UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação
ISSN 0103-2577

The Second Order Impulsive Cauchy Problem

Eduardo Hernández M.

Nº 115

NOTAS

Série Matemática

São Carlos – SP
Mar./2001

Existência de Soluções para Equações Diferenciais de Segunda Ordem com Impulsos.

Eduardo Hernández M.

Departamento de Matemática - ICMC - USP.

Resumo

Neste artigo provamos a existência de soluções fracas e clássicas para um tipo de equações de segunda ordem com impulso modeladas na forma

$$\begin{aligned}
\ddot{u}(t) &= Au(t) + f(t, u(t), \dot{u}(t)), & t \in (-T_0, T_1),\ t \neq t_i,\\
u(0) &= x_0, \quad \dot{u}(0) = y_0, &\\
\triangle u(t_i) &= I_i^1(u(t_i)), \quad \triangle \dot{u}(t_i) = I_i^2(\dot{u}(t_i^+)), &
\end{aligned}$$

onde A é o gerador infinitesimal de uma função coseno fortemente contínua de operadores lineares num espaço de Banach X e f, I_i^1, I_i^2 são funções apropriadas.

The Second Order Impulsive Cauchy Problem.

Eduardo Hernández M.

Keywords: Cosine function, differential equations with impulses.

AMS-Subject Classification: Primary 34A37.

Abstract

We study the existence of mild and classical solutions for a abstract second order impulsive Cauchy problem modeled in the form

$$\begin{aligned} \ddot{u}(t) &= Au(t) + f(t, u(t), \dot{u}(t)), & t \in (-T_0, T_1),\ t \neq t_i, \\ u(0) &= x_0, \quad \dot{u}(0) = y_0, & \\ \triangle u(t_i) &= I_i^1(u(t_i)), \quad \triangle \dot{u}(t_i) = I_i^2(\dot{u}(t_i^+)), & \end{aligned}$$

where A is the infinitesimal generator of a strongly continuous cosine family of linear operators on a Banach space X and f, I_i^1 and I_i^2 are appropriated continuous functions.

1 Introduction

This paper is concern with the second order impulsive Cauchy problem

$$\begin{aligned} &\ddot{u}(t) = Au(t) + f(t, u(t), \dot{u}(t)), && t \in (-T_0, T_1),\ t \neq t_i, \\ &u(0) = x_0, \quad \dot{u}(0) = y_0, && \qquad (1) \\ &\triangle u(t_i) = I_i^1(u(t_i)), \quad \triangle \dot{u}(t_i) = I_i^2(\dot{u}(t_i^+)). && \end{aligned}$$

In the system (1) A is the infinitesimal generator of a strongly continuous cosine family of linear operators $(C(t))_{t\in \mathbb{R}}$ on a Banach space X; $-T_0 < 0 < t_1 < t_2 < .... < t_n < T_1$; $\triangle u(t_i) = u(t_i^+) - u(t_i^-)$, $\triangle \dot{u}(t_i) = \dot{u}(t_i^+) - \dot{u}(t_i^-)$ are the impulsive conditions and $I_j^i : X \to X$, $f : \mathbb{R} \times X \times X \to X$ are appropriated continuous functions.

The theory of impulsive differential equations has become an important area of investigation in recent years. Relative to this theory we only refer to the reader for the works of Rogovchenko [4]-[5]; Liu [3]; Sun [6] and Cabada [1] .

Motivated for numerous applications, recently Liu (see [3]) studied the first order evolution problem

$$\begin{aligned} &\dot{u}(t) = Au(t) + f(t, u(t)), \\ &u(0) = x_0, \\ &\triangle u(t_i) = I_i u(t_i), \end{aligned}$$

where A is the infinitesimal generator of a C_0-semigroup of linear operators on a Banach space X. In the cited paper, Liu apply the semigroup theory to prove the existence of mild, strong and classical solutions for system (2) using usual assumptions on the function f.

Our goal will be give some existence results of mild and classical solutions for the second order impulsive Cauchy problem (1) using the cosine functions theory. It's well know that, in general, the second order abstract Cauchy problem

$$\begin{aligned} \ddot{u}(t) &= Au(t), \\ u(0) &= x_0, \end{aligned}$$

can't be studied by reduce this system to a first order linear equation, therefore, the existence result of this work, aren't consequence of those in Liu ([3]).

Throughout this paper, X will be a Banach space provided with a norm $\|\cdot\|$; $A : D(A) \to X$ will be the infinitesimal generator of strongly continuous cosine function of linear operators $C = (C(t))_{t\in \mathbb{R}}$ on X and $S = (S(t))_{t\in\mathbb{R}}$ will be the associated sine function, that is

$$S(t)x = \int_0^t C(\theta)x d\theta, \qquad x \in X, t \in \mathbb{R}.$$

For the theory of strongly continuous cosine functions, refer Fatorinni [2] and Travis [7]. We will point out here some notations and properties that will be essential for us. We'll denote by $D(A)$ the domain of A, that is

$$D(A) = \{x \in X : C(t)x \text{ is twise continuously differentiable}\}$$

and by E the set

$$E = \{x \in X : C(t)x \text{ is once continuously differentiable}\}.$$

The properties in the next result are well know ([7])

Lemma 1 *In the previous condition, the next properties hold.*

1. *If $x \in X$ then $S(t)x \in E$.*
2. *If $x \in E$ then $S(t)x \in D(A)$, $\frac{d}{dt}C(t)x = AS(t)x$ and $\frac{d^2}{dt^2}S(t)x = AS(t)x$.*
3. *If $x \in D(A)$ then $C(t)x \in D(A)$ and $\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax$.*
4. *If $x \in D(A)$ then $S(t)x \in D(A)$ and $\frac{d^2}{dt^2}S(t)x = AS(t)x = S(t)Ax$.*

This work contain two sections. In the section 2 we discuss the existence of solution for the impulsive system (1). Firstly we introduce the concept of mild and classical solution for the impulsive system (1) and subsequently, employing the contraction mapping principle and the ideas in Travis [7], we prove the existence of mild and classical solutions.

In what follows $B_r(x : Z)$ will denote the closed ball with center at x and radius r in a metric space Z.

For a nonnegative bounded function $\xi : (-T_0, T_1) \to \mathbb{R}$ and $-T_0 \leq s < t \leq T_1$ we employ the notation $\xi_{s,t}$ for

$$\xi_{s,t} = \sup\{\xi(\theta) : \theta \in [s,t]\}$$

and we will write simply ξ_t for $\xi_{-T_0,t}$ when no confusion arises.

Preserving the notation in Liu [3], $PC((-T_0,T_1):X)$ is the space of the functions $u:(-T_0,T_1)\to X$ such that u is continuous in $t\neq t_i$; u is left continuous in each t_i and $u(t_i^+)$ exist; endowed with the norm $\| u \|=\| u \|_{T_1}$.

Finally, $\mathcal{PC}^1((-T_0,T_1):X)$ we'll be the space

$$\mathcal{PC}^1((-T_0,T_1):X)=\left\{(u,w):\begin{array}{l} u,w\in PC((-T_0,T_1):X), \\ \dot{u}=w \text{ on each } \ (t_{i-1},t_i)\end{array}\right\},$$

provided of the norm $|||\,(u,w)\,|||=\| u \|_T+\| w \|_T$.

2 Existence Results.

In this section we discuss the existence of mild and classical solutions for the impulsive system (1). By comparison with the second order abstract Cauchy problem, we introduce the next definitions:

Definition 1 *A function $(u,v)\in\mathcal{PC}^1((-T_0,T_1):X)$ is a mild solution of system (1) if*

$$\begin{aligned} u(t) &= C(t)x_0+S(t)y_0+\int_0^t S(t-s)f(s,u(s),\dot{u}(s))ds \\ &\quad +\Sigma_{0<t_i<t}C(t-t_i)I_i^1(u(t_i))+\Sigma_{0<t_i<t}S(t-t_i)I_i^2(\dot{u}(t_i^+)), \end{aligned}$$

for every $t\in(-T_0,T_1)$.

Definition 2 *A function $(u,v)\in\mathcal{PC}^1((-T_0,T_1))$ is a classical solution of the impulsive system (1) if: $u\in C^2((T_0,T_1)\backslash\{t_1,t_2,...t_n\}:X)$ and u is solution of the impulsive system (1) on $(-T_0,T_1)$.*

In order to establish the existence of mild solutions, we introduce the following technical assumptions.

(A$_1$) $f:\mathbb{R}\times X\times X\to X$ is a continuous function and there exist positive constants $L^1(f)$ and $L^2(f)$ such that

$$\| f(t,u,v)-f(t,u',v') \|\leq L^1(f)\,\| u-u' \| + L^2(f)\,\| v-v' \|,$$

for every u,u',v and $v'\in X$.

(A$_2$) The functions $I_i^2:X\to X$ are continuous and there exist positive constants $L(I_i^2)$ such that

$$\| I_i^2(u)-I_i^2(u') \|\leq L(I_i^2)\,\| u-u' \|,$$

for every $u,u'\in X$.

(A$_3$) There exist closed linear operators $B:D(B)\subset X\to X$, $F:D(F)\subset X\to X$ such that

i) $S(t)(X) \subset D(B)$; the function $BS(\cdot)x$ is continuous for each $x \in X$ and $\frac{d}{dt}C(t)x = BS(t)Fx$ for every $x \in D(F)$.

ii) The functions I_i^1 are $D(F)$-valued and there exist positive constants $L(FI_i^1)$ such that

$$\| FI_i^1(u) - FI_i^1(u') \| \leq L(FI_i^1) \| u - u' \|, \quad \text{for every } u, u' \in X.$$

Theorem 1 *Let $x_0 \in E$, $y_0 \in X$ and assume that assumptions* $(\mathbf{A_1})$-$(\mathbf{A_3})$ *hold. If* $\max\{\Lambda_1, \Lambda_2\} < 1$, *where*

$$\Lambda_1 = L^1(f)\left(\| S \|_{T_1} + \| C \|_{T_1}\right)(T_0 + T_1) + \Sigma_{i=1}^n \left(\| C \|_{T_1} L(I_i^1) + \| BS \|_{T_1} L(FI_i^1)\right),$$

$$\Lambda_2 = L^2(f)\left(\| S \|_{T_1} + \| C \|_{T_1}\right)(T_0 + T_1) + \Sigma_{i=1}^n \left(\| C \|_{T_1} + \| S \|_{T_1}\right) L(I_i^2),$$

then there exists a unique mild solution of system (1.).

Proof: For $(u, w) \in \mathcal{PC}^1((-T_0, T_1) : X)$ we define $\Phi(u, w) = (\Phi_1(u, w), \Phi_2(u, w))$, where

$$\begin{aligned} \Phi_1(u, w)(t) &= C(t)x_0 + S(t)y_0 + \int_0^t S(t - s) f(s, u(s), \dot{u}(s)) ds \\ &\quad + \Sigma_{0<t_i<t} C(t - t_i) I_i^1(u(t_i)) + \Sigma_{0<t_i<t} S(t - t_i) I_i^2(\dot{u}(t_i^+)) \end{aligned}$$

and

$$\begin{aligned} \Phi_2(u, w)(t) &= AS(t)x_0 + C(t)y_0 + \int_0^t C(t - s) f(s, u(s), \dot{u}(s)) ds \\ &\quad + \Sigma_{0<t_i<t} BS(t - t_i) FI_i^1(u(t_i)) + \Sigma_{0<t_i<t} C(t - t_i) I_i^2(\dot{u}(t_i^+)). \end{aligned}$$

Clearly $\Phi(u, w) \in \mathcal{PC}^1((-T_0, T_1) : X)$. Next we'll prove that Φ is a contraction mapping on $\mathcal{PC}^1((-T_0, T_1) : X)$. To prove this property, we take (u, w), (v, z) in $\mathcal{PC}^1((-T_0, T_1) : X)$ and $t \in (-T_0, T_1)$. From assumption $(\mathbf{A_1})$ we see

$$\begin{aligned} \| \Phi_1(u, w)(t) - \Phi_1(v, z)(t) \| &\leq \int_0^t \| S \|_{T_1} \left(L^1(f) \| u - v \|_\theta + L^2(f) \| \dot{u} - \dot{v} \|_\theta\right) d\theta \\ &\quad + \Sigma_{0<t_i<t} \| C \|_{T_1} L(I_i^1) \| u(t_i) - v(t_i) \| \\ &\quad + \Sigma_{0<t_i<t} \| S \|_{T_1} L(I_i^2) \| \dot{u}(t_i{}^+) - \dot{v}(t_i{}^+) \| \end{aligned}$$

thus

$$\begin{aligned} \| \Phi_1(u, w) - \Phi_1(v, z) \|_{T_1} &\leq \left(\| S \|_{T_1} L^1(f)(T_0 + T_1) + \Sigma_{i=1}^n \| C \|_{T_1} L(I_i^1)\right) \| u - v \|_{T_1} \\ &\quad + \left(L^2(f)(T_0 + T_1) + \Sigma_{i=1}^n \| S \|_{T_1} L(I_i^2)\right) \| w - z \|_{T_1}. \end{aligned} \tag{2}$$

Similarly

$$\| \Phi_2((u,w)) - \Phi_2((v,z)) \|$$

$$\leq \left(\| C \|_{T_1} L^1(f)(T_0+T_1) + \| BS \|_{T_1} \Sigma_{i=1}^n L(FI_i^1)\right) \| u-v \|_{T_1}$$
$$+ \| C \|_{T_1} \left(L^2(f)(T_0+T_1) + \Sigma_{i=1}^n L(I_i^2)\right) \| w-z \|_{T_1} . \tag{3}$$

Inequalities (2)-(3) and the assumption $\max\{\Lambda_1, \Lambda_2\} < 1$, implies that Φ has a unique fixed point $(u,w) \in \mathcal{P}C^1((-T_0,T_1) : X)$. Clearly (u,w) is a mild solution for the impulsive system (1). The proof is complete.

Corollary 1 *Let assumption* $(\mathbf{A}_1)$-$(\mathbf{A}_2)$ *be satisfied and assume that* $I_i^1 \equiv 0$ *for* $i = 1,2,...,n$. *If* $\max\{\Lambda_1, \Lambda_2\} < 1$ *then there exists a unique mild solution of system (1).*

Remark 1 *In relation with assumption* $(\mathbf{A}_1)$, *a obvious case , for instance, is given for Lipschitz continuous functions* $I_i^1 : X \to [D(A)]$, *where* $[D(A)]$ *denote the space* $D(A)$ *endowed with the graph norm.*

2.1 Classical Solutions.

Next we will establish the existence of classical solution for the impulsive system (1) under the assumption that f is continuously differentiable. For this purpose, we need of the following lemmas.

Lemma 2 *Let* $g \in C([0,b] : X) \bigcap C_b^1((0,b) : X)$, $b > 0$. *Let* $h : [0,b] \to X$ *the function defined by*

$$h(t) = \int_0^t C(t-s)g(s)ds.$$

Then $h(t) \in E$ *for each* $t \in [0,b]$.

Proof: Let $t \in [0,b]$. Since g is continuously differentiable

$$\begin{aligned}\int_0^t C(t-s)g(s)ds &= \int_0^t C(t-s)g(0)ds + \int_0^t C(t-s)\int_0^s g'(\mu)d\mu ds \\ &= \int_0^t C(t-s)g(0)ds + \int_0^t \int_0^{t-\mu} C(\xi)g'(\mu)d\xi d\mu,\end{aligned}$$

thus

$$\int_0^t C(t-s)g(s)ds = S(t)g(0) + \int_0^t S(t-\mu)g'(\mu)d\mu. \tag{4}$$

Since $S(t)g(0) \in E$, we only need to proof that the second term in the right side of (4) is in E. In relation with this property we observe that for $\rho \in \mathbb{R}$

$$\frac{C(\rho+h) - C(\rho)}{h} \int_0^t S(t-s)g'(s)ds$$

$$\begin{aligned} &= \frac{1}{2h}\int_0^t \left(S(t-s+\rho+h) - S(t-s+\rho)\right) g'(s)ds \\ &+ \frac{1}{2h}\int_0^t \left(S(t-s-\rho-h) - S(t-s-\rho)\right) g'(s)ds, \end{aligned}$$

then

$$\frac{d}{d\rho}C(\rho)\int_0^t S(t-s)g'(s)ds = \frac{1}{2}\int_0^t \left(C(t-s+\rho) - C(t-s-\rho)\right) g'(s)ds$$

and therefore $\int_0^t S(t-s)g'(s)ds \in E$. The proof is finished.

Lemma 3 *Let $x_0 \in D(A)$, $y_0 \in E$ and assume that hypotheses in Theorem 1 hold. If f is continuously differentiable and exists $L^3(f) > 0$ such that*

$$\max\left\{ \| \frac{\partial f}{\partial x_j}(t,x,y) - \frac{\partial f}{\partial x_j}(t,x,\tilde{y}) \| : j = 1,2,3 \right\} \le L^3(f) \| y - \tilde{y} \|, \quad (5)$$

$$L^3(f)(T_0 + T_1) \| C \| T_1 < 1, \quad (6)$$

then there exits a unique classical solution $u(\cdot)$ of

$$\begin{aligned} \ddot{x}(t) &= Ax(t) + f(t, x(t), \dot{x}(t)), \qquad t \in (-T_0, t_{i+1}), \\ x(t_i) &= x_0, \quad \dot{x}(t_i) = y_0. \end{aligned} \quad (7)$$

Moreover $u(t_{i+1}^-) \in D(A)$ and $\dot{u}(t_{i+1}^-) \in E$.

Proof: Let $w : (T_0, T_1) \to X$ the unique mild solution of

$$\begin{aligned} \ddot{x}(t) &= Ax(t) + f(t, x(t), \dot{x}(t)), \qquad t \in (T_0, T_1), \\ x(t_i) &= x_0, \quad \dot{x}(t_{i)} = y_0, \end{aligned}$$

and $u : (-T_0, t_{i+1}) \to X$ the mild solution of (7). From the proof of Proposition 3.3 in [7] and condition (6), we infer that w and u are classical solutions and thus, that $u(t_{i+1}) = w(t_{i+1}) \in D(A)$, since $u = w$ on (T_0, t_{i+1}).

On the other hand,

$$\dot{u}(t_{i+1}^-) = AS(t_{i+1})x_0 + C(t_{i+1})y_0 + \int_0^{t_{i+1}} C(t_{i+1} - s) f(s, u(s), \dot{u}(s))ds = \dot{w}(t_{i+1}). \quad (8)$$

Clearly $AS(t_{i+1})x_0 + C(t_{i+1})y_0 \in E$. Moreover from Lemma 2 the integral term in (8) is in E; therefore $\dot{w}(t_i) \in E$. The proof is complete.

Now we establish the principal result of this work.

Theorem 2 *(Classical solution) Assume that assumptions in Theorem 1 and Lemma 3 hold. Let $(u,v) : (-T_0, T_1) \to X$ the mild solution of system (1). If $I_i^1(u(t_i)) \in D(A)$ and $I_i^2(\dot{u}(t_i^+)) \in E$ for each $i \in \{1, 2, .., n\}$ then (u,v) is a classical solution of system (1).*

Proof: Let $w : (-T_0, t_1) \to X$ the unique classical solution of

$$\begin{aligned} \ddot{x}(t) &= Ax(t) + f(t, x(t), \dot{x}(t)), && t \in (-T_0, t_1), \\ x(0) &= x_0, \quad \dot{x}(0) = y_0 \end{aligned}$$

and $(u_1, v_1) : (-T_0, t_1] \to X \times X$ the function defined by $(u_1, v_1)(t) = (w(t^-), \dot{w}(t^-))$. From Lemma 3 we know that $(u_1(t_1), v_1(t_1)) \in D(A) \times E$, thus there exists a unique classical solution $w(\cdot)$ of the abstract Cauchy problem

$$\begin{aligned} \ddot{x}(t) &= Ax(t) + f(t, x(t), \dot{x}(t)), && t \in (-T_0, t_2), \\ x(t_1) &= u_1(t_1) + I_1^1(u(t_1)), \quad \dot{x}(t_1) = v_1(t_1) + I_1^2(\dot{u}(t_1^+)). \end{aligned}$$

Similarly to the previous case, $(u_2, v_2) : (-T_0, t_2] \to X \times X$ will be the function defined by $(u_2, v_2)(t) = (w(t^-), \dot{w}(t^-))$.

In general, if w is the classical solution of the second order Cauchy problem

$$\begin{aligned} \ddot{x}(t) &= Ax(t) + f(t, x(t), \dot{x}(t)), && t \in (-T_0, t_k) \\ x(t_{k-1}) &= u_{k-1}(t_{k-1}) + I_{k-1}^1(u(t_{k-1})), \\ \dot{x}(t_{k-1}) &= v_{k-1}(t_{k-1}) + I_{k-1}^2(\dot{u}(t_{k-1}^+)), \end{aligned}$$

we denote by (u_k, v_k) the function $(u_k, v_k) : (-T_0, t_k] \to X$ where $u_k(t) = w(t^-)$ and $v_k(t) = \dot{w}(t^-)$.

Let $(\tilde{u}, \tilde{v}) : (-T_0, T_1) \to X$ the function defined by

$$(\tilde{u}(t), \tilde{v}(t)) = \begin{cases} (u_1(t), v_1(t)) & \text{if } \; -T_0 < t \leq t_1, \\ (u_k(t), v_k(t)) & \text{if } \; t_{k-1} < t \leq t_k, \\ (u_{n+1}(t), v_{n+1}(t)) & \text{if } \; t_n < t < T_1. \end{cases}$$

It is easy to see that $(\tilde{u}, \tilde{v})$ is the unique classical solution of the impulsive system

$$\begin{aligned} &\ddot{x}(t) = Ax(t) + f(t, x(t), \dot{x}(t)), && t \in (-T_0, T_1),\ t \neq t_i, \\ &x(0) = x_0, \quad \dot{x}(0) = y_0, \\ &\triangle x(t_i) = I_i^1(u(t_i)), \quad \triangle \dot{x}(t_i) = I_i^2(\dot{u}(t_i^+)). \end{aligned}$$

Next, we show that $(\tilde{u}, \tilde{v}) = (u, v)$. In order to reduce the proof, is only a tedious calculus, we introduce the group of linear operators

$$U(t) = \begin{pmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{pmatrix}_{t \in \mathbb{R}}$$

on $D(A) \times E$. The function $(\tilde{u}, \tilde{v})$ is a classical solution of the first order impulsive Cauchy problem

$$\dot{W}(t) = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} W(t) + F(t, W(t)),$$

$$W(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad \triangle W(t_i) = I_i \begin{pmatrix} u(t_i) \\ \dot{u}(t_i^+) \end{pmatrix},$$

where I_i and F are defined in obvious manner. From the proof of Lemma 2.3 in [3] we have

$$W(t) = U(t)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \int_0^t U(t-s)F(s, W(s))ds + \sum_{t_i<t} U(t-t_i)I_i\begin{pmatrix} u(t_i) \\ \dot{u}(t_i^+) \end{pmatrix},$$

thus

$$\begin{aligned} \tilde{u}(t) &= C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s, \tilde{u}(s), \tilde{v}(s))ds \\ &\quad + \Sigma_{0<t_i<t}C(t-t_i)I_i^1(u(t_i)) + \Sigma_{0<t_i<t}S(t-t_i)I_i^2(\dot{u}(t_i^+)) \end{aligned} \tag{9}$$

and

$$\begin{aligned} \tilde{v}(t) &= AS(t)x_0 + C(t)y_0 + \int_0^t C(t-s)f(s, \tilde{u}(s), \tilde{v}(s))ds \\ &\quad + \Sigma_{0<t_i<t}AS(t-t_i)I_i^1(u(t_i)) + \Sigma_{0<t_i<t}C(t-t_i)I_i^2(\dot{u}(t_i^+)). \end{aligned} \tag{10}$$

Finally, using $\theta(t) = \max\{\| u(t) - \tilde{u}(t) \|, \| v(t) - \tilde{v}(t) \|\}$, from assumption $(\mathbf{A_1})$ and (9)-(10) we obtain that

$$\theta(t) \leq C_1 \int_0^t \theta(s)ds, \tag{11}$$

where C_1 is a constant independent of $t \in (-T_0, T_1)$. The Gronwall's inequality implies that $u = \tilde{u}$. The proof is complete.

Now, we consider briefly the impulsive system

$$\begin{aligned} &\ddot{u}(t) = Au(t) + f(t, u(t)), \qquad t \in (T_0, T_1), \\ &u(0) = x_0, \quad \dot{u}(0) = y_0, \\ &\triangle u(t_i) = I_i^1(u(t_i)), \quad \triangle \dot{u}(t_i) = I_i^2(u(t_i^+)). \end{aligned} \tag{12}$$

Definition 3 *A function $u \in PC((-T_0, T_1) : X)$ is a mild solution of the second order impulsive system (12) if*

$$\begin{aligned} u(t) &= C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s, u(s))ds \\ &\quad + \Sigma_{0<t_i<t}C(t-t_i)I_i^1(u(t_i)) + \Sigma_{0<t_i<t}S(t-t_i)I_i^2(u(t_i^+)) \end{aligned}$$

for each $t \in (-T_0, T_1)$.

Definition 4 *A function $u : (T_0, T_1) \to X$ is a classical solution of the system (12) if: $u \in PC((T_0, T_1) : X) \bigcap C^2((T_0, T_1) \backslash \{t_1, t_2, ...t_n\} : X)$ and the equation (12) is satisfied.*

In order to establish our next result we introduce one more simple assumption:

$(\mathbf{A_4})$ $f : \mathbb{R} \times X \to X$ and $I_i^1 : X \to X$ are continuous functions and exist positive constants $L(f)$ and $L(I_i^1)$ such that

$$\begin{aligned} \| f(t, u) - f(t, u') \| &\leq L^1(f) \| u - u' \|, \\ \| I_i^1(u) - I_i^1(u') \| &\leq L(I_i^1) \| u - u' \| \end{aligned}$$

for every u, $u' \in X$

We omit the proof of the following results:

Theorem 3 *Let* x_0, $y_0 \in X$ *and assume that assumtions* $(\mathbf{A}_2)$, $(\mathbf{A}_4)$ *hold. Suppose, furtermore, that*

$$\| S \|_{T_1} L^1(f)(T_0+T_1) + \Sigma_{i=1}^{n} \Big(\| C \|_{T_1} L(I_i^1) + \| S \|_{T_1} L(I_i^2) \Big) < 1.$$

Then there exists a unique mild solution of system (12).

Lemma 4 *Let* $x_0 \in D(A)$, $y_0 \in E$ *and assume that hypotheses in Theorem 3 hold. If* f *is continuously differentiable, then the mild solution,* $u(\cdot)$, *of*

$$\begin{aligned} \ddot{x}(t) &= Ax(t) + f(t, x(t)), \qquad\qquad t \in (-T_0, t_{i+1}), \\ x(t_i) &= x_0, \quad \dot{x}(t_i) = y_0 \end{aligned} \tag{13}$$

is a classical solution and $u(t_{i+1}^-) \in D(A)$, $\dot{u}(t_{i+1}^-) \in E$.

Theorem 4 *(Classical solution) Assume that assumptions in Theorem 3 and Lemma 4 hold. Let* $(u,v) : (-T_0, T_1) \to X$ *the mild solution of system (12). If* $I_i^1(u(t_i)) \in D(A)$ *and* $I_i^2(t_i^+) \in E$ *for each* i, *then* (u,v) *is a classical solution of system (12).*

References

[1] Cabada, Alberto; Nieto, Juan J.; Franco, Daniel; Trofimchuk, Sergei I. *A generalization of the monotone method for second order periodic boundary value problem with impulses at fixed points.* Dynam. Contin. Discrete Impuls. Systems 7 (2000), no. 1, 145–158.

[2] H.O. Fattorini. *Second Order Linear Differential Equations in Banach Spaces.* North-Holland. Mathematics Studies 108. Notas de Metematica 98.

[3] Liu, James H. *Nonlinear impulsive evolution equations.* Dynam. Contin. Discrete Impuls. Systems 6 (1999), no. 1, 77–85.

[4] Rogovchenko, Yuri V. *Impulsive evolution systems: main results and new trends.* Dynam. Contin. Discrete Impuls. Systems 3 (1997), no. 1, 57–88.

[5] Rogovchenko, Yuri V. *Nonlinear impulse evolution systems and applications to population models.* J. Math. Anal. Appl. 207 (1997), no. 2, 300–315.

[6] Jinli, Sun; Yihai, Ma *Initial value problems for the second order mixed monotone type of impulsive differential equations in Banach spaces.* J. Math. Anal. Appl. 247 (2000), no. 2, 506–516.

[7] Travis, C. C.; Webb, G. F. *Second order differential equations in Banach space. Nonlinear equations in abstract spaces.* (Proc. Internat. Sympos., Univ. Texas, Arlington, Tex., 1977), pp. 331–361, Academic Press, New York, 1978.

Eduardo Hernández M.
Departamento de Matemática
Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo - Campus de São Carlos
Caixa Postal 668
13560-970 São Carlos, SP. Brazil
lalohm@icmc.sc.usp.br

NOTAS DO ICMC

SÉRIE MATEMÁTICA

114/2001 HERNÁNDEZ M., E. – A Variation-of-constants formula for partial functional differential equations with unbounded delay.

113/2001 ZVENGROWSKI, P. – Seminar on the span and parallelizability of manifolds USP at São Carlos.

112/2001 OLIVEIRA, R.D.S.; TARI, F. – On pairs of regular foliations in the plane.

111/2001 NABARRO, A.C. – Projections of hypersurfaces in IR^4 to planes.

110/2000 GUTIERREZ, C.; SARMIENTO, A. – Injectivity of C^1 maps IR^2 IR^4 at infinity and planar vector fields.

109/2000 GUTIERREZ, C.; CORTEZ, M.E.C. – Self-induced interval exchanges and piecewise linear maps of the interval.

108/2000 GUTIERREZ, C.; GUADALUPE, I.; TRIBUZY, R.; GUÍÑEZ, V. – A differential equation for lines of curvature on surfaces immersed in IR^4.

107/2000 GUTIERREZ, C.; GUÍÑEZ, V. – Simple umbilic points on surfaces immersed in IR^4.

106/2000 ARRAUT, J.L.; BIASI, C. – Foliations by planes in the complement of a compact set.

105/2000 BENÁ, M.A.; GODOY, S.M.S. – On the stability in terms of two measures of perturbed neutral functional differential equations.