UNIVERSIDADE DE SÃO PAULO

Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2577

A Variation-of-Constants Formula for Partial Functional Differential Equations with Unbounded Delay

Eduardo Hernández M.

Nº 114

NOTAS

Série Matemática

São Carlos – SP
Mar./2001

Uma Formula de Variação de Constantes para Equações Funcionais com Retardamento não Limitado.

Eduardo Hernández M.

Departamento de Matemática - ICMC - USP.

Resumo

Neste artigo estabelecemos uma fórmula de variação de constantes para um tipo de equação funcional em derivadas parciais com retardamento não limitado modelada na forma $\dot{x}(t) = Ax(t) + f(t, x_t)$, onde A é o gerador infinitesimal de um semigrupo fortemente contínuo de operadores lineares num espaço de Banach X e f é uma função apropriada.

A Variation-of-Constants Formula for Partial Functional Differential Equations with Unbounded Delay.

Eduardo Hernández M.*

AMS-Subject Classification: Primary 35F25, 34K40.

Keywords: Functional differential equations, semigroups of linear operators.

Abstract

We establish a variation-of-constants formula for a class of abstract partial functional differential equations with unbounded delay that can be described in the form $\dot{x}(t) = Ax(t) + f(t, x_t)$, where A is the infinitesimal generator of a strongly continuous semigroup of linear operators on a Banach space X and f is a continuously differentiable function.

1 Introduction

The purpose of this paper is to establish, for the case of partial functional differential equations with unbounded delay, an analogue of a variation-of-constants formula due to Alekseev (see [1]). The Alekseev formula appears in studies of stability and asymptotic behaviour of non-linear systems; see Brauer [3] for the case of ordinary differential equations; Arrieta, Carvalho and Hale [2] for partial evolution equations and Shanhold [10] for functional differential equations with finite delay.

In the case of functional differential equations with delay, Shanholt [10] establishes a variation-of-constants formula for a type of equations modeled as the abstract Cauchy problem

$$\dot{x}(t) = F(t, x_t), \qquad t \geq \sigma, \tag{1}$$

$$x_\sigma = \varphi, \tag{2}$$

where $x(t) \in \mathbb{R}^n$; $\varphi, x_t \in C([-r, 0] : \mathbb{R}^n)$ and F is a continuously differentiable function.

Equation (1) does not include several interesting problems, for example, partial integrodifferential equations with infinite delay, which arise in the study of a number of problems, such as heat conduction in materials with memory and population dynamics for spatially distributed populations. For this reason, in this paper we establish a new

*Research Partially Supported by FAPESP-SP-Brazil, grant # 1999/13394-5

version of the "Alekseev formula " for a class of partial functional differential equations with unbounded delay that can be described in the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(t, x_t), \qquad t \geq \sigma, \\ x_\sigma &= \varphi, \qquad \varphi \in \Omega, \end{aligned} \tag{3}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on a Banach space X; the histories $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; $\sigma \geq 0$; $\Omega \subset \mathcal{B}$ is open and $f : [0, \infty) \times \Omega \to X$ is a continuously differentiable function.

As in Brauer [3], Arrieta [2] and Shanhold [10], the formula will be established using the perturbed system

$$\begin{aligned} \dot{u}(t) &= Au(t) + f(t, u_t) + g(t, u_t), \\ u_\sigma &= \varphi, \qquad \varphi \in \Omega, \end{aligned} \tag{4}$$

where $g : [0, \infty) \times \Omega \to X$ is continuously differentiable.

Equation (3) is called partial functional differential equation with unbounded delay. A complete reference, including existence results of mild, periodic, strong and classical solutions for system (3), are the paper [6] and [7].

Throughout this paper X will be a abstract Banach space provided with norm $\| \cdot \|$ and A will be the infinitesimal generator of a uniformly bounded C_0 semigroup $(T(t))_{t \geq 0}$ defined on X; that is, we assume that there exists $\tilde{M} > 0$, such that

$$\| T(t) \| \leq \tilde{M}, \qquad t \geq 0.$$

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato [5]. To establish the axioms of the space $\mathcal{B}$ we follow the terminology used in Hino-Murakami-Naito [8]. Thus, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a semi-norm $\| \cdot \|_{\mathcal{B}}$. We will assume that $\mathcal{B}$ satisfies the following axioms:

(A) If $x : (-\infty, \sigma + a) \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$, the following conditions hold:

i) x_t is in $\mathcal{B}$.

ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$.

iii) $\| x_t \| \leq K(t - \sigma) \sup\{ \| x(s) \| : \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \|_{\mathcal{B}}$, where $H > 0$ is a constant; $K, M : [0, \infty) \to [0, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in (A), x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + a)$.

(B) The space $\mathcal{B}$ is complete.

Axiom (**A1**) implies that the operator functions $S(\cdot)$, $W(\cdot)$ given by

$$[S(t)\varphi](\theta) := \begin{cases} \varphi(0) & \text{for } -t \leq \theta \leq 0, \\ \varphi(t + \theta) & \text{for } -\infty < \theta \leq -t, \end{cases} \tag{5}$$

and

$$[W(t)\varphi](\theta) := \begin{cases} T(t+\theta)\varphi(0) & \text{for } -t \leq \theta \leq 0, \\ \\ \varphi(t+\theta) & \text{for } -\infty < \theta \leq -t, \end{cases} \tag{6}$$

are strongly continuous semigroups of linear operators on $\mathcal{B}$. We will denote by A_S and A_W the infinitesimal generators of $S(\cdot)$ and $W(\cdot)$ and by $D(A_S)$ and $D(A_W)$ their respective domains.

To obtain some of our results we will need additional properties of the space $\mathcal{B}$, for this reason we introduce the next axiom:

($\mathbf{C_4}$) There exist $r > 0$ such that; if $(\varphi_n)_{n\in N}$ is a uniformly bounded sequence in $C_{00}((\infty, 0] : X)$ which converge uniformly on compact subset of $(\infty, -r]$ and pointwise on $(-r, 0]$ to a function φ, then $\varphi \in \mathcal{B}$ and $\| \varphi_n - \varphi \|_{\mathcal{B}} \to 0$ as $n \to \infty$.

Example 1 *The space $\mathcal{B} = L^p(g; X)$, $1 \leq p < \infty$, (see [8]) is the space of all classes of functions $\varphi : (-\infty, 0] \to X$, such that φ is Lebesgue measurable on $(-\infty, 0]$ and $g(\cdot) \| \varphi(\cdot) \|^p$ is Lebesgue integrable on $(-\infty, 0]$, where $g : (-\infty, 0] \to \mathbb{R}$ is a positive integrable function with $g(0) = 1$. The seminorm in $\mathcal{B}$ is*

$$\| \varphi \| = \| \varphi(0) \| + \left(\int_{-\infty}^{0} g(\theta) \| \varphi(\theta) \|^p \, d\theta \right)^{\frac{1}{p}}.$$

If there exists a nonnegative and locally bounded function γ defined on $(-\infty, 0]$, such that $g(\xi + \theta) \leq \gamma(\xi) g(\theta)$ for $\theta \leq 0$ and $\theta \in (-\infty, 0] \backslash N_\xi$, where $N_\xi \subseteq (-\infty, 0]$ is a set with Lebesgue measure 0, then the space satisfies the axioms $\mathbf{A}, \mathbf{A1}, \mathbf{B}$ *and* $\mathbf{C_4}$.

The terminology and notations are those generally used in operator theory. In particular, if Z, Y are Banach spaces, we indicate by $\mathcal{L}(Z : Y)$ the Banach space of the bounded linear operators of Z into Y and we abbreviate to $\mathcal{L}(Z)$ whenever $Z = Y$. In addition $B_r(z : Z)$ will denote the closed ball in the space Z with center at z and radius r.

On the other hand, for some bounded function $\xi : [0, a] \to \mathbb{R}^+$ and $0 \leq \sigma \leq t \leq a$, we will employ the notation

$$\xi_{\sigma,t} = \sup\{\xi(s) : s \in [\sigma, t]\}$$

and we will write simply ξ_t for $\xi_{0,t}$ when no confusion arises.

Finally, for $x \in X$, we will use the notation $\mathcal{X}_x$ for the function $\mathcal{X}_x : (-\infty, 0] \to X$ where $\mathcal{X}_x(\theta) = 0$ for $\theta < 0$ and $\mathcal{X}_x(0) = x$.

2 A New Variation-of-Constants Formula

The concepts of mild and classical solutions for the model (3) are introduced by Henriquez in [7] and [6] respectively. By completeness we include these definitions

Definition 1 *We will say that a function $x : (-\infty, \sigma + a) \to X$, $a > 0$, is a mild solution of the abstract Cauchy problem (3) if: $x_\sigma = \varphi$; the restriction of $x(\cdot)$ to the interval $[\sigma, \sigma + a)$ is continuous and*

$$x(t) = T(t - \sigma)\varphi(0) + \int_\sigma^t T(t - s)f(s, x_s)ds, \tag{7}$$

for each $\sigma \leq t < \sigma + a$

Definition 2 *A mild solution $x : (-\infty, \sigma + a) \to X$, $a > 0$, is a classical solution of the abstract Cauchy problem (3) if: $x_\sigma = \varphi$; the restriction of $x(\cdot)$ to the interval $[\sigma, \sigma + a)$ is continuous; $\dot{x} \in C((\sigma, \sigma + a) : X)$; $x(t) \in D(A)$ for every $t \in (\sigma, \sigma + a)$ and equation (3) is satisfied on $(\sigma, \sigma + a)$.*

Notation 1 *In what follows we reserve the notations $x(\cdot, \sigma, \varphi)$ and $u(\cdot, \sigma, \varphi)$ for the mild solutions of (3) and (4) and $I(\sigma, \varphi, f)$, $I(\sigma, \varphi, f + g)$ for the maximal interval of definition of $x(\cdot, \sigma, \varphi)$ and $u(\cdot, \sigma, \varphi)$ respectively. For $x(\sigma, \varphi)$ we will use the decomposition*

$$x(\sigma, \varphi) = y(\sigma, \varphi) + z(\sigma, \varphi), \tag{8}$$

where $y_\sigma = \varphi$ and $y(s) = T(s - \sigma)\varphi(0)$ for $s \geq \sigma$.

Our first result is a Lemma essential for our purpose.

Lemma 1 *If $\varphi \in D(A_W)$ and $\sigma \geq 0$ then*

$$\begin{aligned} \| x_{\sigma+h}(\sigma, \varphi) - \varphi \| &= 0(h), \\ \| x_\sigma(\sigma - h, \varphi) - \varphi \| &= 0(h). \end{aligned}$$

Proof: From the continuity of solutions of (3) with respect to σ and φ, we can to fix $C_1 > 0$ and $r_1 > 0$ such that $\| f(t, x_t(\theta, \psi) \| \leq C_1$ for each $t, \theta \in B(\sigma, r_1)$ and $\psi \in B(\varphi, r_1)$. Using the decomposition (8), for $h > 0$ sufficiently small we have

$$\| z(\sigma + h, \sigma, \varphi) \| \leq \int_\sigma^{\sigma+h} \| T(\sigma + h - s)f(s, x_s(\sigma, \varphi) \| ds \leq \tilde{M}C_1 h.$$

Thus, from axiom **(A)**

$$\| z_{\sigma+h}(\sigma, \varphi) \|_{\mathcal{B}} \leq K_{r_1}\tilde{M}C_1 h \leq C_2 h.$$

Similarly

$$\| z_\sigma(\sigma - h, \varphi) \|\|\|_{\mathcal{B}} \leq K_{r_1}\tilde{M}C_1 h \leq C_2 h.$$

Finally, the assertions follow from

$$\begin{aligned} \| x_{\sigma+h}(\sigma, \varphi) - \varphi \|_{\mathcal{B}} &\leq \| W(h)\varphi - \varphi \|_{\mathcal{B}} + \| z_{\sigma+h}(\sigma, \varphi) \|_{\mathcal{B}} \leq C_3 h, \\ \| x_\sigma(\sigma - h, \varphi) - \varphi \|_{\mathcal{B}} &\leq \| W(h)\varphi - \varphi \|_{\mathcal{B}} + \| z_\sigma(\sigma - h, \varphi) \|_{\mathcal{B}} \leq C_4 h, \end{aligned}$$

since $\varphi \in D(A_W)$. The proof is complete.

In the next result, Axiom **(C$_3$)**; introduced originally by Henriquez in [6], is:

($\mathbf{C_3}$) Let $a > 0$. Let $x : (-\infty, \sigma + a] \to X$ be a continuous function such that $x_\sigma \equiv 0$ and the right derivative, denoted $\dot{x}(0^+)$, exists. If the function ψ defined by $\psi(\theta) = 0$ for $\theta < 0$ and $\psi(0) := \dot{x}(0^+)$ belongs to $\mathcal{B}$, then $\| (\frac{1}{h})x_h - \psi \|_{\mathcal{B}} \to 0$ as $h \to 0^+$.

The proof of the next Theorem follows from Theorem 4 and Corollary 2 in Henríquez [6].

Theorem 1 *Let $\varphi \in D(A_W)$. If $f(\sigma, \varphi) = 0$ or $\mathcal{B}$ satisfies Axiom $\mathbf{C_3}$ and $\mathcal{X}_{f(\sigma,\varphi)} \in \mathcal{B}$ then $x(\cdot, \sigma, \varphi)$ is a classical solution.*

The key to obtain the Alekseev representation formula is to prove the differentiability of the function $\varphi \to x_t(\sigma, \varphi)$. For this reason and in order to use the Implicit Function Theorem we introduce some additional notations. For $a > 0$ we define

$$\Phi(\cdot) : \mathcal{B} \times C([\sigma, \sigma + a] : X) \to C([\sigma, \sigma + a] : X)$$

as

$$\Phi(\varphi, u)(t) := T(t - \sigma)\varphi(0) + \int_\sigma^t T(t - s) f(s, \bar{S}(s - \sigma)u + S(s - \sigma)\varphi) ds - u(t)$$

where

$$\bar{S}(s)u(\theta) := \begin{cases} u(s + \theta) - u(0) & \text{if } -s \leq \theta \leq 0, \\ 0 & \text{if } -\infty < \theta \leq -s. \end{cases} \tag{9}$$

Remark 1 *In (9), in a slight abuse of notation, and in order to abbreviate the notations, we have assumed that f is defined on $\mathcal{B}$.*

Notation 2 *For a differentiable function $j : I \times \Omega \to X$, we will use the decomposition*

$$j(t + s, \psi) - j(t, \psi) = D_1 j(t, \psi)s + W_1(j, t, t + s, \psi) \tag{10}$$

and

$$j(t, \psi + \psi_1) - j(t, \psi) = D_2 j(t, \psi)\psi_1 + W_2(j, t, \psi, \psi_1) \tag{11}$$

where

$$\frac{W_1(j, t, t + s, \psi)}{| s |} \to 0, \qquad \textit{as} \quad s \to 0$$

$$\frac{W_2(j, t, \psi, \psi + \psi_1)}{\| \psi_1 \|_{\mathcal{B}}} \to 0, \qquad \textit{as} \quad \psi_1 \to 0$$

and

$$\| W_2(j, t, \psi, \psi_1) - W_2(j, t, \psi, \psi_2) \| \leq \eta(t, \psi, \beta) \| \psi_1 - \psi_2 \|_{\mathcal{B}} \tag{12}$$

for $\| \psi_i \| < \beta$, where η is continuous in (t, ψ, β) and $\eta(t, \psi, 0) = 0$.

Lemma 2 *The function $\Phi(\cdot)$ is continuously differentiable.*

Proof: Let $(\varphi, u) \in \mathcal{B} \times C([\sigma, \sigma + a] : X)$. For $v \in C([\sigma, \sigma + a] : X)$ we have

$$\begin{aligned}\Phi(\varphi, u + v)(t) - \Phi(\varphi, u)(t) &= \int_\sigma^t T(t-s) D_2 f(s, \Lambda(s, \varphi, u)) \bar{S}(s - \sigma) v ds \\ &+ \int_\sigma^t T(t-s) W_2(f, s, \Lambda(s, \varphi, u), \bar{S}(s - \sigma) v) ds - v(t)\end{aligned} \tag{13}$$

where

$$\begin{aligned}&\int_\sigma^t \| T(t-s) W_2(s, \Lambda(s, \varphi, u), \bar{S}(s - \sigma) v) \| ds \\ &\leq \int_\sigma^t \tilde{M} \eta(s, \Lambda(s, \varphi, u), \beta) \| \bar{S}(s - \sigma) v \|_{\mathcal{B}} ds \\ &\leq 2K_{0,a} \| v \|_{[\sigma, \sigma + a]} \tilde{M} \int_\sigma^t \eta(s, \Lambda(s, \varphi, u), \beta) ds\end{aligned}$$

for $\| \bar{S}(s - \sigma) v \| \leq \beta$. The existence of $D_2 \Psi(\varphi, u)$ is consequence of the above inequality and (13). From the previous estimates we infer that for $v \in C([\sigma, \sigma + a] : X)$,

$$[D_2 \Phi(\varphi, u) v](t) = \int_\sigma^t T(t-s) D_2 f(s, \Lambda(s, \varphi, u)) \overline{S}(s - \sigma) v \, ds - v(t). \tag{14}$$

The continuity of $D_2 \Phi(\sigma, \varphi)$ follows from (14) and a routine using the phase space axioms.

Using the previous arguments we can prove that $D_1 \Phi(\varphi, u)$ exists and that

$$[D_1 \Phi(\varphi, u) \psi](t) = T(t - \sigma) \psi(0) + \int_\sigma^t T(t-s) D_2 f(s, \Lambda(s, \varphi, u)) S(s - \sigma) \psi ds.$$

for every $\psi \in \mathcal{B}$. The continuity of $D_1 \Phi(\cdot)$ is clear. The proof is complete.

Lemma 3 *$D_2 \Phi(\varphi, u)$ is an isomorphism.*

Proof. Let $(\varphi, u) \in \mathcal{B} \times C([\sigma, \sigma + a] : X)$ and $v \in C([\sigma, \sigma + a] : X)$ with $D_2 \Phi(\varphi, u) \cdot v \equiv 0$. It follows from (14) that

$$\| v \|_{\sigma, t} \leq 2K_{0,a} \int_\sigma^t \tilde{M} \| D_2 f(s, \Lambda(s, \varphi, u)) \|_{\sigma, \sigma + a} \| v \|_{\sigma, s} ds \tag{15}$$

for all $t \in [\sigma, \sigma + a]$; thus $v \equiv 0$ and $D_2 \Phi(\varphi, u)$ is injective.

To prove that $D_2 \Phi(\varphi, u)$ is surjective we take $w \in C([\sigma, \sigma + a] : X)$ and define the mapping $\Upsilon : C([\sigma, \sigma + a] : X) \to C([\sigma, \sigma + a] : X)$ as

$$\Upsilon(v)(t) = \int_\sigma^t T(t-s) D_2 f(s, \Lambda(s, \varphi, u)) \bar{S}(s - \sigma) v \, ds - w(t).$$

Using axiom **(A)** we can to prove, that for n sufficiently large Υ^n is a contraction. Clearly a fixed point of Υ is a solution of $D_2 \Phi(\varphi, u) v = w$ and therefore $D_2 \Phi(\varphi, u)$ is surjective. This proves the result.

Corollary 1 *Let $\sigma > 0$. The function $\varphi \to x(\sigma, \varphi)$ defined from $\mathcal{B}$ into $C([\sigma, \sigma + a] : X)$ is continuously differentiable.*

Proof. The assertion follows from Implicit Function Theorem and Lemmas (2)-(3).

Corollary 2 *Let $t > \sigma \geq 0$. The function $\varphi \to x_t(\sigma, \varphi)$ is continuously differentiable.*

Notation 3 *Let $t \geq \sigma \geq 0$. In the rest of this paper, $\Psi(t, \cdot) : [0, t) \times \Omega \to \mathcal{B}$ is the function defined by $\Psi(t, \sigma, \varphi) = x_t(\sigma, \varphi)$.*

Proposition 1 *Let $a > 0$; $a + \sigma > t \geq \sigma \geq 0$ and $\varphi \in \Omega \bigcap D(A_W)$. If either of the following conditions hold*

a) $\mathcal{B}$ satisfies axiom (C_4) and $\mathcal{X}_{f(\sigma,\varphi)} \in \mathcal{B}$,

b) $f(\sigma, \varphi) = 0$,

then $D_2\Psi(t, \sigma, \varphi)$ exists and

$$D_2\Psi(t, \sigma, \varphi) = D_3\Psi(t, \sigma, \varphi)(A_W(\varphi) + \mathcal{X}_{f(\sigma,\varphi)}). \tag{16}$$

Proof. Firstly, we observe that $[\frac{\partial \psi}{\partial \varphi}(\cdot, \sigma, \varphi)\psi)](0)$ is the solution of the integral equation

$$\begin{aligned} Y(t, \sigma, \varphi)\xi &= T(t - \sigma)\xi(0) + \int_\sigma^t T(t - s)D_2 f(s, x_s(\sigma, \varphi))(Y(\sigma, \varphi)\xi)_s \, ds \\ (Y(\sigma, \varphi))_\sigma \xi &= \xi. \end{aligned}$$

Thus $[\frac{\partial \psi}{\partial \varphi}(\cdot, \sigma, \varphi)\psi)](0) \in C([\sigma, \sigma + a] : X)$ and $\frac{\partial \psi}{\partial \varphi}$ depend continuously of σ and φ.

For $h > 0$ sufficiently small we get

$$\begin{aligned} x_t(\sigma - h, \varphi) - x_t(\sigma, \varphi) &= x_t(\sigma, x_\sigma(\sigma - h, \varphi)) - x_t(\sigma, \varphi) \\ &= -D_3\Psi(t, \sigma, \varphi)[x_\sigma(\sigma - h, \varphi) - \varphi) \\ &\quad -W_2(\Psi(t, \sigma, \cdot), \varphi, x_\sigma(\sigma - h, \varphi)). \end{aligned} \tag{17}$$

Next, we study each term in the right side of (17) separately. Using that $\varphi \in D(A_W)$ and the decomposition introduced in (8), we see

$$\| \frac{x_\sigma(\sigma - h, \varphi) - \varphi}{h} - A_W(\varphi) - \mathcal{X}_{f(\sigma,\varphi)} \|_{\mathcal{B}}$$

$$\leq \| \frac{W(h)\varphi - \varphi}{h} - A_W(\varphi) \|_{\mathcal{B}} + \| \frac{z_\sigma(\sigma - h, \varphi)}{h} - \mathcal{X}_{f(\sigma,\varphi)} \|_{\mathcal{B}} .$$

Since the functions $\frac{z_\sigma(\sigma-h,\varphi)}{h}$ are uniformly bounded, from condition $a)$ or $b)$ it follows that

$$\frac{x_{\sigma+h}(\sigma, \varphi) - \varphi}{h} \to A_W(\varphi) - \mathcal{X}_{f(\sigma,\varphi)}, \qquad \text{as } h \to 0. \tag{18}$$

On the other hand, from Lemma 1 we find

$$\begin{aligned} &\frac{W_2(\Psi(t, \sigma, \cdot), \varphi, x_\sigma(\sigma - h, \varphi))}{h} \\ &= \frac{W_2(\Psi(t, \sigma, \cdot), \varphi, x_\sigma(\sigma - h, \varphi))}{\| x_\sigma(\sigma - h, \varphi) - \varphi \|_{\mathcal{B}}} \cdot \frac{\| x_\sigma(\sigma - h, \varphi) - \varphi \|_{\mathcal{B}}}{h} \to 0 \text{ as } h \to 0. \end{aligned} \tag{19}$$

It follows from (17), (18) and (19) that

$$\frac{d^-}{d\sigma}x_t(\sigma,\varphi) = A_W(\varphi) + \mathcal{X}_{f(\sigma,\varphi)}. \tag{20}$$

On the other hand

$$\begin{aligned} x_t(\sigma-h,\varphi) - x_t(\sigma,\varphi) &= -D_3\Psi(t,\sigma,\varphi)[x_\sigma(\sigma-h,\varphi)-\varphi) + z_\sigma(\sigma-h,\varphi)] \\ &\quad -W_2(\Psi(t,\cdot),\sigma+h,\varphi,x_\sigma(\sigma-h,\varphi)). \end{aligned} \tag{21}$$

As in the previous case we study the terms in (21) one to one. Since the functions $\frac{z_\sigma(\sigma-h,\varphi)}{h}$ are uniformly bounded in $(-\infty,0]$, axiom C_4 implies that $\frac{z_\sigma(\sigma-h,\varphi)}{h} \to \mathcal{X}_{f(\sigma,\varphi)}$ as $h\to 0$ and then from

$$\begin{aligned} &\| \frac{x_\sigma(\sigma-h,\varphi)-\varphi}{h} - A_W(\varphi) - \mathcal{X}_{f(\sigma,\varphi)} \|_{\mathcal{B}} \\ \le\ &\| \frac{W(h)\varphi-\varphi}{h} - A_W(\varphi) \| + \| \frac{z_\sigma(\sigma-h,\varphi)}{h} - \mathcal{X}_{f(\sigma,\varphi)} \|_{\mathcal{B}}, \end{aligned}$$

we infer that

$$\frac{x_\sigma(\sigma-h,\varphi)-\varphi}{h} \to A_W(\varphi) + \mathcal{X}_{f(\sigma,\varphi)} \qquad \text{as} \qquad h\to 0. \tag{22}$$

Using Lemma (1) we have

$$\begin{aligned} &\frac{W_2(\Psi(t,\cdot),\sigma,\varphi,x_\sigma(\sigma-h,\varphi))}{h} \\ =\ &\frac{W_2(\Psi(t,\cdot),\sigma,\varphi,x_\varphi(\varphi-h,\varphi))}{\| x_\sigma(\sigma-h,\varphi)-\varphi \|_{\mathcal{B}}} \cdot \frac{\| x_\sigma(\sigma-h,\varphi)-\varphi \|_{\mathcal{B}}}{h} \to 0 \end{aligned} \tag{23}$$

as $h\to 0$.

From (23) and (22) we conclude that

$$\frac{d^-}{d\sigma}\Psi(t,\sigma,\varphi) = A_W(\varphi) + \mathcal{X}_{f(\sigma,\varphi)}. \tag{24}$$

Using the same argument as above, we prove that

$$\frac{d^+}{d\sigma}\Psi(t,\sigma,\varphi) = A_W(\varphi) + \mathcal{X}_{f(\sigma,\varphi)}. \tag{25}$$

Finally, the assertion is a consequence of (24) and (25). The proof is finished.

Lemma 4 *Let $\varphi\in D(A_W)$ and $\sigma\ge 0$. If $\mathcal{B}$ satisfies axiom C_4 then $x_\sigma(0,\varphi)\in D(A_W)$; moreover*

$$A_W(y_\sigma(0,\varphi)) = W(\sigma)(A_W(\varphi)) \tag{26}$$

and

$$A_W(z_\sigma(\sigma,\varphi))(\theta) := \begin{cases} 0 & \textit{for} \quad \theta\in(-\infty,-\sigma) \\ Az_\sigma(\theta) + f(\theta+\sigma, x_{\sigma+\theta}(0,\varphi)) & \textit{for} \quad \theta\in[-\sigma,0), \\ Az(\sigma,\varphi) & \textit{for} \quad \theta = 0. \end{cases} \tag{27}$$

Proof. Equation (26) follows directly from the relation $W(\sigma)\varphi = y_\sigma(\cdot, 0, \varphi)$ and (27) is a consequence of axiom C_4, we'll omit details.

Now we establish the main result of this work:

Theorem 2 *Let $\varphi \in D(A_W)$ and $\sigma > 0$. If $\mathcal{B}$ satisfies axiom C_4 then*

$$u_t(\sigma, \varphi) = x_t(\sigma, \varphi) + \int_\sigma^t D_3\Psi(t, s, u_s(\sigma, \varphi))\mathcal{X}_{g(s,u_s)}ds \tag{28}$$

for every $t \in I(\sigma, \varphi, f+g) \bigcap I(\sigma, \varphi, f) \bigcap [\sigma, \infty^+)$.

Proof: In order to simplify the proof we only study the case $\sigma = 0$. Since $\varphi \in D(A_W)$ and $\mathcal{B}$ satisfies axiom C_4, from Lemma (4) $u_\sigma \in D(A_W)$ for $t > s \geq 0$. Moreover, since the assertions in Proposition 1 and Lemma 4 hold for the system (4) we find

$$\begin{aligned} \frac{d}{ds}x_t(s, u_s) &= D_3\Psi(t, s, u_s)(A_W(u_s) + \mathcal{X}_{(f+g)(s,u_s)}) \\ &\quad -D_3\Psi(t, s, u_s)(A_W(u_s) + \mathcal{X}_{(f)(s,u_s)}) \\ &= D_3\Psi(t, s, u_s)\mathcal{X}_{g(s,u_s)}, \end{aligned}$$

and thus

$$\frac{d}{ds}x_t(s, u_s) = D_3\Psi(t, s, u_s)\mathcal{X}_{g(s,u_s)}. \tag{29}$$

Finally, integrating (29) between 0 and t, it follows that

$$u_t(0, \varphi) = x_t(0, \varphi) + \int_0^t D_3\Psi(t, s, u_s)\mathcal{X}_{g(s,u_s)}ds.$$

This completes the proof.

The relation (28) has only been shown for a initial condition $\varphi \in D(A_W)$. In order to establish that this formula hold for all $\varphi \in \mathcal{B}$, we first observe:

Lemma 5 *The subspace $D(A_W)$ is dense in $\mathcal{B}$.*

Let $a, \sigma \geq 0$ and $\varphi \in \Omega$. since for $\xi \in \mathcal{B}$, the function $J(t, \sigma, \varphi)\xi = [D_3\Psi(t, \sigma, \varphi)\xi](0)$ is the unique solution of the integral equation

$$\begin{aligned} Y(t, \sigma, \varphi)\xi &= T(t-\sigma)\xi(0) + \int_\sigma^t T(t-s)D_2 f(s, x_s(\sigma, \varphi))(Y(\sigma, \varphi)\xi)_s\, ds \\ (Y(\sigma, \varphi))_\sigma\xi &= \xi \end{aligned}$$

from axiom (**A**) we can to infer that

$$\| J_t(s, u_s(\varphi))\xi - J_t(s, u_s(\psi))\xi \| \leq K_a \| J(\cdot, s, u_s(\varphi))\xi - J(\cdot, s, u_s(\psi))\xi \|_{\sigma,s} \tag{30}$$

for $\psi, \xi \in \mathcal{B}$ and $\sigma + a \geq t \geq s \geq \sigma$.

Proposition 2 *Let $a > 0$. The following properties hold:*

a) for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\| D_2 f(t, x_t(s, u_s(0,\varphi)) - D_2 f(t, x_t(s, u_s(0,\psi)) \| \leq \epsilon$$

for $t \in [s,a]$, $s \in [0,a]$ and $\| \varphi - \psi \| < \delta$;

b) for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\| J(t, s, u_s(0,\varphi)) - J(t, s, u_s(0,\psi)) \| \leq \epsilon$$

for $t \in [s,a]$, $s \in [0,a]$ and $\| \varphi - \psi \| < \delta$.

Proof: The proof of assertion $a)$ follows from the steps in the proof of Lemma 2 in Shanholt [10].

Now we prove assertion $b)$. Since Df is continuous, we can to assume that it is bounded by a constant C_1 on the set

$$A = \{x_t(\cdot, s, u_s(0,\psi)) : s \in [0,a],\ t \in [s,a] \text{ and } \psi \in B(\varphi, r)\}.$$

Employing (30) and the abbreviated notation $u(\zeta) = u(0,\zeta)$, $\zeta \in \mathcal{B}$; for $s \in [0,a]$, $t \in [s,a]$ and $\xi \in \mathcal{B}$, we obtain

$$\| J(t,s,u_s(\varphi))\xi - J(t,s,u_s(\psi))\xi \|$$

$$\begin{aligned}
&\leq \int_s^t \tilde{M} \| D_2 f(\theta, x_\theta(s, u_s(\varphi))) - D_2 f(\theta, x_\theta(s, u_s(s,\psi))) \| \| J_\theta(\cdot, s, u_s(\varphi)) \| \| \xi \|_{\mathcal{B}}\, d\theta \\
&\quad + \tilde{M} \int_s^t \| D_2 f(\theta, x_\theta(s, u_s(\psi)) \| \| J_\theta(\cdot, s, u_s(\varphi)\xi - J_\theta(\cdot, s, u_s(\psi)\xi \|\, d\theta \\
&\leq \int_s^t \tilde{M}\gamma(\theta, s, \varphi, \psi) \| J_\theta(\cdot, s, u_s(\varphi)) \| \| \xi \|_{\mathcal{B}}\, d\theta \\
&\quad + \tilde{M} K_a C_1 \int_s^t \| J(\cdot, s, u_s(\varphi))\xi - J(\cdot, s, u_s(\psi))\xi \|_{[s,\theta]}\, d\theta.
\end{aligned}$$

Thus

$$\| J(t,s,u_s(\varphi))\xi - J(t,s,u_s(\psi))\xi \|$$

$$\leq \; C_1 \int_s^t \gamma(\theta, s, \varphi, \psi)_{s,\theta} d\theta + C_2 \int_s^t \| J(\cdot, s, u_s(\varphi))\xi - J(\cdot, s, u_s(\psi))\xi \|_{[s,\theta]}\, d\theta$$

where $\gamma(\theta, s, \varphi, \psi) \to 0$ as $\psi \to \varphi$, uniformly for $\theta \in [s,a]$ and $s \in [0,a]$. Finally, the second assertion follows from above inequality and the Gronwall's inequality. This completes the proof.

The proof of the next result is a consequence of Lemma (5) and Proposition (2)

Corollary 3 *The representation formula (28) holds for every $\varphi \in \mathcal{B}$.*

References

[1] Alekseev, V. M. *An estimate for the perturbations of the solutions of ordinary differential equations.* (Russian) Vestnik Moskov. Univ. Ser. I Mat. Meh. 1961 no. 2, 28–36.

[2] Arrieta, José; Carvalho, Alexandre N.; Hale, Jack K. *A damped hyperbolic equation with critical exponent.* Comm. Partial Differential Equations 17 (1992), no. 5-6, 841–866.

[3] F. Brauer, *Perturbation of nonlinear systems of differential equations* , J. Math. Anal and Applications, 14 (1966), 198-206.

[4] J. Hale, *Funtional Differential Equations.* Springer-Verlag, New York 1971.

[5] J. Hale and J. Kato, *Phase Space for retarded Equations with Unboundey Delay.* Funkcial Ekvac, 21 (1978).

[6] H. Henriquez, *Regularity of Solutions of Abstract Retarded Functional Differential Equations With Unbounded Delay.* Nonlinear Analysis, Theory Method and Aplications, vol, 28, N^o 3. pp.551-531, 1997.

[7] H. Henriquez, *Periodic Solutions of quasylinear functional Differential Equations With Unbounded Delay.* Funkcialaj. Ekvacioj, 37(2) 1994, 329-343.

[8] Y. Hino, S.Murakami and T. Naito, *Funtional Differential Equations With Unbounded Delay.* Lectures and Notes in Mathematics, vol 1473. Springer, Berlin (1991).

[9] A. Pazy, *Semigroups of linear operator and applications to partial differential equations.* Springer, New York (1983).

[10] G. Shanholt, *A Nonlinear Variation-of-constants formula for Funtional Differential Equations.* Math. Systems Theory 6 (1972/73), 343-352.

Eduardo Hernández M.
Departamento de Matemática
Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo - Campus de São Carlos
Caixa Postal 668
13560-970 São Carlos, SP. Brazil
lalohm@icmc.sc.usp.br

NOTAS DO ICMC

SÉRIE MATEMÁTICA

113/2001 ZVENGROWSKI, P. – Seminar on the span and parallelizability of manifolds USP at São Carlos.

112/2001 OLIVEIRA, R.D.S.; TARI, F. – On pairs of regular foliations in the plane.

111/2001 NABARRO, A.C. – Projections of hypersurfaces in IR^4 to planes.

110/2000 GUTIERREZ, C.; SARMIENTO, A. – Injectivity of C^1 maps IR^2 IR^4 at infinity and planar vector fields.

109/2000 GUTIERREZ, C.; CORTEZ, M.E.C. – Self-induced interval exchanges and piecewise linear maps of the interval.

108/2000 GUTIERREZ, C.; GUADALUPE, I.; TRIBUZY, R.; GUÍÑEZ, V. – A differential equation for lines of curvature on surfaces immersed in IR^4.

107/2000 GUTIERREZ, C.; GUÍÑEZ, V. – Simple umbilic points on surfaces immersed in IR^4.

106/2000 ARRAUT, J.L.; BIASI, C. – Foliations by planes in the complement of a compact set.

105/2000 BENÁ, M.A.; GODOY, S.M.S. – On the stability in terms of two measures of perturbed neutral functional differential equations.

104/2000 CARVALHO, A.N.; CHOLEWA, J.W.; DLOTKO, T. – Abstract parabolic problems in ordered Banach spaces.