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Seminar on the Span and Parallelizability of Manifolds
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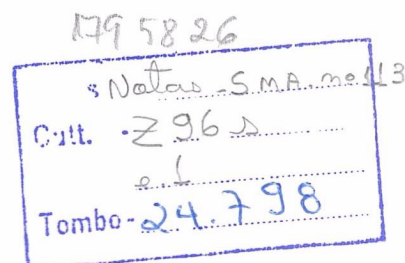
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NOTAS

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SEMINAR ON THE SPAN AND PARALLELIZABILITY OF MANIFOLDS USP at SÃO CARLOS

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Abstract

In these notes we present the state of the span problem for manifolds. The techniques involve Lie groups theory, characteristic classes, spectral sequences and representation rings of classical groups. We touch on the main points of these too. In particular, the parallelizability question for flag manifolds and related manifolds is solved.

Resumo

Nestas notas apresentamos o estado atual do problema do posto linear para as variedades até o momento. As técnicas envolvem a teoria dos grupos de Lie, classes características, seqüências espectrais e os anéis de representações dos grupos clássicos. Nós tocamos em cada um dos pontos principais destes. Em particular, a questão da paralelizabilidade para variedades flâmula e outras relacionadas a estas, é resolvida.

Introduction

The first part of these Lectures (1-8) deals mainly with flag manifolds and the question of finding which are parallelizable. Lectures 1-5 are more or less self contained. However Lectures 6-8 use the representation theory of compact Lie groups, so a reader not familiar with this material would probably want to look at say [10] or [24] in conjunction with Lectures 6-8. A working knowledge of spectral sequences is also useful in reading these notas, as well as some basics of differential topology and vector bundles.

Although the primary goal of Lectures 1-8, the parallelizability (and stable parallelizability) of flag manifolds, is fairly specific, the lectures include a variety of techniques that are applicable to many other questions. These are vector bundle techniques, multilinear algebra, characteristic classes, homotopy theory, K-theory (the Atiyah-Hirzebruch and the Hodgkin

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spectral sequences), and others. One method not treated in these notas (but mentioned briefly in Lecture 3) is the application of Lie algebras.

The second part, Lectures 9-11, deals with the more general question of determining the span of smooth manifolds. Lectures 9-10 cover this material from a general (and historical) point of view, and Lecture 11 applies these ideas in specific cases. These three lectures are more or less independent of the first eight and can be read separately.

1 Lecture

Flag Manifolds and Their Tangent Bundles

The main reference for this lecture is Lam [36]. To set up notation, let k be the field $\mathbb{R}$ or $\mathbb{C}$ ($\mathbb{H}$ can also be treated with a little extra care). The vector space k^n has the Hermitian inner product $\langle x, y \rangle = \sum_{i=1}^n x_i \bar{y}_i$, and $G_k(n)$ is the group of k -linear transformations preserving the inner product.

k	d	$G_k(n)$	$\dim_{\mathbb{R}} G_k(n)$
$\mathbb{R}$	1	$O(n)$	$\binom{n}{2}$
$\mathbb{C}$	2	$U(n)$	n^2

$(d = \dim_{\mathbb{R}} k)$

1.1 Definition: $F = F_k(n_1, \dots, n_s)$, $n_1 + \dots + n_s = n$, is the flag manifold (variedade flâmula) whose elements are "flags" $\sigma = (\sigma_1, \dots, \sigma_s)$ where $\sigma_1, \dots, \sigma_s$ are mutually orthogonal subspaces of k^n and $\dim \sigma_i = n_i$.

Of course this implies $\sigma_1 \oplus \dots \oplus \sigma_s = k^n$. As examples of flag manifolds one has the Grassmann manifolds $F_k(n_1, n_2) = G_k(n_1, n_2) = G_{n, n_1}^k$ of n_1 -planes in k^n (in particular $F_k(1, n-1) = kP^{n-1}$, and the "classical" flag manifolds $F_k(1, 1, \dots, 1)$).

1.2 Let $A \in G_k(n)$ and write it in terms of its (mutually orthonormal) row vectors $A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. Then $\psi(A) = (\sigma_1, \dots, \sigma_s)$, where $\sigma_1 = \text{span}\{a_1, \dots, a_{n_1}\}$, $\sigma_2 = \text{span}\{a_{n_1+1}, \dots, a_{n_1+n_2}\}$, ... gives a map $\psi : G_k(n) \rightarrow F$ with isotropy $H = G_k(n_1) \times \dots \times G_k(n_s)$, and this induces a bijection φ as shown by the commutative diagram

$$\begin{array}{ccc} G_k(n) & \xrightarrow{\psi} & F \\ & \searrow \quad \swarrow \varphi & \\ & G_k(n)/H & \end{array}$$

We can use φ to topologize F , indeed this makes F into a smooth manifold called a homogeneous space.

1.3 It is easy to calculate $\dim_{\mathbb{R}} F = \dim G_k(n) - \dim H = (d/2)(n^2 - \sum n_i^2)$, whence $\dim_k(F) = (1/2)(n^2 - \sum n_i^2)$. Note also that $n^2 = (\sum n_i)(\sum n_j) = \sum n_i^2 + 2 \sum_{i < j} n_i n_j$, hence $\sum_{i < j} n_i n_j = \dim_k F$.

1.4 There are natural k -vector bundles $\xi_1, \dots, \xi_s$ over F of ranks respectively $n_1, \dots, n_s$. The fibre of ξ_i , at a point $\sigma = (\sigma_1, \dots, \sigma_s) \in F$, is simply σ_i .

1.5 Proposition: $\xi_1 \oplus \dots \oplus \xi_s \approx n\varepsilon$, where $n\varepsilon$ denotes the trivial bundle of rank n .

Proof: $(\sigma, v_1, \dots, v_s) \mapsto (\sigma, v_1 + \dots + v_s)$, where $\sigma \in F$ and $v_i \in \sigma_i$, gives a linear isomorphism of the two vector bundles.

1.6 Recall, from elementary linear algebra, the row echelon form (REF) and reduced row echelon form (RREF) obtained by Gaussian elimination on any $m \times n$ matrix A . And one has the following basic theorem from linear algebra.

Theorem: Let A, B be $m \times n$ matrices over any field k . Then $\text{row span}(A) = \text{row span}(B)$ iff $\text{RREF}(A) = \text{RREF}(B)$, where $\text{row span}(A)$ is the subspace of k^n spanned by the m row vectors of A , and similarly for B .

1.7 For a $\mathbb{C}$ -vector space V , the conjugate $\bar{V}$ is given $\bar{V} = V$ with the scalar product $z * v = \bar{z} \cdot v$, $z \in \mathbb{C}, v \in V$. For an $\mathbb{R}$ -vector space V , $\bar{V} = V$ with the same scalar product.

Similarly one defines $\bar{\alpha}$ for a k -vector bundle α over a space X . We assume throughout that all the vector bundles we deal with have a hermitian metric, for example this is always true if the base space X is paracompact ([46], p.23).

Proposition: For vector bundles α, β and $k = \mathbb{R}, \mathbb{C}$,

$$\text{Hom}_k(\alpha, \beta) \approx \bar{\alpha} \otimes \beta.$$

Proof: For an inner product space V (hermitian inner product) let $\varphi : \bar{V} \otimes_k W \rightarrow \text{Hom}_k(V, W)$ be given by $\varphi(v \otimes w)(v') = \langle v', v \rangle w$. Note that φ is linear in v and in w . To see that φ is an isomorphism, let $v_1, \dots, v_m$ be an orthonormal basis of V and $w_1, \dots, w_n$ any basis of W . A basis for $\text{Hom}_k(V, W)$ is $\{f_{ij}\}$, where $f_{ij}(v_\ell) = \delta_{i\ell} w_j$. Since $f_{ij} = \varphi(v_i \otimes w_j)$, φ is surjective and hence an isomorphism (the spaces are finite dimensional). One should also note that $\varphi(v \otimes w)$ is linear in v' , thus $\varphi(v \otimes w) \in \text{Hom}_k(V, W)$.

Now φ is clearly natural with respect to inner product preserving linear transformations $f : V \rightarrow V'$, and any linear transformations $g : W \rightarrow W'$. So using the methods of [46] §3 or [24] Ch.5, φ extends to a vector bundle isomorphism $\bar{\alpha} \otimes_k \beta \xrightarrow{\sim} \text{Hom}_k(\alpha, \beta)$ as long as α is endowed with a hermitian metric. $\square$

1.8 We now turn to the tangent bundle of $F = F_k(n_1, \dots, n_s)$. Some intuition for the main result, Theorem 1.10, can be obtained by imagining a “small” change in $\sigma = (\sigma_1, \dots, \sigma_s) \in F$. For any vector $v \in \sigma_i$, the “small” change in v will be a vector orthogonal to σ_i , which “gives” a linear transformation in $\text{Hom}_k(\sigma_i, \sigma_i^\perp) = \bigoplus_{j \neq i} \text{Hom}_k(\sigma_i, \sigma_j)$. To make this precise, let $i < j$ and $f : \sigma_i \rightarrow \sigma_j$ linear. For convenience take $i = 1, j = 2$. For any $t \in \mathbb{R}$,

$$tf \in \text{Hom}_k(\sigma_1, \sigma_2) = \text{Hom}_k(\xi_1, \xi_2)_\sigma.$$

Let its graph be $\gamma(tf) = \{(v, t \cdot f(v)) : v \in \sigma_1, \} \subset \sigma_1 \oplus \sigma_2$. Note that $\dim_k \gamma(tf) = n_1$ and $\gamma(0f) = \sigma_1 \oplus 0$. Also let $\gamma^\perp(tf)$ be the orthogonal complement of $\gamma(tf)$ in $\sigma_1 \oplus \sigma_2$, clearly $\dim \gamma^\perp(tf) = n_2$ and $\gamma^\perp(0f) = 0 \oplus \sigma_2$.

Definition: Set $\sigma(t) = (\gamma(tf), \gamma^\perp(tf), \sigma_3, \dots, \sigma_s)$. Clearly this defines a smooth curve in F with $\sigma(0) = \sigma$.

Definition: $L(f) = D_t \sigma(0) \in (TF)_\sigma$, where TF is the total space of the tangent bundle T_F of F , and $(TF)_\sigma$ is the fibre over σ .

1.9 Our main task is to prove the following lemma.

Lemma: $L : \text{Hom}_k(\xi_i, \xi_j) \rightarrow TF, i < j$, is injective.

Proof: As above set $i = 1, j = 2$ for convenience. Choose an orthonormal basis $v_1, \dots, v_n$ for k^n such that

$$\begin{aligned} v_1, \dots, v_{n_1} &= \text{basis for } \sigma_1, \\ v_{n_1+1}, \dots, v_{n_1+n_2} &= \text{basis for } \sigma_2, \text{ etc.} \end{aligned}$$

For any $n \times n$ matrix A , we write A in block form using $n = n_1 + n_2 + \dots + n_s$, both in rows and columns. We consider now those A having block form

$$A = \begin{bmatrix} I_{n_1} & * & * & * & \dots & * \\ 0 & I_{n_2} & * & * & \dots & * \\ 0 & 0 & I_{n_1} & * & \dots & * \\ & & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & I_{n_s} \end{bmatrix}.$$

Note first that any such A is in REF (not in general RREF). Let $V_i = \text{span of first } n_1 + \dots + n_i \text{ rows of } A \text{ with respect to the chosen basis } v_1, \dots, v_n$. Since A is in REF, $\dim V_i = n_1 + \dots + n_i$. Next set $\sigma_1^A = V_1$ and, for $i > 1$, $\sigma_i^A = V_i \cap V_{i-1}^\perp$. Clearly $\dim \sigma_i^A = n_i$ and $\sigma_1^A, \dots, \sigma_s^A$ are mutually orthogonal. So define $\Sigma(A) := (\sigma_1^A, \dots, \sigma_s^A) \in F$. For exempo, $\Sigma(I_n) = \sigma$.

We now claim Σ is injective. For write A in the form

$$A = \begin{bmatrix} A_1 & & & & \\ 0 & A_2 & & & \\ 0 & 0 & A_3 & & \\ & \vdots & & & \\ 0 & 0 & 0 & \dots & A_s \end{bmatrix}$$

where

$$\begin{aligned} A_1 &= \text{first } n_1 \text{ rows of } A = [I_{n_1} * * \dots *] \text{ is } n_1 \times n, \\ A_2 &= \text{next } n_2 \text{ rows of } A = [I_{n_2} * \dots *] \text{ is } n_2 \times (n - n_1), \\ &\vdots \\ A_s &= \text{last } n_s \text{ rows of } A = [I_s] \text{ is } n_s \times (n - n_1 - \dots - n_{s-1}). \end{aligned}$$

Notice that each A_i is in RREF, $1 \leq i \leq s$. So by 1.6, A_1 is uniquely determined by $V_1 = \sigma_1^A$, hence also by $\sum(A)$. Now consider $V_2 = \text{span}$ of first $n_1 + n_2$ rows of A . Since $A_1 = [I_{n_1} * \dots *]$, it is clear that $\sigma_2^A = V_2 \cap V_1^\perp$ satisfies $\sigma_2^A \subset \text{span} \{v_{n_1+1}, \dots, v_{n_1+n_2}\}$. In this space A_2 is in RREF, hence σ_2^A (and also $\sum(A)$) determines A_2 uniquely, again by 1.6. Thus $\sum(A)$ determines A_1, A_2 uniquely and continuing this way clearly determines $A_1, A_2, \dots, A_s$ uniquely, i.e. determines A uniquely.

Thus $\sum$ is injective and clearly smooth. Further, we have

$$\begin{aligned} \dim \text{Im } \sum &= n_1(n - n_1) + n_2(n - n_1 - n_2) + \dots + n_{s-1}(n - n_1 - \dots - n_{s-1}) \\ &\quad + n_s(n - n_1 - \dots - n_s) \\ &= n^2 - \sum n_i^2 - n_2 n_1 - n_3(n_1 + n_2) - \dots - n_s(n_1 + \dots + n_{s-1}) \\ &= n^2 - \sum n_i^2 - \sum_{i>j} n_i n_j = n^2 - \sum n_i^2 - \sum_{i<j} n_i n_j \\ &= \dim_k F \quad \text{by 1.3.} \end{aligned}$$

By the inverse function theorem $\sum$ is locally a diffeomorphism, hence is a homeomorphism onto $\text{Im } \sum$. By the invariance of domain, $\text{Im } \sum$ is open in F . Since $\sigma \in \text{Im } \sum$, this forms a coordinate chart for the open neighborhood $U = \text{Im } \sum$ of σ , with $0 \in k^n$ corresponding to σ . Now let

$$A_t = \begin{bmatrix} I_{n_1} & t \cdot M(f) & 0 & \dots & 0 \\ 0 & I_{n_2} & 0 & \dots & 0 \\ 0 & 0 & I_{n_3} & \dots & 0 \\ & \vdots & & & \\ 0 & 0 & 0 & \dots & I_{n_s} \end{bmatrix},$$

where $M(f)$ is the matrix of f with respect to $v_1, \dots, v_{n_1+n_2}$. From the above definitions $\sum(A_t) = \sigma_t$, and by differentiating with respect to t it is clear that $L(f)$ corresponds to

$$\begin{bmatrix} 0 & M(f) & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & \vdots & & & \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

and thus L is injective. □

1.10 Theorem: $\tau_F \approx \bigoplus_{1 \leq i < j \leq s} \text{Hom}_k(\xi_i, \xi_j)$.

Proof: Using the neighborhood U of σ as in the proof of 1.9, we see that L embeds $\bigoplus_{i < j} \text{Hom}(\xi_i, \xi_j)$ into τ_F , since $L(\text{Hom}_k(\xi_i, \xi_j))$ is in the ij block. Again, since $\sum n_i n_j = \dim_k F$ by 1.3, this map is an isomorphism. □

Corollary: $\tau_F \approx \bigoplus_{1 \leq i < j \leq s} \overline{\xi_i} \otimes \xi_j$.

We now make a few applications with $k = \mathbb{R}$, in which case $\tau_F \approx \bigoplus_{i < j} \xi_i \otimes \xi_j$. First recall a few properties of exterior powers of a vector bundle. One has $\text{rank } \lambda^i \alpha = \binom{\text{rank } \alpha}{i}$,

$\lambda^0 \alpha \approx \varepsilon$, $\lambda^1 \alpha \approx \alpha$, $\lambda^i(\alpha \oplus \beta) \approx \oplus_{p+q=i}(\lambda^p \alpha \otimes \lambda^q \beta)$. Also, for a rank n orientable bundle α , there is the Hodge duality $\lambda^i \alpha \approx \lambda^{n-i} \alpha$.

1.11 Application to immersions: From 1.5, and with $k = \mathbb{R}$, $\xi_1 \oplus \cdots \oplus \xi_s \approx n\varepsilon$,

$$\binom{n}{2} \varepsilon \approx \lambda^2(\xi_1 \oplus \cdots \oplus \xi_s) \approx \lambda^2 \xi_1 \oplus \cdots \oplus \lambda^2 \xi_s \oplus \sum_{i < j} \xi_i \otimes \xi_j,$$

$$\binom{n}{2} \varepsilon \approx \eta \oplus \tau, \text{ where } \eta = \sum_{i=1}^s \lambda^2 \xi_i.$$

By the Hirsch immersion theorem [19] F immerses in $\mathbb{R}^{\binom{n}{2}}$, with codimension = rank $\eta = \sum \binom{n_i}{2}$ (as long as $\sum \binom{n_i}{2} > 0$, i.e. $n_i \geq 2$ for some i). For further information about this, especially with regard to when $\binom{n}{2}$ is the best possible immersion dimension, see also [12], [44], [57].

1.12 We now consider the span of manifolds and state some of the basic definitions and theorems. The attached historical summary 1.16 may also be of interest, and gives references for the theorems below.

(a) Write $\alpha \approx \beta$ for isomorphism of vector bundles, and $\alpha \sim \beta$ for stable isomorphism (or stable equivalence), meaning $\alpha \oplus m\varepsilon \approx \beta \oplus n\varepsilon$ for some m, n .

Definition: $\text{span}(\alpha) = \max\{r : \alpha \approx r\varepsilon \oplus \gamma \text{ for some } \gamma\}$. The rank of γ , i.e. rank $\alpha - \text{span } \alpha$, is often called the geometric dimension (gd) of α .

(b) **Definition:** For a smooth M^n , $\text{span } M = \text{span } \tau_M$. Clearly $0 \leq \text{span } M \leq n$.

(c) The lower extreme, $\text{span } M = 0$, is settled by Hopf's theorem: For M connected, smooth, compact, without boundary, $\text{span } M = 0$ iff $X(M) \neq 0$. Equivalently, $\text{span } M > 0$ iff $X(M) = 0$. Since any odd dimensional manifold has Euler characteristic $\chi(M) = 0$, positive span is the case for "over half" the manifolds, in an approximate sense.

(d) The upper extreme is $\text{span } M = n$, M is then called parallelizable. Equivalently, $\tau_M \approx n\varepsilon$ is trivial. A more general condition is $\tau_M \sim 0$, i.e. τ_M is stably trivial; M is then called stably parallelizable. An example of a stably parallelizable manifold that is not parallelizable is S^2 . Our main interest in these lectures will be determining which flag manifolds are parallelizable (or stably parallelizable).

(e) For $n = 2^{4a+b}$ (odd), where $0 \leq b \leq 3$ (a, b are unique), let $\rho(n) = 8a + 2^b$, the Hurwitz-Radon function of n . Then $\text{span } S^{n-1} = \rho(n) - 1$.

Exercise: Show $\text{span } S^{n-1} < \frac{n-1}{2}$ unless $n = 2, 4, 8, 16$, and hence only S^1, S^3, S^7 are parallelizable.

(f) **Theorem** (Bredon-Kosinski): If M^n is stably parallelizable, and $n \neq 1, 3, 7$, then either $\text{span } M^n = \text{span } S^n (= \rho(n+1) - 1)$, or M is parallelizable.

The theorem contains further information to distinguish the two cases, which we omit here.

(g) Although the Bredon-Kosinski theorem effectively gives the span of stably parallelizable manifolds, we shall see in the work that follows that relatively few flag manifolds (and other homogeneous spaces) are stably parallelizable. For such manifolds (i.e. “most” manifolds), little is known about the span.

1.13 The classical flag manifold $F = F_{\mathbb{R}}(1, 1, \dots, 1)$.

Proceeding as in 1.11 and noting $\text{rank } \xi_i = 1$, hence $\lambda^2 \xi_i = 0$, we get $\tau_F \approx \binom{n}{2} \varepsilon$ and thus F is parallelizable.

Remark: $F_{\mathbb{C}}(1, \dots, 1)$ is stably parallelizable but not parallelizable, this is proved in [36] using a variant of the λ^2 construction for the complex case.

We will also be interested, in case $k = \mathbb{R}$, in the case where each (or later some) subspace σ_i carries an orientation. This is written $\tilde{F}(n_1, \dots, n_s)$, clearly $\tilde{F}(n_1, \dots, n_s) \cong SO(n)/H$, where $H = SO(n_1) \times \dots \times SO(n_s)$ (we also require that the orientations here on $\sigma_1, \dots, \sigma_s$ induce the standard orientation on $\mathbb{R}^n$, otherwise the space would have two components). Note that $\tilde{F}(n_1, \dots, n_s)$ is thus a 2^{s-1} -fold covering space of $F(n_1, \dots, n_s)$, hence it again follows that $\tau_{\tilde{F}} \approx \oplus_{i < j} \xi_i \otimes \xi_j$ is true here, where ξ_i, ξ_j are now oriented vector bundles.

1.14 The Grassmann manifold $\tilde{F}_{\mathbb{R}}(2, 2) = \tilde{G}_{4,2}$

Here $\xi_1 \oplus \xi_2 \approx 4\varepsilon$, each ξ_i is rank 2 and orientable. So,

$$6\varepsilon \approx \lambda^2(\xi_1) \oplus \lambda^2(\xi_2) \oplus \tau \approx \lambda^0(\xi_1) \oplus \lambda^0(\xi_2) \oplus \tau \quad (\text{by Hodge duality})$$

which reduces to $6\varepsilon \approx 2\varepsilon \oplus \tau$. Thus $\tilde{G}_{4,2}$ is stably parallelizable (but not parallelizable, indeed it has a non-zero Euler characteristic $\chi(\tilde{G}_{4,2}) = 4$).

Exercise: Show $\tilde{G}_{4,2} \cong S^2 \times S^2$ (Plücker coordinates are useful).

1.15 The Grassmann manifold $\tilde{F}_{\mathbb{R}}(3, 3) = \tilde{G}_{6,3}$

Here $\xi_1 \oplus \xi_2 \approx 6\varepsilon$, each ξ_i is rank 3 and orientable. So,

$$15\varepsilon \approx \lambda^2 \xi_1 \oplus \lambda^2 \xi_2 \oplus \tau \approx \xi_1 \oplus \xi_2 \oplus \tau \approx 6\varepsilon \oplus \tau,$$

and $\tilde{G}_{6,3}$ is therefore stably parallelizable. By the Bredon-Kosinski theorem, since its dimension is 9, its span is either 9 (i.e. it is parallelizable) or equal to $\text{span}(S^9) = 1$. But work of Leite and Miatello [38] implies $\text{span}(\tilde{G}_{6,3}) > 1$, thus it must be parallelizable.

1.16 An historical summary of the span of smooth manifolds

Early history .

- 1885 H. Poincaré, ideas of vector fields and span arise from differential equations
- 1912 L.E.J. Brouwer, the Brouwer fixed point theorem, index of singularities
- 1927 H. Hopf [22], $\text{span}(M) > 0$ if and only if $\chi(M) = 0$
- 1936 E. Stiefel [55], any orientable 3-manifold is parallelizable

Progress on span of spheres .

- 1942 B. Eckmann [15] (also 1923 Hurwitz [23] and 1923 Radon [47]), $\text{span}(S^{n-1}) \geq \rho(n) - 1$, (cf. 1.12(e))
1955-1960 upper bounds improved by N. Steenrod and J.H.C. Whitehead, by H. Toda, and by I.M. James; M. Kervaire and (independently) J. Milnor showed only S^1, S^3, S^7 are parallelizable
1962 J.F. Adams [1]: $\text{span}(S^{n-1}) = \rho(n) - 1$. Corollary: $\text{span}(\mathbb{R}P^{n-1}) = \rho(n) - 1$.

Progress on span of other smooth manifolds .

- 1964 W. Sutherland [58]: $V_{n,r}$ is parallelizable for $r \geq 2$
1966 G. Bredon and A. Kosinski [9], cf. 1.12(f)
1965-1970 progress on conditions for $\text{span}(M) \geq k$, $k = 2, 3$, Atiyah, Audin, Crabb, Dupont, Frank, Steer, Thomas
1969 E. Thomas [60] wrote a survey paper, Bull. AMS 75, 643-683
1967-1973 span of other quotients of S^n , J.C. Becker [8], T. Yoshida [63], also 1980 book of Mahammed, Piccinini, Suter [41]

1975-present, parallelizability question for smooth manifolds .

Grassmann manifolds $G_{n,r} (= G_{n,n-r})$: $G_{2,1} = \mathbb{R}P^1$, $G_{4,1} = \mathbb{R}P^3$, $G_{8,1} = \mathbb{R}P^7$ are parallelizable, no other Grassmann manifold is even stably parallelizable - Yoshida [62], Hiller and Stong [18], Bartík and Korbaš [4], Trew and Zvengrowski [61]

Oriented Grassmann manifolds $\tilde{G}_{n,r} (= \tilde{G}_{n,n-r})$: $\tilde{G}_{2,1} = S^1$, $\tilde{G}_{4,1} = S^3$, $\tilde{G}_{8,1} = S^7$, and $\tilde{G}_{6,3}$ are parallelizable, $\tilde{G}_{4,2}$ is stably parallelizable but not parallelizable, and no other is even stably parallelizable - Miatello and Miatello [45], Sankaran (dissertation), Kultze [35], Sankaran and Zvengrowski [49]

Projective Stiefel manifolds $X_{n,r}$: Antoniano, Gitler, Ucci, and Zvengrowski showed [3] $X_{4,r}$, $X_{8,r}$, $X_{16,8}$, $X_{n,n-1}$, $X_{2m,2m-2}$ are all parallelizable, and none of the others is even stably parallelizable with possible exception of $X_{12,8}$

Flag manifolds (also oriented, partially oriented flag manifolds): J. Korbaš [29], H. Hiller and R.E. Stong, P. Sankaran and P. Zvengrowski [49], [51]

Other homogeneous spaces: W. Singhof, W. Singhof and D. Wemmer [52], [53]

1980-present, further progress on span of smooth manifolds .

- 1979-present D. Randall, Ng Tse-Beng, new $\text{span} \geq k$ type results
1981 U. Koschorke [34] - many advances e.g. on $\text{span} \geq k$ conditions, on conditions for $\text{span} = \text{stable span}$ (span^0), counterexamples to the Eagle conjecture (a generalization of the Bredon-Kosinski theorem)
1984-present specific calculations of $\text{span}(G_{n,r})$, $\text{span}(X_{n,r})$, Korbaš, Sankaran, and Zvengrowski
1994 survey paper by J. Korbaš and P. Zvengrowski [32], see also [33] for an updated version of same
1998 Junod and Suter [27], span of product manifolds

2 Lecture

Stiefel Manifolds

We make a small digression from the main path of the lectures here to consider one specific family of manifolds, the Stiefel manifolds, in more detail. The Stiefel manifolds, $V_{n,r}$ are flag manifolds in the sense of Lect. 4, that is, “partially oriented” flag manifolds. Our goal is to show that for $r \geq 2$, $V_{n,r}$ is parallelizable. For references see [58], [17], [36], [64], [25] (in historical order).

2.1 Definition: $V_{n,r} \cdots$, for $1 \leq r \leq n$, is the space of orthonormal r -frames of vectors in $\mathbb{R}^n$.

We denote a point of $V_{n,r}$ as $\begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}$, where $a_i \in \mathbb{R}^n$ and $\langle a_i, a_j \rangle = \delta_{ij}$. One gives this space the structure of a smooth manifold by writing it as a homogeneous space

$$V_{n,r} = O(n) / (I_r \times O(n-r))$$

Familiar examples are $V_{n,1} = S^{n-1}$, $V_{n,n} = O(n)$, $V_{n,n-1} \cong SO(n)$. There are smooth fibrations $S^{n-r-1} \hookrightarrow V_{n,r+1} \xrightarrow{\pi} V_{n,r}$, where

$$\pi \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ a_{r+1} \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}.$$

2.2 There is also a smooth fibration, in fact a 2^r -fold covering projection, $\varphi : V_{n,r} \twoheadrightarrow F_{\mathbb{R}}(1, \dots, 1, n-r)$, where

$$\varphi \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} = (\text{span } \{a_1\}, \dots, \text{span } \{a_r\}, \text{span}^\perp \{a_1, \dots, a_r\}).$$

Now $F_{\mathbb{R}}(1, \dots, 1, n-r)$ has the canonical bundles $\xi_1, \dots, \xi_r, \xi_{r+1}$. Clearly $\varphi^*(\xi_i) \approx \varepsilon$, $1 \leq i \leq r$, and we call $\beta_{n,r} := \varphi^*(\xi_{r+1})$ the orthogonal complement bundle (of rank $n-r$). The next lemma is an easy application of 1.3, 1.5, and Cor. 1.10, its proof is left as an exercise. Here $\tau_{n,r} = \tau(V_{n,r})$.

Lemma:

- (a) $\beta_{n,r} \oplus r\varepsilon \approx n\varepsilon$,
- (b) $\tau_{n,r} \approx r\beta_{n,r} \oplus \binom{r}{2}\varepsilon$,
- (c) $\dim V_{n,r} = nr - \binom{r+1}{2}$.

Remark: The complex and quaternionic Stiefel manifolds of orthonormal r -frames in $\mathbb{C}^n, \mathbb{H}^n$ can be defined similarly. There is no change in the above lemma except for (c), which becomes real $\dim = dnr - r - d\binom{r}{2}$, $d = 1, 2, 4$ respectively.

2.3 Theorem: $V_{n,r}$ is parallelizable, $r \geq 2$.

This theorem was first proved by Sutherland [58]. A little later Handel [17] and Lam [36] noticed that for $r \geq 3$ the theorem could be proved by elementary vector bundle techniques. Alternatively, notice that Lemma 2.2 trivially shows $V_{n,r}$ is stably parallelizable, $r \geq 1$, and the Bredon-Kosinski theorem will then give a quick proof that $V_{n,r}$ is parallelizable, $r \geq 2$. Here, however, we shall follow [64] to give a complete proof of Theorem 2.3 using nothing more than a basic theorem of homotopy theory, the Hopf classification theorem: for $n \geq 2$, $\dim X \leq n$, and Y $(n-1)$ -connected,

$$[X, Y] \approx H^n(X; \pi_n(Y)) .$$

First, an alternative description of the tangent bundle $\tau_{n,r}$.

2.4 Lemma: Let $A = \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} \in V_{n,r}$. Then

$$(\tau_{n,r})_A = \left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} : v_i \in \mathbb{R}^n, \langle a_i, v_j \rangle + \langle v_i, a_j \rangle = 0, 1 \leq i, j \leq r \right\} .$$

Proof: Consider a smooth curve $A(t) = \begin{pmatrix} a_1(t) \\ \vdots \\ a_r(t) \end{pmatrix}$, with $a_i(0) = v_i$. Differentiating the equations $\langle a_i(t), a_j(t) \rangle = 0$ with respect to t and setting $a_i(0) = v_i$ gives the desired result, so all such $\begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} \in (\tau_{n,r})_A$. On the other hand these equations reduce to the $r + \binom{r}{2} = \binom{r+1}{2}$ equations $\langle a_i, v_i \rangle = 0$, $1 \leq i \leq r$, and $\langle a_i, v_j \rangle + \langle a_j, v_i \rangle = 0$, $1 \leq i < j \leq r$, so the solution space has dimension $\geq nr - \binom{r+1}{2} = \dim V_{n,r}$ by 2.2 (c). This shows that the solution space is $(\tau_{n,r})_A$, and also that the set of equations are linearly independent, i.e. have rank $\binom{r+1}{2}$. $\square$

Corollary: For $r \geq 2$, $\tau_{n,r}$ admits $\binom{r}{2}$ linear sections

$$w_{ij}(A) = \begin{pmatrix} v_1(A) \\ \vdots \\ v_r(A) \end{pmatrix}, \text{ where } v_k(A) = \begin{cases} 0 & , \quad k \neq i, j \\ -a_j & , \quad k = i, \quad i < j \\ a_i & , \quad k = j \end{cases} .$$

2.5 We now turn to the case $r = 2$, and follow some ideas of Eckmann [14]. The 3 equations $\langle a_1, v_1 \rangle + \langle a_2, v_2 \rangle = \langle a_1, v_2 \rangle = \langle a_2, v_1 \rangle = 0$ are considered as a map $f : V_{n,2} \rightarrow V_{2n,3}$,

$$f \left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \\ a_2/\sqrt{2} & a_1/\sqrt{2} \end{pmatrix} ,$$

and a unit vector field on $V_{n,2}$ is simply a lifting g of f to $V_{2n,4}$,

$$\text{i.e. } g \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \\ a_2/\sqrt{2} & a_1/\sqrt{2} \\ v_1 & v_2 \end{pmatrix},$$

indeed an r -field (orthonormal) will be a lifting g as in

$$\begin{array}{ccc} & & V_{2n,3+r} \\ & \nearrow g & \downarrow \pi \\ V_{n,2} & \xrightarrow{f} & V_{2n,3} \end{array}.$$

Since π is a Serre fibration, such a lifting always exists (for any $r \leq \dim V_{n,2} = 2n - 3$) if $f \simeq 0$, and this is what we shall show, thus proving $V_{n,2}$ parallelizable.

2.6 Theorem: $V_{n,2}$ is parallelizable.

Proof: As shown in 2.5, it suffices to prove f null homotopic. Using some basic topology and the $Z/2$ -cohomology of the Stiefel manifolds, one knows $V_{2n,3}$ is $(2n - 4)$ -connected, $\dim V_{n,2} = 2n - 3$, so the Hopf classification theorem applies. Also $\pi_{2n-3}(V_{2n,3}) \approx H_{2n-3}(V_{2n,3}) \approx H_{2n-3}(\mathbb{R}P^{2n-1}/\mathbb{R}P^{2n-4}) \approx Z/2$. So by the Hopf classification it suffices to show

$$0 = f^* : H^{2n-3}(V_{2n,3}; Z/2) \rightarrow H^{2n-3}(V_{n,2}; Z/2).$$

Using the Cor. to 2.4, we have a lift g

$$\begin{array}{ccc} & & V_{2n,4} \\ & \nearrow g & \downarrow \pi \\ V_{n,2} & \xrightarrow{f} & V_{2n,3} \end{array}, \quad \text{where } g \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \\ a_2/\sqrt{2} & a_1/\sqrt{2} \\ -a_1/\sqrt{2} & a_2/\sqrt{2} \end{pmatrix}.$$

Now $H^{2n-3}(V_{2n,3}; Z/2) \approx Z/2$, generated by a class we write y_{2n-3} . Since $\pi^*(y_{2n-3}) = y_{2n-3} \in H^{2n-3}(V_{2n,4}) \approx Z/2$, it suffices to show $g^*(y_{2n-3}) = 0$. Now fairly obvious homotopies of $V_{2n,4}$ show $g \simeq g_1 \simeq g_2$, where

$$g_1(A) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \\ a_2 & 0 \\ 0 & a_1 \end{pmatrix}, \quad g_2(A) = \begin{pmatrix} a_1 & 0 \\ a_2 & 0 \\ 0 & a_1 \\ 0 & a_2 \end{pmatrix}$$

Let $\sigma : V_{n,2} \times V_{n,2} \rightarrow V_{2n,4}$, $\sigma(A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, so that $g_2 = \sigma \Delta$, where $\Delta : V_{n,2} \rightarrow V_{n,2} \times V_{n,2}$ is the diagonal. Consider the involution $\omega : V_{2n,4} \rightarrow V_{2n,4}$ which interchanges the first two rows with the last two and similarly for the columns. Clearly $\omega \sigma = \sigma \tau$,

where $\tau : V_{n,2} \times V_{n,2} \rightarrow V_{n,2} \times V_{n,2}$ switches the two factors. Now ω being an involution and $H^{2n-3}(V_{2n,4}; Z/2) \approx Z/2$, generated by y_{2n-3} , implies $\omega^*(y_{2n-3}) = y_{2n-3}$. It follows that $\tau^*\sigma^*(y_{2n-3}) = \sigma^*\omega^*(y_{2n-3}) = \sigma^*(y_{2n-3})$, i.e. $\sigma^*(y_{2n-3})$ is invariant under τ^* . But $H^*(V_{n,2}; Z/2)$ has only the classes $x_{n-2}, x_{n-1}, x_{n-1}x_{n-2}$ in respective dimensions $n-2, n-1, 2n-3$. Thus $\sigma^*(y_{2n-3}) = \alpha(x_{n-2} \times x_{n-1} + x_{n-1} \times x_{n-2}) + \beta(x_{n-1}x_{n-2} \times 1 + 1 \times x_{n-1}x_{n-2})$, for some $\alpha, \beta \in Z/2$. Taking Δ^* of this gives $0 = \Delta^*\sigma^*(y_{2n-3}) = g_2^*(y_{2n-3}) = g^*(y_{2n-3})$, as required. $\square$

2.7 Lemma: Under $p : V_{n,r+1} \rightarrow V_{n,r}$, $p^*(\tau_{n,r}) \oplus \beta_{n,r+1} \approx \tau_{n,r+1}$.

The proof is left as an exercise, using either 2.2 or 2.4.

2.8 Theorem: $V_{n,r}$ is parallelizable, $r \geq 2$.

Proof: Note $n \geq r+1$ and $r \geq 2$ imply $2r \geq r+2$ and hence $2nr \geq (r+2)(r+1)$, or equivalently $d_{n,r} = nr - \binom{r+1}{2} \geq r+1$.

Now use induction, $2 \leq r \leq n-1$, and suppose $\tau_{n,r}$ trivial.

Theorem 2.6 starts the induction, and

$$\begin{aligned} \tau_{n,r+1} \approx p^*(\tau_{n,r}) \oplus \beta_{n,r+1} &\approx d_{n,r}\varepsilon \oplus \beta_{n,r+1} \text{ (inductive hypothesis)} \\ &\approx (d_{n,r} - r - 1)\varepsilon \oplus (r+1)\varepsilon \oplus \beta_{n,r+1} \\ &\approx (d_{n,r} - r - 1)\varepsilon \oplus n\varepsilon \approx d_{n,r+1}\varepsilon \end{aligned}$$

using that $d_{n,r} - r - 1 \geq 0$ as shown above. $\square$

Remark: The proofs for $k = \mathbb{C}, \mathbb{H}$ are similar, except that the case $r = 2$ (Theorem 2.6) turns out to be much easier as far as showing $f^* = 0$. See [63] for details.

2.9 Questions: (a) Can one give a fully elementary proof that $V_{n,2}$ is parallelizable? For $n \equiv 0 \pmod{2}$, yes (cf. [36] for the case $n \equiv 0 \pmod{4}$), or try as an Exercise).

(b) Can one exhibit the vector fields on $V_{n,2}$, $V_{n,r}$, say with polynomial formulae? For $n = 4, 8$, done in [64].

(c) (Daciberg Lima Gonçalves): When does $\beta_{n,r}$ admit a section. Answer: $r = 1$ and n even, $r = n-1$, or $(n, r) = (7, 2), (8, 3)$.

(d) (Daciberg Lima Gonçalves): When does $\ell\beta_{n,r}$ admit a section, $\ell > 1$? What is $\text{span}(\ell\beta_{n,r})$?

2.10 Definition: $X_{n,r} = V_{n,r}/(A \sim -A)$, $1 \leq r \leq n-1$ is the projective Stiefel manifold. A very familiar example is $X_{n,1} = \mathbb{R}P^{n-1}$. One has obvious covering projections

$$V_{n,r} \xrightarrow{(2)} X_{n,r} \xrightarrow[(2^{r-1})]{q} F(1, \dots, 1, n-r),$$

so $\dim X_{n,r} = \dim V_{n,r} = nr - \binom{r+1}{2}$, also there is a Hopf line bundle $\xi_{n,r}$ over $X_{n,r}$. One has $q^*(\xi_i) \approx \xi_{n,r}$, $1 \leq i \leq r$, and defines $q^*(\xi_{r+1}) := \beta'_{n,r}$ (rank $n-r$). For $r = 1$, one usually uses the convention $\xi_{n,1} = \xi_{n-1}$.

3 Lecture

Parallelizability of Standard Flag Manifolds

The main references here are [30], [48]. For results about the parallelizability of other homogeneous spaces, using Lie algebra techniques and especially the adjoint representation, cf. [52], [53].

3.1 Preliminary facts (to be dealt with in Lecture 5).

(a) For $k = \mathbb{R}$, all Grassmann manifolds $F_{\mathbb{R}}(p, q) = G_{p+q, p} \cong G_{p+q, q}$ are not stably parallelizable, except for $p = 1, q = 1, 3, 7$ (or vice-versa) which are parallelizable, namely $G_{2,1} = \mathbb{R}P^1$, $G_{4,1} = \mathbb{R}P^3$, $G_{8,1} = \mathbb{R}P^7$. For $k = \mathbb{C}, \mathbb{H}$, only $F_k(1, 1)$ are stably parallelizable (and not parallelizable).

(b) The projective Stiefel manifolds $X_{n,2}$ are not stably parallelizable unless $n = 3, 4, 8$, and $X_{3,2}, X_{4,2}, X_{8,2}$ are parallelizable.

3.2 Inclusion method Many inclusion and fibration maps exist between flag manifolds. The inclusions are obtained by increasing the size of the ambient space k^n , and the fibrations by collapsing some of the subspaces of a flag into their (direct) sum, i.e. into the single subspace they span.

Example: Let $F' = F_k(n_1, \dots, n_{s-1})$, $F = F_k(n_1, \dots, n_s)$, and θ be the span $\{e_{n-n_s+1}, \dots, e_n\} \subseteq k^n$. Then $i : F' \hookrightarrow F$, where $i(\sigma_1, \dots, \sigma_{s-1}) = (\sigma_1, \dots, \sigma_{s-1}, \theta)$ is an inclusion of flag manifolds.

Let $p : F \rightarrow F(m, n_s)$, where $m = n_1 + \dots + n_{s-1} = n - n_s$, be given by $p(\sigma_1, \dots, \sigma_{s-1}, \sigma_s) = (\sigma_1 \oplus \dots \oplus \sigma_{s-1}, \sigma_s)$. Then p is a fibration and its fibre is precisely F' with inclusion i .

Exercise: Show this same fact using the homogeneous space definition for $F', F, F(m, n_s)$. The key to using the inclusion method is the next lemma.

3.3 Lemma: $i^*(\tau_F) \sim \tau_{F'}$.

Proof: Clearly $i^*(\xi_i) \approx \xi'_i$, $1 \leq i \leq s-1$, while $i^*(\xi_s) \approx n_s \varepsilon$. By Theorem 1.10,

$$\begin{aligned} i^*(\tau_F) &= i^* \left(\sum_{1 \leq i < j \leq s-1} \text{Hom}_k(\xi_i, \xi_j) \oplus \sum_{i=1}^{s-1} \text{Hom}_k(\xi_i, \xi_s) \right) \\ &\approx \sum_{1 \leq i < j \leq s-1} \text{Hom}_k(\xi'_i, \xi'_j) \oplus \sum_{i=1}^{s-1} \text{Hom}_k(\xi'_i, n_s \varepsilon) \\ &\approx \tau_{F'} \oplus \text{Hom}_k \left(\sum_{i=1}^{s-1} \xi'_i, n_s \varepsilon \right) \approx \tau_{F'} \oplus \text{Hom}_k((n - n_s) \varepsilon, n_s \varepsilon) \sim \tau_{F'}. \square \end{aligned}$$

Remark: This is a special case of the fact that for any smooth fibration, the inclusion of the fibre has trivial normal bundle (for consider a small enough coordinate neighborhood U of a point $x \in X$, the base space, then $p^{-1}(U)$ is diffeomorphic to $U \times \text{fibre}$).

3.4 Theorem: Let $s \geq 3$, $k = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

- (i) $F_k(1, \dots, 1)$ is stably parallelizable, and is parallelizable iff $k = \mathbb{R}$,
- (ii) If $n_i > 1$ for some i , then $F_k(n_1, \dots, n_s)$ is not stably parallelizable.

Proof: Without loss of generality suppose $n_1 \geq n_2 \geq \dots \geq n_s$. For (i) the case $k = \mathbb{R}$ is already done in 1.13, and it was remarked there that for $k = \mathbb{C}$ (also $\mathbb{H}$) $F_k(1, \dots, 1)$ is stably parallelizable. In the cases $k = \mathbb{C}, \mathbb{H}$, $\chi(F_k(1, \dots, 1)) \neq 0$ so it cannot be parallelizable, indeed has $\text{span} = 0$ by 1.12 (c).

For (ii), suppose $F_k(n_1, \dots, n_s)$ is stably parallelizable, then by the inclusion method $F_k(n_1, n_2)$ is also stably parallelizable. By 3.1(a), the only possibilities are $k = \mathbb{R}$, $(n_1, n_2) = (1, 1), (3, 1), (7, 1)$, or $k = \mathbb{C}, \mathbb{H}$ and $(n_1, n_2) = (1, 1)$. So, if also $n_1 > 1$, it is only possible that $k = \mathbb{R}$ and $(n_1, n_2) = (3, 1), (7, 1)$. The proof will then be completed, again by the inclusion method, if we can show $F_{\mathbb{R}}(3, 1, 1)$ and $F_{\mathbb{R}}(7, 1, 1)$ not stably parallelizable. The double cover $X_{5,2} \rightarrow F_{\mathbb{R}}(3, 1, 1)$ and $X_{5,2}$ not stably parallelizable (see 3.1 (b)) shows $F_{\mathbb{R}}(3, 1, 1)$ not stably parallelizable, and similarly for $F_{\mathbb{R}}(7, 1, 1)$ using $X_{9,2}$. $\square$

4 Lecture

Parallelizability of Partially Oriented Flag Manifolds

The main reference for this lecture is [49], and $k = \mathbb{R}$ throughout.

4.1 Definition: $F(n_1, \dots, n_r \mid n_{r+1}, \dots, n_s)$ is the usual flag manifold $F(n_1, \dots, n_s)$ except that the subspaces $\sigma_1, \dots, \sigma_r$ also carry an orientation, with the convention that if $r = s$ then the orientations on $\sigma_1, \dots, \sigma_s$ induce the standard orientation of $\mathbb{R}^n$.

Thus $F(n_1, \dots, n_r \mid n_{r+1}, \dots, n_s)$ is a 2^r covering space of the standard flag manifold $F(n_1, \dots, n_s)$, $r < s$ (or a 2^{r-1} fold covering if $r = s$). Note that with the above convention $F(n_1, \dots, n_{s-1} \mid n_s) \cong F(n_1, \dots, n_s \mid)$, and we write this space $\tilde{F}(n_1, \dots, n_s)$ in analogy to the oriented Grassmann manifolds $\tilde{G}_{n,r}$. As a homogeneous space $F(n_1, \dots, n_r \mid n_{r+1}, \dots, n_s)$, for $r \leq s - 2$, is $O(n)/SO(n_1) \times \dots \times SO(n_r) \times O(n_{r+1}) \times \dots \times O(n_s)$, while $\tilde{F}(n_1, \dots, n_s) \cong SO/SO(n_1) \times \dots \times SO(n_s)$.

- 4.2 Examples:** (a) $\tilde{F}(n_1, n_2) = \tilde{G}_{n,n_1} = \tilde{G}_{n,n_2}$, in particular $\tilde{F}(1, n-1) = S^{n-1}$,
 (b) $F(1, \dots, 1 \mid n-r) = \tilde{F}(1, \dots, 1, n-r) = V_{n,r}$,
 (c) $F(m \mid 1, 1) \cong X_{m+2,2}$.

4.3 The tangent bundle τ of $F(n_1, \dots, n_r \mid n_{r+1}, \dots, n_s)$ is given by the same formula as 1.10, Cor., since it pulls back under the covering projection. Thus $\tau \approx \sum_{i < j} \xi_i \otimes \xi_j$, with the extra information that ξ_i is now an oriented bundle of rank n_i , $1 \leq i \leq r$.

The techniques used now are for the most part (but not entirely) the same as in the previous work: the λ^2 -construction for positive results, and the inclusion method for negative results.

4.4 Preliminary facts (to be dealt with in Lec. 5)

(a) $\tilde{F}(p, q) = \tilde{G}_{p+q, p} = \tilde{G}_{p+q, q}$ is stably parallelizable iff $p = 1$, or $q = 1$, or $(p, q) = (2, 2), (3, 3)$, and of these only $\tilde{G}_{2,1} = S^1$, $\tilde{G}_{4,1} = S^3$, $\tilde{G}_{8,1} = S^1$, $\tilde{G}_{6,3}$ are parallelizable.

(b) $X_{n,2}$ is not stably parallelizable unless $n = 3, 4, 8$ (as in 3.1 (b), also $X_{10,4}$ is not stably parallelizable. $X_{3,2}, X_{4,2}, X_{8,2}$ are all parallelizable.

We now study the parallelizability (stable parallelizability) of these manifolds. We consider only $s \geq 3$, since the case $s = 2$ is already done in 4.4 (a). And, without loss of generality, we always suppose $n_1 \geq \dots \geq n_r$, $n_{r+1} \geq \dots \geq n_s$.

4.5 Theorem: Suppose $n_2 > 1$. Then $\tilde{F}(n_1, n_2, \dots, n_s)$ is not stably parallelizable unless $\{n_1, \dots, n_s\} \subseteq \{3, 1\}$ or $\{n_1, \dots, n_s\} \subseteq \{2, 1\}$.

Proof: Write $n_1 \geq n_2 \geq \dots \geq n_p > n_{p+1} = \dots = n_s = 1$. By the inclusion method, $\tilde{F}(n_1, \dots, n_s)$ sp implies $\tilde{F}(n_i, n_j)$ sp for any $1 \leq i < j \leq s$. In particular, for $1 \leq i < j \leq p$, using 4.4(a), we see $n_i = n_j = 2$ or $n_i = n_j = 3$. This completes the proof. $\square$

Corollary: Suppose at least two of $n_1, \dots, n_s$ are > 1 . Then $F(n_1, \dots, n_r \mid n_{r+1}, \dots, n_s)$ is not stably parallelizable unless $\{n_1, \dots, n_s\} \subseteq \{3, 1\}$ or $\{2, 1\}$.

This follows because the stable parallelizability of $F(n_1, \dots, n_r \mid n_{r+1}, \dots, n_s)$ would imply the stable parallelizability of its covering space $\tilde{F}(n_1, \dots, n_s)$.

Remark: We are always assuming $r \geq 1$, otherwise we are back in the case of standard flag manifolds dealt with in Lec. 3.

4.6 Lemma: The following are stably parallelizable: (a) $F(1, \dots, 1 \mid 1, \dots, 1)$,

(b) $F(2, \dots, 2, 1, \dots, 1 \mid 1, \dots, 1)$,

(c) $\tilde{F}(3, \dots, 3, 1, \dots, 1)$.

Proof: (a) Here we have a covering space of $F(1, \dots, 1)$ so it is in fact parallelizable by 1.13.

(b) Left as an exercise - use λ^2 .

(c) Here $\xi_1 \oplus \dots \oplus \xi_k \oplus \xi_{k+1} \oplus \dots \oplus \xi_s \approx n\varepsilon$, with $\xi_1, \dots, \xi_k$ oriented rank 3, $\xi_{k+1}, \dots, \xi_s$ oriented rank 1, i.e. trivial line bundles. So $\xi_1 \oplus \dots \oplus \xi_k \oplus (s-k)\varepsilon \approx n\varepsilon$. The λ^2 construction gives

$$\tau \oplus \lambda^2 \xi_1 \oplus \dots \oplus \lambda^2 \xi_k \oplus \binom{s-k}{2} \varepsilon \approx \binom{n}{2} \varepsilon, \text{ now Hodge duality gives}$$

$$\tau \oplus \lambda^1 \xi_1 \oplus \dots \oplus \lambda^1 \xi_k \oplus \binom{s-k}{2} \varepsilon \approx \binom{n}{2} \varepsilon, \text{ and } \lambda^1 \xi_i \approx \xi_i \text{ gives}$$

$$\tau \oplus (\xi_1 \oplus \dots \oplus \xi_k) \oplus \binom{s-k}{2} \varepsilon \approx \binom{n}{2} \varepsilon,$$

$$\tau \oplus (\xi_1 \oplus \dots \oplus \xi_k \oplus (s-k)\varepsilon) \oplus \binom{s-k}{2} \varepsilon \approx \binom{n}{2} \varepsilon,$$

$$\tau \oplus n\varepsilon \oplus \binom{s-k}{2}\varepsilon \approx \binom{n}{2}\varepsilon, \text{ i.e. } \tau \sim 0. \quad \square$$

Notice that 4.4 and 4.5 completely determine which $\tilde{F}(n_1, \dots, n_s)$ are stably parallelizable, so we may suppose $1 \leq r \leq s-2$ henceforth. The next theorem gives most of the remaining negative results.

4.7 Theorem: Consider $F = F(n_1, \dots, n_r \mid n_{r+1}, \dots, n_s)$, $1 \leq r \leq s-2$, and suppose either precisely one $n_i > 1$, or $\{n_1, \dots, n_s\} = \{3, 1\}$, or $\{n_1, \dots, n_s\} = \{2, 1\}$ with $n_{r+1} = 2$. Then F is not stably parallelizable in the following cases.

- (i) $F(n_1, \dots, n_r \mid 1, 1, \dots, 1)$, $n_i \neq 1, 2, 6$ for some $i \leq r$,
- (ii) $F(n_1, \dots, n_r \mid 1, 1, \dots, 1)$, $r \leq s-4$, $n_i \neq 1, 2$ for some $i \leq r$,
- (iii) $n_{r+1} > 1$, $r < s-2$,
- (iv) $n_{r+1} > 1$, $r = s-2$, and $(n_{s-1}, n_s) \neq (3, 1), (7, 1)$.

Proof: For (i), the inclusion $F(n_i \mid 1, 1) = X_{n_i+2,2} \hookrightarrow F$ shows F is not stably parallelizable unless $n_i = 1, 2, 6$ (using 4.4 (b)).

For (ii), we may suppose using (i) that $n_i = 6$, so by inclusion it suffices to show $F(6 \mid 1, 1, 1, 1)$ not stably parallelizable. But $X_{10,4} \rightarrow F(6 \mid 1, 1, 1, 1)$ is a (8-fold) covering and $X_{10,4}$ is not stably parallelizable (4.4(b)), hence $F(6 \mid 1, 1, 1, 1)$ is not stably parallelizable.

For (iii) the inclusion $F(n_{r+1}, n_{r+2}, \dots, n_s) \hookrightarrow F$ and Theorem 3.4 suffices.

For (iv), if $(n_{s-1}, n_s) \neq (3, 1), (7, 1)$, the inclusion $F(n_{s-1}, n_s) \hookrightarrow F$ shows F not stably parallelizable. $\square$

4.8 If $r = s-2$, and $(n_{s-1}, n_s) = (7, 1)$, and $n_1 > 1$ then F is not stably parallelizable by Cor. 4.5. However we need one extra negative result if $(n_{s-1}, n_s) = (3, 1)$ (Cor. 4.5 does not help here).

Theorem: For $k \geq 1$, $F = F(3, \dots, 3, 1, \dots, 1 \mid 3, 1)$ is not stably parallelizable.

Proof: One shows $F(3 \mid 3, 1)$ not stably parallelizable by computing its Stiefel-Whitney classes, see [49] for details. The inclusion method then completes the proof.

4.9 Proposition: $F(6 \mid 1, 1)$ and $F(6, 1 \mid 1, 1)$ are parallelizable.

Proof: The first equals $X_{8,2}$ and is covered by 4.4 (b). The second is more delicate, see [49] for details.

At this stage only a few questions remain. First, in the positive cases 4.6 (b), (c), to determine which are parallelizable (in 4.6(a) all were observed to be parallelizable). Using results of Miatello-Miatello [45] and Stong [56], one can see that all manifolds in 4.6(b), (c) are parallelizable except for $\tilde{F}(2, 2, \dots, 2)$ and $\tilde{F}(2, \dots, 2, 1)$. As for the negative results (for stably parallelizable), there are four unsolved families, as follows.

$$\begin{aligned}
4.10 \text{ Unsolved: } & F(1, \dots, 1 \mid 3, 1) \quad , \quad s \geq 3 , \\
& F(1, \dots, 1 \mid 7, 1) \quad , \quad s \geq 3 , \\
& F(6, 1, \dots, 1 \mid 1, 1) \quad , \quad s \geq 5 , \\
& F(6, 1, \dots, 1 \mid 1, 1, 1) \quad , \quad s \geq 4 .
\end{aligned}$$

To see this, first consider any case with at least two $n_i, n_j > 1$. Using 4.5, $\{n_1, \dots, n_s\} =$ either $\{2, 1\}$ or $\{3, 1\}$. If $\{2, 1\}$, then $n_{r+1} = 2$ is eliminated by 4.7 (iii), (iv), leaving $F(2, \dots, 2, 1, \dots, 1 \mid 1, \dots, 1)$ which is stably parallelizable by 4.6. If $\{n_1, \dots, n_s\} = \{3, 1\}$, then 4.7 (i), (iii) and 4.8 eliminate all possibilities except $\tilde{F}(3, \dots, 3, 1, \dots, 1)$, and this is known to be stably parallelizable by 4.6.

This leaves only the cases where precisely one $n_q > 1$, and these are easily seen to be the families listed, using 4.7.

4.11 For completeness, we list what is now known.

Theorem: Let $F = F(n_1, \dots, n_r \mid n_{r+1}, \dots, n_s)$, with $r \geq 1$ and $s \geq 3$. The following are all stably parallelizable and apart from $\tilde{F}(2, 2, \dots, 2)$, $\tilde{F}(2, \dots, 2, 1)$ are parallelizable:

- (i) $F(1, \dots, 1 \mid 1, \dots, 1)$,
- (ii) $\tilde{F}(n_1, 1, \dots, 1) = V_{n_1+1, s}$,
- (iii) $\tilde{F}(3, \dots, 3, 1, \dots, 1)$,
- (iv) $F(2, \dots, 2, 1, \dots, 1 \mid 1, \dots, 1)$,
- (v) $F(6 \mid 1, 1) = X_{8,2}$, $F(6, 1 \mid 1, 1)$.

Any other flag manifold of this type is not stably parallelizable, apart from the four undecided families listed in 4.10. Our main goal in the following lectures is to introduce the tool of K -theory and show how it leads to solutions for the undecided families.

5 Lecture

Introducing K -theory

5.1 In this lecture we shall consider the Stiefel and projective Stiefel manifolds, the Grassmann and oriented Grassman manifolds. These were already used in the “preliminaries” of Lec. 3,4. We shall now see how the use of K -theory can give quick proofs of parallelizability results for these manifolds. The Stiefel manifolds $V_{n,r}$ will be omitted, they have already been dealt with in Lec. 2 and in any case are all stably parallelizable (parallelizable for $r \geq 2$).

5.2 To define K -theory (see also [24], Ch. 9), one starts with the Grothendieck completion of an abelian semi-group A , the same process by which one obtains the integers from the non-negative integers in elementary algebra. So let A be an abelian semi-group.

Definition: $\bar{A} = \{(a, b) : a, b \in A\} / \sim$, where $(a, b) \sim (a', b')$ iff $a + b' = a' + b$. One checks that $\sim$ is an equivalence relation. Then define $(a, b) + (c, d) = (a + c, b + d)$, check this is well defined, and $\bar{A}$ is now an abelian group with $0 = (a, a)$ for any $a \in A$ and $-(a, b) = (b, a)$. Further $A \subseteq \bar{A}$ with $a \mapsto (a, 0)$.

5.3 Now consider $A =$ abelian semigroup of isomorphism classes of (constant rank) F -vector bundles over a space X , with $[\alpha] + [\beta] = [\alpha \oplus \beta]$.



Definition: $\overline{A} = K_F(X)$.

Remarks: (a) The requirement of constant rank is redundant if X is path connected.

(b) $\text{rank} : A \rightarrow Z$ induces a well defined homomorphism $\text{rank} : K(X) \rightarrow Z$.

(c) Recall that for vector bundles of finite type (for X paracompact all vector bundles are of finite type), say α , there exists a vector bundle β such that $\alpha \oplus \beta$ is trivial (cf. [24], p. 32).

5.4 Definition: $\widetilde{K}_F(X) = \ker(\text{rank} : K_F(X) \rightarrow Z)$.

Notice that $K_F(X)$ also has a ring structure given by $[\alpha] \cdot [\beta] = [\alpha \otimes \beta]$, this induces a ring structure on $\widetilde{K}_F(X)$ as well. Since $\alpha \otimes \beta \approx \beta \otimes \alpha$, it is a commutative ring.

Remark: $\widetilde{K}_F(X)$ can equivalently be defined as the equivalence classes of stable vector bundles over X , but this definition needs care for multiplication since $\alpha_1 \sim \alpha_2, \beta_1 \sim \beta_2$ does not in general imply $\alpha_1 \otimes \beta_1 \sim \alpha_2 \otimes \beta_2$.

Remark: Notice that for a vector bundle α of rank n , $\alpha \sim 0$ iff $[\alpha] = 0 \in \widetilde{K}_F(X)$ iff $[\alpha] - n = 0 \in K_F(X)$. Thus K -theory is an ideal tool for detecting stable parallelizability.

5.5 One makes K -theory into a cohomology theory by defining $\widetilde{K}_F^{-p}(X) = \widetilde{K}_F(\Sigma^p X)$, where ΣX is the reduced suspension of X . According to the remarkable Bott periodicity theorems, $\widetilde{K}^{-p}(X) \approx \widetilde{K}^{-p+2}(X)$ and $\widetilde{KO}^{-p}(X) \approx \widetilde{KO}^{-p+8}(X)$; these allow us to define $\widetilde{K}_F^i(X)$ and hence also $K_F^i(X)$ for all $i \in Z$. Also, for $f : X \rightarrow Y$, the induced bundle construction $f^*(\beta)$, for any vector bundle β over Y , gives rise to a well defined homomorphism

$$\begin{aligned} f^* : K_F(Y) &\longrightarrow K_F(X) \\ [\beta] &\longmapsto [f^*\beta] \end{aligned} ,$$

similarly $\Sigma^p f$ gives rise to $f^* : K_F^{-p}(Y) \longrightarrow K_F^{-p}(X)$. Altogether, these definitions turn the graded object $K_F^* = \{, \dots, K_F^{-1}, K_F, K_F^1, K_F^2, \dots, \}$ into a cohomology theory.

5.6 Recall (Lec. 3) the projective Stiefel manifold $X_{n,r}$, the covering projections $V_{n,r} \xrightarrow{(2)} X_{n,r} \xrightarrow{(2^{r-1})} F(1, \dots, 1, n-r)$, the Hopf bundle $\xi_{n,r} \approx q^*(\xi_i)$, $1 \leq i \leq r$, and $\beta'_{n,r} = q^*(\xi_{r+1})$ of rank $n-r$. Also define $\beta_{n,r} = \beta'_{n,r} \otimes \xi_{n,r}$ (also rank $n-r$), and recall $\xi \otimes \xi \approx \varepsilon$ for any real line bundle ξ . We have the following basic isomorphisms.

- Lemma:**
- (a) $\beta'_{n,r} \oplus r\varepsilon \approx n\xi_{n,r}$,
 - (b) $\beta_{n,r} \oplus r\xi_{n,r} \approx n\varepsilon$,
 - (c) $\tau_{n,r} \approx r\beta'_{n,r} \oplus \binom{r}{2}\varepsilon$,
 - (d) $\tau_{n,r} \oplus \binom{r+1}{2}\varepsilon \approx nr\xi_{n,r} \quad (\implies \tau_{n,r} \sim nr\xi_{n,r})$
 - (e) $\tau_{n,r} \oplus \lambda^2\beta_{n,r} \approx \binom{n}{2}\varepsilon$.

Exercise: Derive (b) from (a), and derive (e) by taking λ^2 of (b), then using (c).

Deriving (a), (c), (d) is also not difficult using the known facts about $F(1, \dots, 1, n-r)$, cf. [14] or [32] for details.

5.7 One can already show that many $X_{n,r}$ are not stably parallelizable, using Stiefel-Whitney classes and the $Z/2$ -cohomology, $H^*(X_{n,r}; Z/2)$, first found by Gitler-Handel [8]. There is always a generator $x \in H^1(X_{n,r}; Z/2)$, and the Stiefel-Whitney class of $\xi_{n,r}$ is $w(\xi_{n,r}) = 1 + x$. Then, since stably equivalent vector bundles have the same Stiefel-Whitney classes,

$$w(\tau_{n,r}) = w(nr\xi_{n,r}) = (w(\xi_{n,r}))^{nr} = (1+x)^{nr}.$$

To compute the binomial coefficients mod 2 arising this way, the following result of Lucas [19], for any prime p , is useful.

Theorem: Let $n = \sum_{i \geq 0} d_i p^i$, $r = \sum_{i \geq 0} c_i p^i$, where $0 \leq c_i, d_i < p$, be the p -ary expansions of n, r . Then

$$\binom{n}{r} \equiv \prod_{i \geq 0} \binom{d_i}{c_i} \pmod{p}.$$

Proof: $(1+t)^n = (1+t)^{\sum d_i p^i} = \prod_{i \geq 0} (1+t)^{d_i p^i} \equiv \prod_{i \geq 0} (1+t^{p^i})^{d_i}$, since $(a+b)^{p^i} \equiv (a^{p^i} + b^{p^i}) \pmod{p}$. Now simply compare the coefficient of t^r on both sides. $\square$

5.8 Examples: (a) For $X_{10,3}$, the work of Gitler-Handel [16] shows $H^*(X_{10,3}; Z/2)$ contains $x, x^2, \dots, x^7 \neq 0$, $x^8 = 0$, and other generators x_8, x_9 that do not concern us here. Then

$$w(\tau_{10,3}) = (1+x)^{30} = (1+x)^{2 \cdot 15} = (1+x^2)^{15} = 1 + x^2 + x^4 + x^6$$

since $15 = 1111$ in binary expansion. So $w(\tau) \neq 1$, hence $X_{10,3}$ not stably parallelizable.

(b) For $X_{10,4}$, we similarly have $x^8 = 0$, so here $w(\tau_{10,4}) = (1+x)^{40} = (1+x^8)^5 = 1^5 = 1$, giving no information. Cases like $X_{10,4}$, being more delicate, were not settled until the K -theory of $X_{n,r}$ was calculated, cf. [3], [5], [6]. We will not go into $K^*(X_{n,r})$ here, but will quote the final answer.

5.9 Theorem: $X_{n,r}$ is parallelizable for $(n, n-1)$, $(2m, 2m-2)$, $(4, r)$, $(8, r)$, and $(16, 8)$. No other is stably parallelizable except possibly $X_{12,8}$ which is undecided.

5.10 We now turn to the Grassmann manifolds $G_{n,r} = F_{\mathbb{R}}(r, n-r)$. Since $G_{n,r} \cong G_{n,n-r}$, we assume without loss of generality that $r \leq n/2$. The next result is just a repeat of 3.1 (a).

Theorem: $G_{n,r}$ is parallelizable for $r = 1$ and $n = 2, 4, 8$. Any other $G_{n,r}$ (with $r \leq n/2$) is not stably parallelizable.

This theorem was first proved in [62] and a little later in [18], [4], and [61]. We will study the proof given in [61] since it is a very easy application of K -theory and also covers the complex and quaternionic Grassmann manifolds.

5.11 First recall the result of Adams (cf. [1], or [24] p. 237) :

$$\widetilde{KO}(\mathbb{R}P^m) \approx Z/2^{\varphi(m)}, \text{ where } \varphi(m) = |\{i : 1 \leq i \leq m, i \equiv 0, 1, 2, 4 \pmod{8}\}|.$$

Furthermore the generator of this cyclic group is $[\xi_m] - 1$, or equivalently $2^{\varphi(m)}\xi_m \sim 0$ (as mentioned in 2.10, ξ_m denotes the Hopf line bundle). Here are some sample values.

m	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$m-1$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\varphi(m-1)$	1	2	2	3	3	3	3	4	5	6	6	7	7	7	7	8	9

Notice that $\varphi(n)$ is approximately $n/2$, so that $2^{\varphi(m-1)}$ is much larger than $m-1$ except for the small values $m-1 \leq 8$. This makes it easy to prove the next lemma, we omit the proof.

Lemma: $m\xi_{m-1} \sim 0$ iff $m = 2, 4, 8$, and $(m-i)\xi_{m-1} \not\sim 0$, $i \geq 1$.

5.12 Proof of Theorem 5.10: For convenience we write the canonical bundles ξ_1, ξ_2 on $G_{n,r} = F_{\mathbb{R}}(r, n-r)$ as $\xi_1 = \beta$, $\xi_2 = \gamma$. On $G_{n-1,r-1}$ write these as β', γ' , and under the inclusion $i : G_{n-1,r-1} \hookrightarrow G_{n,r}$, where $i(A) = \text{span}(A, e_n)$, observe that $i^*(\gamma) \approx \gamma'$, $i^*(\beta) \approx \beta' \oplus \varepsilon$. It follows that

$$i^*(\tau_{n,r}) \approx i^*(\beta \otimes \gamma) \approx i^*\beta \otimes i^*\gamma \approx \tau_{n-1,r-1} \oplus \gamma'.$$

Iterating the inclusion i , $(r-1)$ times, gives an inclusion

$$\mathbb{R}P^{n-r} = F_{\mathbb{R}}(1, n-r) \xrightarrow{j} G_{n,r} \quad \text{with}$$

$$j^*(\tau_{n,r}) \approx \tau_{\mathbb{R}P^{n-r}} \oplus (r-1)\gamma_{\mathbb{R}P^{n-r}} \approx \xi_{n-r} \otimes \gamma_{n-r} \oplus (r-1)\gamma_{n-r},$$

where we write $\gamma_{n-r} = \gamma_{\mathbb{R}P^{n-r}}$.

Now recall (Lec. 1) $\xi_{n-r} \oplus \gamma_{n-r} \approx (n-r+1)\varepsilon$ implies $\varepsilon \oplus (\xi_{n-r} \otimes \gamma_{n-r}) \approx (n-r+1)\xi_{n-r}$, i.e. $\xi_{n-r} \otimes \gamma_{n-r} \sim (n-r+1)\xi_{n-r}$ and $\gamma_{n-r} \sim -\xi_{n-r}$. Substituting,

$$j^*(\tau_{n,r}) \approx (n-r+1)\xi \oplus (r-1)(-\xi) = (n-r+1-(r-1))\xi_{n-r}.$$

By Lemma 5.11 this can be stably equivalent to 0 iff $r = 1$ and $n-r+1 = 2, 4, 8$, i.e. $n = 2, 4, 8$.

Remark: The cases $k = \mathbb{C}, \mathbb{H}$ can be handled in the same way, and the final step is in fact easier because the Hopf bundle ξ_{n-r} turns out to have infinite order, in $\widetilde{K}(kP^{n-r})$.

5.13 We now turn to $\tilde{G}_{n,k}$, the oriented Grassmann manifolds. Since $\tilde{G}_{n,1} \cong S^{n-1} \cong \tilde{G}_{n,n-1}$, we write $\tilde{G}_{n,k} = \tilde{F}(k, \ell)$ with $k + \ell = n$ and henceforth suppose $k, \ell > 1$ to dispense of the spheres. In Lec. 1 we saw that $\tilde{F}(2, 2)$ was stably parallelizable and $\tilde{F}(3, 3)$ parallelizable, our aim now is to show that no other $\tilde{F}(k, \ell)$, $k, \ell > 1$, is stably parallelizable. This theorem was first stated and partially proved in [45], fully proved in [49], and again proved (using characteristic classes, the Pontrjagin classes) in [35]. In [50] $K^*(\tilde{G}_{n,k})$ was computed and this leads to a very short proof of this theorem, which we now present at least in part. The work in [50] also settled which $\tilde{G}_{n,k}$ admit an almost complex structure, a problem going back to the 50's, but we only consider the parallelizability here.

5.14 Let $k = 2s$ or $2s + 1$, $\ell = 2t$ or $2t + 1$. The description of $K(\tilde{G}_{n,k}) = K^o(\tilde{G}_{n,k})$ shows that this ring always contains a subalgebra $H_{s,t} \subseteq K(\tilde{G}_{n,k})$ defined as

$$H_{s,t} = Z[p_1, \dots, p_s, q_1, \dots, q_t] / ((1 + p_1 + \dots + p_s)(1 + q_1 + \dots + q_t) = 1) .$$

Further $H_{s,t}$ is graded via $\deg p_i = \deg q_i = i$. Thus one obtains $p_1 + q_1 = 0$ in degree 1, $p_2 + p_1 q_1 + q_2 = 0$ in degree 2, etc.

Remark: In the case k, ℓ both odd, $H_{s,t} = K(\tilde{G}_{n,k})$, for other parities $H_{s,t} \subset K(\tilde{G}_{n,k})$.

5.15 Lemma: $[c\tau] = p_1 q_1 + k q_1 + \ell p_1 + k \ell \in H_{s,t} \subseteq K(\tilde{G}_{n,k})$.

Proof: Two technical things must be assumed here. The first is that the “ K -theory Pontrjagin classes” p_1, q_1 are related to the exterior power representations $\lambda_k^1 \in R_{\mathbb{R}}\text{Spin}(k)$, $\lambda_\ell^1 \in R_{\mathbb{R}}\text{Spin}(\ell)$, by $c\lambda_k^1 = p_1 + k$, $c\lambda_\ell^1 = q_1 + \ell$, and this is a simple consequence of the definition of the Pontrjagin classes (cf. [3]). The second is that the α -construction here satisfies $\alpha(\lambda_k^1) = [\beta]$, $\alpha(\lambda_\ell^1) = [\gamma]$ in $KO(\tilde{G}_{n,k})$, for information about the α -construction (also called the mixing construction) see [20] §9, especially p.70.

Then $\alpha(\lambda_k^1 \otimes \lambda_\ell^1) = [\beta \otimes \gamma] = [\tau] \in KO(\tilde{G}_{n,k})$, so

$$\alpha(c\lambda_k^1 \otimes c\lambda_\ell^1) = [c\tau] \in K(\tilde{G}_{n,k}) .$$

Thus $[c\tau] = \alpha(c\lambda_k^1) \cdot \alpha(c\lambda_\ell^1)$

$$= (p_1 + k)(q_1 + \ell) = p_1 q_1 + k q_1 + \ell p_1 + k \ell .$$

□

Remark: If one is not familiar with the technicalities in the above proof, Lemma 5.15 can just be assumed and Theorem 5.16 can be directly studied.

5.16 Theorem: For $k, \ell \geq 2$, $\tilde{G}_{n,k}$ stably parallelizable $\implies (n, k) = (2, 2)$ or $(3, 3)$.

Proof: From 5.15, if also $\tilde{G}_{n,k}$ is stably parallelizable, we have (since $\dim \tilde{G}_{n,k} = k\ell$) $k\ell = [\tau]$, so $k\ell = [c\tau] = p_1 q_1 + k q_1 + \ell p_1 + k \ell$, giving $k q_1 + \ell p_1 = 0$ in degree 1, and $p_1 q_1 = 0$ in degree 2. The second relation gives $0 = p_2 + p_1 q_1 + q_2 = p_2 + q_2$ which can only hold in $H_{s,t}$ if $p_2 = q_2 = 0$, i.e. $s = t = 1$. Since $p_1 + q_1 = 0$ in $H_{s,t}$, the first relation gives $(k - \ell)q_1 = 0$ which implies $k = \ell$. This leaves only $k = \ell = 2$ or $k = \ell = 3$. □

6 Lecture

Outline of the Hodgkin Spectral Sequence

6.1 The Hodgkin spectral sequence is a powerful tool to compute the (complex) K -theory of a homogeneous space X . Unfortunately it is not also available for KO -theory. The basic reference for it is the book [20]. In this lecture we give an 8-step “user’s guide” to employing this tool. Steps 4-6 involve some homological algebra, and details for these steps are given in Lec. 7.

Step 1 Write $X = G/H$ where G = compact, connected Lie group with $\pi_1(G) = 0$ and H is a closed subgroup. Frequently $G = \text{Spin}(n)$ is used. We remark that $\pi_1(G)$ has no 2-torsion suffices, and write $i : H \hookrightarrow G$.

Step 2 Determine RH, RG , the (complex) representation rings of H, G . It is useful if RG is a polynomial algebra, this is the case for $G = \text{Spin}(n)$. Excellent references for this step and Step 3 are the books [10], [24].

Step 3 Calculate the “restriction” homomorphism $i^* : RG \rightarrow RH$. This also makes RH into an RG -module.

Step 4 $E_2 = \text{Tor}_{RG}(RH, Z)$ gives the E_2 term of the Hodgkin spectral sequence.

Step 5 Simplify the Tor calculation, using the “change of rings” theorem, to get rid of redundant terms. One gets $\text{Tor}_{RG}(RH, Z) \approx \text{Tor}_A(B, Z)$, with A, B having fewer generators, and tries to retain $A =$ polynomial algebra.

Step 6 Use the Koszul resolution for $\text{Tor}_A(B, Z)$ calculation.

Step 7 Now having E_2 , calculate the differentials in the spectral sequence to get $E_3, \dots, E_\infty$. We remark that in all examples known to the author the spectral sequence has “collapsed”, i.e. $0 = d_2 = d_3 = \dots$, whence $E_2 = E_3 = \dots = E_\infty$ and thus this step is trivial in these circumstances. See also 6.2 below.

Step 8 Reassemble $K^*(X)$, as a $\mathbb{Z}/2$ -graded algebra (due to Bott periodicity), from E_∞ . An excellent source for this step is [3], 6.1-6.5.

6.2 Remark: Here are a few further details about the Hodgkin spectral sequence.

(a) $E_2^p = \text{Tor}_{RG}^p(RH, Z)$ gives the p -grading, the q -grading is the $\mathbb{Z}/2$ -grading (periodic) induced by regarding $(RG)^0 = RG$, $(RG)^1 = 0$, similarly for RH and Z .

(b) $d_r : E_r^{p-r} \rightarrow E_r^p$, and $d_r = 0$ for r even.

(c) $E_2^{*,*}$ gives rise to a multiplicative spectral sequence, in particular if $\text{Tor}_{RG}^*(RH, Z)$ is generated as an algebra by its elements in degree ≤ 2 then the spectral sequence collapses.

7 Lecture

Some Homological Algebra

In this lecture we review some homological algebra that is very useful towards performing Steps 4,5,6 of the Hodgkin spectral sequence calculation, as outlined in Lecture 6. Let R be a commutative ring with unit and $A, B, \dots$ (left) R -modules.

7.1 Definition: A free resolution P of A is an exact sequence

$$\dots \longrightarrow P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0 ,$$

where each P_i is a free R -module.

Note that ε , called the augmentation, is necessarily surjective. One then obtains a chain complex

$$P \otimes_R B = \dots \xrightarrow{d_3 \otimes 1} P_2 \otimes_R B \xrightarrow{d_2 \otimes 1} P_1 \otimes_R B \xrightarrow{d_1 \otimes 1} P_0 \otimes_R B ,$$

but in general this sequence is not exact.

7.2 Definition: $\text{Tor}_R^n(A, B) = H_n(P \otimes_R B)$, $n \geq 0$.

7.3 Remarks: (a) It must be verified that this is well defined, i.e. independent of the resolution P .

(b) The modules in P can be taken to be projective, a generalization of free.

(c) $\text{Tor}_R(A, B)$ can also be computed by a proj. resolution Q of B , using $A \otimes_R Q$, giving the same result up to isomorphism.

All of 7.1 – 7.3 are basic facts and may be found in any text of homological algebra. In general the main difficulty in a Tor calculation is to find a convenient projective resolution P . When $R =$ a polynomial algebra, say $R = K[x_1, \dots, x_m]$ where $K =$ comm ring with 1 (usually $K = \mathbb{Z}$ in applications), there is a convenient free resolution called the Koszul resolution, of K . Suppose each x_i has degree 0, so $\varepsilon : R \twoheadrightarrow K$ with $\varepsilon(1) = 1$, $\varepsilon(x_i) = 0$, gives an augmentation to start the resolution.

Let $E = E_R[\xi_1, \dots, \xi_m]$ be the exterior algebra over R on generators $\xi_1, \dots, \xi_m$ of degree 1. Thus each E_r is the free R -module on $\{\xi_{i_1} \cdots \xi_{i_r} = i_1 < \cdots < i_r\}$, and $E_0 = R$, $E_m \approx R$, $E_r = 0$ for $r > m$. Setting $\partial(\xi_j) = x_j$ makes E into a differential graded algebra over R , i.e. the property

$$\partial(uv) = \partial u \cdot v + (-1)^p u \cdot \partial v ,$$

for u homogeneous of degree p , and linearity, defines ∂ on all of E .

Example: Take $u = \xi_1$, $v = \xi_2$, so $\deg u = 1$.

$$\partial(uv) = \partial(\xi_1 \xi_2) = \partial \xi_1 \cdot \xi_2 - \xi_1 \cdot \partial \xi_2 = x_1 \xi_2 - \xi_1 x_2 = x_1 \xi_2 - x_2 \xi_1 .$$

7.4 Proposition: $0 \longrightarrow E_m \xrightarrow{\partial} E_{m-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} E_1 \xrightarrow{\partial} E_0 \xrightarrow{\varepsilon} K$ is a free R -resolution of K (here K is regarded as a trivial R -module, i.e. $x_i \cdot k = k \quad \forall k \in K$, $1 \leq i \leq m$).

For a proof see [40], p. 204.

The next theorem serves to simplify the calculation $\text{Tor}_{RG}(RH, Z)$ occurring in the Hodgkin spectral sequence by “cancelling” common parts of RH, RG as far as possible (we make this precise in Remark 7.6(b)). It is a straightforward application of techniques in [11] involving supplemented algebras and the change of rings theorem.

Given $i : H \hookrightarrow G$, suppose $RG \approx Z[\gamma_1, \dots, \gamma_n]$ with $\varepsilon(\gamma_i) = 0$. Let $\Lambda = Z[\gamma_1, \dots, \gamma_r]$ for some $r \leq n$, and $i^*(\gamma_i) = h_i \in RH$, $1 \leq i \leq r$. Also let $A = RG/(\gamma_1, \dots, \gamma_r)$, $B = RG/(h_1, \dots, h_r)$ and note that just as i^* makes RH an RG -module, it induces an A -module structure on B .

7.5 Theorem: If RH is free as a Λ -module, then $\text{Tor}_{RG}(RH, Z) \approx \text{Tor}_A(B, Z)$.

7.6 Remarks: (a) RH flat as a Λ -module suffices (free $\Rightarrow$ flat),

(b) the idea of “cancelling as far as possible” means taking r as large as possible with RH free as a Λ -module.

8 Lecture

Completion of Parallelizability Theorem

8.1 Recall from 4.9 that there were four families of partially oriented flag manifolds whose parallelizability was still undecided:

$$\begin{aligned} F(1, \dots, 1 \mid 3, 1) & , \quad s \geq 3 , \\ F(1, \dots, 1 \mid 7, 1) & , \quad s \geq 3 , \\ F(6, 1, \dots, 1 \mid 1, 1) & , \quad s \geq 5 , \\ F(6, 1, \dots, 1 \mid 1, 1, 1) & , \quad s \geq 4 . \end{aligned}$$

The object in this lecture is to apply K -theory to show that none of these manifolds is even stably parallelizable, although in the end there is a single undecided case $F(6, 1, 1, \mid 1, 1)$ (of dim 30).

8.2 By the inclusion method, to prove all the assertions in 8.1, it will suffice to show that none of the following are stably parallelizable:

$$\begin{aligned} X &= F(1 \mid 3, 1) , \\ Y &= F(1 \mid 7, 1) , \\ Z &= F(6, 1, 1, 1 \mid 1, 1) , \\ W &= F(6 \mid 1, 1, 1) , \end{aligned}$$

with respective dimensions 7, 15, 40, 21. In this lecture we will outline the proofs that X, W are not stably parallelizable, using the Atiyah-Hirzebruch spectral sequence for X , and for W using the Hodgkin spectral sequence. The proof for Y is similar to that for X , and the proof for Z similar (actually much easier) to that for W . The main reference is [51].

8.3 Remark: The above conclusions together with those in Lectures 3, 4 decide the stable parallelizability and parallelizability of all standard and all partially oriented flag manifolds, apart from the one exception $F(6, 1, 1, \mid 1, 1)$.

8.4 We determine first the tangent bundles of X, W . Using the usual notation, we have

$$\begin{aligned} \textbf{Proposition:} \quad [\tau_x] &= 4[\xi_3] + 3 \in KO(X) , \\ [\tau_w] &= 8[\xi_2 \oplus \xi_3 \oplus \xi_4] - 3 \in KO(W) . \end{aligned}$$

Proof: For X we have $\xi_1 = \varepsilon$, ξ_2 rank 3, ξ_3 rank 1, and $\xi_1 \oplus \xi_2 \oplus \xi_3 \approx 5\varepsilon$. Since $\xi_1 = \varepsilon$ this gives $[\xi_2] + [\xi_3] = 4$. Now $\tau_x \approx \xi_1 \otimes \xi_2 \oplus \xi_1 \otimes \xi_3 \oplus \xi_2 \otimes \xi_3 \approx \xi_2 \oplus \xi_3 \oplus \xi_2 \otimes \xi_3$. So in $KO(X)$,

$$[\tau_x] = [\xi_2] + [\xi_3] + (4 - [\xi_3])[\xi_3] = 4 + 4[\xi_3] - 1 = 4[\xi_3] + 3 .$$

For W , let $\xi := \xi_2 \oplus \xi_3 \oplus \xi_4$, it is rank 3 orientable. Also $\xi_1 \oplus \xi \approx 9\varepsilon$. Then $\xi \approx \lambda^2 \xi \approx \xi_2 \otimes \xi_3 \oplus \xi_3 \otimes \xi_4 \oplus \xi_4 \otimes \xi_2$, from which $\xi^2 = \xi \otimes \xi \approx \xi_2^2 \oplus \xi_3^2 \oplus \xi_4^2 \oplus 2(\xi_2 \otimes \xi_3 \oplus \xi_3 \otimes \xi_4 \oplus \xi_4 \otimes \xi_2) \approx 3\varepsilon \oplus 2\xi$.

$$\begin{aligned} \text{Thus} \quad [\tau_w] &= [\xi_1 \otimes \xi] + [\lambda^2 \xi] \\ &= [9 - \xi][\xi] + [\xi] = 9[\xi] - [\xi^2] + [\xi] \\ &= 10[\xi] - 3 - 2[\xi] = 8[\xi] - 3 . \end{aligned}$$

□

Exercise: Show similarly $[\tau_Y] = 8[\xi_3] + 7$, $[\tau_Z] = 16[\xi_6] + 24$ (for Z first show $\xi_5 \approx \xi_6$).

8.5 We now turn to the proof that X is not stably parallelizable. By 8.4 it will suffice to show $4\xi_3 \not\sim 0$. Note the fibre map $q : X = F(1 \mid 3, 1) \rightarrow F(4, 1) = P^4$ (cf. 3.2), clearly $q^*(\xi_{P^4}) \approx \xi_3$ (writing ξ_{P^4} now for the Hopf bundle over $\mathbb{R}P^4$ to avoid confusion). Since $8\xi_{P^4} \sim 0$ (cf. 5.11) it follows that $8\xi_3 \sim 0$, our goal will be to use the Atiyah-Hirzebruch spectral sequence to show $4\xi_{P^4} \not\sim 0$ implies the same for $4\xi_3$, which will then complete the proof.

To use the Atiyah-Hirzebruch spectral sequence for KO -theory one needs to know both $H^*(X; Z)$ and $H^*(X; Z/2)$. For this there is another useful fibration

$$F(3, 1) = P^3 \hookrightarrow X = F(1 \mid 3, 1) \xrightarrow{\tilde{q}} \tilde{F}(1, 4) = S^4 ,$$

to which the Serre spectral sequence may be applied. We omit the details and simply give the results, see [51] for all the details.

dim	$H^*(X; Z) \xleftarrow{q^*} H^*(P^4; Z)$	$H^*(X; Z/2) \xleftarrow{q^*} H^*(P^4; Z/2)$
0	$Z \xleftarrow{\approx} Z$	$Z/2 \xleftarrow{\approx} Z/2$
1		$Z/2 \xleftarrow{\approx} Z/2$
2	$Z/2 \xleftarrow{\approx} Z/2$	$Z/2 \xleftarrow{\approx} Z/2$
3		$Z/2 \xleftarrow{\approx} Z/2$
4	$Z/4 \xleftarrow{\times 2} Z/2$	$Z/2 \xleftarrow{0} Z/2$
5		$Z/2$
6	$Z/2$	$Z/2$
7	Z	$Z/2$

8.6 The Atiyah-Hirzebruch spectral sequence is summarized by

$$E_2^{p, -q} = H^p(Y; KO^q(pt)) \implies KO^{-p-q}(Y) ,$$

which means that the E_2 term is the ordinary cohomology of Y with coefficients in $KO^*(pt)$, converging to E_∞ which represents the successive quotients in a filtration of $KO^{-p-q}(Y)$. Further facts are as follows.

- (a) d_r has bidegree $(r, r-1)$,
- (b) the spectral sequence lies in quadrants 1, 4,
- (c) the coefficients are $KO^0(pt) \approx KO^4(pt) \approx Z$, $KO^1(pt) \approx KO^2(pt) \approx Z/2$, $KO^3(pt) = KO^5(pt) = KO^6(pt) = KO^7(pt) = 0$, $KO^{i+8} \approx KO^i$ (so the spectral sequence is periodic in q).

In general it is not too difficult to find the E_2 term, but determining the d_r and the possible group extensions of the E_∞ terms may be quite tricky. We are primarily interested in $K^0(Y)$, so will consider the diagonal $p+q=0$. This always (for Y connected) has the term $Z = H^0(Y; KO^0(pt)) = E_2^{0,0}$, which always lasts to E_∞ and gives rise to the direct summand Z in $KO^0(Y) \approx Z \oplus \widetilde{KO}^0(Y)$, so it is the $E_\infty^{p, -p}$ terms with $p > 0$ that are of interest.

Remark: There is a similar (and simpler) spectral sequence for $K^*(Y)$, with $K^0(pt) = Z$, $K^1(pt) = 0$, $K^{i+2} \approx K^i$ giving periodicity of order 2 in q .

8.7 The real and complex K -theory of $\mathbb{R}P^n$ and $\mathbb{C}P^n$ was found in 1961 by J.F. Adams [1] (or cf. [24], p.237 or 5.11 in these notes). In particular all differentials in the Atiyah-Hirzebruch spectral sequence are known to vanish, and the groups $K^0(\mathbb{R}P^n)$ are cyclic so all extensions of the $E_\infty^{p,-p}$ terms are non-trivial. For example, for $\mathbb{R}P^4 = P^4$, we have already seen in 5.11 that $\widehat{KO}^0(P^4) \approx Z/2^{\varphi(4)} = Z/8$. Since (ignoring the $E_2^{0,0} = Z$ term as discussed in 8.6) the diagonal $E_2^{p,-p}$ has just three non-zero entries, $E_2^{1,-1} \approx E_2^{2,-2} \approx E_2^{4,-4} \approx Z/2$, these three $Z/2$'s must last to E_∞ and then recombine to give $Z/8$ by the non-trivial extensions

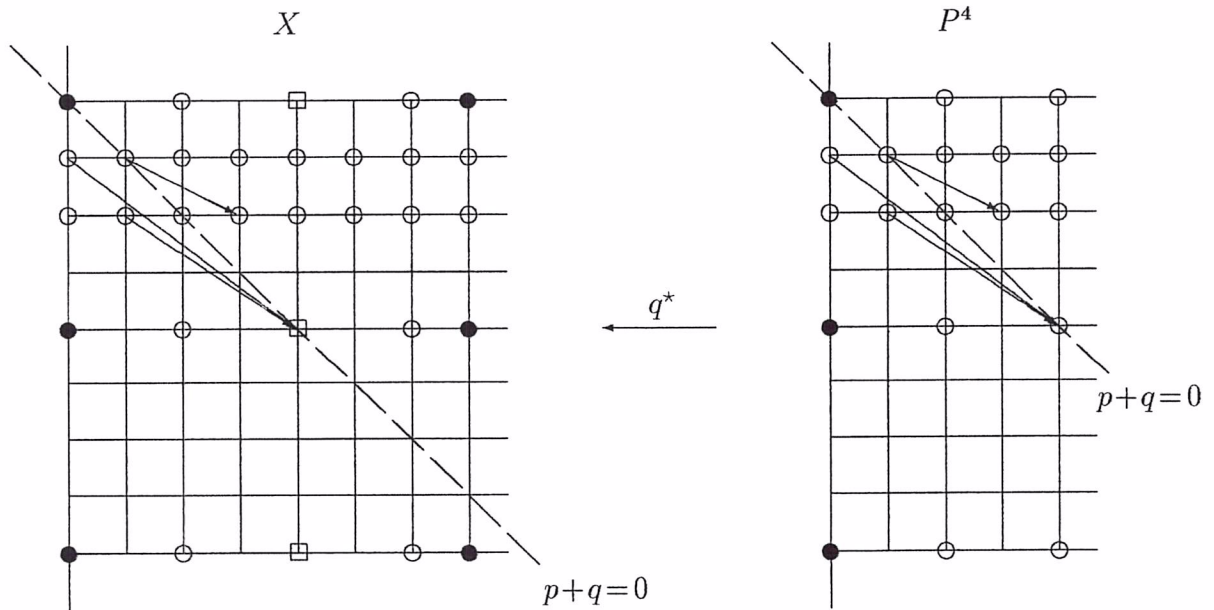
$$\begin{array}{ccccccccc} 0 & \rightarrow & 0 & \rightarrow & Z/2 & \rightarrow & Z/2 & \rightarrow & 0 \\ 0 & \rightarrow & Z/2 & \rightarrow & Z/4 & \rightarrow & Z/2 & \rightarrow & 0 \\ 0 & \rightarrow & Z/4 & \rightarrow & Z/8 & \rightarrow & Z/2 & \rightarrow & 0. \end{array}$$

This is shown in the right hand spectral sequence in Diagram 8.8 below, the E_2 term for $Y = P^4$. The notation $\bullet = Z$, $\circ = Z/2$, $\square = Z/4$ is used, the arrows indicate all possible d_r into or from $E_r^{p,-p}$ terms, $r \geq 2$, and these are all 0 as mentioned above.

The left hand spectral sequence is the E_2 term for X . Further, the map $q : X \rightarrow P^4$ induces a map of the spectral sequences, and this is completely described at the E_2 level by 8.5.

8.8 Diagram

$KO^*(X) \xleftarrow{q^*} KO^*(P^4)$ at E_2 level.



The only possible non-zero differentials coming from or going to $E_r^{p,-p}$ are shown, there are 3 such in each spectral sequence. As described in 8.7 these 3 differentials equal 0 in the P^4 spectral sequence. Then using naturality and the fact that $q^* : H^i(P^4; Z/2) \xrightarrow{\sim} H^i(X; Z/2)$, $i = 1, 2$, it follows that the 3 differentials also vanish in the X spectral sequence. Thus $E_\infty^{p,-p} = E_2^{p,-p}$ for both spectral sequences. The maps $q^* : E_\infty^{p,-p}(P^4) \rightarrow E_\infty^{p,-p}(X)$ are therefore the same as q^* at the E_2 level, so are isomorphisms $Z/2 \xrightarrow{\sim} Z/2$ for $p = 1, 2$ and the injection $Z/2 \xrightarrow{2} Z/4$ for $p = 4$. It follows that $q^* : KO^0(P^4) \rightarrow KO^0(X)$ is injective, so ξ_3 has order 8, $4\xi_3 \neq 0$, and X is not stably parallelizable.

8.9 We now turn to $W = F(6 | 1, 1, 1)$, and will outline (for full details cf. [51]) the 8 steps for the Hodgkin spectral sequence determining the complex K -theory $K^*(W)$.

Step 1: Recall $W = F(6 | 1, 1, 1) = SO(9)/SO(6) \times D$, where $D = \{\text{diag}(1, 1, 1), \text{diag}(1, -1, -1), \text{diag}(-1, 1, -1), \text{diag}(-1, -1, 1)\} \subset SO(3)$ is the diagonal subgroup of $SO(3)$. Let $\phi : \text{Spin}(9) \twoheadrightarrow SO(9)$ be the usual double cover, $Q = \phi^{-1}(D)$, then $W = \text{Spin}(9)/H$ where $H = \phi^{-1}(SO(6) \times D) \xrightarrow{j} \text{Spin}(9)$.

8.10 Step 2: $R \text{Spin}(9) = Z[P_1, P_2, P_3, \Delta_4]$, and $R \text{Spin}(6) = Z[p_1, \Delta_3^+, \Delta_3^-]$ are standard (cf. [24]). Here P_1, P_2, P_3, p_1 are the “ K -theory pontrjagin classes” which are certain linear combinations of the corresponding exterior powers $\lambda^1, \lambda^2, \lambda^3$ adjusted to have augmentation (or total degree) 0, e.g. $P_1 = \lambda^1 - \begin{cases} 2r \\ 2r+1 \end{cases}$ in $R(SO(2r)), R(SO(2r+1))$ resp, similarly in $R \text{Spin}(2r), R \text{Spin}(2r+1)$, and in general $P_k = \lambda^k + \text{lin. comb. of } \lambda^{k-1}, \dots, \lambda^1, \lambda^0 = 1$. Using the standard description of $\text{Spin}(9)$ in terms of the Clifford algebra C_9 ([24], p.169-170), we see

$$Q = \{\pm 1, \pm e_7 e_8, \pm e_8 e_9, \pm e_9 e_7\} \subset \text{Spin}(9).$$

Since $(e_7 e_8)^2 = e_7 e_8 e_7 e_8 = -e_7 e_7 e_8 e_8 = (-1)^3 = -1$, and $(e_7 e_8)(e_8 e_9) = e_7 e_8^2 e_9 = -e_7 e_9 = e_9 e_7$, etc., Q is clearly isomorphic to the group $\{\pm 1, \pm i, \pm j, \pm k\}$ of unit quaternions. The representation ring RQ is well known, one has $RQ = Z[x, y, q] / \sim$, where $\sim$ is the relations $x^2 = y^2 = 1$, $q^2 = (1+x)(1+y)$, $qx = q = qy$.

Here x, y are the 1-dimensional representations of the Klein 4-group D , composed with $Q \twoheadrightarrow D$, while q is the obvious representation of Q on the quaternions $\mathbb{H}$, regarded as a 2-dimensional complex representation.

Now $Z/2$ acts on $\text{Spin}(6) \times Q$ via $(a, b) \mapsto (-a, -b)$, and clearly $H \approx \text{Spin}(6) \times_{\mathbb{Z}/2} Q$. Thus

$$\begin{aligned} RH &= R(\text{Spin}(6) \times Q)^{Z/2} \\ &\approx (R\text{Spin}(6) \otimes RQ)^{Z/2}. \end{aligned}$$

A calculation of this gives

$$RH \approx Z[p_1, p_2, p_3] \left[\left(\Delta_3^+ \right)^2, \Delta_3^+ q, \Delta_3^- q, x, y \right] / \sim,$$

where $\sim$ consists of 9 relations, for details see [51].

Caution: $q \notin RH$, so $\Delta_3^+ q$ and $\Delta_3^- q$ are not products in RH , similarly for $(\Delta_3^+)^2$.

8.11 Step 3: Using the maximal torus of $\text{Spin}(9)$ one can compute the restriction homomorphism $j^\#$. The result is

$$\begin{aligned} j^\#(P_1) &= p_1 + \omega := p_1' \quad (\text{where } \omega = x + y + xy - 3 = q^2 - 4 \in RH), \\ j^\#(P_2) &= p_2 + \omega p_1 := p_2', \\ j^\#(P_3) &= p_3 + \omega p_2 := p_3', \\ j^\#(P_4) &= p_3 \omega, \\ j^\#(\Delta_4) &= \Delta_3^+ q + \Delta_3^- q. \end{aligned}$$

8.12 Step 4,5: To apply the change of rings as in 7.5, let

$$\Lambda = Z[P_1, P_2, P_3]. \text{ Also let}$$

$$A = R\text{Spin}(9)/(\Lambda^+) = Z[\Delta_4]$$

$$B = RH/(p'_1, p'_2, p'_3) .$$

With some difficulty one can show that RH is Λ -free, so $\text{Tor}_{RG}(RH, Z) \approx \text{Tor}_A(B, Z)$.

8.13 Step 6: Since A is a polynomial algebra on just a single generator Δ_4 , the Koszul resolution (and this step) are in this case quite easy, the resolution consisting of chains in degrees 1,0 only. After tensoring with B one obtains the chain complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & BD & \xrightarrow{d} & B & \longrightarrow & 0 \\ & & D & \xrightarrow{\quad} & \Delta_3^+ q + \Delta_3^- q - 16 & & \end{array} \quad (BD \approx B)$$

In particular only $\text{Tor}_A^0(B, Z), \text{Tor}_A^1(B, Z) \neq 0$.

8.14 Step 7-8: These are also quite easy here. Indeed for any Hodgkin spectral sequence with only $\text{Tor}_0, \text{Tor}_1$ possibly non-zero, the spectral sequence automatically collapses, $E_2 = E_\infty$, and furthermore $K^0 = \text{Tor}^0, K^1 = \text{Tor}^1$. So for the case at hand we find

$$\begin{aligned} K^0(W) = \text{Tor}^0 = B / \text{Im}(d) &= B / (\Delta_3^+ q + \Delta_3^- q - 16) \\ &\approx Z \left[(\Delta_3^+)^2, \Delta_3^+ q, x, y \right] / \sim \end{aligned}$$

for a suitable set of relations $\sim$.

8.15 To complete the proof it remains to show, from 8.4, that $8\omega \neq 0 \in K^0(W)$ (see also the definition of ω in 8.11). To do this one constructs an epimorphism

$$\begin{array}{rcl} r : K^0(W) & \longrightarrow & R := (Z/16)[u, v] / (u^2 + 2u, v^2 + v) \\ \left(\Delta_3^+ \right)^2 & \longmapsto & 0 \\ \Delta_3^+ q & \longmapsto & 0 \\ x & \longmapsto & u + 1 \\ y & \longmapsto & v + 1 . \end{array}$$

Of course one must check that all the relations $\sim$ map to 0 under r , this is routine. Next, noting that $r(\omega) = r(x + y + xy - 3) = u + 1 + v + 1 + (u + 1)(v + 1) - 3 = uv + 2u + 2v$, it suffices to show $0 \neq r(8\omega) = 8r(\omega) = 8uv$. This is done by a representation of R in $M_{4,4}(Z/16)$:

$$u \longmapsto U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \quad v \longmapsto V = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix} .$$

One checks this respects $U^2 + 2U = 0, V^2 + 2V = 0$, also $UV = VU = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -2 & -2 & 4 \end{bmatrix}$.

Clearly $8UV \neq 0$, completing the proof that W is not stably parallelizable. $\square$

9 Lecture

Generalities About Span

We start with a few basic definitions, some of which were already given in 1.12 but are repeated here for completeness. Let α be a vector bundle (usually real) of rank n over a paracompact space X , and let $r\varepsilon$ denote the trivial bundle of rank r over X .

- 9.1 Definition:** (a) $\text{span } \alpha = \max\{r : \alpha \approx r\varepsilon \oplus \beta\}$ for some β ,
 (b) $gd \alpha = n - \text{span } \alpha$ (geometric dimension),
 (c) $\text{span}^0 \alpha = \max\{-q + \text{span}(\alpha \oplus q\varepsilon) : q \geq 0\}$ (stable span),
 (d) $gd^0 \alpha = n - \text{span}^0 \alpha \geq 0$, (stable geometric dimension).

- 9.2 Remarks:** (a) $0 \leq \text{span } \alpha \leq \text{span}^0 \alpha \leq n$,
 (b) $n \geq gd \alpha \geq gd^0 \alpha \geq 0$,
 (c) α is trivial $\iff \text{span } \alpha = n \iff gd \alpha = 0$,
 (d) α is stably trivial $\iff \text{span}^0 \alpha = n \iff gd^0 \alpha = 0$,
 (e) in the definition of $\text{span}^0 \alpha$, if X is a CW-complex of dimension d , then (for real vector bundles) $n + q > d$ (i.e. $\dots q > d - n$) suffices to achieve the stable span, cf. [24] Ch. 9, with analogous results for complex vector bundles.

Next, we specialize the above to the case $X = M$, a smooth, connected n -manifold, and $\alpha = \tau_M$, the tangent bundle of M . In this case we write $\text{span } M = \text{span } \tau_M$, and $\text{span}^0 M = \text{span}^0 \tau_M$. Thus $0 \leq \text{span } M \leq \text{span}^0 M \leq n$, and from 9.2(e) it follows that $\text{span}^0 M = -1 + \text{span}(\tau_M \oplus \varepsilon)$.

9.3 Definition: If $\text{span } M = n$ (equivalently $gd \tau_M = 0$) one calls M parallelizable. If $\text{span}^0 M = n$ (equivalently $gd^0 \tau_M = 0$) one calls M stably parallelizable.

Of course, a (smooth) section of τ_M is called a vector field on M , and the study of vector fields goes back to the work of Poincaré in about 1890 on differential equations. In this terminology, $\text{span } M \geq 1$ is the same as saying that M admits a nowhere vanishing vector field, and $\text{span } M \geq r$ is the same as saying M admits r vector fields which are pointwise linearly independent, an “ r -field”.

To get a little intuition let us start with two simple examples, S^1 and S^2 . Here S^{n-1} is taken as the unit sphere in $\mathbb{R}^n$ with the standard smoothness structure inherited from $\mathbb{R}^n$. For $S^1 = \{(x, y) : x^2 + y^2 = 1\}$, clearly the tangent bundle τ is trivial with non-zero section $v(x, y) = (-y, x)$. Thus $\text{span } S^1 = 1$, i.e. it is parallelizable. This implies the n -torus $T^n = S^1 \times \dots \times S^1$ also is parallelizable. For S^2 , however, a well known theorem of elementary topology says $\text{span } S^2 = 0$ (“you can’t comb flat a hairy basketball” or equivalently “at some point on the Earth the wind is not blowing”). On the other hand any sphere is stably parallelizable, e.g. $\text{span}^0 S^2 = 2$. This is because the normal bundle ν of S^{n-1} embedded in $\mathbb{R}^n$ is trivial (at any point $X \in S^{n-1}$ the vector x itself gives a nowhere zero section of ν), and clearly $\tau \oplus \nu \approx n\varepsilon$. This example shows that $\text{span}^0 M$ can be strictly greater than $\text{span } M$.

The term parallelizable is related to the idea of parallel transport in an obvious sense: if $v_1(x), \dots, v_n(x)$ form a basis for the tangent space T_x of M at each $x \in M$, then any vector

$\sum_{i=1}^n \lambda_i v_i(x) \in T_x$ is “parallel” to $\sum_{i=1}^n \lambda_i v_i(y) \in T_y$, $y \in M$. A parallelizable manifold M is obviously orientable, indeed the ordered basis $(v_1(x), \dots, v_n(x))$ defines an orientation for T_x that is clearly consistent for y “near” to x (more precisely for x, y in the same chart of some atlas for M), and thus orients M . In fact a bit more is true.

9.4 Proposition: If M is stably parallelizable then it must be orientable.

Proof: The fact $\tau_M \oplus \varepsilon \approx (n+1)\varepsilon$ implies $0 = w_1(\tau_M) = w_2(\tau_M) = \dots$, since the Stiefel-Whitney classes are stable invariants of a vector bundle. But $w_1(\tau_M) = 0$ is equivalent to τ_M being orientable. $\square$

We remark that the term π -manifold is frequently used for stably parallelizable. It is convenient to refer to the two main problems dealing with span as follows.

VFI Find span M for a specific M or for a specific family of smooth manifolds.

VFII For a fixed $k \geq 1$ (usually k will be small) find necessary and sufficient conditions that $\text{span } M \geq k$.

Of course VFII can help in solving VFI by giving a lower bound for $\text{span } M$ when the conditions are satisfied, and an upper bound when they are not satisfied. Using this terminology, some of the history of the vector field problem is next described.

1927, Hopf [22]: $\text{span } M \geq 1$ iff $\chi(M) = 0$, where $\chi(M)$ is the euler characteristic of M .

This is a beautiful example of a VFII result. In particular it implies $\text{span } M \geq 1$ if $n = \dim M$ is odd. As another application, consider $n = 2$. The euler characteristic of a surface M^2 is 0 iff $M^2 = T^2$ or $M^2 = K$, the Klein bottle. We have already seen $\text{span } T^2 = 2$, and since K is non-orientable $1 \leq \text{span } K < 2$ implies $\text{span } K = 1$. For any other $M^2 \neq T^2, K$, $\text{span } M^2 = 0$.

1936, Stiefel [55]: Any oriented 3-manifold is parallelizable.

This is a nice example of a VFI result. We remark that for M^3 non-orientable, the Hopf theorem and 9.4 above imply $1 \leq \text{span } M \leq 2$, and both cases are possible.

Proceeding dimension by dimension is not very practicable, and instead topologists turned a great deal of effort to the VFI problem for the spheres, over the period 1920-1962.

1923, Hurwitz [23], 1923 Radon [47], 1942 Eckmann [15]: $\text{span } S^{n-1} \geq \rho(n) - 1$, where the so-called Hurwitz-Radon function $\rho(n)$ is defined as follows. Write $n = 2^{4a+b} \cdot S$, where S is odd and $0 \leq b \leq 3$. Then $\rho(n) = 2^b + 8a$.

The r -fields constructed on S^{n-1} by these methods are linear in the coordinates $(x_1, \dots, x_n)$. In particular they are also then skew, meaning $v(-x) = -v(x)$ and therefore pass to r -fields on $\mathbb{R}P^{n-1}$. So $\text{span}(\mathbb{R}P^{n-1}) \geq \rho(n) - 1$ follows as a corollary. For quite a while topologists made progress towards showing that in fact $\text{span } S^{n-1} = \rho(n) - 1$, and it would take too much space to give all the details. The coup de grace:

1962, Adams [1]: $\text{span } S^{n-1} = \rho(n) - 1$.

Corollary: $\text{span } \mathbb{R}P^{n-1} = \rho(n) - 1$.

Proof (of the corollary): We just saw above that $\text{span } \mathbb{R}P^{n-1} = \rho(n) - 1 = \text{span } S^{n-1}$, but the double covering map $S^{n-1} \rightarrow \mathbb{R}P^{n-1}$ shows (as is the case for any smooth fibration) $\text{span } S^{n-1} \geq \text{span } \mathbb{R}P^{n-1}$. $\square$

The family of spheres is generalized by the family of Stiefel manifolds $V_{n,r}$ ($V_{n,1} = S^{n-1}$).

1964, Sutherland [58]: For $r > 1$, $V_{n,r}$ is parallelizable.

The next theorem effectively settles the span of all stably parallelizable manifolds. Some parts of this theorem had been anticipated by Thomas and Kervaire.

1966, Bredon-Kosinski: If M^n is stably parallelizable, then $\text{span } M = \text{span } S^n$ or $\text{span } M = n$. Furthermore,

- (a) for n even, $\text{span } M = n \iff \chi(M) = 0$,
- (b) for $n = 2r+1$ odd, $n \neq 1, 3, 7$, $\text{span } M = n \iff \tilde{\chi}_2(M) = 0$, where the mod 2 Kervaire semi-characteristic $\tilde{\chi}_2$ is defined by $\tilde{\chi}_2(M) = \sum_{i=0}^r \dim(H_i(M; \mathbb{Z}_2))$.

The remainder of this lecture will have two parts. The first part will sketch the history of VFI and VFII after the Bredon-Kosinski theorem. The second part will return to the basic theorems mentioned above and say a little about their proofs. Then, in Lecture 10, various approaches available to enable further progress on the vector field problem (for manifolds that are not stably parallelizable) will be outlined. Finally, in Lecture 11, a few specific examples of successful applications of these techniques are given.

Before returning to the history, it might be worth mentioning a comparison with the immersion problem for smooth manifolds. From the above, we can interpret the span problem as finding the $gd(\tau_M)$. The inverse of the tangent bundle τ_M is ν_M , the normal bundle of M . For any immersion of M^n in $\mathbb{R}^{n+k}$ clearly $\tau_M \oplus \nu_M \approx (n+k)\varepsilon$. The normal bundle ν_M is however only a stable vector bundle, and the immersion problem (to find the minimal k so that M immerses in $\mathbb{R}^{n+k}$) can be thought of as finding $gd^0(\nu_M)$. This is due to Hirsch's theorem [19]: M^n immerses in $\mathbb{R}^{n+k}$, $k > 0$, iff $gd^0(\nu_M) \leq k$. Since stable invariants tend to be easier to compute, this makes the immersion problem somewhat more tractable than the span problem. For another interesting relation between immersion dimension, span^0 , and Lusternik-Schnirelman category, cf. Korbaš and Szücs [31]. Specifically, if the smooth manifold M has L-S category c (i.e. $1+c$ is the minimal number of open sets, each contractible in M , that cover M), then $gd^0 \tau_M \leq c \cdot gd^0 \nu_M$ and $gd^0 \nu_M \leq c \cdot gd^0 \tau_M$.

Returning to the historical development, the VFII problem saw good progress for $k = 2, 3$ between 1965-70, in work of Atiyah, Audin, Crabb, Dupont, Frank, Steer and Thomas. Later work in the 80's and 90's for $k = 4$ and in some special cases $k = 5, \dots, 9$ is due to Koschorke, Randall and Ng Tse Beng.

As for as the VFI problem, the special case of parallelizable manifolds saw a great deal of success starting with Yoshida's results [62] for Grassmann manifolds in 1975; these are outlined on p.8 of these notes and indeed Lectures 1-8 of these notes are devoted to the parallelizability results for flag manifolds and associated manifolds such as Stiefel and projective

Stiefel manifolds. One consequence of all this work is that the parallelizable, or even stably parallelizable, manifolds are relatively rare, i.e. the vast majority of manifolds are not stably parallelizable.

One can rewrite the Bredon-Kosinski theorem (first part) as follows. If M^n is stably parallelizable, then $\text{span } M = \text{span}(S^n)$ or $\text{span } M = \text{span}^0 M$ (which of course equals n here). This suggests the following.

Eagle Conjecture: For any manifold M , $\text{span } M = \text{span}(S^n)$ or $\text{span } M = \text{span}^0 M$.

Since $\text{span}^0 M$ is generally easier to compute, this conjecture would simplify the calculation of $\text{span } M$. Unfortunately it is false, indeed Koschorke [34] produced many counterexamples, e.g. $M^{13} = S^{11} \times \mathbb{R}P^2$. Clearly $\text{span}^0 M \geq \text{span}^0 S^{11} = 11$. On the other hand $w(M) = w(S^{11}) : w(\mathbb{R}P^2) = 1 \cdot (1 + x + x^2) \implies w_2(M) \neq 0 \implies \text{span}^0(M) \leq 11$. Thus $\text{span}^0 M = 11$. Similarly, $\text{span}(M) \geq \text{span}(S^{11}) = \rho(12) - 1 = 3$, and Koschorke shows that in fact $\text{span}(M) = 3$. Since $\text{span}(S^{13}) = \rho(14) - 1 = 1$, $\text{span } M \neq \text{span } S^{13}$ and $\text{span } M \neq \text{span}^0 M$.

The difficulties in finding the span of manifolds that are not stably parallelizable are therefore numerous (of course we are supposing $\chi(M) = 0$ as well, otherwise Hopf's theorem gives the answer). One example of a successful solution is $\text{span}(\mathbb{R}P^{n-1}) = \rho(n) - 1$, as shown earlier in this lecture. Other solutions for quotient spaces of spheres (e.g. lens spaces) are due to Yoshida [63] in 1967, Becker [8] in 1973, and see also the book of Mahammed, Piccinini, Suter [41] in 1980. More recently substantial progress has been made on the span of Grassmann manifolds and projective Stiefel manifolds; Lecture 11 is devoted to this while Lecture 10 outlines the various methods available to solve VFI. Another recent work, of Junod and Suter [27] in 1998, determines the span of certain families of product manifolds.

Now, in the last part of this lecture, we return to the proofs of the above theorems, through the Bredon-Kosinski theorem, and sketch the proofs. Starting with Hopf's theorem, it's not hard to see that $\chi(M) = 0$ is a necessary condition for $\text{span } M \geq 1$. For, a nowhere zero vector field on M gives rise to a fixed point free map $f : M \rightarrow M$ which satisfies $f \simeq 1_M$, f is just the "flow" induced by the vector field. Since f has no fixed point its Lefschetz number $L(f) = 0$. Since $f \simeq 1_M$, $L(f) = \chi(M)$. Thus $\chi(M) = 0$. The converse is more difficult, and a careful proof can be found in Steenrod [54], Ch.3, using obstruction theory. The obstruction turns out to be an "index" equal to $\chi(M)$.

Stiefel's theorem for oriented M^3 is an easy consequence of obstruction theory, cf.[46] p.148 where it is given as an exercise! The key fact is that the obstruction in question has coefficients in $\pi_2(SO(3)) = \pi_2(\mathbb{R}P^3) \approx \pi_2(S^3) = 0$, hence the obstruction vanishes.

The Hurwitz-Radon-Eckmann theorem has roots going back 100 years earlier. Indeed, just like complex numbers $\mathbb{C}$ define a bilinear multiplication $\mathbb{R}^2 \otimes \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which is also norm preserving ($\|zw\| = \|z\| \cdot \|w\|$), W.R. Hamilton in 1843 discovered the quaternions $\mathbb{H}$ which give a norm preserving bilinear multiplication $\mathbb{R}^4 \otimes \mathbb{R}^4 \rightarrow \mathbb{R}^4$. Two months later Hamilton's friend J. Graves discovered the octonions $\mathbb{O}$, which give a similar multiplication in $\mathbb{R}^8$. Graves also worked on the $\mathbb{R}^{16}$ case but in vain. In 1923 Hurwitz, and independently Radon, proved via matrix theory that there exists a bilinear norm preserving multiplication $\mathbb{R}^r \otimes \mathbb{R}^n \rightarrow \mathbb{R}^n$ iff $r \leq \rho(n)$. A new proof was given in 1942 by Eckmann, using representation theory. Later it was realized that the Clifford algebras are intrinsically related to this theorem, cf. Adams, Lax, Phillips [2], Husemoller [24] Ch.12, Zvengrowski [67].

To see the relation with vector fields on spheres, we illustrate the case $n = 4$. Writing $\alpha = (x_1, x_2, x_3, x_4) = x_1 + x_2 i + x_3 j + x_4 k$ for a quaternion α , and say $\|\alpha\| = 1$ so $\alpha \in S^3$,

consider

$$\begin{aligned}\alpha &= 1\alpha = (x_1, x_2, x_3, x_4), \\ i\alpha &= (-x_2, x_1, -x_4, x_3), \\ j\alpha &= (-x_3, x_4, x_1, -x_2), \\ k\alpha &= (-x_4, -x_3, x_2, x_1).\end{aligned}$$

Clearly $i\alpha, j\alpha, k\alpha$ form an orthonormal 3-field on S^3 . Similarly, a norm preserving multiplication $\mathbb{R}^r \otimes \mathbb{R}^n \rightarrow \mathbb{R}^n$ induces an $(r-1)$ -field on S^{n-1} . Thus $\text{span } S^{n-1} \geq \rho(n) - 1$.

The reverse inequality $\text{span } S^{n-1} \leq \rho(n) - 1$ was much harder to prove. By 1960, using techniques from homotopy theory, it was known to be true for $n \leq 1,024$. Finally, in 1962, J.F. Adams was able to apply the newly discovered technique of K -theory to complete the proof. Good references are Adam's original paper [1] and Husemoller [24].

The parallelizability of $V_{n,r}$, $r \geq 2$, is covered in Lecture 2 of these notes, so we turn to the Bredon-Kosinski theorem. Supposing M^n stably parallelizable, then $gd^0\tau_M = 0$ so $\exists M \xrightarrow{f} \mathbb{R}^{n+1}$, an immersion, by Hirsch theory as described above. From 9.4 above, it is clear that the normal bundle ν of this immersion is a trivial line bundle. For each $x \in M$ we can then choose a unit vector $u(x) \in \mathbb{R}^{n+1}$ giving a section of ν . This gives a "Gauss" map $u : M \rightarrow S^n$. Since the tangent plane to S^n at any $u(x) \in S^n$ is the orthogonal complement $\langle u(x) \rangle^\perp$, which is the same as the tangent plane to M at x , it follows that $u^*(\tau_{S^n}) \approx \tau_M$. Thus τ_M is closely related to τ_{S^n} , in particular $\text{span}(M) \geq \text{span}(S^n)$ is already proved.

Now suppose n is even. If $\chi(M) \neq 0$ then, from Hopf's theorem, $\text{span}(M) = 0 = \text{span } S^n$. If $\chi(M) = 0$ then the euler class $e(\tau_M) = 0$ (cf. Milnor-Stasheff [46], p.130). Thus τ_M and the trivial bundle $n\varepsilon$ have the same euler class. Since the euler class classifies all rank n bundles over M that are stably equivalent to τ_M , $\tau_M \approx \varepsilon$ and $\text{span } M = n$. This completes the proof of the Bredon-Kosinski theorem for n even. For n odd further techniques are needed, such as surgery.

10 Lecture

Techniques to Determine $\text{Span}(M)$

In this lecture various techniques that can lead to solutions of the VFI and VFII problems are discussed (in no special order). In most cases a full discussion would require several pages, so only an outline will be given here. Much of this material overlaps with the survey papers [60], [32], [33].

10.1 Characteristic classes: These can be applied in various ways, the most obvious being an easy way to get lower bounds for $gd^0\alpha$ of any vector bundle α (and hence upper bounds for $\text{span}^0 M$). As a simple example, consider the Stiefel-Whitney classes of $\mathbb{R}P^5$. It is well known (cf. [46]) that

$$w(\mathbb{R}P^5) = w(\tau_{\mathbb{R}P^5}) = (1+x)^6 = 1 + x^2 + x^4$$

where x is the generator of $H^1(\mathbb{R}P^5; \mathbb{Z}_2) = \mathbb{Z}_2$. Thus $gd^0(\tau) \geq 4$, giving $\text{span}(\mathbb{R}P^5) \leq \text{span}^0(\mathbb{R}P^5) \leq 1$. On the other hand $\text{span}(\mathbb{R}P^5) \geq \rho(6) - 1 = 1$ (cf. Lecture 9), proving $\text{span}(\mathbb{R}P^5) = 1$. Needless to say this method will not always work as well as in the example just shown, e.g. $w(\mathbb{R}P^{15}) = (1+x)^{16} = 1$, giving only $gd^0(\tau) \geq 0$. But in fact $gd^0\tau = 7$ here.

10.2 Postnikov towers: This is a classical homotopy theoretic method. For example one considers the classifying map $\tau : M \rightarrow BO(n)$ for the tangent bundle of M (also written τ by a slight abuse of notation). An r -field on M is then the same as a lifting, up to homotopy, of τ to $BO(n-r)$ as in the diagram

$$\begin{array}{ccc} & & BO(n-r) \\ & \nearrow & \downarrow \\ M & \xrightarrow{\tau} & BO(n) \end{array} .$$

The vertical map is a well known fibration, and the obstructions to lifting τ are found by using the Postnikov resolution of this map. This method can be especially useful in VF II questions, for small r .

10.3 Singularity approach: This method is due mainly to Koschorke [28], see also [32] for an outline of this method. It gives another type of obstruction to $\text{span } M \geq k$, hence is useful for VF II results. In addition many theorems of the type $\text{span } M = \text{span}^0 M$ are proved. For example, if n is even and $\chi(M) = 0$, then $\text{span } M = \text{span}^0 M$. Conditions are also given implying $\text{span } M = \text{span}^0 M$ for $n \equiv 1 \pmod{4}$, then for $n \equiv 3 \pmod{8}$ (naturally the conditions become more complicated).

10.4 Signature: For a definition of $\sigma(M)$, the signature of M , cf. [46] p.224. Note $\sigma(M) \in \mathbb{Z}$ and $\sigma(M) = 0$ if $\dim M \neq 4m$. In case $n = 4m$ and $\sigma(M) \neq 0$, the following theorem of Mayer [42] can give strong upper bounds for $\text{span}^0 M$.

Theorem: For M^n oriented, $n = 4m$, and $\text{span}^0 M \geq r$, $\sigma(M)$ is divisible by b_r where $b_{r+8} = 16b_r$ and $b_1 = 2, b_2 = 4, b_3 = 8, b_4 = b_5 = b_6 = b_7 = 16, b_8 = 32$.

Recently Mayer [43] has defined a twisted signature of M , which may possibly extend these results.

Remarks: (a) If $\sigma(M) = 0$ the theorem gives no information.

(b) $b_r = 2^{\psi(r)}$, where $\psi(r) = |\{i : 1 \leq i \leq r, i \equiv 1, 2, 3, 4 \pmod{8}\}|$. Note ψ is similar, but not equal, to Adams' φ function, where $\varphi(r) = |\{i : 1 \leq i \leq r, i \equiv 0, 1, 2, 4 \pmod{8}\}|$.

10.5 Normal cobordism Euler class: An interesting formulation of the span question is due to J.-P. Dax [13] (also see C.R. Acad. Sci. Paris 279 (1974), 545-548). In theory this could solve most questions about span, but the invariant seems difficult to compute. Let ξ_{k-1} be the Hopf line bundle over $\mathbb{R}P^{k-1}$. Define the external tensor product $\tau_M \hat{\otimes} \xi_{k-1} = \pi_1^*(\tau_M) \otimes \pi_2^*(\xi_{k-1})$, a rank n vector bundle over $M \times \mathbb{R}P^{k-1}$, where π_1, π_2 are the obvious projections. The first proposition reduces the problem of finding k sections of τ_M to finding a single section of $\tau_M \hat{\otimes} \xi_{k-1}$.

Proposition: Let $1 \leq k \leq n$. If $\text{span } M \geq k$ then $\text{span } \tau_M \hat{\otimes} \xi_{k-1} \geq 1$. If also $2k < n$ then $\text{span } M \geq k$ iff $\text{span } \tau_M \hat{\otimes} \xi_{k-1} \geq 1$.

Secondly, for any rank r vector bundle α over a finite CW-complex X , a normal Euler cobordism class $\sigma(\alpha) \in \Omega^r(X; \bar{\alpha})$ is defined, where $\Omega^r(X; \bar{\alpha})$ is a certain normal cobordism group of X , in dimension r , with coefficients in the stable vector bundle $\bar{\alpha}$ represented by α .

Proposition: Let $1 \leq k \leq n$. If $\text{span } \tau_M \hat{\otimes} \xi_{k-1} \geq 1$, then $\sigma(\tau_M \hat{\otimes} \xi_{k-1}) = 0$. If $k \leq n - 2$ and $\sigma(\tau_M \hat{\otimes} \xi_{k-1}) = 0$, then $\text{span } \tau_M \hat{\otimes} \xi_{k-1} \geq 1$.

10.6 Index method: This method goes back at least to Hopf, a good introduction can be found in Thomas' survey paper [60]. The idea is to consider a family of r vector fields $\{v_1, \dots, v_r\}$ on M which are pointwise linearly independent except at a finite number of points $x_1, \dots, x_m \in M$. Surround each x_i with a small enough ball B_i^n so that $x_j \notin B_i^n$, $j \neq i$. For each $x \in \partial B_i^n = S_i^{n-1}$ the r vector fields are then linearly independent (orthonormal without loss of generality via the Gram-Schmidt procedure), so form an orthonormal r -frame in $\mathbb{R}^n$. Thus one has a map $f_i : S_i^{n-1} \rightarrow V_{n,r}$, giving $[f_i] \in \pi_{n-1}(V_{n,r})$. Clearly the r vector fields on S_i^{n-1} can be extended orthonormally to all of B_i^n iff $[f_i] = 0$. But more generally, one considers $\sum_{i=1}^m [f_i] \in \pi_{n-1}(V_{n,r})$, and it can be shown that there exists an r -field on M , which

agrees with $\{v_1, \dots, v_r\}$ on the $(n-2)$ -skeleton $M^{(n-2)}$, iff the index $\sum_{i=1}^m [f_i] = 0$.

Most of the earlier results on the VF II problem were obtained by this method.

10.7 Elementary considerations arising from fibrations and cofibrations: To illustrate this for fibrations, let $F \xrightarrow{i} E \xrightarrow{p} B$ be a smooth fibration. Then $p^*(\tau_B)$ is a subbundle of τ_E , and by taking its orthogonal complement with respect to some Riemannian metric for τ_E , one has $\tau_E \approx p^*(\tau_B) \oplus \eta$. Here η is called the "vertical" bundle or the "bundle along the fibres", and clearly $i^*(\eta) \approx \tau_F$. Here $\dim B > 0$ and $\dim F \geq 0$ are assumed.

10.8 Proposition: (a) $\text{span } B \leq \text{span } E$, $\text{span}^0 B \leq \text{span}^0 E$,

(b) $i^*(\tau_E) \approx \tau_F \oplus b\varepsilon$, where $b = \dim B$, and $\text{span}^0 B \leq \text{span}^0 E \leq b + \text{span}^0 F$,

(c) for a principal fibration $\tau_F \approx \dim F \cdot \varepsilon$ is trivial and η is trivial. If also $\dim F > 0$, then B stably parallelizable will imply E parallelizable.

Proof: For (a), $\text{span } B \leq \text{span } E$ follows from $\tau_E \approx p^*(\tau_B) \oplus \eta$, while $\text{span}^0 B \leq \text{span}^0 E$ follows from $\tau_E \oplus \varepsilon \approx p^*(\tau_B) \oplus \eta \oplus \varepsilon \approx p^*(\tau_B \oplus \varepsilon) \oplus \eta$.

For (b), $\tau_E \approx p^*(\tau_B) \oplus \eta$ implies $i^*(\tau_E) \approx i^*(\eta) \oplus i^*p^*(\tau_B) \approx \tau_F \oplus b\varepsilon$, since $p \circ i = \text{constant}$ map: $F \rightarrow \{b_0\} \subset B$ so $i^*p^*(\tau_B)$ is trivial. Also, $\text{span}^0 E \leq \text{span}^0 i^*(\tau_E) = \text{span}^0(\tau_F \oplus b\varepsilon) = b + \text{span}^0 F$.

For (c), write $F = G$, the group of the principal bundle. Any smooth manifold that is also a topological group is well known to be parallelizable, in fact (almost) the same proof shows η is trivial. For let $\mu : G \times E \rightarrow E$ be the map giving the action of G on E , and let $\dim G (= \dim F) = d > 0$ (if $d = 0$ then $\text{rank } \eta = d = 0$ so η is automatically trivial). Take a basis $v_1, \dots, v_f$ for the tangent space of G at $e \in G$, the identity element. Let $x \in E$, $y = p(x) \in B$, and $F_y = p^{-1}(y)$ the fibre over y . Also let $\mu_x : G \rightarrow F_y$ be the

diffeomorphism given by $\mu_x(g) = \mu(g, x)$, $g \in G$. Then $\mu_x(e) = x$, and $(\mu_x)_*(v_i)$, $1 \leq i \leq d$ gives a trivialization of η .

Thus one always has $\tau_E \approx p^*(\tau_B) \oplus d\varepsilon$ in this case. So if B is stably parallelizable and $d \geq 1$, it's clear that E is parallelizable (write $\tau_E \approx p^*(\tau_B \oplus d\varepsilon)$). $\square$

For any cofibration (embedding, or immersion will suffice) $i : M^m \hookrightarrow N^n$ of smooth manifolds,

$$i^*(\tau_N) \approx \tau_M \oplus \nu,$$

where here ν is the normal bundle of M in N . Often one can identify ν and this will give information about the spans, e.g. if $\nu \approx (n - m)\varepsilon$ is trivial it follows that $\text{span } N \leq \text{span } i^*(\tau_N) = \text{span}^0 M + n - m$.

10.9 James-Thomas number and the Browder-Dupont invariant:

From all of the above it is clear that it is very useful to be able to decide if $\text{span } M$ and $\text{span}^0 M$ agree. For $n = \dim M$ even this is easy, cf. 10.3 above. So now suppose n is odd. James-Thomas [26] prove that for n odd the number of isomorphism classes of rank n vector bundles α over M , such that $\alpha \sim \tau_M$, is 1 or 2. This number is called the James-Thomas number of M , or $JT\#(M)$. In case $JT\#(M) = 1$ there is no problem and $\text{span } M = \text{span}^0 M$. The projective spaces $\mathbb{R}P^n$ are an example of $JT\# = 1$, as proved in [26]. However it seems that $JT\# = 2$ will more often be the case. In this case Sutherland [59] defines a mod 2 invariant $b_B(\alpha)$, the Browder-Dupont invariant, which distinguishes the two isomorphism classes. Also, $b_B(\tau_M) = \hat{\chi}_2(M)$, the mod 2 Kervaire semi-characteristic, is known. Thus one theoretically has a tool to decide if $\text{span } M = \text{span}^0 M$ agree, in all cases. Unfortunately the computation of b_B , to date, seems quite difficult in general.

10.10 Methods specific to a given manifold or family of manifolds:

This will be left to Lecture 11, which will give some nice examples of using the general methods described above and some methods specific to the case at hand.

11 Lecture

Examples of $\text{Span}(M)$

In Lecture 9, as an example of manifolds that are not stably parallelizable, we determined $\text{span } \mathbb{R}P^{n-1} = \rho(n) - 1$. The real projective spaces are generalized by the Grassmann manifolds $G_{n,r} = G(r, n - r)$ of r -planes in $\mathbb{R}^n$, and also by the projective Stiefel manifolds $X_{n,r} = V_{n,r}/Z_2$. Here $G(1, n - 1) = G(n - 1, n) = \mathbb{R}P^{n-1}$, and $X_{n,1} = \mathbb{R}P^{n-1}$. In both families, apart from a small number of exceptional cases, these manifolds are not stably parallelizable, with full details of this given in Lecture 5. Also, in both cases there is a standard double cover (resp. $\tilde{G}(r, n - r)$ and $V_{n,r}$), hence a Hopf line bundle over $G_{n,r}$ and over $X_{n,r}$ that by a small abuse of notation we shall write as $\xi_{n,r}$ or simply ξ in all cases.

Two useful propositions are given to start. The first is a stable version of Hopf's theorem. The second is the statement of Hodge duality for vector bundles, which to the best of the author's knowledge only appears in the literature for oriented vector bundles.

11.1 Proposition: $\text{span}^0 M > 0$ iff $\chi(M)$ is even. For the proof cf. [32], p.9.

11.2 Proposition: Let α be a rank n vector bundle over a paracompact space X . Then $\lambda^i \alpha \approx \lambda^{n-i} \alpha \otimes \lambda^n \alpha$.

Proof: The oriented version, $\lambda^i \beta \approx \lambda^{n-i} \beta$ for β a rank n oriented vector bundle over X , is well known and in many references; so we assume this to start. In fact, for the oriented case, the proof proceeds by starting with a dimension n oriented inner product vector space V , and defining an isomorphism $\varphi_B : \lambda^r V \rightarrow \lambda^{n-r} V$ as follows, for a given oriented orthonormal basis $B = \{v_1, \dots, v_n\}$ of V . For any multi-index $H = (i_1, \dots, i_r)$, $1 \leq i_1 < i_2 \dots < i_r \leq n$, define $v_H = v_{i_1} \wedge \dots \wedge v_{i_r}$. Let $K = (j_1, \dots, j_{n-r})$ be the complementary multi-index, i.e. $j_1 < \dots < j_{n-r}$ and $\{i_1, \dots, i_r, j_1, \dots, j_{n-r}\} = \{1, 2, \dots, n\}$. Then $\varphi_B(v_H) = \varepsilon(\sigma) v_K$ defines φ_H , where $\varepsilon(\sigma)$ is the sign of the “shuffle” permutation $(i_1, \dots, i_r, j_1, \dots, j_{n-r}) \rightarrow (1, 2, \dots, n) := N$. One checks that if B' is another orthonormal basis with the same orientation as B , then $\varphi_{B'} = \varphi_B$. Thus $\varphi = \varphi_B$ is well defined and functorial for inner-product, orientation preserving isomorphisms of vector spaces, and by standard methods of vector bundle constructions ([24], p. 67 or [46], §3) it passes to an isomorphism $\varphi : \lambda^r \beta \rightarrow \lambda^{n-r} \beta$ of rank n oriented vector bundles with euclidean metrics (for which the base space X paracompact suffices).

With a slight modification of this proof it gives the isomorphism we want. For an n -dimensional inner product space V (without an orientation) and ordered orthonormal basis B as above, now define $\varphi_B : \lambda^r V \rightarrow \lambda^{n-r} V \otimes \lambda^n V$ by $\varphi_B(v_H) = \varepsilon(\sigma) v_K \otimes v_N$. As above, it suffices to show φ_B is well defined. If B' is another ON basis with the same orientation as B , then clearly $v_N = v'_N$, so $\varphi_B = \varphi_{B'}$ by the oriented case above.

Now consider the basis $B'' = \{-v_1, v_2, \dots, v_n\}$ having opposite orientation to B . If $1 \in H$ then

$$\begin{aligned} \varphi_{B''}(v_H) &= \varphi_{B''}(v_1 \wedge v_{i_2} \wedge \dots \wedge v_{i_r}) = \varphi_{B''}((-v_1) \wedge v_{i_2} \wedge \dots \wedge v_{i_r}) = -\varphi_{B''}(v''_H) \\ &= -\varepsilon(\sigma) v''_K \otimes v''_N = -\varepsilon(\sigma) v_K \otimes (-v_N) = \varphi_B(v_H). \end{aligned}$$

Similarly, if $1 \notin H$ then $1 \in K$ giving $v''_H = v_H$, $v''_K = -v_K$, $v''_N = -v_N$ and again $\varphi_{B''}(v_H) = \varphi_B(v_H)$. So $\varphi_{B''} = \varphi_B$.

It follows that if B' is any ON basis with opposite orientation to B , then B' and B'' have the same orientation so $\varphi_{B'} = \varphi_{B''}$ by the orientation preserving case. Hence $\varphi_{B'} = \varphi_B$, in all cases. $\square$

Grassmann manifolds: Writing $G_{n,r} = G(r, s)$ where $s = n - r$, one has the rank r universal bundle γ over $G(r, s)$, its rank s orthogonal complement $\gamma^\perp$, and the Hopf line bundle ξ . Of course $\gamma \oplus \gamma^\perp \approx n\xi$, and in Lecture 1 we saw that $\tau_{G(r,s)} \approx \gamma \otimes \gamma^\perp$.

The euler characteristic $\chi(G(r, s))$ is known to be 0 if rs is odd, and otherwise equals $\binom{[n/2]}{[r/2]}$. Thus, for span considerations (and using Hopf’s theorem), only the case rs odd, i.e. both r, s odd, need be considered. We also need only consider $r, s > 1$, otherwise $G(r, s)$ reduces to the known case $\mathbb{R}P^{n-1}$. Thus, the first interesting cases are $G(3, 3)$ and $G(5, 3) = G(3, 5)$, with dimensions 9, 15 respectively. References for $\chi(G(r, s))$ are Hoggar [21], Korbaš-Zvengrowski [33], and it can also be worked out directly from Schubert calculus [46].

The “linear” lower bound $\text{span } G(r, s) \geq p(n) - 1$. (also called the Hurwitz-Radon lower bound) was first derived by Leite-Miatello [38] for $\tilde{G}(r, s)$. We give an elementary proof of this for $G(r, s)$, also see Korbaš [30] for a generalization along similar lines.

11.3 Proposition: $\text{span } \tilde{G}(r, s) \geq \text{span } G(r, s) \geq \rho(n) - 1$.

Proof: Since $\tilde{G}(r, s) \rightarrow G(r, s)$ is a smooth fibration, the first inequality holds (also for span^0). Recall $\tau \approx \gamma \otimes \gamma^\perp \approx \text{hom}_{\mathbb{R}}(\gamma, \gamma^\perp)$, so at each point $A \in G(r, s)$, A being an r -plane in $\mathbb{R}^n$, $\tau_A = \text{hom}_{\mathbb{R}}(A, A^\perp)$. Using the Hurwitz-Radon-Eckmann result, one sees that $\exists \varphi_1, \dots, \varphi_{\rho(n)-1} \in O(n)$ with the property that for any $(a_1, \dots, a_{\rho(n)-1})$ with $\sum_{i=1}^{\rho(n)-1} a_i^2 = 1$, one also has

$$\sum_{i=1}^{\rho(n)-1} a_i \varphi_i \in O(n). \text{ Further } \varphi_i + \varphi_i^t = 0.$$

Define vector fields $v_1, \dots, v_{\rho(n)-1}$ on $G(r, s)$ by taking $v_i(A)$ to be the composition

$$A \hookrightarrow \mathbb{R}^n \xrightarrow{\varphi_i} \mathbb{R}^n \xrightarrow{\pi} A^\perp,$$

where π is the orthogonal projection onto $A^\perp$. Then $\sum a_i v_i(A)$ is the composition

$$A \hookrightarrow \mathbb{R}^n \xrightarrow{\sum a_i \varphi_i} \mathbb{R}^n \xrightarrow{\pi} A^\perp,$$

and it suffices to show that $\sum a_i v_i(A) \neq 0$ whenever $\sum a_i^2 = 1$. Now $\sum a_i \varphi_i$ is orthogonal, hence non-singular, and skew symmetric. Also $\sum a_i v_i(A) = 0$ is equivalent to the assertion that A is an invariant subspace of $\sum a_i \varphi_i$. But $\dim A = r$ is odd, and a non-singular skew symmetric linear transformation has only even dimensional (real) invariant subspaces. Thus $\sum a_i v_i(A) \neq 0$ and the vector fields $v_1, \dots, v_{\rho(n)-1}$ are linearly independent. $\square$

Upper bounds can be obtained from the Z_2 -cohomology and the Stiefel-Whitney classes. For example, using [4] for calculations of these classes in $G(r, s)$, and denoting the “universal” Stiefel-Whitney classes of γ by $w_1, \dots, w_r$, we have

$$\begin{aligned} w G(3, 3) &= 1 + w_1^2 \\ w G(5, 3) &= 1 + (w_1^4 + w_2^2) + (w_1^2 w_2^2 + w_3^2) + (w_2^4). \end{aligned}$$

$$\begin{aligned} \text{It follows that } \text{span } G(3, 3) &\leq 9 - 2 = 7, \\ \text{span } G(5, 3) &\leq 15 - 8 = 7. \end{aligned}$$

11.4 Proposition: $\text{span } G(3, 3) = 7, \text{span } G(5, 3) = 7$.

Proof: In both cases $\text{span} \leq 7$ is known, as seen just above. Using 11.3, $\text{span } G(3, 3) \geq \rho(6) - 1 = 1$, $\text{span } G(5, 3) \geq \rho(8) - 1 = 7$. So $\text{span } G(5, 3) = 7$ is proved, but only $1 \leq \text{span } G(3, 3) \leq 7$. To obtain the lower bound 7 here, Korbaš and Sankaran (cf. Proc. Indian Acad. Sci. (Math. Sci.) 101 (1991), 111-120) proceed as follows.

First, one shows $\text{span } G(3, 3) \geq 2$ using the appropriate VF II theorem, e.g. from Thomas [60] (and noting $G(r, s)$ is oriented iff $r \equiv s \pmod{2}$, so $G(3, 3)$ is oriented), it suffices to know $w_8 G(3, 3) = 0$ (which was seen above) and $\hat{\chi}_2 G(3, 3) = 0$ (which follows easily once $H^*(G(3, 3); Z_2)$ is known).

Second, since $w_1 G(3, 3) = 0$ and its dimension $9 \equiv 1 \pmod{4}$, results of Koschorke ([34], Ch. 20) imply $s = s G(3, 3) = 1$ as well as $\text{span } G(3, 3) = s$ or $\text{span } G(3, 3) = \text{span}^0 G(3, 3)$. Since $\text{span } G(3, 3) > s = 1$, we have $\text{span } G(3, 3) = \text{span}^0 G(3, 3)$.

It remains to prove $\text{span}^0 G(3, 3) \geq 7$, or equivalently $gd^0(\tau) \leq 9 - 7 = 2$ for the tangent bundle τ . Now $\gamma \oplus \gamma^\perp \approx 6\varepsilon$, and $\tau \approx \gamma \otimes \gamma^\perp$. Taking λ^2 , the second exterior power, gives

$$\begin{aligned}\lambda^2(\gamma) \oplus \gamma \otimes \gamma^\perp \oplus \lambda^2(\gamma^\perp) &\approx 15\varepsilon, \\ \tau \oplus \lambda^2(\gamma) \oplus \lambda^2(\gamma^\perp) &\approx 15\varepsilon.\end{aligned}$$

Applying Hodge duality 11.2, and noting $\lambda^3(\gamma) \approx \lambda^3(\gamma^\perp) \approx \xi$ since both are non-orientable rank 3,

$$\begin{aligned}\tau \oplus \gamma \otimes \xi \oplus \gamma^\perp \otimes \xi &\approx 15\varepsilon, \\ \tau \oplus (\gamma \oplus \gamma^\perp) \otimes \xi &\approx 15\varepsilon, \\ \tau \oplus 6\varepsilon \otimes \xi &\approx 15\varepsilon, \\ \tau \oplus 6\xi &\approx 15\varepsilon.\end{aligned}$$

Stably, then, $\tau \sim -6\xi$. Here $\xi = \xi_{3,3}$ is the Hopf line bundle over $G(3, 3)$. In fact $8\xi \sim 0$, which is seen using the composition

$$G(3, 3) \xrightarrow{i} G(3, 5) = G_{8,3} \xrightarrow{m} \mathbb{R}P^7.$$

Here i is the standard inclusion coming from $\mathbb{R}^6 \hookrightarrow \mathbb{R}^8$, while m is induced by the vector triple product M in $\mathbb{R}^8$ [68]; choose an ON basis v_1, v_2, v_3 for a 3-plane x in $\mathbb{R}^8$, then $w = M(v_1, v_2, v_3) \in S^7$ and is unique up to sign, so $m(x) = [w] \in \mathbb{R}P^7$ is well defined. Clearly m is covered by $\tilde{m} : \tilde{G}_{8,3} \rightarrow S^7$, where $\tilde{m}$ is defined as for m but using an oriented ON basis v_1, v_2, v_3 for an oriented 3-plane $\tilde{x}$. It follows that

$$(mi)^*\xi_7 \approx i^*m^*\xi_7 \approx i^*\xi_{3,5} \approx \xi_{3,3} = \xi.$$

Since $8\xi_7 \sim 0$ from Adams [1], $8\xi \sim 0$ follows.

So, $\tau \sim -6\xi \sim 2\xi$ shows $gd^0\tau \leq 2$, as required. $\square$

Many further properties of $\text{span } G(r, s)$ and $\text{span } \tilde{G}(r, s)$, as well as their stable spans, are summarized in the next theorem (taken from [32]). Most of the proofs are outlined in [32], and we will only prove (6) (one half is already done in 11.4) and (7) (in which we have strengthened the corresponding statement in [32]). We also define the sequence $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 7, a_{i+4} = 8 + a_i$ (clearly related to the signature sequence b_i in 10.4).

11.5 Theorem:

- (1) If $\binom{s+t}{s} \equiv 1 \pmod{2}$, then $\text{span } \tilde{G}(4s+1, 4t+1) = \text{span } G(4s+1, 4t+1) = 1$. Also $\text{span}^0 \tilde{G}(4s+1, 4t+1) \leq 4(s+t) + 2$, $\text{span}^0 G(4s+1, 4t+1) \leq 4(s+t) + 1$.
- (2) If $\binom{s+t}{s} \equiv 0 \pmod{2}$ then

$$\begin{aligned}\text{span } \tilde{G}(4s+1, 4t+1) &= \text{span}^0 \tilde{G}(4s+1, 4t+1), \\ 2 &\leq \text{span } \tilde{G}(4s+1, 4t+1) \leq 4(s+t) + 1 + \text{span}^0 \tilde{G}(4s, 4t), \\ \text{span}^0 \tilde{G}(4s, 4t) &\leq a_{i+1}, \\ 1 &\leq (\text{stable}) \text{span } G(4s+1, 4t+1) \leq 4(s+t+1) + \text{span}^0 G(4s, 4t), \\ \text{span}^0 G(4s, 4t) &\leq a_i.\end{aligned}$$

(3) $\text{span } \tilde{G}(4s+3, 4t+3) = \text{span}^0 \tilde{G}(4s+3, 4t+3)$, and

$$2 \leq \text{span } \tilde{G}(4s+3, 4t+3) \leq 8(s+t+1) + \text{span}^0 \tilde{G}(4s+1, 4t+1),$$

$$1 \leq (\text{stable}) \text{span } G(4s+3, 4t+3) \leq 8(s+t+1) + \text{span}^0 G(4s+1, 4t+1).$$

(4) $\text{span } \tilde{G}(4s+1, 4t+3) \geq \text{span } G(4s+1, 4t+3) \geq \rho(4(s+t+1)) - 1 \geq 3$, similarly for $(4s+3, 4t+1)$.

If $\binom{s+t}{s} \equiv 0 \pmod{2}$ then

$$(\text{stable}) \text{span } \tilde{G}(4s+1, 4t+3) \leq 4(s+t) + 3 + \text{span}^0 \tilde{G}(4s, 4t+2),$$

$$\text{span}^0 \tilde{G}(4s, 4t+2) \leq a_{i+1}, \text{ similarly for } (4s+2, 4t), \text{ and}$$

$$(\text{stable}) \text{span } G(4s+1, 4t+3) \leq 4(s+t) + 3 + a_i, \text{ similarly for } (4s+3, 4t+1).$$

If $\binom{s+t}{s} \equiv 1 \pmod{2}$ then

$$(\text{stable}) \text{span } G(4s+1, 4t+3) \leq 4(s+t) + 3, \text{ similarly for } (4s+3, 4t+1).$$

(5) If $u \geq 1$ then $\text{span}^0 G(3, 4 \cdot 2^u + 1) = \text{span } G(3, 4 \cdot 2^u + 1) = 3$.

For $m = 0$ or $m = 2^u$, $u \geq 1$, $\text{span}^0 G(3, 8m+5) = \text{span } G(3, 8m+5) = 7$.

(6) $\text{span } G(3, 3) = 7$, $\text{span}^0 G(2, 2) = 2$.

(7) $\text{span}^0 G(r, s) = 0$ iff $\binom{n}{r} \equiv 1 \pmod{2}$, $n = r + s$.

Proof of (6): As mentioned, $\text{span } G(3, 3)$ is done above in 11.4, also see Lecture 1 for $\tilde{G}(3, 3)$ being parallelizable, so has $\text{span } 9$. For $G(2, 2)$, $w G(2, 2) = 1 + w_1^2$ so $\text{span}^0 G(2, 2) \leq 4 - 2 = 2$. Again $\tau \approx \gamma \otimes \gamma^\perp$, $\gamma \oplus \gamma^\perp \approx 4\epsilon$ and $\gamma, \gamma^\perp$ both rank 2 non-orientable. Then $\lambda^2 \gamma \approx \lambda^2 \gamma^\perp \approx \xi$. Now $6\epsilon \approx \lambda^2(\gamma \oplus \gamma^\perp) \approx \xi \oplus \tau \oplus \xi \implies \tau \sim -2\xi$. Also $4\epsilon \approx \lambda^3(\gamma \oplus \gamma^\perp) \approx 0 \oplus \xi \otimes \gamma^\perp \oplus \gamma \otimes \xi \oplus 0 \approx (\gamma \oplus \gamma^\perp) \otimes \xi \approx 4\xi$. Hence $-2\xi \sim 2\xi$, so $gd^0 \tau \leq 2$ and $\text{span}^0 G(2, 2) \geq 2$.

Proof of (7): Applying 11.1, it will suffice to show $\chi = \chi G(r, s)$ is odd iff $\binom{n}{r} \equiv 1 \pmod{2}$. If r, s are both odd then $\chi = 0$ is even and $\binom{n}{r} = \binom{\text{even}}{\text{odd}} \equiv 0 \pmod{2}$ by Lucas' theorem, so this case never occurs for χ or for $\binom{n}{r}$.

Consider then the cases r odd, s even or both r, s even. In either case $\chi = \binom{[n/2]}{[r/2]}$.

In the first case write $r = 2a + 1$, $s = 2b$. Then $\chi = \binom{a+b}{a} \equiv \binom{2a+2b}{2a} \equiv \binom{2a+2b+1}{2a+1} = \binom{n}{r}$, by Lucas' theorem.

In the second case $r + 2a, s = 2b$. Now $\chi = \binom{a+b}{a} \equiv \binom{2a+2b}{2a} = \binom{n}{r}$. So in either case χ is odd iff $\binom{n}{r}$ is odd. $\square$

One should not suppose from this theorem that the span of Grassmann manifolds is well understood, in fact nearly the opposite is true. For example using the above and Stiefel-Whitney classes only shows $1 \leq \text{span}(G(3, 7)) \leq 7$. In contrast to this the span of the projective Stiefel manifolds $X_{n,r}$ is very well understood, in the sense that for any given n, r

the span can be determined to within a very narrow range, and often exactly. (apart from the single case n odd, $r = 2$). We will outline the methods for obtaining lower and upper bounds, and refer to [32], [33], [66] for further details.

Lower bounds: The isomorphism $\tau_{n,r} \oplus \binom{r+1}{2} \varepsilon \approx nr\xi_{n,r}$ holds, so stably $\tau \sim nr\xi_{n,r}$. Defining

$$k_{n,r} = -\binom{r+1}{2} + \text{span}(nr\xi_{n-1}),$$

where $\xi_{n-1} = \xi_{n,1}$ is the Hopf line bundle on $\mathbb{R}P^{n-1}$, it is easy to show now that $\text{span}^0 X_{n,r} \geq k_{n,r}$. This gives strong lower bounds for $\text{span}^0 X_{n,r}$, which have been effectively computed by Lam [37] and others. Using Koschorke's theorems one can show that $\text{span}^0 X_{n,r} = \text{span} X_{n,r}$ in at least 70% (in an asymptotic sense) of the n, r values, so $\text{span} X_{n,r} \geq k_{n,r}$ is known in these cases. Even better, a result of Korbaš-Zvengrowski (cf. [33], p.100) shows $\text{span} X_{n,r} \geq k_{n,r}$ holds for all n, r except possibly n odd, $r = 2$.

Remark: A reasonable conjecture is $\text{span} X_{n,r} = \text{span}^0 X_{n,r}$ in all cases. Secondly, cases are known where $\text{span} X_{n,r} > k_{n,r}$, so these numbers are not always best possible.

Upper bounds: Let $\binom{r+1}{2} = R$. Using a universal property (cf. [16]) of $X_{n,r}$ for n -fold multiples of a line bundle that admit r independent sections, one has the following easy consequence.

11.6 Proposition: $\text{span}^0 X_{n,r} \geq t$ iff there is a map $f : X_{n,r} \rightarrow X_{nr, R+t}$ such that $f^*(\xi_{nr, R+t} = \xi_{n,r})$.

This proposition is utilized by showing that if t is too large then no such map f can exist, and hence $\text{span} X_{n,r} \leq \text{span}^0 X_{n,r} < t$ for such t . One can use any cohomology theory T , for which $T^*(X_{n,r})$ is computable, to get non-existence results of this and thus upper bounds. To date $H^*(X_{n,r}; \mathbb{Z}_2)$ and $K^*(X_{n,r})$ have been used, as well as the (more elementary) method of Stiefel-Whitney classes.

In addition to the tangent bundle $\tau_{n,r}$ and Hopf line bundle $\xi_{n,r}$ over $X_{n,r}$, there is the orthogonal complement bundle $\beta_{n,r}$ of rank $n - r$ which takes as fibre the orthogonal complement of each $\bar{x} \in X_{n,r}$, where $\bar{x}$ is the equivalence class (i.e. $\pm$) of an orthonormal r -frame $x \in V_{n,r}$. Note $\text{rank } \tau_{n,r} = \dim X_{n,r} = nr - \binom{r+1}{2}$. Also define $\beta'_{n,r} = \beta_{n,r} \otimes \xi_{n,r}$, of rank $n - r$. There are many isomorphisms connecting these bundles. For a complete list cf. [33], p.98, here we just mention

$$\tau_{n,r} \approx r \beta'_{n,r} \oplus \binom{r}{2} \varepsilon.$$

This gives the "linear" lower bound $\text{span} X_{n,r} \geq \binom{r}{2}$, which is in general much weaker than the lower bound $k_{n,r}$ discussed above. However, it does give a quick proof that $\text{span} X_{n,r} \geq 1$ for $r \geq 2$, and hence $\chi(X_{n,r}) = 0$ for $r \geq 2$.

11.7 Example: $\text{span} X_{14,4} = 38$.

Proof: $\dim X_{14,4} = 14 \cdot 4 - \binom{5}{2} = 46$. From 10.3 and the fact that $X_{14,4}$ is even dimensional with $\chi(X_{14,4}) = 0$, it follows that $\text{span } X_{14,4} = \text{span}^0 X_{14,4} \geq k_{14,4}$ (as mentioned in the lower bound description above, this is also directly known without using span^0). Following Lam [37] and the definition of $k_{n,r}$,

$$\text{span } X_{14,4} \geq -10 + \text{span}(56 \xi_{13}) \geq -10 + 56 - 13 + 5 = 38.$$

The cohomology $H^*(X_{n,r}; Z_2)$ has been known since the first papers [7], [16] on the subject. In this case it will suffice to know that there is a single class $x \in H^1(X_{14,4}; Z_2)$, indeed $x = w_1(\xi_{14,4})$, with height 12 (i.e. $x^{12} = 0$, $x^{11} \neq 0$). Since $\tau \sim 14 \cdot 4 \xi = 56 \xi$,

$$w(X_{14,4}) = w(56 \xi) = (w(\xi))^{56} = (1 + x)^{56} = (1 + x^8)^7 = 1 + x^8.$$

Hence $gd^0 \tau \geq 8$, or $\text{span } X_{14,4} \leq \text{span}^0 X_{14,4} \leq 46 - 8 = 38$. □

The reader may think that the above example was specially selected to work out nicely, but in fact for a “randomly” selected $X_{n,r}$ (apart from the odd, 2 case) the same methods as used in 11.7 will give very narrow, if not exact, bounds for the span. For example, $X_{58,51}$ has dimension 1,632 and one finds.

$$1618 \leq \text{span } X_{58,51} \leq 1625,$$

while $X_{314,159}$ has dimension 37,206, and

$$36,897 \leq \text{span } X_{314,159} \leq 37,056.$$

Many more refined techniques have been applied to improve both the lower and upper bounds, including primary and secondary cohomology operations, K -theory and its ring structure, plus results on orders of line bundles. For example, it has long been known that $X_{n,n-1}$ and $X_{2m,2n-2}$ are parallelizable. Recently Zvengrowski [66] has determined $\text{span } X_{2m+1,2m-1}$ and $\text{span } X_{n,n-3}$. Thus $\text{span } X_{n,n-r}$ is known for $r \leq 3$.

Nevertheless, much remains to be done to completely determine $\text{span } X_{n,r}$. As two examples of unsolved questions, there is first $X_{12,8}$ for which parallelizability is still unsettled. It is known that $55 \leq \text{span } X_{12,8} \leq 60 = \dim X_{12,8}$. Second, there is the lower bound question for the $X_{2m+1,2}$ case. It is known that $\text{span } X_{2m+1,2} \geq 4$ for $m \geq 2$, but this is quite weak and $\text{span } X_{2m+1,2} \geq 2m - 1$ seems likely but unproved.

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