$ which surrounds the origin. Suppose also that $\nabla f^{\#}$ has an internal tangency with C at the point q; then, by observing the trajectories of $\nabla f^{\#}$ around q, we may see that there exist closed subintervals $[p,q]$ $[q,r]$ of C, with $[p,q] \cap [q,r] = \{q\}$, and a homeomorphism $T : [p,q] \to [q,r]$ such that,*

(a1) $Tp = r, Tq = q$ and, for every $x \in (p,q]$, there is an arc of trajectory $[x, Tx]_f \subset I\!R^2 \setminus D(C)$ of $\nabla f^{\#}$, starting at x, ending at Tx and meeting C exactly and transversally at $\{x, Tx\}$,

(a2) the family $[x,Tx]_f : x \in (p,q]$ depends continuously on x and tends to $\{q\}$ as $x \to q$.

Lemma 3. *Let suppose that $\nabla f^{\#}$ is in general position with a smooth circle $C \subset I\!R^2 \setminus \overline{D}_r$ which surrounds the origin. Suppose also that $\nabla f^{\#}$ has an internal tangency with C at the point q. Given any pair of subintervals $[p,q]$, $[q,r]$ of C (generated by q) as in Remark 1, the family $\{[x,Tx]_f : x \in (p,q]\}$ tends continuously to the compact arc of trajectory $[p,Tp]_f \subset I\!R^2 \setminus D(C)$ which either meets C exactly and transversally at $\{p, Tp\}$ or meets C with a quadratic external tangency; this second alternative happens if, and only if, $(p,q]$ is the maximal interval with properties (a1) and (a2) of Remark 1.*

Proof: If $(p, q]$ is not the maximal interval with properties (a1) and (a2) of Remark 1, then $[p, Tp]_f \subset I\!R^2 \setminus D(C)$ meets C exactly and transversally at $\{p, Tp\}$.

Otherwise, as $I\!R^2 \setminus \overline{D}(C)$ is not bounded, the closure of $(p, q] \cup [p, r)$ cannot be the whole circle. Therefore, there are two possibilities. The first one is that the positive (resp. negative) half-trajectory γ_p^+ (resp. γ_r^-) of $\nabla f^{\#}$ starting at p (resp. at r) does not meet C and so it must accumulate at the point ∞ of the Riemann sphere $I\!R^2 \cup \infty$. Under these circunstances, the subinterval $[p, q] \cup [q, r]$ is the compact edge of non-bounded *HRC* of $\nabla f^{\#}$ made up of $\gamma_p^+ \cup \gamma_r^-$ together with the union of the arcs $[x, Tx]_f$, with $x \in (p, q]$. This contradiction with Proposition 1 shows that the second possibility must happen: this lemma is true. $\square$

Lemma 4. *There exists a smooth circle $C \subset I\!R^2 \setminus \overline{D}_r$, surrounding the origin, in general position with $\nabla f^{\#}$, and such that*

(i) if a trajectory γ of $\nabla f^{\#}$ meets C, with an external tangency, say p, then $\gamma \cap C = \{p\}$;

(ii) As a consequence of (i), every tangency of the Hamiltonian vector field $\nabla f^{\#}$ with C is quadratic and external. In particular, there exists a correspondence between tangencies and HRCs (which- by Proposition 1- are contained in the disc of $I\!R^2$ bounded by C).

Proof:

By a small C^2 perturbation of f, we may assume that $\nabla f^{\#}$ is in general position with C; in particular, every tangency of $\nabla f^{\#}$ with C is quadratic. If (i) of this lemma is not satisfied, we may use Lema 2 to obtain a new circle C_1 such that $\nabla f^{\#}$ is in general position with C_1 and the number of tangencies of $\nabla f^{\#}$ with C_1 is smaller than that of $\nabla f^{\#}$ with C. Using this procedure, as many times as necessary, we will be able to obtain a circle as required to prove (i)

As (i) is true, (ii) follows from Lemma 3. $\square$

5 Main Proposition

This section is devoted to the proof of the following

Proposition 3. *There exists a smooth circle $C \subset I\!R^2 \setminus \overline{D}_r$, surrounding the origin, such that $X(C)$ is also an embedded circle and, for some exterior collar neighborhood $U \subset I\!R^2 \setminus \overline{D}_r$ of C, $X(U)$ is also an exterior collar neighborhood of $X(C)$.*

The proof of this proposition will be completed at the end of this section after some preparatory lemmas.

We say that a smooth circle $C \subset I\!R^2 \setminus \overline{D}_r$ is *of ETT (i.e. external tangency type) for* $\nabla f^{\#}$ if the following is satisfied: C surrounds the origin, there are two points $a, b \in C$, with $f(a) < f(b)$, and there are points $a_1, a_2, \cdots, a_n \in C_-$ and $b_1, b_2, \cdots, b_n \in C_+$, where C_- and C_+ are the connected components of $C \setminus \{a, b\}$, such that:

(a1) $\nabla f^{\#}$ is tangent to C exactly at a and b; also, these tangencies are quadratic and external;

(a2) $f(a) = \inf\{f(x) : x \in C\} < \sup\{f(x) : x \in C\} = f(b)$;

(a3) f takes diffeomorphically each C_i, with $i \in \{-, +\}$, onto the open interval $(f(a), f(b))$ (i.e., $X(C_i)$ is the graph of a map $(f(a), f(b)) \to I\!R$);

(a4) X restricted to $C \setminus \{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$ is an embedding, and also, $X(C_-)$ and $X(C_+)$ meet transversally to each other

(a5) $(X(a_1), X(a_2), \cdots, X(a_n)) = (X(b_1), X(b_2), \cdots, X(b_n))$ and $f(a) < f(a_1) = f(b_1) < f(a_2) = f(b_2) < \cdots < f(a_n) = f(b_n) < f(b)$.

(a6) there are sequences $x_n \to a$ and $y_n \to b$ of points x_n and y_n in $I\!R^2 \setminus \overline{D}(C)$ such that, for all n, $f(x_n) < f(a) < f(b) < f(y_n)$. This means that the local exterior of C around a (resp. around b) is taken to the unbounded connected component of $I\!R^2 \setminus X(C)$. In particular, $n \geq 0$ is an even number.

(a7) If $x \in I\!R^2 \setminus D(C)$ is close enough to $y \in C_+$ (resp. $y \in C_-$) and $f(x) = f(y)$, then $g(y) < g(x)$ (resp. $g(y) > g(x)$).

(a8) If $\bar{a}_1, \bar{a}_n \in C_-$ and $\bar{b}_1, \bar{b}_n \in C_+$ are close enough to a_1, a_n and b_1, b_n, respectively, and $[a_1, a_n] \subset (\bar{a}_1, \bar{a}_n)$, $[b_1, b_n] \subset (\bar{b}_1, \bar{b}_n)$ then, $X([\bar{a}_1, a_1) \cup (a_n, \bar{a}_n])$ is below $X([\bar{b}_1, b_1) \cup (b_n, \bar{b}_n])$ (i.e. if $a' \in [\bar{a}_1, a_1) \cup (a_n, \bar{a}_n]$ and $b' \in [\bar{b}_1, b_1) \cup (b_n, \bar{b}_n]$ are such that $f(a') = f(b')$ then $g(a') < g(b')$).

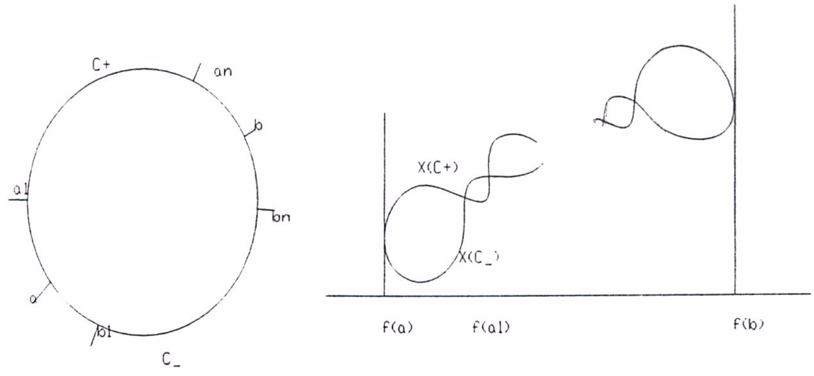

fig. 3

Lemma 5. *There is a smooth circle $C \subset I\!R^2 \setminus \overline{D}_r$ of ETT for $\nabla f^{\#}$.*

Proof:

In fact, by Lemma 4, we may take a smooth circle $C \subset I\!R^2 \setminus \overline{D}_r$, surrounding the origin, such that every tangency of the Hamiltonian vector field $\nabla f^{\#}$ with C is quadratic and external. Therefore, by the index formula of Theorem 3 and by Corollary 1, there are two points $a, b \in C$, with $f(a) < f(b)$, such that (a1) above is satisfied. This implies that (a2) and (a3) of definition above are also satisfied. Furthermore, by a small perturbation of X, we may assume that (a4) and (a5) of definition above are satisfied too. Item (a6) follows directly from the preceding properties. As $X(C)$ is tangent to the vertical foliation at the points $X(a)$ and $X(b)$, and by using (a6), The connected components C_- and C_+ can be named to satisfy (a7). Item (a7) implies (a8). $\square$

In the following of this section, C will be a smooth circle of ETT for $\nabla f^{\#}$ and we shall use all corresponding introduced notation.

Given $\alpha, \beta \in C_-$ (resp. $\alpha, \beta \in C_+$), $[\alpha, \beta], (\alpha, \beta), [\alpha, \beta)$ will denote subintervals of C_- (resp. of C_+) with endpoints α, β. Let $\mathcal{L}$ be the foliation of $I\!R^2$ by straight lines parallel to the line L which passes through the points $X(a_1)$ e $X(a_n)$. By a small perturbation of $X(C)$ with support in $X([a_1, a_n] \cup [b_1, b_n])$, we may assume that

(b) every point of tangency of $X([a_1, a_n])$ with $\mathcal{L}$ is quadratic, $X([a_1, a_n])$ and $X([b_1, b_n])$ are transversal to L.

Also, by taking $\bar{a}_1, \bar{a}_n \in C_-$ and $\bar{b}_1, \bar{b}_n \in C_+$ close to a_1, a_n and b_1, b_n, respectively, and $[a_1, a_n] \subset (\bar{a}_1, \bar{a}_n)$, $[b_1, b_n] \subset (\bar{b}_1, \bar{b}_n)$, we may suppose as well that

(c) $X([\bar{a}_1, a_1) \cup (a_n, \bar{a}_n])$ and $X([\bar{b}_1, b_1) \cup (b_n, \bar{b}_n])$ are disjoint of L.

Let $\theta \in I\!R$ be such that $R_\theta(\mathcal{L})$ is made up of vertical lines, where R_θ is the rigid rotation of angle θ. Recall that $(f_\theta, g_\theta) = X_\theta = R_\theta \circ X \circ R_\theta^{-1}$. By means of a small C^2-perturbation of f, we may assume that

(d) $\nabla f_\theta{}^{\#}$ is in general position with $R_\theta(C)$.

Then we have that

Lemma 6. *Remark 1 and Lemma 3 are also valid when referred to the vector field*

$$Y = (R_{-\theta})_* \nabla f_\theta{}^{\#} = R_{-\theta} \circ \nabla f_\theta{}^{\#} \circ R_\theta.$$

Also X takes any trajectory of Y into a subinterval of a leaf of $\mathcal{L}$.

Proof: If γ is a trajectory of Y then $R_\theta(\gamma)$ is a trajectory of $\nabla f_\theta{}^{\#}$. Therefore, $X_\theta \circ R_\theta(\gamma)$ is a subinterval of a vertical line and so $R_{-\theta} \circ X_\theta \circ R_\theta(\gamma)$ is a subinterval of a leaf of $\mathcal{L}$. However,

$$R_{-\theta} \circ X_\theta \circ R_\theta = X.$$

This implies that X takes any trajectory of Y into a subinterval of a leaf of $\mathcal{L}$. On the other hand, as $(f_\theta, g_\theta) = X_\theta = R_\theta \circ X \circ R_\theta^{-1}$ satisfies the assumptions of Theorem 1, Remark 1 and Lemma 3 are valid for the vector field $\nabla f_\theta{}^{\#}$. By definition of Y, we obtain the remainning conclusion of this lemma. $\square$

We claim that

Lemma 7. *If $X([\bar{a}_1, a_1) \cup (a_n, \bar{a}_n])$ is below L and $X([\bar{b}_1, b_1) \cup (b_n, \bar{b}_n])$ is above L, then there is a smooth circle $C_1 \subset I\!R^2 \setminus D(C)$, surrounding the origin, obtained from C by a deformation which fixes $C \setminus ((\bar{a}_1, \bar{a}_n) \cup (\bar{b}_1, \bar{b}_n))$ and takes $[\bar{a}_1, \bar{a}_n] \subset C$ and $[\bar{b}_1, \bar{b}_n] \subset C)$ to the closed sub-intervals $[\bar{a}_1, \bar{a}_n]_{C_1} \subset C_1$ and $[\bar{b}_1, \bar{b}_n]_{C_1} \subset C_1$, respectively, which satisfy $X([\bar{a}_1, \bar{a}_n]_{C_1})$ is below L and $X([\bar{b}_1, \bar{b}_n]_{C_1})$ is above L. In particular, C_1 is as requested to prove Proposition 3.*

Proof: Suppose that Y has an internal tangency with C at the point $q \in (a_1, a_n)$. By (d) and Lemma 6 we may proceed as in Remark 1 (applied to Y and considering the notation introduced there) to obtain sub-intervals $[p, q]$, $[q, r]$ of C (generated by q), determined by the condition that $(p, q]$ is the maximal subinterval of $[a_1, a_n]$ satisfying properties (a1) and (a2) of Remark 1. By Lemma 3, every element of the family $\{[x, Tx]_Y : x \in (p, q]\}$ is an arc of trajectory of Y. Notice that the maximality criterium for $(p.q]$ right above is different from that of Lemma 3.

To perform a sequence of adequate deformations, we meet three possible cases:

The first one is that $\{p, r\} \cap \{a_1, a_n\} \neq \emptyset$. Consider only the case in which $p = a_1$ and $r \neq a_n$. We may deform C into a new circle C_1 in such a way that: the deformation fixes $C \setminus (\bar{a}_1, \bar{a}_n)$ and takes $[\bar{a}_1, \bar{a}_n] \subset C$ to a closed sub-interval $[\bar{a}_1, \bar{a}_n]_{C_1} \subset C_1$ such that

(e) the cardinality of $L \cap [\bar{a}_1, \bar{a}_n]_{C_1}$ is less than that of (the finite set) $L \cap [\bar{a}_1, \bar{a}_n]$; and, concerning tangencies with $\mathcal{L}$, that are above L, $[\bar{a}_1, \bar{a}_n]_{C_1}$ has less ones than $[\bar{a}_1, \bar{a}_n]$.

In this deformation the arc $[p, Tp] \subset C$ has been taken to an interval whose image by X is below L. This deformation takes place inside a small neighborhood of $\{[x, Tx]_Y : x \in [p, q]\})$ and so C_1 surrounds the origin. Also as both $[\bar{a}_1, \bar{a}_n] \subset C$ and $\mathcal{L}$ are transversal to the vertical foliation, $[\bar{a}_1, \bar{a}_n]_{C_1} \subset C_1$ can be obtained to be transversal to the vertical foliation. We do not care if C_1 has more self-intersections than C.

The second case happens when $\{p, r\} \subset (a_1, a_n)$ and the arc of trajectory $[p, r]_Y$ of Y meets C according to the conditions (i) of Lemma 3 The arguments of such lemma imply that we may deform of C into a new circle C_1 according to the following conditions. The deformation fixes $C \setminus (\bar{a}_1, \bar{a}_n)$ and takes $[\bar{a}_1, \bar{a}_n] \subset C$ to a closed sub-interval $[\bar{a}_1, \bar{a}_n]_{C_1} \subset C_1$ such that

(f) $L \cap [\bar{a}_1, \bar{a}_n]_{C_1}$ has the same number of elements than $L \cap [\bar{a}_1, \bar{a}_n]$; and, concerning tangencies with $\mathcal{L}$, that are above L, $X([\bar{a}_1, \bar{a}_n]_{C_1}])$ has one less than $X([\bar{a}_1, \bar{a}_n])$.

As above, this deformation takes place inside a small neighborhood of $\cup\{[x, Tx]_Y : x \in [p, q]\})$ and so C_1 surrounds the origin. Also as $X([\bar{a}_1, \bar{a}_n])$ and $\mathcal{L}$ are transversal to the vertical foliation, $[\bar{a}_1, \bar{a}_n]_{C_1} \subset C_1$ can be obtained so that $X([\bar{a}_1, \bar{a}_n]_{C_1})$ is transversal to the vertical foliation. Again, as in case above, we do not care if C_1 has more self- intersections than C.

The third case occurs when $\{p, r\} \subset (a_1, a_n)$ and the arc of trajectory $[p, r]_Y$ of Y does not meet C according to the condition (i) of Lemma 3. We shall show now that this case is not possible. In fact, otherwise, this supposition and (d) imply that the open subinterval of trajectory (p, r) meets tangentially C exactly once, say at s.

Let $[p, s]$ and $[s, r]$ be the subintervals of C such that $[p, s]_Y \cup [p, s]$ and $[s, r]_Y \cup [s, r]$ bound discs $\bar{D}([p.s]_Y \cup [p, s])$ and $\bar{D}([s, r]_Y \cup [s, r])$ contained in $I\!R^2 \setminus D(C)$. Then, either $[s, r]_Y$ is contained in $\bar{D}([p.s]_Y \cup [p, s])$ or $[p, s]_Y$ is contained in $\bar{D}([s, r]_Y \cup [s, r])$. If $[s, r]_Y$ is contained in $\bar{D}([p.s]_Y \cup [p, s])$, then the circle $(C \setminus [p, s]) \cup [p, s]_Y$ can be approximated by a circle C_1 such that $X(C_1)$ has exactly two tangencies with the vertical foliation: $\{X(a), X(s)\}$. It follows from (a7) that X meets C_1 with an internal tangency at s. As X meets C_1 with an external tangency at a, we conclude, by Theorem 3 that $j_X(C_1) = 1$. This contradiction with Corollary 1 shows that $[s, r]_Y$ is not contained in $\bar{D}([p.s]_Y \cup [p, s])$. Similarly, $[p, s]_Y$ is not contained in $\bar{D}([s, r]_Y \cup [s, r])$. This contradiction implies that the third case is not possible.

As cases 1 and 2 above are the only possible ones, and thanks to (e)-(f), we only need to perform finitely many times the process (just described above) of obtaining new circles, of ETT for $\nabla f^{\#}$, in order to finally obtain a smooth circle, say C_2, such that $X([\bar{a}_1, \bar{a}_n]_{C_2})$ is below L. Similarly, by a deformation that fixes $C_2 \setminus [\bar{b}_1, \bar{b}_n]$ we shall finally obtain one circle as requested in this lemma. $\square$

Proof of Proposition 3

By (a8), $X([\bar{a}_1, a_1) \cup (a_n, \bar{a}_n])$ is below $X([\bar{b}_1, b_1) \cup (b_n, \bar{b}_n])$. It is easy to see that we may deform C, locally around $\{a_1, b_1, a_n, b_n\}$ so that the new circle of ETT for $\nabla f^{\#}$ satisfies the conditions of Lemma 7 and so it can be deformed into one as requested to prove this proposition. $\square$

As a direct consequence of this proposition and Proposition 2 we obtain:

Corollary 2. *Under the assumptions of Theorem 1, there exists an embedded circle $C \subset I\!R^2 \setminus \overline{D}_r$ such that X restricted to $I\!R^2 \setminus \overline{D}(C)$ can be extended to an orientation preserving embedding from $I\!R^2$ into $I\!R^2$.*

6 Proof of Theorem 1

Item (i) of Theorem 1 follows directly from Corollary 2.

The proof of (ii) of theorem 1 follows directly from Corollary 2 and from the following result of Gutierrez and Teixeira [G-T]

Theorem 4. *Let $r > 0$ and $X =: I\!R^2 \to I\!R^2$ be a C^1 vector field satisfying the following conditions: (i) X has a singularity, say S; (ii) for all $p \in I\!R^2 \setminus \overline{D}_r$, no eigenvalue of $DX(p)$ belongs to $\{z \in \mathbb{C} : \Re(z) \geq 0\}$; (iii) for all $p \in I\!R^2$,* $\mathrm{Det}(DX(p)) > 0$.

If $\int_{I\!R^2} \mathrm{Trace}(DX)$ is less than 0 (resp. greater of equal than 0), then the point ∞ of the Riemann sphere $I\!R^2 \cup \{\infty\}$ is a repellor (resp. an attractor) of X. In particular if X has no closed trajectories, then either the stable manifold, $W^s(S)$, of S is equal to $I\!R^2$ or the unstable manifold, $W^u(S)$, of S is equal to $I\!R^2$.

We should comment that there are vector fields of $I\!R^2$, as in Theorem 4, having either attracting or repelling behaviour at ∞ [G-T].

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