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The NonLocal Cauchy Problem for Partial Neutral Functional Differential Equations with Unbounded Delay

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O Problema não Local de Cauchy para Equações Diferenciais Funcionais do Tipo Neutro com Retardamento não Limitado.

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Resumo

Neste artigo estudamos a existência de soluções fracas, fortes e clássicas para o problema de Cauchy não local associado a uma classe de equações diferenciais funcionais do tipo neutro com retardamento não limitado modeladas na forma $\frac{d}{dt}(x(t) + F(t, x_t)) = Ax(t) + G(t, x_t)$, onde A é o gerador de um semigrupo fortemente continuo de operadores lineares num espaço de Banach X e F, G são funções apropriadas.

The NonLocal Cauchy Problem for Partial Neutral Functional Differential Equations with Unbounded Delay.

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Abstract

We study the nonlocal Cauchy problem for a class of quasi-linear partial neutral functional differential equations with unbounded delay described in the form $\frac{d}{dt}(x(t) + F(t, x_t)) = Ax(t) + G(t, x_t)$, where A is the infinitesimal generator of a strongly continuous semigroup of linear operators on a Banach space X and F, G are appropriated continuous functions.

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1 Introduction

The purpose of this paper is to prove the existence of mild, strong and classical solutions for a nonlocal Cauchy problem with unbounded delay modeled in the form

$$\begin{aligned} \tfrac{d}{dt}(x(t) + F(t, x_t)) &= Ax(t) + G(t, x_t), \qquad \sigma \leq t < T, \\ x_\sigma &= \varphi + q(u_{t_1}, u_{t_2}, u_{t_3}, ..., u_{t_n}) \in \mathcal{B}, \end{aligned} \tag{1}$$

where A is the infinitesimal generator of an analytic semigroup of linear operators $(T(t))_{t\geq 0}$ on a Banach space X; the histories $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; $\Omega \subset$ is open; $0 \leq \sigma < T$; $\sigma < t_0 < t_1 < t_2 < t_3 .... < t_n \leq T$; and $q : \mathcal{B}^n \to \mathcal{B}$; $F, G : [\sigma, T] \times \Omega \to X$ are appropriate continuous functions.

The neutral partial differential equation with unbounded delay

$$\begin{aligned} \tfrac{d}{dt}(x(t) + F(t, x_t)) &= Ax(t) + G(t, x_t), \qquad t \geq \sigma, \\ x_\sigma &= \varphi \in \mathcal{B}, \end{aligned} \tag{2}$$

has been studied by Hernández and Henríquez in [6] and [7]. Using classical point fixed theorems and the ideas contained in [10], the authors prove the existence of mild, strong and periodical solutions for the system (2).

Motivated by physical applications, Byszewski studied the nonlocal Cauchy problem for semi-linear evolution equations. The problem considered by Byszewski is modeled in the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(t, x(t)), \\ x(0) &= x_0 + q(t_1, t_2, t_3, ..., t_n, u(\cdot)) \in X, \end{aligned} \tag{3}$$

where A is the infinitesimal generator of a C_0-semigroup of linear operators on X and f, q are appropriated continuous functions. A complete reference including existence results of mild, strong and classical solutions for system (3) is the paper [1]. For another models of equations for which the nonlocal Cauchy problem is considered refer to [2]-[4] and [8].

The nonlocal Cauchy problem for functional differential equations with delay is also studied by Byszewski. In the paper [4], Byszewski discuss the existence, uniqueness and continuous dependence on initial data of solutions to the nonlocal Cauchy problem

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(t, x_t), \qquad t \geq \sigma, \\ x_\sigma &= \varphi + q(u_{t_1}, u_{t_2}, u_{t_3}, ..., u_{t_n}), \end{aligned}$$

where A is the infinitesimal generator of a strongly continuous semigroup of linear operators; the histories, x_t, belongs to $C([-r, 0] : X)$ and $q : (C([-r, 0] : X))^n \to X$, $f : [0, T] \times C([-r, 0] : X) \to X$ are appropriated continuous functions.

The result obtained in this paper are generalizations of the results given by Hernández and Henríquez in [6]-[7] and Byszewski [1], [4]. The existence of classical solutions for the neutral system (2), a non considered problem in [7], will be consequence of Theorem 3.

Throughout this paper, X will be a Banach space provided with norm $\| \cdot \|$ and $A : D(A) \to X$ will be the infinitesimal generator of an analytic semigroup of linear operators, $T = (T(t))_{t \geq 0}$, on X. For the theory of strongly continuous semigroup, refer to Pazy [10] and Goldstein [15]. We will point out here some notations and properties that will be used in this work. It is well know that there exist constants $\tilde{M}$ and $w \in \mathbb{R}$ such that

$$\| T(t) \| \leq \tilde{M} e^{wt}, \qquad t \geq 0. \tag{4}$$

If T is uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X and the expression

$$\| x \|_\alpha = \| (-A)^\alpha x \|,$$

define a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\| \cdot \|_\alpha$, then the following properties are well known ([10], pp. 74):

Lemma 1 *If the previous conditions hold then*

1) *Let* $0 < \alpha \leq 1$. *Then* X_α *is a Banach space.*

2) *If* $0 < \beta \leq \alpha$ *then* $X_\alpha \to X_\beta$ *is continuous.*

3) *For every constant* $a > 0$ *there exists* $C_a > 0$ *such that*

$$\| (-A)^\alpha T(t) \| \leq \frac{C_a}{t^\alpha}, \qquad 0 < t \leq a.$$

4) For every $a > 0$ there exists a positive constant C'_a such that

$$\| (T(t) - I)(-A)^{-\alpha} \| \leq C'_a t^{\alpha}, \qquad 0 < t \leq a.$$

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato [12]. To establish the axioms of the space $\mathcal{B}$ we follow the terminology used in Hino-Murakami-Naito [11]. Thus, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\| \cdot \|_{\mathcal{B}}$. We will assume that $\mathcal{B}$ satisfies the following axioms:

(A) If $x : (-\infty, \sigma + a) \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$ the following conditions hold:

i) x_t is in $\mathcal{B}$.

ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$.

iii) $\| x_t \| \leq K(t - \sigma) \sup\{\| x(s) \| : \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \|_{\mathcal{B}}$, where $H > 0$ is a constant; $K, M : [0, \infty) \to [0, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in **(A)**, x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + a)$.

(B) The space $\mathcal{B}$ is complete.

For the literature on phase space, we refer to the reader to Hino [11]. While noting here that from the axiom **(A1)**, it follows that the operator functions $S(\cdot)$, $W(\cdot)$, given by

$$[S(t)\varphi](\theta) := \begin{cases} \varphi(0) & \text{for } -t \leq \theta \leq 0, \\ \varphi(t + \theta) & \text{for } -\infty < \theta < -t, \end{cases} \tag{5}$$

and

$$[W(t)\varphi](\theta) := \begin{cases} T(t + \theta)\varphi(0) & \text{for } -t \leq \theta \leq 0, \\ \varphi(t + \theta) & \text{for } -\infty < \theta < -t, \end{cases} \tag{6}$$

are strongly continuous semigroups of linear operators on $\mathcal{B}$. We will denote by A_S and A_W the infinitesimal generators of $S(\cdot)$ and $W(\cdot)$ and by $D(A_S)$ and $D(A_W)$ their respective domains.

To obtain some of our results we will require additional properties for the phase space $\mathcal{B}$, in particular we consider the following axiom (see [13], pp. 526 for details);

(C$_3$) Let $a > 0$. Let $x : (-\infty, \sigma + a] \to X$ be a continuous function such that $x_\sigma \equiv 0$ and the right derivative, denoted $\dot{x}(0^+)$, exists. If the function ψ defined by $\psi(\theta) = 0$ for $\theta < 0$ and $\psi(0) := \dot{x}(0^+)$ belongs to $\mathcal{B}$, then $\| (\frac{1}{h})x_h - \psi \|_{\mathcal{B}} \to 0$ as $h \to 0^+$.

The paper contain two sections. In section 2 we define the different concepts used in this work and discuss the existence of mild, strong and classical solutions for the abstract nonlocal Cauchy problem (1). In general, the results are obtained using the contraction mapping principle and the techniques in Pazy [10] and Hernández and Henriquez [7].

Throughout this work we assume that X is a abstract Banach space. The terminology and notations are those generally used in operator theory. In particular, if Z and Y are Banach spaces, we indicate by $\mathcal{L}(Z:Y)$ the Banach space of the bounded linear operators of Z in Y and we abbreviate to $\mathcal{L}(Z)$ whenever $Z=Y$. In addition $B_r(x:Z)$ will denote the closed ball in the space Z with center at x and radius r.

For some bounded function $\xi:[\sigma,T]\to X$ and $\sigma\leq s<t\leq T$ we employ the notation

$$\| \xi(\cdot) \|_{[s,t]}= \sup\{\| \xi(\theta) \|: \theta\in[s,t]\} \tag{7}$$

and we will write simply ξ_t for $\| \xi(\cdot) \|_{[\sigma,t]}$ when no confusion arises.

For $x\in X$, we will use the notation $\mathcal{X}_x$ for the function $\mathcal{X}_x:(-\infty,0]\to X$ where $\chi_x(\theta)=0$ for $\theta<0$ and $\chi_x(0)=x$.

Finally, for a differentiable function $j:[0,T]\times\mathcal{B}\to X$ we will employ the following decompositions

$$j(s,\psi)-j(t,\psi) = D_1 j(t,\psi)(s-t)+W_1(j,t,s,\psi), \tag{8}$$

$$j(t,\psi_1)-j(t,\psi) = D_2 j(t,\psi)(\psi_1-\psi)+W_2(j,t,\psi,\psi_1) \tag{9}$$

where

$$\frac{W_1(j,t,s,\psi)}{|\, t-s\,|}\to 0, \qquad as \quad s\to t,$$
$$\frac{W_1(j,t,\psi,\psi_1)}{\| \psi-\psi_1 \|_{\mathcal{B}}}\to 0, \qquad as \quad \psi_1\to\psi.$$

2 Existence Results

In this section we discuss the existence of mild and regular solutions for the abstract nonlocal Cauchy problem (1). In order to define the concept of mild solutions, we associate to problem (3) the integral equation

$$\begin{aligned} u(t) &= T(t-\sigma)(\varphi(0)+F(\sigma,u_0)+h(u_{t_1},u_{t_2},u_{t_3},..,u_{t_n})(0)\,)-F(t,u_t) \\ &\quad -\int_\sigma^t AT(t-s)F(s,u_s)ds+\int_\sigma^t T(t-s)G(s,u_s)ds. \end{aligned} \tag{10}$$

Definition 1 *A function $u:(-\infty,T]\to X$, is a mild solution of the abstract Cauchy problem (2) if: $u_\sigma=\varphi+h(u_{t_1},u_{t_2},u_{t_3},...,u_{t_n})$; the restriction of $u(\cdot)$ to the interval $[\sigma,T]$ is continuous; for each $\sigma\leq t\leq T$ the function $AT(t-s)F(s,u_s)$, $s\in[\sigma,t)$, is integrable and (11) hold on $[\sigma,T]$.*

Definition 2 *A function $u:(-\infty,T]\to X$, is a strong solution of the abstract nonlocal Cauchy problem (1) if: $u_\sigma=\varphi+h(u_{t_1},u_{t_2},u_{t_3},,,u_{t_n})$; $u(t)$ and $F(t,u_t)$ are differentiable a.e. on $[\sigma,T]$; $u'(t)$ and $F'(t,u_t)$ belong to $L^1([\sigma,T]:X)$ and $u(\cdot)$ satisfies a.e. (1) on $[\sigma,T]$.*

Definition 3 *A function* $u : (-\infty, T] \to X$, *is a classical solution of the abstract nonlocal Cauchy problem (1) if:* $u_\sigma = \varphi + h(u_{t_1}, u_{t_2}, u_{t_3},,, u_{t_n})$, *the restriction of* $x(\cdot)$ *to the interval* $[\sigma, T]$ *is continuous,* $x(t) \in D(A)$ *for all* $t \in (\sigma, T)$, $\dot{u}$ *is continuous in* (σ, T) *and* $u(\cdot)$ *satisfies equation (1) on* (σ, T).

Next, we show the existence and uniqueness of mild solutions for the system (1) using the contraction mapping principle. In the next assumption r is a fixed positive real number.

(H₁) $q : \mathcal{B}^n \to \mathcal{B}$ is continuous and exist positive constants $L(h_i)$ such that

$$\| q(\psi_1, \psi_2, \psi_3, ..., \psi_n) - q(\varphi_1, \varphi_2, \varphi_3, ..., \varphi_n) \| \leq \sum_{i=1}^{n} L_i(q) \| \psi_i - \varphi_i \|_{\mathcal{B}} \qquad (11)$$

for every $\psi_i, \varphi_j \in B_r(0, \mathcal{B}) \subset \Omega$.

(H₂) There exist $\beta \in (0,1)$ and positive constants $L(G)$, $L_\beta(F)$ such that, F is X_β-valued and the following Lipschitz conditions hold

$$\begin{aligned} \| (-A)^\beta F(t, \psi_1) - (-A)^\beta F(s, \psi_2) \| &\leq L_\beta(F)(| t - s | + \| \psi_1 - \psi_2 \|_{\mathcal{B}}) \\ \| G(t, \psi_1) - G(s, \psi_2) \| &\leq L(G)(| t - s | + \| \psi_1 - \psi_2 \|_{\mathcal{B}}) \end{aligned}$$

for every $\sigma \leq s, t \leq T$ and $\psi_1, \psi_2 \in B_r(0, \mathcal{B}) \subset \Omega$.

Remark 1 *In the rest of this work, to simplify notations, we only consider the case* $\sigma = 0$.

Remark 2 *In the next result we'll use the notations*

$$N_q = \sup\{\| q(\psi) \| : \psi \in B_r(0, \mathcal{B})\}$$

and $N_{(-A)^\beta F}$, N_F, N_G *for the* sup *of* $(-A)^\beta F$, F *and* N_G *on* $[0, T] \times B_r(0, \mathcal{B})$.

2.1 Mild Solution.

Theorem 1 *Let assumptions* (**H₁**) *and* (**H₂**) *be satisfied. Let* $\varphi \in B_r(0, \mathcal{B})$ *and assume that*

$$\max\left\{\Lambda_1 + \sum_{i=1}^{n} L_i(q) M_T, \ \Lambda_2 + \sum_{i=1}^{n} L_i(q) K_T\right\} < 1 \qquad (12)$$

and

$$(\tilde{M}H + 1) \| \varphi \| + N_q(1 + \tilde{M}H) + N_F(\tilde{M} + 1) + \frac{C_{1-\beta}}{\beta} N_{(-A)^\beta F} T^{1-\beta} + \tilde{M} N_G T < r, \quad (13)$$

where

$$\begin{aligned} \Lambda_1 &= M_T\left(\tilde{M}H \sum_{i=1}^{n} L_i(q) + \| (-A)^{-\beta} \| L_\beta(F) + \frac{C_{1-\beta}}{\beta} L_\beta(F) T^\beta + \tilde{M} L(G) T\right) \\ &\quad + \tilde{M} \| (-A)^{-\beta} \| L_\beta(F) \end{aligned}$$

and

$$\Lambda_1 = K_T\left(\tilde{M}H\sum_{i=1}^{n}L_i(q)+ \parallel (-A)^{-\beta} \parallel L_\beta(F) + \frac{C_{1-\beta}L_\beta(F)}{\beta}T^\beta + \tilde{M}L(G)T\right).$$

Then there exists a unique mild solution $u(\cdot,\varphi)$ of system (1) defined on $(-\infty, T]$.

Proof: Let $\mathcal{B}C([0,T]:X)$ the space

$$\mathcal{B}C([0,T]:X) = \{u:(-\infty,T]\to X:\ u_0\in\mathcal{B};\ u\in C([0,T]:X)\}$$

endowed with the norm

$$\parallel u \parallel = \parallel u_0 \parallel_{\mathcal{B}} + \parallel u \parallel_T .$$

On $Y = B_r(0, \mathcal{B}C([0,T]:X))$ we define the mapping $\Upsilon : Y \to \mathcal{B}C([0,T]:X)$ by

$$\begin{aligned}\Upsilon(u)(t) &= T(t)(\varphi(0) + F(0,u_0) + h(u_{t_1},u_{t_2},u_{t_3},..,u_{t_n})(0)) - F(t,u_t)\\ &\quad + \int_0^t (-A)^{1-\beta}T(t-s)(-A)^\beta F(s,u_s)ds\\ &\quad + \int_0^t T(t-s)G(s,u_s)ds, \qquad \text{for } 0\le t\le T,\\ \Upsilon(u)_0 &= \varphi + q(u_{t_1},u_{t_2},u_{t_3},..,u_{t_n}).\end{aligned}$$

It's follow from the proof of Theorem 2.1 in [7], that the mapping Υ is well defined. Next we will prove that Υ maps Y into Y and that Υ is a contraction on Y. To prove the first assertion, we observe that for $u\in Y$ and $t\in[0,T]$

$$\parallel \Upsilon(u)(t) \parallel \le \tilde{M}(H(\parallel \varphi \parallel + N_q) + N_F) + N_F + \int_0^t\left(\frac{C_{1-\beta}N_{(-A)^\beta F}}{(t-s)^{1-\beta}} + \tilde{M}N_G\right)ds$$

and

$$\parallel \Upsilon(u)_0 \parallel \le \parallel \varphi \parallel_{\mathcal{B}} + N_q,$$

thus

$$\parallel \Upsilon(u) \parallel_{\mathcal{B}C} \le \parallel \varphi \parallel_{\mathcal{B}} + N_q + \tilde{M}(H(\parallel \varphi \parallel_{\mathcal{B}} + N_q) + N_F) + N_F + \frac{C_{1-\beta}}{\beta}N_{(-A)^\beta F}T^\beta + \tilde{M}N_GT. \tag{14}$$

It's follow from (14) and (13) that $\Upsilon(u)\in Y$. In order to prove that Υ satisfies a Lipschitz condition we take $u,v\in Y$. For $t\in[0,T]$ we see that can to compute

$$\parallel \Upsilon(u)(t) - \Upsilon(v)(t) \parallel$$

$$\begin{aligned}&\le \tilde{M}\left(H\sum_{i=1}^{n}L_i(q)\parallel u_{t_i}-v_{t_i}\parallel_{\mathcal{B}} + L_\beta(F)\parallel(-A)^{-\beta}\parallel\parallel u_0 - v_0\parallel_{\mathcal{B}}\right)\\ &\quad + \parallel(-A)^{-\beta}\parallel L_\beta(F)\parallel u_t - v_t\parallel_{\mathcal{B}}\\ &\quad + \int_0^t \frac{C_{1-\beta}L_\beta(F)}{(t-s)^{1-\beta}}\parallel u_s - v_s\parallel_{\mathcal{B}}ds + \int_0^t \tilde{M}L(G)\parallel u_s-v_s\parallel_{\mathcal{B}}ds\\ &\le \Lambda_1\parallel u_0-v_0\parallel_{\mathcal{B}} + \Lambda_2\parallel u-v\parallel_T,\end{aligned}$$

where the estimate

$$\| u_s - v_s \|_{\mathcal{B}} \leq M_T \| u_0 - v_0 \|_{\mathcal{B}} + K_T \| u - v \|_T$$

has been used. Thus

$$\| \Upsilon(u) - \Upsilon(v) \|_T \leq \Lambda_1 \| u_0 - v_0 \|_{\mathcal{B}} + \Lambda_2 \| u - v \|_T . \tag{15}$$

On the other hand, a simple calculus prove that

$$\| \Upsilon(u)_0 - \Upsilon(v)_0 \|_{\mathcal{B}} \leq \sum_{i=1}^{n} L_i(q) M_T \| u_0 - v_0 \| + \sum_{i=1}^{n} L_i(q) K_T \| u - v \|_T . \tag{16}$$

The inequalities (15)-(16) and assumption (12) implies that Υ is a contraction on Y. Clearly a fixed point of Υ is the unique mild solution of the system (1). The proof is finished.

2.2 Strong solution.

The proof of the existence of strong solutions for the system (1) follows, with obvious modifications, the steeps in the proofs of Lemma 3.1 and Theorem 3.1 in [7], for this reason we'll omit the proofs of the next results.

Proposition 1 *Assume that assumptions in Theorem 1 hold. Let $u(\cdot) = u(\cdot, \varphi)$ the mild solution of (1). If F is $D(A)$-valued; AF is continuous; $S(t)(\varphi + q(u_{t_1}, u_{t_2}, u_{t_3},,, u_{t_n}))$ is Lipschitz on $[0, T]$ and $u(0, \varphi) \in D(A)$ then $t \to u_t(\cdot, \varphi)$ is Lipschitz continuous on $[0, T]$.*

Theorem 2 *Let assumption in Proposition 1 be satisfied. If X is a reflexive space then $u(\cdot, \varphi)$ is a strong solution.*

2.3 Classical solution.

Now we establish the principal result of this work.

Theorem 3 *Assume that assumptions in Theorem 1 hold. Let $u(\cdot) = u(\cdot, \varphi)$ the mild solution of (1) and assume, furthermore, that*

1) The functions $(-A)^{\beta} F$ and G are continuously differentiable, F is $D(A)$-valued and AF is continuous.

2) $u_0 \in D(A_W)$, $D_2 F(0, u_0) \equiv 0$ and

$$\begin{aligned} \mu &= K_T \| D_2 F(t, u_t) \|_T + K_T \frac{C_{1-\beta}}{\beta} L_{\beta}(F) \| D_2 (-A)^{\beta} F(t, u_t) \|_T T^{\beta} \\ &\quad + K_T \tilde{M} L(G) \| D_2 G(t, u_t) \|_T T < 1. \end{aligned} \tag{17}$$

If $\mathcal{X}_{G(0,u_0)} \equiv 0$ or $\mathcal{X}_{G(0,u_0)} \in \mathcal{B}$ and $\mathcal{B}$ satisfies axiom $(\mathbf{C_3})$ *then $u(\cdot,\varphi)$ is a classical solution.*

Proof. Let $z(\cdot)$ be a solution of the integral equation

$$\begin{aligned} z(t) &= T(t)(Au(0) + G(t,u_t)) + p(t) - D_2F(t,u_t)\cdot z_t \\ &+ \int_0^t (-A)^{1-\beta} T(t-s) D_2(-A)^\beta F(s,u_s)\cdot z_s\, ds \\ &+ \int_0^t T(t-s) D_2 G(s,u_s)\cdot z_s\, ds, \qquad t \geq 0, \end{aligned} \tag{18}$$

where

$$z_0 = A_W(u_0) + \mathcal{X}_{G(0,u_0)} \tag{19}$$

and

$$\begin{aligned} p(t) &= -D_1F(t,u_t) + \int_0^t (-A)^{1-\beta} T(t-s) D_1(-A)^\beta F(s,u_s) ds \\ &+ \int_0^t T(t-s) D_1 G(s,u_s) ds. \end{aligned}$$

The existence and uniqueness of solution to the integral equation (18)-(19) is consequence of the contraction mapping principle and (17). In what follows, we prove that $\dot{u}(\cdot) = z(\cdot)$ on $[0,T]$. Using the decomposition introduced in (8)-(9), for $t \in [0,T]$ and $h > 0$ sufficiently small we get

$$\| \xi(t,h) \| = \| \frac{u(t+h) - u(t)}{h} - z(t) \|$$

$$\begin{aligned} \leq\ & \| T(t)(\frac{T(h)-I}{h}u(0) - Au(0)) \| \\ &+ \| T(t)(\frac{T(h)-I}{h})F(0,u_0) + \frac{1}{h}\int_0^h (-A)^{1-\beta}T(t+h-s)(-A)^\beta F(s,u_s) ds \| \\ &+ \| D_1F(t,u_{t+h}) - D_1F(t,u_t) \| + \| D_2F(t,u_t)\xi_t(\cdot,h) \| \\ &+ \| \frac{W_1(F,t,t+h,u_{t+h})}{h} \| + \| \frac{W_2(F,t,u_t,u_{t+h})}{h} \| \\ &+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| D_1(-A)^\beta F(s,u_{s+h}) - D_1(-A)^\beta F(s,u_s) \| ds \\ &+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| D_2(-A)^\beta F(s,u_s)\cdot \xi_s(\cdot,h) \| ds \\ &+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \left[\| \frac{W_1((-A)^\beta F, s, s+h, u_{s+h})}{h} \| + \| \frac{W_2((-A)^\beta F, s, u_s, u_{s+h})}{h} \| \right] ds \\ &+ \| \frac{1}{h}\int_0^h T(t+h-s)G(s,u_s)ds - T(t)G(0,u_0) \| \\ &+ \int_0^t \tilde{M} \| D_1G(s,u_{s+h}) - D_1G(s,u_s) \| ds + \int_0^t \tilde{M} \| D_2G(s,u_s)\xi_s(\cdot,h) \| ds \\ &+ \tilde{M}\int_0^t \left(\| \frac{W_1(G,s,s+h,u_{s+h})}{h} \| + \| \frac{W_2(G,s,u_s,u_{s+h})}{h} \| \right) ds \end{aligned}$$

Since $u(0) \in D(A)$ and AF is continuous, it's follows from Proposition (1) that $t \to u_t(\cdot, \varphi)$ is Lipschitz continuous thus

$$\frac{1}{h} \| W_2(G, s, u_s, u_{s+h}) \| = \frac{\| W_2(G, s, u_s, u_{s+h}) \|}{\| u_{s+h} - u_s \|_{\mathcal{B}}} \cdot \frac{\| u_{s+h} - u_s \|_{\mathcal{B}}}{h} \to 0, \tag{20}$$

and

$$\frac{1}{h} \| W_2((-A)^\beta F, s, u_s, u_{s+h}) \| = \frac{\| W_2((-A)^\beta F, s, u_s, u_{s+h}) \|}{\| u_{s+h} - u_s \|_{\mathcal{B}}} \cdot \frac{\| u_{s+h} - u_s \|_{\mathcal{B}}}{h} \to 0 \tag{21}$$

uniformly for $s \in [0, T]$. Using (20) and (21), we can to rewrite the last inequality in the form

$$\begin{aligned}
\| \xi(t,h) \| \;\leq\;& \rho(t,h) + \frac{1}{h}\int_0^h \frac{C_{1-\beta}}{(t+h-s)^{1-\beta}} \| (-A)^\beta F(s,u_s) - (-A)^\beta F(0,u_0) \| \, ds \\
&+ \| D_2 F(t,u_t) \| \| \xi_t(\cdot,h) \| \\
&+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| D_2(-A)^\beta F(s,u_s) \| \| \xi_s(\cdot,h) \|_{\mathcal{B}} \, ds \\
&+ \tilde{M} \int_0^t \| D_2 G(s,u_s) \| \| \xi_s(\cdot,h) \|_{\mathcal{B}} \, ds,
\end{aligned}$$

where $\rho(t,h) \to 0$ as $h \to 0$, uniformly for $t \in [0,T]$. Employing axiom (**A**), (17) and the Lipschitz continuity of $s \to u_s$, we see

$$\| \xi(t,h) \|_t \;\leq\; \frac{\rho(t,h)}{1-\mu} + \frac{C_{1-\beta} L_\beta(F) C h^\beta}{(1-\mu)\beta} + \frac{\mu M_T}{(1-\mu)K_T} \left\| \frac{u_h - u_0}{h} - z_0 \right\|_{\mathcal{B}} .$$

It's clear that $\dot{u}(\cdot) = z(\cdot)$ if $\| \frac{u_h - u_0}{h} - z_0 \|_{\mathcal{B}}$ as $h \to 0$. To show this convergence we introduce the decomposition

$$u(\cdot) = y(\cdot) + \sum_{i=1}^{3} z_i$$

where $y(\theta) = T(\theta)(u(0) + q(u_{t_1}, u_{t_2}, u_{t_3}, \ldots, u_{t_n})(0)$ with $y_0 = \varphi + q(u_{t_1}, u_{t_2}, u_{t_3}, \ldots, u_{t_n})$ and

$$\begin{aligned}
z^1(\theta) &= T(\theta)F(0,u_0) - F(\theta, u_\theta) \\
z^2(\theta) &= \int_0^\theta (-A)^{1-\beta} T(\theta - s)(-A)^\beta F(s,u_s) ds \\
z^3(\theta) &= \int_0^\theta T(\theta - s) G(s,u_s) ds.
\end{aligned}$$

with $z_0^i = 0$. According the above notations we see

$$\begin{aligned}
\left\| \frac{u_h - u_0}{h} - A_W(u_0) - \mathcal{X}_{G(0,u_0)} \right\|_{\mathcal{B}} \;\leq\;& \left\| \frac{W(h)u_0 - u_0}{h} - A_W(u_0) \right\|_{\mathcal{B}} + \left\| \frac{z_h^1 + z_h^2}{h} \right\|_{\mathcal{B}} \\
&+ \left\| \frac{z_h^3}{h} - \mathcal{X}_{G(0,u_0)} \right\|_{\mathcal{B}} .
\end{aligned} \tag{22}$$

Next we estimate each term of (22) separately. Clearly

$$I_1(h) \to 0 \qquad \text{as} \qquad h \to 0, \tag{23}$$

since $u_0 \in D(A_W)$. On the other hand, if either, $\mathcal{X}_{G(0,u_0)} \equiv 0$ or $\mathcal{X}_{G(0,u_0)} \in \mathcal{B}$ and $\mathcal{B}$ satisfies axiom $(\mathbf{C_3})$, it's follow that

$$I_3(h) \to 0 \qquad \text{as} \qquad h \to 0. \tag{24}$$

In relation with the second term in the right side of (22), from Axiom $(\mathbf{A})$ we find

$$\begin{aligned} I_2(h) \; & \leq K_b \tfrac{1}{h} \parallel (T(\theta) - I)F(0,u_0) + \textstyle\int_0^\theta (-A)^{1-\beta} T(\theta - s)(-A)^\beta F(s,u_s) ds \parallel_h \\ & + K_b \tfrac{1}{h} \parallel F(0,u_0) - F(\theta, u_\theta) \parallel_h . \end{aligned} \tag{25}$$

Moreover for $\theta \in [0,h]$,

$$\begin{aligned} \frac{1}{h} & \parallel \; (T(\theta) - I)F(0,u_0) + \int_0^\theta (-A)^{1-\beta} T(\theta - s)(-A)^\beta F(s,u_s) ds \parallel \\ & \leq \frac{1}{h} \int_0^\theta \parallel (-A)^{1-\beta} T(\theta - s)\{(-A)^\beta F(s,u_s) - (-A)^\beta F(0,u_0)\} \parallel ds \\ & \leq \frac{1}{h} \int_0^\theta \frac{C_{1-\beta} L_\beta(F)}{(\theta - s)^{1-\beta}} Chds \leq C_{1-\beta} L_\beta(F) C \frac{h^\beta}{\beta} \end{aligned}$$

thus

$$\frac{1}{h} \parallel (T(\theta) - I)F(0,u_0) + \int_0^\theta (-A)^{1-\beta} T(\theta - s)(-A)^\beta F(s,u_s) ds \parallel_h \to 0 \tag{26}$$

as $h \to 0$. On the other hand

$$\parallel \frac{F(0,u_0) - F(\theta, u_\theta)}{\theta} \cdot \frac{\theta}{h} \parallel_h \to 0 \tag{27}$$

as $h \to 0$, since $DF(0,u_0) \equiv 0$. Using (26) and (27) in (25), we conclude that

$$I_2(h) \to 0 \qquad h \to 0. \tag{28}$$

Inequalities (23), (24) (28) and (22) implies that $\frac{u_h - u_0}{h} \to z_0$ as $h \to 0$ and thus $\dot{u}(\cdot) = z(\cdot)$. Finally the condition $u(t) \in D(A)$ for $t \in [0,b)$, is consequence of Lemma 2 below.

The proof of the next result is analogous to the proof of Theorem (2.4.1) in [15]. However there are some differences that require attention and we include the principal ideas of the proof for completeness.

Lemma 2 *Let $0 < \beta < 1$, $g \in C([0,T] : X_{1-\beta})$, $f \in C([0,T] : X)$ and $y_i : [0,T] \to X$, $i = 1,2$, the functions defined by*

$$\begin{aligned} y_1(t) \; &= \; \int_0^t (-A)^{1-\beta} T(t-s) g(s) ds, \\ y_2(t) \; &= \; \int_0^t T(t-s) f(s) ds. \end{aligned}$$

If either, g, f are respectively θ_g and θ_f Hölder continuous on $[0,T]$ with $\beta + \theta_g > 1$ or g, f are Lipschitz continuous on $[0,T]$, then $y_i(t) \in D(A)$ for every $t \in [0,T)$ and $\dot{y}_i \in C((0,T) : X)$.

Proof. We only proof the assertions for the function y_1. Assume that g is θ_g-Hölder continuous. For $t \in [0, T)$ we rewrite $y_1(t)$ in the form

$$\int_0^t (-A)^{1-\beta} T(t-s)(g(s) - g(t))ds + \int_0^t (-A)^{1-\beta} T(t-s) g(t) ds = v(t) + w(t). \qquad (29)$$

Clearly, $Aw(t) = T(t)(-A)^{1-\beta} g(t) - (-A)^{1-\beta} g(t) \in C([0, T] : X)$. Now for $\epsilon > 0$ sufficiently small we define the function

$$v_\epsilon(t) := \begin{cases} \int_0^{t-\epsilon} (-A)^{1-\beta} T(t-s)(g(s) - g(t))ds, & \text{for } t \in [\epsilon, T), \\ 0 & \text{for } t \in [0, \epsilon]. \end{cases} \qquad (30)$$

It is clear that $v_\epsilon(t) \in D(A)$. Moreover for $0 < \delta_1 < \delta_2$

$$\begin{aligned} \| Av_{\delta_2}(t) - Av_{\delta_1}(t) \| &\leq \int_{t-\delta_2}^{t-\delta_1} \| (-A)^{2-\beta} T(t-s)(g(s) - g(t)) \| ds \\ &\leq \int_{t-\delta_2}^{t-\delta_1} \frac{C_{2-\beta}}{(t-s)^{2-\beta}} |s-t|^{\theta_g} ds \\ &\leq C_{2-\beta} (\delta_2^{\beta+\theta_g-1} - \delta_1^{\beta+\theta_g-1}). \end{aligned} \qquad (31)$$

The last inequality proves that $A(v_\delta)(t)$ is convergent, $\beta + \theta_g > 1$, and therefore

$$A(v(t)) = \int_0^t A^{2-\beta} T(t-s)(g(s) - g(t))ds \qquad (32)$$

since A is a closed operator. The continuity of $\partial_t y$ follows as in [15], Theorem 2.4.1.

If g is Lipchitz the proof is similar. This complete the proof.

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