UNIVERSIDADE DE SÃO PAULO

Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2577

Projections of Hypersurfaces in IR^4 to Planes

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Nº 111

NOTAS

Série Matemática

São Carlos – SP
Nov./2000

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November 16, 2000

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Abstract

We study in this paper orthogonal projections of hypersurfaces in $\mathbb{R}^4$ to 2-spaces. We list all the generic local singularities of such projections and establish geometric criteria for recognition of the singularities of codimension ≤ 1 and for their versal unfoldings. We also determine a stratification of the set of singular planes of projections.

Resumo

Estudamos projeções ortogonais, de hipersuperfícies de $\mathbb{R}^4$, em planos. Listamos todas as singularidades locais de tais projeções e estabelecemos critérios geométricos para o reconhecimento das singularidades de codimensão ≤ 1 e para o reconhecimento de seus desdobramentos versais. Nós também determinamos uma estratificação do conjunto dos planos singulares das projeções.

Mathematical Subject Classifications: 58K05, 58K40.

1 Introduction

We study in this paper the $\mathcal{A}$-singularities of the orthogonal projections of a hypersurface M in $\mathbb{R}^4$ to a 2-dimensional plane. The singularities of such maps measure, for instance, the contact of M with 2-dimensional planes.

The contact of smooth varieties with degenerate objects (lines, planes, circles, spheres, etc.) is measured locally by the $\mathcal{K}$-singularities of some map-germs [19], although in practice the (right-left) $\mathcal{A}$-singularities are sought as they yield a finer classification and more geometrical information. This approach allowed the discovery of beautiful geometric results of surfaces in $\mathbb{R}^3$ (for example, the classification of the umbilic points and the structure of the focal set at such points, the flat geometry of the focal set, the ridge and subparabolic curves, the changes on these set, etc.; see [1] for references). These results are being extended to surfaces in $\mathbb{R}^4$ in [4, 7, 16, 17, 23].

The work in this paper is part of our investigation in [21] of the flat geometry of hypersurfaces in $\mathbb{R}^4$. In [22], we study the contact of M with hyperplanes and lines, and here we deal with the contact with planes.

There is a natural family of projections of M to planes, parametrised by the Grassmanian of 2-planes in $\mathbb{R}^4$, $G(2,4)$ (see [5]). If a and b are the generators of the plane of projection $u \in G(2,4)$ then the family of projections to 2-spaces is given by

$$\begin{array}{rrcl} \Pi: & M \times G(2,4) & \to & \mathbb{R}^2 \\ & (p,u) & \mapsto & (<p,a>,<p,b>) \end{array}$$

[5]. Given $u \in G(2,4)$ the map Π_u measures the contact between M and the plane orthogonal to u, the kernel of Π_u. (Note that Π_u is of corank 1.) Bruce and Nogueira showed in [5] that Π_u does not depend on the vectors a and b but only on the plane u, so in fact Π is a 4 parameter family. Therefore the generic $\mathcal{A}$-singularities of Π_u are those of $\mathcal{A}_e$-codimension ≤ 4. This fact is a direct consequence of Montaldi's Theorem in [20], and follows from the fact that the family Π defined on the ambient space is $\mathcal{A}$-versal. So the first task is to determine the normal forms of the singularities of corank 1 and codimension ≤ 4 of germs $\mathbb{R}^3,0 \to \mathbb{R}^2,0$. Those that are $\mathcal{A}$-equivalent to $(x, f(x,y) \pm z^2)$ are classified by Rieger and Ruas in [25]. We classify in section 2 those germs of $\mathcal{A}_e$-codimension ≤ 4 that are not covered in [25]. The main tools used are the complete transversal technique in [2, 6] and the software "Transversal" elaborated by Neil Kirk [13] (Transversal package available from Singularity Theory Page at www.liverpool.ac.uk/PureMaths).

In section 3, we give geometric criteria for recognition of the codimension ≤ 1 singularities of germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ and in section 4 criteria for determining when a deformation of a codimension 1 singularity is versal. Finally, we determine in section 5 a partition of the set of singular planes by the singularity type of Π_u at a fixed point on M.

Acknowledgements: This work is part of my Ph.D. thesis. I would like to thank FAPESP for financial support. I am also very grateful to Neil Kirk for letting me use his software "Transversal" and especially to Farid Tari for his supervision.

2 Classification results

As highlighted in the introduction, the generic singularities of Π are those of codimension ≤ 4. Before giving a complete list of such singularities we need some notation and preliminaries. (Although our interest is in germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ we set these notation in more general terms.)

Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a smooth map-germ and denote the Lie group of right-left equivalences by $\mathcal{A} = Diff(\mathbb{R}^n, 0) \times Diff(\mathbb{R}^p, 0)$ which acts on the space of smooth germs $\mathcal{E}(n,p)$ as follows: $(h,k).f = h \circ f \circ k^{-1}$, where $(h,k) \in \mathcal{A}$. We say that f is $\mathcal{A}$-equivalent to g if there is $(h,k) \in \mathcal{A}$ such that $g = h \circ f \circ k^{-1}$. Let $J^k(n,p)$ denote the space of kth order Taylor polynomials without constant terms, and write $j^k f$ for the k-jet of f. A germ f is said to be k-determined if, for any g such that $j^k f = j^k g$, it follows that f is equivalent to g.

Denote by $\mathcal{E}(n)$ and $\mathcal{E}(p)$ the local rings of function-germs in source and target whose respective maximal ideals are $m(n)$ and $m(p)$. Let θ_f denote the $\mathcal{E}(n)$-module of vector fields over f, and set $\theta_n = \theta_{(1_{\mathbb{R}^n})}$ and $\theta_p = \theta_{(1_{\mathbb{R}^p})}$ where $1_{\mathbb{R}^n}$ and $1_{\mathbb{R}^p}$ denote respectively the identity mapping of $\mathbb{R}^n$ and $\mathbb{R}^p$. One can define a $\mathcal{E}(n)$-homomorphism

$$\begin{aligned} tf : \theta_n &\to \theta_f \\ \phi &\mapsto df \circ \phi \end{aligned}$$

and a $\mathcal{E}(p)$-homomorphism

$$\begin{aligned} wf : \theta_p &\to \theta_f \\ \psi &\mapsto \psi \circ f. \end{aligned}$$

The tangent space to the $\mathcal{A}$-orbit at f is then defined to be $L\mathcal{A} \cdot f = tf(m(n).\theta_n) + wf(m(p).\theta_p)$. The $\mathcal{A}$-codimension of f is given by $cod(\mathcal{A}, f) = dim_{\mathbb{R}} m(n).\theta_f / L\mathcal{A} \cdot f$

and the $\mathcal{A}_e$-codimension by

$$\mathcal{A}_e\text{–}codim(f) = dim_{\mathbb{R}}(\mathcal{E}(n,p)/L\mathcal{A}_e \cdot f),$$

where

$$L\mathcal{A}_e \cdot f = tf(\theta_n) + wf(\theta_p).$$

The space $L\mathcal{A}_e \cdot f$ is called the extended tangent space.

The more recent advances in determinacy and classification theory are given in the articles by Bruce, Kirk, du Plessis and Wall [2, 6]. We state below tree results from these papers that can be used to list singularities of map germs $\mathbb{R}^n, 0 \to \mathbb{R}^p, 0$.

Theorem 2.1. *Let $\mathcal{G}$ be one of the Mather's group ($\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ or $\mathcal{K}$) and let $\mathcal{H}$ be a strongly Z-closed subgroup of $\mathcal{G}$. Then, for any k, $f \in \mathcal{E}(n,p)$ is $k - \mathcal{H}$-determined if and only if there is a strongly closed subgroup $\mathcal{U} \subset \mathcal{H}$ of $\mathcal{G}$, with $J^1\mathcal{U}$ unilpotent, such that*

$$m(n)^{k+1}.\theta_f \subset L\mathcal{U}.f.$$

Let $\mathcal{G}$ be a subgroup of one of the Mather's group and define $\mathcal{G}_s$ to be the subgroup of $\mathcal{G}$ whose elements have the s-jet equal to the identity.

Corollary 2.2. *[6] f is $k - \mathcal{G}_s$-determined ($\mathcal{G} = \mathcal{L}$ or $\mathcal{A}$, $s \geq 1$) if and only if*

$$m_n^{k+1}.\theta_f \subset L\mathcal{G}_s.f + m_n^{k+1}.f^*(m_p).\mathcal{E}_n + m_n^{2k+2}.\mathcal{E}(n,p).$$

Finite determinacy means that the classification problem can be reduced to the space of k-jets.

Let $H^{k+1}(n,p)$ be the space of germs of homogeneous polynomials of degree $k+1$ in $\mathcal{E}(n,p)$.

Proposition 2.3 (Complete Transversal to jets [2]). *Let $\mathcal{G}$ be a Mather's group and denote the tangent space to the $J^{k+1}\mathcal{G}_1$-orbit of $j^{k+1}f$ by $L(J^{k+1}\mathcal{G}_1).j^{k+1}f$. Then given $f \in m(n).\mathcal{E}(n,p)$ and $T \subset H^{k+1}(n,p)$ such that*

$$H^{k+1}(n,p) \subset L(J^{k+1}\mathcal{G}_1).j^{k+1}f + T$$

any $(k+1)$-jet $j^{k+1}g$, with $j^kg = j^kf$, is in the same $\mathcal{G}_1$-orbit of $j^{k+1}f + t$, for some $t \in T$.

The techniques developed in [2, 6] provide a very efficient, wide-ranging classification scheme involving algebraic calculations which may be reduced to finite dimensional symbolic problems. However, the calculations can become very intensive and repetitive. In [13], Kirk produced a package, called "Transversal", which consists of a collection of procedures which run under the symbolic algebra system Maple and deals with problems in classification and unfolding theory for these equivalence relations. This is a very powerful tool, and according to Kirk [13] the results of Mond [18] and Rieger [24] were all reproduced in a matter of hours using "Transversal".

Recent applications of "Transversal" [3, 8, 9, 10, 11, 12, 28] represent some of the most extensive classifications carried out to date. In this paper we use "Transversal" to classify germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$.

Germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ of corank 1 that can be written in the form $(x, f(x, y) \pm z^2)$ in some coordinate system are classified by Rieger and Ruas in [25] and their list is given below.

Theorem 2.4. *[25] The germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ of corank 1 and $\mathcal{A}_e$-codim ≤ 4, that are equivalent to one of the form $(x, f(x, y) \pm z^2)$ are given in Table 1.*

Table 1:

Name	*Normal form*	$\mathcal{A}_e$-*cod*	*Simple (y/n)*
1	(x, y)	0	y
2 *(fold)*	$(x, y^2 \pm z^2)$	0	y
3 *(cusp)*	$(x, xy + y^3 \pm z^2)$	0	y
4_k *($k = 2$, lips/beaks)*	$(x, y^3 + (\pm 1)^{k-1} x^k y \pm z^2)$, $2 \leq k \leq 5$	$k-1$	y
5 *(swallowtail)*	$(x, xy + y^4 \pm z^2)$	1	y
6	$(x, xy + y^5 \pm y^7 \pm z^2)$	2	y
7	$(x, xy + y^5 \pm z^2)$	3	y
8	$(x, xy + y^6 \pm y^8 + ay^9 \pm z^2)$	4	n
11_{2k+1}	$(x, xy^2 + y^4 + y^{2k+1} \pm z^2)$, $2 \leq k \leq 4$	k	y
12	$(x, xy^2 + y^5 + y^6 \pm z^2)$	3	y
13	$(x, xy^2 + y^5 \pm y^9 \pm z^2)$	4	y
16	$(x, x^2y + y^4 \pm y^5 \pm z^2)$	3	y
17	$(x, x^2y + y^4 \pm z^2)$	4	y

We complete in this section the list in Theorem 2.4 and obtain the remaining germs of codimension ≤ 4. These germs are those that can not be written in the form $(x, f(x,y) \pm z^2)$.

Theorem 2.5. *The $\mathcal{A}$-classes of the singularities $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ of corank 1 and $\mathcal{A}_e$-codimension (or codimension of the stratum in the presence of moduli) ≤ 4 that are not listed in Theorem 2.4 are given in Table 2.*

Table 2:

Name	*Normal form*	$\mathcal{A}_e$-*cod*	δ
N_1	$(x, xy + y^3 + ay^2z + z^3 \pm z^5)$ $a \neq 0$, $4a^3 \pm 27 \neq 0$	3 (2†)	5
N_2	$(x, xy + y^3 + ay^2z + z^3)$, $a \neq 0$, $4a^3 + 27 \neq 0$	4 (3†)	5
N_3	$(x, xy + y^3 + z^3 \pm y^3z)$	3	4
N_4	$(x, xy \pm y^2z + z^3 \pm z^5)$	3	5
N_5	$(x, xy \pm y^3 + yz^2 + z^5)$	3	5
N_6	$(x, xy \pm y^2z + z^3)$	4	5
N_7	$(x, xy + y^3 + z^3 \pm y^4z)$	4	5
N_8	$(x, xy + z^3 \pm y^4 + y^3z + ay^4z + b(y^6 + \lambda y^5z))$, $b \neq 0$	6 (4†)	6
N_9	$(x, xy + yz^2 \pm y^4 + z^5 + ay^6)$	5 (4†)	6
N_{10}	$(x, xy + y^2z + yz^3 \pm z^4 + az^6)$	5 (4†)	6
N_{11}	$(x, xy \pm y^3 + yz^2 + z^7)$	4	7
N_{12}	$(x, xyz \pm y^2z + z^3 + ax^2y + bx^2z + cyz^3 + z^4)$	7* (4†)	4

where λ is a constant, and δ denotes the determinacy degree of the germs.

† codimension of the estratum

$*$ $4b - 1 \neq 0$ and $4b - 1 + 6ac \neq 0$.

Proof: We are interested in germs of corank 1, so we can write $F(x, y, z) = (x, g(x, y, z))$. Let $j^2g = x\phi(x, y, z) + \psi(y, z)$. Then the germs that can not be written in the form $(x, f(x, y) \pm z^2)$ are those with $\psi \equiv 0$. Therefore to complete the list in Theorem 2.4, we need to consider the cases where $j^2g = x\phi(x, y, z)$. It is not difficult to show that we can change coordinates so that j^2F is given by (x, xy) or $(x, 0)$. We now follow these germs and carry out the classification inductively on the jet level, using the complete transversal method [2] and the "Transversal" package [13]. We start by considering the 2-jet (x, xy).

(1) $j^2F = (x, xy)$.

We have $F_x = (1, y)$, $F_y = (0, x)$, $F_z = (0, 0)$. Therefore,

$$H^3(3, 2) \subset T(J^3\mathcal{A}_1).j^3F + \mathbb{R}.\{y^2z, y^3, yz^2, z^3\}.$$

and by Proposition 2.3 any 3-jet with a 2-jet (x, xy) is equivalent to $(x, xy + ay^2z + by^3 + cyz^2 + dz^3)$ for some $a, b, c, d \in \mathbb{R}$. We can now make linear changes of coordinates and obtain the following orbits in $J^3(3,2)$:

$$\begin{aligned}
&(x, xy + y^3 + ay^2z + z^3),\\
&(x, xy + y^3 + z^3),\\
&(x, xy \pm y^2z + z^3),\\
&(x, xy + z^3),\\
&(x, xy \pm y^3 + yz^2),\\
&(x, xy + yz^2),\\
&(x, xy + y^2z),\\
&(x, xy + y^3),\\
&(x, xy),
\end{aligned}$$

where a in the first orbit is a smooth modulus. Using "Transversal" we show that the last 2 cases lead to germs of $\mathcal{A}_e$- codimension ≥ 5 and therefore will not be followed.

- The 3-jet $(x, xy + y^3 + ay^2z + z^3)$:

  Using "Transversal" we show that the 4-transversal is empty and the 5-transversal is given by $\mathbb{R}.\{z^5\}$. The orbits in $J^5(3,2)$ are $(x, xy+y^3+ay^2z+z^3 \pm z^5)$ if $4a^3 \pm 27 \neq 0$ and $a \neq 0$, and $(x, xy + y^3 + ay^2z + z^3)$ if $4a^3 + 27 \neq 0$ and $a \neq 0$. Both germs are 5-determined, the first (N_1) has codimension 3 (the codimension of the stratum is 2) and the second (N_2) is of codimension 4 (the codimension of the stratum is 3). The exceptional case $4a^3 \pm 27 = 0$ will not be followed as the current version of "Transversal" does deal with constant such as $(27/4)^{2/3}$. We shall only consider the case $a = 0$.

- The 3-jet $(x, xy + y^3 + z^3)$:

  A 4-transversal is given by $\mathbb{R}.\{y^3z\}$. The orbits in $J^4(3,2)$ are $(x, xy+y^3+z^3 \pm y^3z)$ and $(x, xy + y^3 + z^3)$. The first case (N_3) is 5-determined and has codimension 3. For the second case, a 5-transversal is given by $\mathbb{R}.\{y^4z\}$ so in $J^5(3,2)$ we have two orbits, namely $(x, xy + y^3 + z^3 \pm y^4z)$ and $(x, xy + y^3 + z^3)$. The form N_7 is 6-determined and has codimension 4. The later yields germs of codimension ≥ 5.

- The 3 jet $(x, xy \pm y^2z + z^3)$:

  The 4-transversal is empty and a 5-transversal is given by $\mathbb{R}.\{z^5\}$. The orbits in $J^5(3,2)$ are $(x, xy \pm y^2z + z^3 \pm z^5)$ and $(x, xy \pm y^2z + z^3)$. Both germs are 5-determined with the first (N_4) having codimension 3 and the second (N_6) is of codimension 4.

- The 3 jet $(x, xy + z^3)$:

A 4-transversal is given by $\mathbb{R}.\{y^4, y^3z\}$ so the orbits in $J^4(3,2)$ are $(x, xy + z^3 + y^3z \pm y^4)$ and $(x, xy + z^3 + y^3z)$. A 5-transversal of the first germ is given by $\mathbb{R}.\{y^4z\}$ so any germ in $J^5(3,2)$ with a 4-jet $(x, xy + z^3 + y^3z \pm y^4)$ is equivalent to $(x, xy + z^3 + y^3z \pm y^4 + ay^4z)$ for some $a \in \mathbb{R}$. Here the parameter a is a smooth modulus. The orbits in $J^6(3,2)$ are given by $(x, xy + z^3 \pm y^4 + y^3z + ay^4z + b(y^6 + \lambda y^5z))$ where b is also a smooth modulus and λ is a constant. These germs (N_8) are 6-determined and have codimension 6 (the codimension of the stratum is 4) provided $b \neq 0$.

The orbit in $J^5(3,2)$ whose 4-jet is equal to $(x, xy+z^3+y^3z)$ are $(x, xy+z^3+y^3z\pm y^5)$ and $(x, xy + z^3 + y^3z)$. These lead to germs of codimension greater or equal to 5.

- The 3 jet $(x, xy \pm y^3 + yz^2)$:

The 4-transversal is empty and a 5-transversal is given by $\mathbb{R}.\{z^5\}$, so the orbits in $J^5(3,2)$ are $(x, xy \pm y^3 + yz^2 + z^5)$ and $(x, xy \pm y^3 + yz^2)$. The first germ (N_5) is 5-determined and has codimension 3. The second germ has an empty 6-transversal but a 7-transversal is given by $\mathbb{R}.\{z^7\}$. In $J^7(3,2)$, the orbit (N_{11}) $(x, xy \pm y^3 + yz^2 + z^7)$ is 7-determined and has codimension 4. The remaining orbit leads to germs of higher codimensions.

- The 3-jet $(x, xy + yz^2)$:

A 4-transversal is given by $\mathbb{R}.\{y^4\}$ and a 5-transversal of the orbit $(x, xy+yz^2\pm y^4)$ is given by $\mathbb{R}.\{z^5\}$. The orbits in $J^5(3,2)$ are $(x, xy+yz^2\pm y^4+z^5)$ and $(x, xy+yz^2\pm y^4)$. This second orbit generates germs of codimension ≥ 5. A 6-transversal of the first germ is given by $\mathbb{R}.\{y^6\}$, therefore the germs in $J^5(3,2)$ with a 5-jet as above are equivalent to $(x, xy + yz^2 \pm y^4 + z^5 + ay^6)$, for some $a \in \mathbb{R}$. In this case a is a smooth modulus. These germs (N_9) are 6-determined and have codimension 5 (the codimension of the stratum equal to 4).

- The 3-jet $(x, xy + y^2z)$:

A 4-transversal is given by $\mathbb{R}.\{z^4, yz^3\}$ and the orbits in $J^4(3,2)$ are $(x, xy + y^2z + yz^3\pm z^4)$ and $(x, xy+y^2z\pm z^4)$. The second orbit generates germs of higher codimension. A 5-transversal of the first orbit is empty and a 6-transversal is given by $\mathbb{R}.\{z^6\}$. Therefore any germ in $J^6(3,2)$ with a 5-jet as above is equivalent to $(x, xy + y^2z + yz^3 \pm z^4 + az^6)$ for some $a \in \mathbb{R}$. Here a is a smooth modulus. These germs (N_{10}) are 6-determined and have codimension 5 (the codimension of the stratum is equal to 4).

(2) $j^2F = (x, 0)$.

A 4-transversal is given by $\mathbb{R}.\{xyz, z^3, y^2z, x^2y, x^2z\}$. There are various cases to consider here but the least degenerate case (we are seeking strata of codimension ≤ 4) is $(x, xyz \pm y^2z + z^3 + ax^2y + bx^2z)$. Then a 5-transversal is given by $\mathbb{R}.\{yz^3, z^4\}$. The least degenerate case is given by $(x, xyz \pm y^2z + z^3 + ax^2y + bx^2z + cyz^3 + z^4)$ (N_{12}) which is 4-determined and has codimension 7 (the codimension of the stratum is 4), provided $4b - 1 \neq 0$ and $4b - 1 + 6ac \neq 0$. Any other case is part of (or leads to) a stratum with $\mathcal{A}_e$-codimension greater or equal to 5. ∎

We also use "Transversal" to obtain normal forms for the versal unfoldings of the germs in Table 2.

Proposition 2.6. *Versal unfoldings of the germs listed in Table 2 are given in Table 3.*

Table 3:

Name	Versal unfolding
N_1	$(x, xy + y^3 + ay^2z + z^3 \pm z^5 + u_1z + u_2y^2)$
N_2	$(x, xy + y^3 + ay^2z + z^3 + u_1z + u_2y^2 + u_3y^5)$,
N_3	$(x, xy + y^3 + z^3 \pm y^3z + u_1z + u_2yz + u_3y^2z)$
N_4	$(x, xy \pm y^2z + z^3 \pm z^5 + u_1z + u_2y^2 + u_3y^3)$
N_5	$(x, xy \pm y^3 + yz^2 + z^5 + u_1z + u_2y^2 + u_3z^3)$
N_6	$(x, xy \pm y^2z + z^3 + u_1z + u_2y^2 + u_3y^3 + u_4y^4z)$
N_7	$(x, xy + y^3 + z^3 \pm y^4z + u_1z + u_2yz + u_3y^2z + u_4y^3z)$
N_8	$(x, xy + z^3 \pm y^4 + y^3z + ay^4z + by^6 + cy^5z + u_1z + u_2yz + u_3y^2 + u_4y^3)$
N_9	$(x, xy + yz^2 \pm y^4 + z^5 + ay^6 + u_1z + u_2y^2 + u_3y^3 + u_4z^3)$
N_{10}	$(x, xy + y^2z + yz^3 \pm z^4 + az^6 + u_1z + u_2z^2 + u_3yz^2 + u_4z^3)$
N_{11}	$(x, xy \pm y^3 + yz^2 + z^7 + u_1z + u_2y^2 + u_3y^3 + u_4y^5)$
N_{12}	$(x, xyz \pm y^2z + z^3 + ax^2y + bx^2z + cyz^3 + z^4 + u_1y + u_2z + u_3xy + u_4y^2)$

3 Geometric criteria for recognition of the singularities of $\mathcal{A}_e$-codim ≤ 1

Geometric criteria for recognition of the singularities of $\mathcal{A}_e$-codim ≤ 1 of germs $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ are given in [14] and [27]. We adapt these criteria to germs $F : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$. When F has corank 1, we can change coordinates and write $F(x, y, z) = (x, f(x, y, z))$.

The differential of F at (x, y, z) is then given by

$$DF(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial f}{\partial x}(x,y,z) & \frac{\partial f}{\partial y}(x,y,z) & \frac{\partial f}{\partial z}(x,y,z) \end{pmatrix}$$

so that the critical set of F is given by

$$\Sigma = \{(x,y,z) : \frac{\partial f}{\partial y}(x,y,z) = \frac{\partial f}{\partial z}(x,y,z) = 0\}.$$

Let $G : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ such that $G(x,y,z) = (f_y(x,y,z), f_z(x,y,z))$. We say that $\Sigma = G^{-1}(0)$ is smooth if G is regular, that is, G has maximal rank at $(0,0,0)$.

With this definition it is easy to show that if Σ is smooth, then F can be written in some coordinate systems in the form $(x, f(x,y) \pm z^2)$. In this case $\Sigma = \{(x,y,0); f_y(x,y) = 0\}$ is a smooth curve. Let $\phi : I, 0 \to \mathbb{R}^3, 0$ be a parametrization of Σ, where I is a neighbourhood of 0 in $\mathbb{R}$. The order of contact of Σ with the kernel of $DF(0,0,0)$ $(ker(DF(0,0,0)))$ is the order of the vanishing of the derivatives of $F \circ \phi$ at 0. One can show that this order of contact is independent of the parametrization of Σ and is an $\mathcal{A}$-invariant of the map F.

We shall use this order of contact to recognize geometrically the fold, cusp and swallowtail singularities in Table 1 of Theorem 2.4. When the critical set is smooth, we can set $F(x,y,z) = (x, f(x,y) \pm z^2)$ as the order of contact is invariant.

Proposition 3.1. *Let $F : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ a singular germ with a smooth critical set Σ. Then*

(i) F is a fold if and only if $\frac{\partial}{\partial t}(F \circ \phi)(0) \neq 0$.

(ii) F is a cusp if and only if $\frac{\partial}{\partial t}(F \circ \phi)(0) = 0$ and $\frac{\partial^2}{\partial t^2}(F \circ \phi)(0) \neq 0$.

Proof: Let $F(x,y,z) = (x, f(x,y) \pm z^2)$ and ϕ a local parametrization of $\Sigma = \{(x,y,0); f_y(x,y) = 0\}$. So $\phi(t) = (\phi_1(t), 0)$ with $\phi_1(0) = 0$ and $\frac{\partial}{\partial t}\phi(t) = (-\frac{\partial^2 f}{\partial y^2}(\phi_1(t)), \frac{\partial^2 f}{\partial x \partial y}(\phi_1(t)), 0)$.

Then

$$\begin{aligned}
\frac{\partial}{\partial t}(F \circ \phi)(t) &= DF(\phi(t)).\frac{\partial}{\partial t}\phi(t) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial f}{\partial x}(\phi_1(t)) & \frac{\partial f}{\partial y}(\phi_1(t)) & 0 \end{pmatrix} \begin{pmatrix} -\frac{\partial^2 f}{\partial y^2}(\phi_1(t)) \\ \frac{\partial^2 f}{\partial x \partial y}(\phi_1(t)) \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\partial^2 f}{\partial y^2}(\phi_1(t)) \\ -\frac{\partial f}{\partial x}(\phi_1(t)).\frac{\partial^2 f}{\partial y^2}(\phi_1(t)) + \frac{\partial f}{\partial y}(\phi_1(t)).\frac{\partial^2 f}{\partial x \partial y}(\phi_1(t)) \end{pmatrix} \\
&= -\frac{\partial^2 f}{\partial y^2}(\phi_1(t)). \begin{pmatrix} 1 \\ \frac{\partial f}{\partial x}(\phi_1(t)) \end{pmatrix} \qquad (*)
\end{aligned}$$

and at the origin $\frac{\partial}{\partial t}(F \circ \phi)(0) = -\frac{\partial^2 f}{\partial y^2}(0,0). \begin{pmatrix} 1 \\ \frac{\partial f}{\partial x}(0,0) \end{pmatrix}$. Therefore $\frac{\partial}{\partial t}(F \circ \phi)(0) \neq 0$ if and only if $\frac{\partial^2 f}{\partial y^2}(0,0) \neq 0$, if and only if F is a fold.

Differentiating $(*)$ we get:

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}(F \circ \phi)(t) &= \frac{\partial}{\partial t}\left(-\frac{\partial^2 f}{\partial y^2}(\phi_1(t)). \begin{pmatrix} 1 \\ \frac{\partial f}{\partial x}(\phi_1(t)) \end{pmatrix}\right) \\
&= -\left\{-\frac{\partial^3 f}{\partial x \partial y^2}(\phi_1(t)).\frac{\partial^2 f}{\partial y^2}(\phi_1(t)) + \frac{\partial^3 f}{\partial y^3}(\phi_1(t)).\frac{\partial^2 f}{\partial x \partial y}(\phi_1(t))\right\} \begin{pmatrix} 1 \\ \frac{\partial f}{\partial x}(\phi_1(t)) \end{pmatrix} \\
&\quad -\frac{\partial^2 f}{\partial y^2}(\phi_1(t)). \begin{pmatrix} 1 \\ \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}(\phi_1(t))\right) \end{pmatrix} \qquad (**)
\end{aligned}$$

If $\frac{\partial}{\partial t}(F \circ \phi)(0) = 0$, that is $\frac{\partial^2 f}{\partial y^2}(0,0) = 0$, then

$$\frac{\partial^2}{\partial t^2}(F \circ \phi)(0) = -\frac{\partial^3 f}{\partial y^3}(0,0).\frac{\partial^2 f}{\partial x \partial y}(0,0) \begin{pmatrix} 1 \\ \frac{\partial f}{\partial x}(0,0) \end{pmatrix}.$$

Hence $\frac{\partial}{\partial t}(F \circ \phi)(0) = 0$ and $\frac{\partial^2}{\partial t^2}(F \circ \phi)(0) \neq 0$ if and only if $\frac{\partial^2 f}{\partial y^2}(0,0) = 0$ and $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq 0$ and $\frac{\partial^3 f}{\partial y^3}(0,0) \neq 0$, if and only if F is a cusp. ■

The conditions in Proposition 3.1 reflect the order of contact of Σ with the kernel of $DF(0,0,0)$.

Corollary 3.2. *A singular germ $F : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ of corank 1 and with a smooth critical set is a fold if and only if Σ is transversal to the set $ker(DF(0,0,0))$ at the origin. It is a cusp if and only if Σ and $ker(DF(0,0,0))$ have 2-point contact at the origin (see Figure 1).*

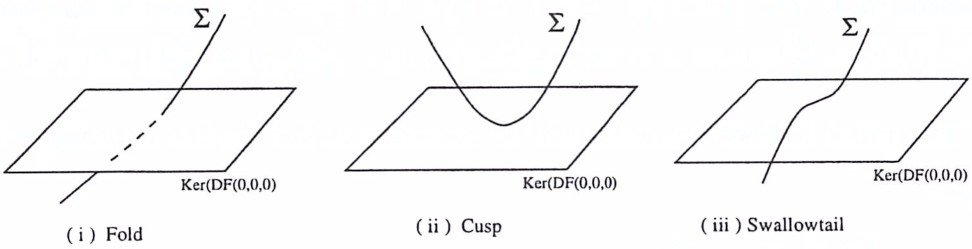

Figure 1: Recognition of fold, cusp and swallowtail singularities

Proposition 3.3. *Let $F : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ be a singular germ with a smooth critical set. Then F is a swallowtail if and only if $\frac{\partial}{\partial t}(F \circ \phi)(0) = \frac{\partial^2}{\partial t^2}(F \circ \phi)(0) = 0$ and $\frac{\partial^3}{\partial t^3}(F \circ \phi)(0) \neq 0$.*

Proof: From the proof of Proposition 3.1 we have $\frac{\partial}{\partial t}(F \circ \phi)(0) = \frac{\partial^2}{\partial t^2}(F \circ \phi)(0) = 0$ if and only if $\frac{\partial^2 f}{\partial y^2}(0,0) = \frac{\partial^3 f}{\partial y^3}(0,0).\frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$.

Differentiating the expression $(**)$ and setting $\frac{\partial^2 f}{\partial y^2}(0,0) = \frac{\partial^3 f}{\partial y^3}(0,0).\frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$, yield

$$\frac{\partial^3}{\partial t^3}(F \circ \phi)(0) = -\frac{\partial^4 f}{\partial y^4}(0,0).(\frac{\partial^2 f}{\partial x \partial y}(0,0))^2 \begin{pmatrix} 1 \\ \frac{\partial f}{\partial x}(0,0) \end{pmatrix}.$$

Then, $\frac{\partial}{\partial t}(F \circ \phi)(0) = \frac{\partial^2}{\partial t^2}(F \circ \phi)(0) = 0$ and $\frac{\partial^3}{\partial t^3}(F \circ \phi)(0) \neq 0$ if and only if $\frac{\partial^2 f}{\partial y^2}(0,0) = \frac{\partial^3 f}{\partial y^3}(0,0) = 0$, $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq 0$, and $\frac{\partial^4 f}{\partial y^4}(0,0) \neq 0$, if and only if F is a swallowtail. ■

Again the conditions in Proposition 3.3 express the order contact of the critical set Σ with the $ker(DF(0,0,0))$.

Corollary 3.4. *A singular germ $F : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ of corank 1 and with a smooth critical set is a swallowtail if and only if the critical set Σ has contact 3 with the kernel of $DF(0,0,0)$ at the origin (see Figure 1).*

When F has a lips/beaks singularity, its critical set Σ is singular. Then we need additional algebraic conditions for recognizing these singularities.

Proposition 3.5. *A singular germ that can be written in the form $F = (x, f(x,y) \pm z^2)$ is a lips/beaks if and only if $\frac{\partial^2 f}{\partial y^2}(0,0) = \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$, $\frac{\partial^3 f}{\partial y^3}(0,0) \neq 0$ and $(\frac{\partial^3 f}{\partial x \partial y^2}(0,0))^2 - \frac{\partial^3 f}{\partial x^2 \partial y}(0,0).\frac{\partial^3 f}{\partial y^3}(0,0) \neq 0$.*

Proof: The proof follows by a direct calculation. ■

Proposition 3.6. *The germ $\frac{\partial f}{\partial y} : \mathbb{R}^2, 0 \to \mathbb{R}, 0$ is a germ of a Morse function if and only if $\frac{\partial^2 f}{\partial y^2}(0,0) = \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$ and $(\frac{\partial^3 f}{\partial x \partial y^2})^2(0,0) - \frac{\partial^3 f}{\partial x^2 \partial y}(0,0).\frac{\partial^3 f}{\partial y^3}(0,0) \neq 0$.*

Proof: The proof follows immediately by considering the Taylor expression of $\frac{\partial f}{\partial y}$ at the origin,

$$\begin{aligned} \frac{\partial f}{\partial y}(x,y) = \ & \frac{\partial^2 f}{\partial x \partial y}(0,0).x + \frac{\partial^2 f}{\partial y^2}(0,0).y + \\ & \frac{1}{2}\{\frac{\partial^3 f}{\partial x^2 \partial y}(0,0).x^2 + 2\frac{\partial^3 f}{\partial x \partial y^2}(0,0).xy + \frac{\partial^3 f}{\partial y^3}.y^2\} + O_3(x,y). \end{aligned}$$ ■

It is not difficult to show that if two germs F and G are $\mathcal{A}$-equivalents then the critical set of F is the zero set of a Morse function if and only if the critical set of G is also the zero set of a Morse function. Therefore we have the following.

Corollary 3.7. *Let $F(x,y,z) = (x, f(x,y) \pm z^2))$. If $\frac{\partial^3 f}{\partial y^3}(0,0) \neq 0$, then F is a lips/beaks if an only if the critical set is the zero of a Morse function (see Figure 2).*

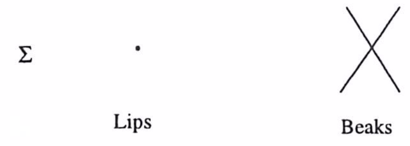

Figure 2: Lips/beaks singularities

Remark 3.1. *The above geometric criteria do not extend to higher codimension singularities. The reason is that when Σ is smooth its order of contact k with the kernel of $DF(0,0,0)$ determines the A_{k-1} $\mathcal{K}$-class of the projection. When $k \geq 4$, there are several $\mathcal{A}$-orbits of the projection inside the A_{k-1}-orbit. One needs algebraic conditions to distinguish between these $\mathcal{A}$-orbits.*

4 Geometric criteria for recognition of the versal unfoldings of the singularities of $\mathcal{A}_e$-codim $= 1$

Geometric criteria for determining when a deformation of a lips/beaks or swallowtail singularity of a germ $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ are given in [26]. We extend these results to germs $F : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$. We deal in more details with the lips/beaks case.

Let $F : \mathbb{R}^3 \times \mathbb{R}, 0 \to \mathbb{R}^2 \times \mathbb{R}, 0$ be an unfolding of a germ $F_0(x,y,z,0) = (x, f_0(x,y) \pm z^2)$ that has a lips/beaks singularity at the origin. Then the critical set of F_0 can be considered as a plane curve with a Morse singularity and the unfolding F induces an unfolding of this Morse singularity.

Theorem 4.1. *An unfolding F of a lips/beaks singularity of F_0 is versal if and only if induces a versal unfolding of the Morse singularity of the critical set of F_0.*

Proof: Let $\tilde{F}$ be an unfolding of a lips/beaks and write $\tilde{F}(x,y,z,u) = (\tilde{F}_1(x,y,z,u), \tilde{F}_2(x,y,z,u), u)$.

Claim: $\tilde{F}$ is equivalent to an unfolding written in the form $F(x,y,z,u) = (x, F_2(x,y,z,u), u)$.

Proof of the claim: without loss of generality we can assume that $\frac{\partial \tilde{F}_1}{\partial x} \neq 0$ at the origin. Then the local diffeomorphism

$$\begin{aligned} H: \quad \mathbb{R}^3 \times \mathbb{R} &\to \mathbb{R}^3 \times \mathbb{R} \\ (x,y,z,u) &\mapsto (\tilde{F}_1(x,y,z,u), y, z, u) \end{aligned}$$

makes the following diagram commute

$$\begin{array}{ccc}
\mathbb{R}^3 \times \mathbb{R} & \xrightarrow{\tilde{F}} & \mathbb{R}^2 \times \mathbb{R} \\
(x, y, z, u) & & (\tilde{F_1}(x, y, z, u), \tilde{F_2}(x, y, z, u), u) \\
\downarrow H & & \downarrow id \\
\mathbb{R}^3 \times \mathbb{R} & \xrightarrow{F} & \mathbb{R}^2 \times \mathbb{R} \\
((\tilde{F_1}(x, y, z, u), y, z, u) & & (\tilde{F_1}(x, y, z, u), \tilde{F_2}(x, y, z, u), u)
\end{array}$$

where $F_2(x, y, z, u) = \tilde{F_2}(\pi_1 \circ \tilde{F_1}^{-1}(x, y, z, u), y, z, u)$ and π_1 is the projection to the first coordinate. Observe that $\pi_1 \circ \tilde{F} \circ H^{-1} = \pi_1$, therefore $\tilde{F} \circ H^{-1} = (x, \tilde{F_2}(x, y, z, u), u)$ and hence F and $\tilde{F}$ are equivalents unfoldings (see [15]). Therefore we can set

$$F(x, y, z, u) = (x, g_1(x, y, z) + ug_2(x, y, z, u), u).$$

The germ $(x, g_1(x, y, z))$ has a lips/beaks singularity at the origin, consequently, changes of coordinates in the source and target reduce F_0 to the form $(x, y^3 \pm x^2 y \pm z^2)$. In this new system of coordinates

$$F(x, y, z, u) = (x, y^3 \pm x^2 y \pm z^2 + u\bar{g}_2(x, y, z, u), u).$$

Since a lips/beaks singularity is 3-determined, we can work in the $J^3(x, y, z, u)$, and write

$$j^3 F(x, y, z, u) = (x, y^3 \pm x^2 y \pm z^2 + \alpha(x, z, u)u + \beta(z, u)uy + a_1 uy^2 + a_2 uxy, u).$$

Now successive explicit changes of coordinates in the source and target reduce this 3-jet to the form

$$j^3 F(x, y, z, u) = (x, y^3 \pm x^2 y \pm z^2 + \psi(u)y, u).$$

Then F is a versal unfolding of F_0 if and only if $\psi'(0) \neq 0$ (see [15]). We have

$$\Sigma = \{(x, y, z, u) : z = 0 \text{ and } \sigma(x, y, z, u) = 3y^2 \pm x^2 + \psi(u) + \cdots = 0\}.$$

Then $\frac{\partial \sigma}{\partial u}(0) = \psi'(0)$, and hence F is versal if and only if $\frac{\partial \sigma}{\partial u}(0) \neq 0$, if and only if σ is a versal unfolding of the Morse singularity of Σ. ■

Let $\Sigma^{1,1}$ denotes the critical set of the restriction of F to Σ. Then proceeding as above, we show the following.

Theorem 4.2. *Let $F(x, y, z, u)$ be an unfolding of a swallowtail singularity. Then F is versal if and only if $\Sigma_F^{1,1}$ is a smooth curve.*

5 A stratification of the space of singular planes

Recall from the introduction that there is a natural 4-parameter family of projections

$$\begin{array}{rcl}\Pi: \; M \times G(2,4) & \to & \mathbb{R}^2 \\ (p,u) & \mapsto & (<p,a>,<p,b>)\end{array}$$

where a and b are the generators of the plane of projection $u \in G(2,4)$ (the projection Π_u does not dependent on the vectors a and b, but only on the the plane u [5]). We start with an elementary lemma that provides a simple way of writing the family Π.

Lemma 5.1. *The map Π can be written locally in the form*

$$\begin{array}{rcl}\Pi: \mathbb{R}^3 \times \mathbb{R}^4, 0 & \to & \mathbb{R}^2, 0 \\ (x,y,z,\beta_1,\gamma_1,\beta_2,\gamma_2) & \mapsto & (x+\beta_1 y+\gamma_1 z, \beta_2 y+\gamma_2 z+f(x,y,z)).\end{array}$$

Proof: We can take the point in consideration on M as the origin, and write M locally as the graph of a function $w = f(x,y,z)$. We assume, without loss of generality, that the initial plane of projection u_0 is generated by the vectors $a = (1,0,0,0)$ and $b = (0,0,0,1)$. Then vectors $(1,\bar{\alpha_1},\bar{\beta_1},\bar{\gamma_1})$ and $(\bar{\alpha_2},\bar{\beta_2},\bar{\gamma_2},1)$ generate all the planes near u_0. We claim that there are always linearly independent vectors of the form $(1,\beta_1,\gamma_1,0)$ and $(0,\beta_2,\gamma_2,1)$ that generate these planes. Indeed, by setting $\xi_1 = \xi_4 = 1/(1-\bar{\alpha_2}\bar{\gamma_1})$, $\xi_2 = -\bar{\gamma_1}/(1-\bar{\alpha_2}\bar{\gamma_1})$, and $\xi_3 = -\bar{\alpha_2}/(1-\bar{\alpha_2}\bar{\gamma_1})$, we have

$$\xi_1(1,\bar{\alpha_1},\bar{\beta_1},\bar{\gamma_1})+\xi_2(\bar{\alpha_2},\bar{\beta_2},\bar{\gamma_2},1) = (1,(\bar{\alpha_1}-\bar{\gamma_1}\bar{\beta_2})/(1-\bar{\alpha_2}\bar{\gamma_1}),(\bar{\beta_1}-\bar{\gamma_1}\bar{\gamma_2})/(1-\bar{\alpha_2}\bar{\gamma_1}),0),$$

$$\xi_3(1,\bar{\alpha_1},\bar{\beta_1},\bar{\gamma_1})+\xi_4(\bar{\alpha_2},\bar{\beta_2},\bar{\gamma_2},1) = (0,(\bar{\beta_2}-\bar{\alpha_2}\bar{\alpha_1})/(1-\bar{\alpha_2}\bar{\gamma_1}),(\bar{\gamma_2}-\bar{\beta_1}\bar{\alpha_2})/(1-\bar{\alpha_2}\bar{\gamma_1}),1).$$

Since Π_u depends only on the plane u we can take $a = (1,\beta_1,\gamma_1,0)$ and $b = (0,\beta_2,\gamma_2,1)$ as generators of u and obtain the required result. ∎

Given M in Monge form $(x,y,z,f(x,y,z))$ (then $f = f_x = f_y = f_z = 0$ at the origin), we can find the algebraic conditions on the coefficients of the Taylor expansion of f for Π_u to have one of the singularities in Theorems 2.4 and 2.5 and for Π to be a versal unfolding of these singularities (see [21]). These algebraic conditions are in general too complicated and difficult to interpret geometrically. However for the singularities of codimension 1, we have the following corollary of the results in the previous section.

Corollary 5.2. *The family Π is always a versal unfolding of the lips/beaks and swallowtail singularities of Π_u.*

If M is given in Monge form $w = f(x, y, z)$, then Π_u is singular if and only if the normal direction to M at the origin, $(0, 0, 0, 1)$, belongs to the plane u. So we can take $a = (0, 0, 0, 1)$ and $b = (\alpha_1, \alpha_2, \alpha_3, 0)$, with $||b|| = 1$, as the generators of the plane u. Therefore, the singular planes of projections can be identified with the sphere $S^2 = \{b = (\alpha_1, \alpha_2, \alpha_3, 0), ||b|| = 1\}$. We shall stratify this sphere according to the singularity type of the projection Π_u.

We rotate the coordinates axis so that $j^2 f = a_1 x^2 + a_2 y^2 + a_3 z^2$, where $a_i = \frac{\kappa_i}{2}$, $i = 1, 2, 3$, with κ_i the principal curvatures of M at the origin.

Proposition 5.3. *Let M be as above at the origin p, and u a singular plane of the projection determined by a direction $b = (\alpha_1, \alpha_2, \alpha_3, 0) \in S^2$.*

(i) If p is an elliptic point, then the projection Π_u has a singularity of type fold for any direction on S^2.

(ii) If p is a hyperbolic point, then Π_u has a fold singularity for any direction on S^2 except on two circles where the singularity is in general of cusp type. On these circles there are up to $2k$ directions, $k = 0, \cdots 6$, where the singularities are of swallowtail type. These directions could yield singularities of higher codimensions (generically of type 6, 7, 8 in Table 1) if conditions are imposed on M at p.

(iii) If p is a parabolic point (say $\kappa_2 = 0$), then Π_u is a fold except on the circle $\alpha_2 = 0$ where it has generically a singularity of type 4_2. For 0 or 2 directions on this circle the singularity is of type 4_3. These directions could yield singularities of type 4_4 and 4_5 if conditions are imposed on M. If an extra condition is imposed on M at p (beside $\kappa_2 = 0$), the singularity is 11_5 on the circle $\alpha_2 = 0$. Then for 0, 4 or 8 directions on this circle the singularity is of type 11_7, for 0 or 4 directions the singularity is of type 12 and for 0 or 2 directions it is of type 16. Imposing a further condition on M at p the singularity of the projection becomes of type 11_9, 13 or 17. If $\kappa_1\kappa_3 < 0$ there are 2 additional directions on $\alpha_2 = 0$ where the singularity is of type N_1. These exceptional directions could yield singularities of higher codimension (generically of type $N_2, \cdots, N_{11}$) if conditions are imposed on M at p.

(iv) If p is a partial umbilic point (say $\kappa_1 = \kappa_3 = 0$), then the projection has generically a singularity of type 4_2. There are 0 or 2 circles where the singularity is of type 4_3, and 0 or 2 directions on these circles where the singularity is 4_4. There are also 0, 1 or 3 other circles where the singularity is of type 11_5 and on these circles there are 0, 2,

6, 18 or 30 directions where the singularity is of type 11_7*; 0, 4, 8, 12 or 24 directions where the singularity is of type* 12*; and 0, 4 or 12 directions where the singularity is of type* 16*. There is a unique direction (not on these circles) where the singularity is of type* N_{12}*.*

Proof: Let $b = (\alpha_1, \alpha_2, \alpha_3, 0) \in S^2$. Then the projection to the plane u generated by b and $(0,0,0,1)$ is given by

$$\Pi_u = (\alpha_1 x + \alpha_2 y + \alpha_3 z, f(x,y,z)).$$

We can assume without loss of generality that $\alpha_1 \neq 0$. We can then change coordinates in the source and target and write $\Pi_u = (x, \alpha_1^2 f((x - \alpha_2 y - \alpha_3 z)/\alpha_1, y, z)$, so that

$$j^2\Pi_u = (x, a_1 x^2 + (a_2\alpha_1^2 + a_1\alpha_2^2)y^2 + (a_1\alpha_3^2 + a_3\alpha_1^2)z^2 + 2a_1\alpha_2\alpha_3 yz - 2a_1\alpha_3 xz - 2a_1\alpha_2 xy).$$

Let $q(y,z) = (a_2\alpha_1^2 + a_1\alpha_2^2)y^2 + 2a_1\alpha_2\alpha_3 yz + (a_1\alpha_3^2 + a_3\alpha_1^2)z^2$ be the quadratic form obtained by setting $x = 0$ in the second component of $j^2\Pi_u$. This form is degenerate if and only if

$$a_2 a_3 \alpha_1^2 + a_1 a_3 \alpha_2^2 + a_1 a_2 \alpha_3^2 = 0. \tag{1}$$

At an elliptic point (case (i)), the coefficients a_i have the same sign, so equation (1) does not hold. Therefore $\Pi_u \sim (x, y^2 \pm z^2)$ which is a fold singularity.

At a hyperbolic point (case (ii)) equation (1) determines a cone. This cone intersects the sphere S^2 in two circles (we only need to consider one of them as b and $-b$ define the same projection). Away from these circles, the singularity of Π_u is clearly of type fold. On the circles the quadratic q is degenerate. As the point in consideration is not parabolic, we can suppose without loss of generality that the coefficient of z^2 in q is different from zero. Then the change coordinates $Z = z + \frac{a_1\alpha_2\alpha_3}{a_1\alpha_3^2 + a_3\alpha_1^2} y$ and further changes of coordinates in the source and target yield

$$j^2\Pi_u \sim (x, -\frac{2a_1 a_3 \alpha_1^2 \alpha_2}{a_1\alpha_3^2 + a_3\alpha_1^2} xy + (a_1\alpha_3^2 + a_3\alpha_1^2)z^2).$$

As the point is hyperbolic, the coefficient of xy is different from zero (note that we assumed $\alpha_1 \neq 0$, and if $\alpha_2 = 0$ then the coefficient of z^2 in q must be zero, which is not the case from our assumption above.)

We now need to analyze the coefficient of y^3 in the second component of Π_u. To simplify the calculations, we set $\alpha_1 = 1$, so equation (1) determines a circle (a curve of degree 2). Then the coefficient of y^3 is a polynomial of degree 6 in α_2 and α_3. Therefore this polynomial can vanish at $0, 2, 4, 6, 8, 10$, or 12 directions on the circle. Away from these directions the singularity of Π_u is a cusp. At one of these directions we need to analyze the higher jets of the second component of Π_u which depend only on the Taylor expansion of the function f giving the surface M in Monge form. (Note that the values of α_i, $i = 1, 2, 3$ are now fixed). Without imposing any condition on the surface M, the coefficient of y^4 in Π_u does not vanish and the singularity of Π_u is of type swallowtail. When the coefficient of y^4 vanishes (this determines a 2-dimensional surface on M) the singularity of Π_u is generically of type 6 (Table 1). There might be a curve on this surface where the singularity becomes of type 7 and a point on this curve where it is of type 8.

The calculations for parts (iii) and (iv) follow in the same way. ■

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