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APPLICATIONS OF ROBUST SYNCHRONIZATION TO COMMUNICATION SYSTEMS

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ABSTRACT. In this work, using chaotic systems, we study the role of synchronization on codification and decodification of messages. We first present a general result that is useful to prove uniform dissipativeness for nonautonomous systems of ordinary differential equations. Then some theorems are established to give sufficient conditions to obtain synchronization of coupled systems. The above results are applied to some specific coupled systems, namely, coupled Lorenz systems, coupled Duffing's equations, coupled Chua's systems, etc., showing how to code and decode messages using chaotic systems. One of our main results is to obtain the robustness of the synchronization, with respect to parameter variation.

1. INTRODUCTION

The main purpose of this paper is to develop mathematical methods for studying synchronization. We consider two coupled systems of nonlinear differential equations, depending on parameters, and show that the difference between the components of the solutions of these systems becomes small, when the time is sufficiently large and the parameters are sufficiently close. The class of systems considered in this work has the property that if the parameters are identical the difference between the components of the solutions goes to zero, and in many cases it goes to zero exponentially. We consider a type of unilateral coupling, where the second system is coupled with the first but the first is independent of the second. This type

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of synchronization using chaotic systems was originally studied by Carroll-Pecora[27]. This is known as *Master-Slave Synchronization.* Cuomo-Oppenheim[12] in 1993 use *Master-Slave Synchronization* in chaotic systems for codification and decodification of messages. Also Tresser-Worfolk[33] consider a similar problem. They suppose that the message is a continuous and bounded function $m(t)$. Then from a system of differential equations, they construct two new systems with identical parameters which are used for codification and decodification of the message. These systems are called *Master* and *Slave* systems, respectively. However, in practical applications, it is impossible to construct systems with identical parameters. It is a realistic assumption to assume that the corresponding parameters of both systems are close and to have an estimate of the error. This is the approach we take to obtain one of our main results related to the robustness with respect to parameter variation and to perturbation on the message. We also consider applications to codification and decodification of messages.

We consider the messages as a function $m \in C_K(\mathbb{R}, \mathbb{R}^s)$, where $C_K(\mathbb{R}, \mathbb{R}^s)$ is the ball of radius K in the Banach space of continuous and bounded functions $\alpha : \mathbb{R} \to \mathbb{R}^s$ with $\|\alpha\| = \sup_{t \in \mathbb{R}} \|\alpha(t)\|$, that is, $C_K(\mathbb{R}, \mathbb{R}^s)$ is the set of all continuous and bonded functions $\alpha : \mathbb{R} \to \mathbb{R}^s$ with $\|\alpha\| \leq K$.

Given $f \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^s \times \Gamma, \mathbb{R}^n)$ satisfying a Lipschitz condition on bounded sets (2.1) with respect to $(x, y) \in \mathbb{R}^n \times \mathbb{R}^s$, where $\Gamma \subset \mathbb{R}^m$ and $s \leq n$, and given $m, \bar{m} \in C_K(\mathbb{R}, \mathbb{R}^s)$ we construct the system

$$\begin{cases} \dot{z} = f\big(t, z, x + m(t), \gamma_1\big) \\ \dot{w} = f\big(t, w, x + \bar{m}(t), \gamma_2\big) \end{cases} \tag{1.1}$$

where $z = (x, y), w = (u, v) \in \mathbb{R}^s \times \mathbb{R}^{n-s} = \mathbb{R}^n$.

Below, we explain how synchronization of the above system can be used for codification and decodification of messages. Given a signal $m(t)$ to be codified we construct the first equation of (1.1). Solving it for an arbitrary chosen initial condition, we codify and transmit the message as $x(t) + m(t)$. In the transmission channel there may occur a perturbation of the message. As consequence the received signal will be $x(t) + \bar{m}(t)$, where $\bar{m}(t)$ is close to $m(t)$. With the received information, $x(t) + \bar{m}(t)$, we construct the second equation of (1.1).

Choosing an arbitrary initial condition we solve the second equation and obtain the solution $\left(u(t), v(t)\right)$. The recovered message is given by $m_r(t) := x(t) + \bar{m}(t) - u(t)$. If system (1.1) synchronizes then the recovered message $m_r(t)$ is close to original message $m(t)$ for large values of t.

The method we present to prove synchronization of system (1.1) is based in two fundamental ideas. We first prove that the system is uniform dissipative, that is, there exists a bounded set $\mathcal{B}$ independent of parameters such that every solution of (1.1) enters in $\mathcal{B}$ in finite time and remains there. In this paper this is proved by using a Liapunov like function associated to system (1.1). After this, we use integral inequalities and exponential estimates or Liapunov like functions to prove the synchronization in the bounded set $\mathcal{B}$.

In Section 2 we present some theoretical results to prove uniform dissipativeness and synchronization and in Section 3 these results are applied to the study of codification and decodification of messages.

Mathematical Methods to study synchronization between chaotic systems were presented in Fujisaka-Yamada [16], in Afraimovich et all[3]. Abstract results and the robustness with respect to parameters variation and uniform dissipativeness were obtained in Rodrigues[29] and Afraimovich-Rodrigues[1]. It was also observed and studied in lasers systems by Prof. R. Roy and his group as shown in [15], [18] and [19].

In some respects this paper improves Gameiro[17]. In [31] and [32], but with different techniques and different application, Rodrigues-Alberto-Bretas studied applications of synchronization to stability of power systems. Other applications can be found in Rodrigues[29]. Applications of synchronization to parabolic partial differential equations are discussed by Carvalho-Dlotko-Rodrigues[8].

2. Dissipativeness and Synchronization.

In this section we state and prove some results, in a convenient form, that will be very useful on the applications of the next section.

Let E be a Banach space and $\Lambda \subset E$. Let $f \in C(\mathbb{R} \times \mathbb{R}^n \times \Lambda, \mathbb{R}^n)$ satisfying the following Lipschitz condition, with respect to $x \in \mathbb{R}^n$: for each $\lambda \in \Lambda$ and each bounded set $B \subset \mathbb{R} \times \mathbb{R}^n$, there exists a constant $L \geq 0$ such that

$$\|f(t, x_1, \lambda) - f(t, x_2, \lambda)\| \leq L\|x_1 - x_2\|, \quad \forall (t, x_1), (t, x_2) \in B. \tag{2.1}$$

Consider the equation

$$\dot{x} = f(t, x, \lambda). \tag{2.2}$$

The next theorem gives sufficient conditions for the uniform dissipativeness of the above equation.

Theorem 2.1. *Let $V \in C^1(\mathbb{R} \times \mathbb{R}^n \times \Lambda, \mathbb{R})$. Suppose there are $a, b, c \in C(\mathbb{R}^n, \mathbb{R})$ such that $a(x) \to +\infty$ as $\|x\| \to +\infty$ and*

$$a(x) \leq V(t, x, \lambda) \leq b(x), \quad -\dot{V}(t, x, \lambda) \geq c(x)$$

for every $(t, x, \lambda) \in \mathbb{R} \times \mathbb{R}^n \times \Lambda$, where

$$\dot{V}(t, x, \lambda) := \frac{\partial}{\partial t} V(t, x, \lambda) + \left[\frac{\partial}{\partial x} V(t, x, \lambda)\right] f(t, x, \lambda).$$

We also assume that there exists $\rho > 0$, such that the set $C_\rho := \left\{x \in \mathbb{R}^n : c(x) \leq \rho\right\}$ is nonempty and bounded. Let $r > 0$ sufficiently large in such a way that $r > \sup_{x \in C_\rho} b(x)$. Let $\mathcal{A}_r := \left\{x \in \mathbb{R}^n : a(x) \leq r\right\}$. Then the following hold for equation (2.2):

(i) *Given $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and $\lambda \in \Lambda$, the solution $x(t, t_0, x_0, \lambda)$ of (2.2) is defined in $[t_0, +\infty)$ and there exists $t_1 \geq t_0$ such that $x(t, t_0, x_0, \lambda) \in \mathcal{A}_r$ for every $t \geq t_1$.*

(ii) *If $x(t)$ is a solution of (2.2) defined for every $t \in \mathbb{R}$, with $\sup_{t \in \mathbb{R}} \|x(t)\| < +\infty$, then $x(t) \in \mathcal{A}_r$ for every $t \in \mathbb{R}$.*

Proof: (i) Under the above conditions, given $(t_0, x_0, \lambda) \in \mathbb{R} \times \mathbb{R}^n \times \Lambda$, one can show that the solution $x(t, t_0, x_0, \lambda)$ is defined for t in $[t_0, +\infty)$. Now we claim that there exists $t_1 \geq t_0$ such that $x(t_1, t_0, x_0, \lambda) \in C_\rho$. If this is not the case, then

$$\dot{V}\big(t, x(t, t_0, x_0, \lambda), \lambda\big) < -\rho < 0, \quad \forall t \geq t_0.$$

This implies that the map $t \in [t_0, +\infty) \mapsto V\big(t, x(t, t_0, x_0, \lambda), \lambda\big)$ is decreasing and bounded from below, since a is bounded from below. Therefore the limit

$$\lim_{t \to +\infty} V\big(t, x(t, t_0, x_0, \lambda), \lambda\big)$$

exists and is finite. From the mean value theorem, it follows that, for every $t \geq t_0$, there exists $s_t \in (t, t+1)$ such that

$$V\big(t+1, x(t+1, t_0, x_0, \lambda), \lambda\big) - V\big(t, x(t, t_0, x_0, \lambda), \lambda\big) = \dot{V}\big(s_t, x(s_t, t_0, x_0, \lambda), \lambda\big) < -\rho.$$

If we let $t \to +\infty$ we obtain a contradiction. Therefore there exists $t_1 \geq t_0$ such that $x(t_1, t_0, x_0, \lambda) \in C_\rho$. Now we prove that $x(t, t_0, x_0, \lambda) \in \mathcal{A}_r$ for every $t \geq t_1$. If this is not true, we can find $t_2 > t_1$ such that $x(t_2, t_0, x_0, \lambda) \notin \mathcal{A}_r$. Then

$$V\big(t_2, x(t_2, t_0, x_0, \lambda), \lambda\big) \geq a\big(x(t_2, t_0, x_0, \lambda)\big) > r$$

and so

$$V\big(t_1, x(t_1, t_0, x_0, \lambda), \lambda\big) \leq b\big(x(t_1, t_0, x_0, \lambda)\big) < r,$$

since $x(t_1, t_0, x_0, \lambda) \in C_\rho$. This implies that there exists $\tau \in (t_1, t_2)$ such that

$$V\big(\tau, x(\tau, t_0, x_0, \lambda), \lambda\big) = r \quad \text{and} \quad V\big(t, x(t, t_0, x_0, \lambda), \lambda\big) > r, \quad \forall t \in (\tau, t_2).$$

Thus we have that $x(t, t_0, x_0, \lambda) \notin C_\rho$, for $t \in (\tau, t_2)$. Hence

$$\dot{V}\big(t, x(t, t_0, x_0, \lambda), \lambda\big) < -\rho, \quad \forall t \in (\tau, t_2)$$

and so the function $t \in (\tau, t_2) \mapsto V\big(t, x(t, t_0, x, \lambda), \lambda\big)$ is decreasing. This is a contradiction. Therefore we conclude that $x(t, t_0, x_0, \lambda) \in \mathcal{A}_r, \quad \forall t \geq t_1$.

(ii) Suppose that $x(t)$ is a solution of (2.2) defined for $t \in \mathbb{R}$ and such that $\sup_{t \in \mathbb{R}} \|x(t)\| < +\infty$. Then we will show that $x(t) \in \mathcal{A}_r$ for every $t \in \mathbb{R}$. In fact, suppose that there exists $\tau \in \mathbb{R}$ such that $x(\tau) \notin \mathcal{A}_r$. Then from item (i) we conclude that $x(t) \notin C_\rho$ for $t \leq \tau$. Therefore $\dot{V}\big(t, x(t), \lambda\big) < -\rho$ for every $t \leq \tau$. Then for any $t \leq \tau$, there exists $s_t \in (t-1, t)$ such that

$$V\big(t, x(t), \lambda\big) - V\big(t-1, x(t-1), \lambda\big) = \dot{V}\big(s_t, x(s_t), \lambda\big) \leq -\rho.$$

Therefore,

$$V\big(t, x(t), \lambda\big) + \rho \le V\big(t-1, x(t-1), \lambda\big)$$

From the above inequality, choosing convenient values of t, we obtain

$$V\big(\tau, x(\tau), \lambda\big) + 2\rho \le V\big(\tau-1, x(\tau-1), \lambda\big) + \rho \le V\big(\tau-2, x(\tau-2), \lambda\big).$$

Using this procedure, recursively we obtain

$$V\big(\tau, x(\tau), \lambda\big) + n\rho \le V\big(\tau-n, x(\tau-n), \lambda\big).$$

The last inequality implies that $V\big(\tau-n, x(\tau-n), \lambda\big) \to +\infty$, as $n \to +\infty$. This is a contradiction because $V\big(t, x(t), \lambda\big) \le b\big(x(t)\big)$ and b is bounded on bounded sets. Therefore $x(t) \in \mathcal{A}_r, \quad \forall t \in \mathbb{R}$. ■

Remark 2.2. *In the above theorem if $c(x) \to \infty$, as $\|x\| \to \infty$, then there exists $\rho > 0$, such that the set $C_\rho := \big\{x \in \mathbb{R}^n : c(x) \le \rho\big\}$ is nonempty and bounded.*

In the next theorems we give sufficient conditions to obtain the synchronization of two systems of ordinary differential equations.

Let $f, g \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \Lambda, \mathbb{R}^n)$ satisfying the Lipschitz condition on bounded sets (2.1), with respect to $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. For $\lambda_1, \lambda_2 \in \Lambda$, consider the following system

$$\begin{cases} \dot{x} = f(t, x, y, \lambda_1) \\ \dot{y} = g(t, x, y, \lambda_2) \end{cases} \tag{2.3}$$

Definition 2.1. We say that system (2.3) **synchronizes globally** (or simply **synchronizes**) if given $\epsilon > 0$ there exists $\delta > 0$, such that, if $\|\lambda_2 - \lambda_1\| < \delta$ then for every $(t_0, x_0, y_0) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ we have

$$\limsup_{t \to +\infty} \|y(t, t_0, x_0, y_0, \lambda_1, \lambda_2) - x(t, t_0, x_0, y_0, \lambda_1, \lambda_2)\| < \epsilon.$$

Theorem 2.3. *We assume the following hypotheses:*

(i) *There exists a bounded set $\mathcal{B} \subset \mathbb{R}^n \times \mathbb{R}^n$ such that, for each $(t_0, x_0, y_0) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda_1, \lambda_2 \in \Lambda$, the solution $\big(x(t), y(t)\big) := \big(x(t, t_0, x_0, y_0, \lambda_1, \lambda_2), y(t, t_0, x_0, y_0, \lambda_1, \lambda_2)\big)$ of (2.3) is defined for $t \in [t_0, +\infty)$ and there exists $t_1 \ge t_0$ such that:*
$\big(x(t), y(t)\big) \in \mathcal{B}, \quad$ for every $t \ge t_1$.

(ii) *There exists a continuous function* $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \Lambda \to \mathcal{L}(\mathbb{R}^n)$, *such that*

$g(t,x,y,\lambda) - f(t,x,y,\lambda) = F\big(t,x,y,\lambda\big)(y-x)$

for every $(t,x,y,\lambda) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \Lambda$.

(iii) *There exist a constant* $k \geq 0$ *such that, for any* $(t_0, x_0, y_0) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ *and* $\lambda_1, \lambda_2 \in \Lambda$, *there exists* $\alpha > 0$ *and* $M \geq 0$ *such that* $F\big(t, x(t), y(t), \lambda_2\big)$ *defines an evolution operator* $T(t,s,\lambda_2)$ *satisfying* $\|T(t,s,\lambda_2)\| \leq Me^{-\alpha(t-s)}$ *and* $\int_{t_1}^{t} \|T(t,s,\lambda_2)\| ds \leq k$ *for* $t_1 \leq s \leq t$.

(iv) *There exists a function* $H : \mathbb{R}_+ \to \mathbb{R}_+$, *with* $H(0) = 0$, H *continuous at* $0 \in \mathbb{R}_+$, *such that* $\|f(t,x,y,\lambda_2) - f(t,x,y,\lambda_1)\| \leq H(\|\lambda_2 - \lambda_1\|)$, *for every* $t \in \mathbb{R}$, $(x,y) \in \mathcal{B}$ *and* $\lambda_1, \lambda_2 \in \Lambda$.

Then we have

$$\|y(t) - x(t)\| \leq M\|y(t_1) - x(t_1)\|e^{-\alpha(t-t_1)} + kH(\|\lambda_2 - \lambda_1\|), \quad \forall t \geq t_1.$$

Proof: Given $(t_0, x_0, y_0) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda_1, \lambda_2 \in \Lambda$ let $z(t) := y(t) - x(t)$. We have

$$\begin{aligned} \dot{y} - \dot{x} &= g(t,x,y,\lambda_2) - f(t,x,y,\lambda_1) = \\ &= g(t,x,y,\lambda_2) - f(t,x,y,\lambda_2) + f(t,x,y,\lambda_2) - f(t,x,y,\lambda_1) = \\ &= F\big(t,x,y,\lambda_2\big)(y-x) + \big(f(t,x,y,\lambda_2) - f(t,x,y,\lambda_1)\big). \end{aligned}$$

Then

$$\dot{z}(t) = F\big(t, x(t), y(t), \lambda_2\big)z(t) + \Big(f\big(t, x(t), y(t), \lambda_2\big) - f\big(t, x(t), y(t), \lambda_1\big)\Big).$$

Consider the initial value problem

$$\begin{cases} \dot{w} = F\big(t, x(t), y(t), \lambda_2\big)w + \Big(f\big(t, x(t), y(t), \lambda_2\big) - f\big(t, x(t), y(t), \lambda_1\big)\Big) \\ w(t_1) = z(t_1) = y(t_1) - x(t_1) \end{cases}$$

Let $T(t,s,\lambda_2)$ the evolution operator of $\dot{u} = F\big(t, x(t), y(t), \lambda_2\big)u$. Then the solution of the above equation is

$$w(t) = T(t,t_1,\lambda_2)w(t_1) + \int_{t_1}^{t} T(t,\tau,\lambda_2)\Big(f\big(t, x(t), y(t), \lambda_2\big) - f\big(t, x(t), y(t), \lambda_1\big)\Big) d\tau.$$

Since $\|T(t,s,\lambda_2)\| \leq Me^{-\alpha(t-s)}$, $\int_{t_1}^{t} \|T(t,s,\lambda_2)\| ds \leq k$ and

$$\|f\big(t, x(t), y(t), \lambda_2\big) - f\big(t, x(t), y(t), \lambda_1\big)\| \leq H(\|\lambda_2 - \lambda_1\|)$$

for $t_1 \leq s \leq t$, it follows that

$$\begin{aligned}\|y(t)-x(t)\| = \|w(t)\| &\leq \|T(t,t_1,\lambda_2)w(t_1)\| + \\ &\int_{t_1}^{t} \|T(t,\tau,\lambda_2)\| \, \|f\big(t,x(t),y(t),\lambda_2\big) - f\big(t,x(t),y(t),\lambda_1\big)\| d\tau \\ &\leq M\|w(t_1)\|e^{-\alpha(t-t_1)} + H(\|\lambda_2-\lambda_1\|)\int_{t_1}^{t}\|T(t,\tau,\lambda_2)\|d\tau \\ &\leq M\|y(t_1)-x(t_1)\|e^{-\alpha(t-t_1)} + kH(\|\lambda_2-\lambda_1\|).\end{aligned}$$

■

Theorem 2.4. *We assume the following hypotheses:*

(i) *There exists a bounded set $\mathcal{B} \subset \mathbb{R}^n \times \mathbb{R}^n$ such that, for each $(t_0, x_0, y_0) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda_1, \lambda_2 \in \Lambda$, the solution $\big(x(t), y(t)\big) := \big(x(t,t_0,x_0,y_0,\lambda_1,\lambda_2), y(t,t_0,x_0,y_0,\lambda_1,\lambda_2)\big)$ of (2.3) is defined for $t \in [t_0, +\infty)$ and there exists $t_1 \geq t_0$ such that*

$$\big(x(t), y(t)\big) \in \mathcal{B}, \quad \textit{for every } t \geq t_1.$$

(ii) *There exists $V \in C^1(\mathbb{R} \times \mathbb{R}^n \times \Lambda, \mathbb{R})$ and positive constants k_1, ρ, α_1, α_2 and $\beta \geq 1$, such that*

$$\frac{\partial}{\partial t}V(t,y-x,\lambda) + \Big\langle \nabla V(t,y-x,\lambda), g(t,x,y,\lambda) - f(t,x,y,\lambda)\Big\rangle \leq -\rho V(t,y-x,\lambda),$$

$$\alpha_1\|y-x\|^\beta \leq V(t,y-x,\lambda) \leq \alpha_2\|y-x\|^\beta \quad \textit{and} \quad \big\|\nabla V\big(t,y-x,\lambda\big)\big\| \leq k_1 \tag{2.4}$$

for every $(x,y) \in \mathcal{B}$, $t \in \mathbb{R}$ and $\lambda \in \Lambda$, where

$$\nabla V(t,x,\lambda) = \left(\frac{\partial V}{\partial x_1}(t,x,\lambda), \ldots, \frac{\partial V}{\partial x_n}(t,x,\lambda)\right)$$

and $\nabla V(t,y-x,\lambda) = \nabla V(t,z,\lambda)\big|_{z=y-x}$.

(iv) *There exists a function $H_1 : \mathbb{R}_+ \to \mathbb{R}_+$, with $H_1(0) = 0$, H_1 continuous at $0 \in \mathbb{R}_+$, such that*

$$\|f(t,x,y,\lambda_2) - f(t,x,y,\lambda_1)\| \leq H_1(\|\lambda_2 - \lambda_1\|)$$

for every $t \in \mathbb{R}$, $(x,y) \in \mathcal{B}$ and $\lambda_1, \lambda_2 \in \Lambda$.

Then we have

$$\|y(t)-x(t)\| \leq M\|y(t_1)-x(t_1)\|e^{-\alpha(t-t_1)} + kH(\|\lambda_2-\lambda_1\|), \quad \forall t \geq t_1.$$

where $\alpha := \rho/\beta$, $k := (\frac{k_1}{\rho\alpha_1})^{1/\beta}$, $M := (\alpha_2/\alpha_1)^{1/\beta}$ *and* $H : \mathbb{R}_+ \to \mathbb{R}_+$ *is given by* $H(r) := \big(H_1(r)\big)^{1/\beta}$.

Proof: Given $(t_0, x_0, y_0)) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda_1, \lambda_2 \in \Lambda$ we have

$$\begin{aligned} \dot{y}-\dot{x} &= g(t,x,y,\lambda_2) - f(t,x,y,\lambda_1) = \\ &= \big(g(t,x,y,\lambda_2) - f(t,x,y,\lambda_2)\big) + \big(f(t,x,y,\lambda_2) - f(t,x,y,\lambda_1)\big). \end{aligned}$$

The derivative of $V(t, y-x, \lambda_2)$ along the solution of the above system is

$$\begin{aligned} \dot{V}(t,y-x,\lambda_2) &= \tfrac{\partial}{\partial t}V(t,y-x,\lambda_2) + \big\langle \nabla V(t,y-x,\lambda_2), g(t,x,y,\lambda_2) - f(t,x,y,\lambda_2)\big\rangle \\ &\quad + \big\langle \nabla V(t,y-x,\lambda_2), f(t,x,y,\lambda_2) - f(t,x,y,\lambda_1)\big\rangle. \end{aligned}$$

Then

$$\begin{aligned} \dot{V}\big(t,y(t)-x(t),\lambda_2\big) &= \tfrac{\partial}{\partial \tau}V\big(\tau,y(t)-x(t),\lambda_2\big)\Big|_{\tau=t} \\ &\quad + \big\langle \nabla V\big(t,y(t)-x(t),\lambda_2\big), g\big(t,x(t),y(t),\lambda_2\big) - f\big(t,x(t),y(t),\lambda_2\big)\big\rangle \\ &\quad + \big\langle \nabla V\big(t,y(t)-x(t),\lambda_2\big), f\big(t,x(t),y(t),\lambda_2\big) - f\big(t,x(t),y(t),\lambda_1\big)\big\rangle. \end{aligned}$$

For $t \geq t_1$ we have $\big(x(t), y(t)\big) \in \mathcal{B}$, and then

$$\begin{aligned} &\tfrac{\partial}{\partial \tau}V\big(\tau,y(t)-x(t),\lambda_2\big)\Big|_{\tau=t} \\ &+ \big\langle \nabla V\big(t,y(t)-x(t),\lambda_2\big), g\big(t,x(t),y(t),\lambda_2\big) - f\big(t,x(t),y(t),\lambda_2\big)\big\rangle \\ &\leq -\rho V\big(t,y(t)-x(t),\lambda_2\big) \end{aligned}$$

and

$$\begin{aligned} &\big\langle \nabla V\big(t,y(t)-x(t),\lambda_2\big), f\big(t,x(t),y(t),\lambda_2\big) - f\big(t,x(t),y(t),\lambda_1\big)\big\rangle \\ &\leq \big\|\nabla V\big(t,y(t)-x(t),\lambda_2\big)\big\| \, \big\|f\big(t,x(t),y(t),\lambda_2\big) - f\big(t,x(t),y(t),\lambda_1\big)\big\| \\ &\leq k_1 H_1(\|\lambda_2-\lambda_1\|). \end{aligned}$$

Therefore

$$\dot{V}\big(t,y(t)-x(t),\lambda_2\big) \leq -\rho V\big(t,y(t)-x(t),\lambda_2\big) + k_1 H_1(\|\lambda_2-\lambda_1\|).$$

Multiplying this inequality by $e^{\rho t}$ and integrating from t_1 to t we have

$$\begin{aligned} e^{\rho t} V\big(t, y(t) - x(t), \lambda_2\big) &\leq e^{\rho t_1} V\big(t_1, y(t_1) - x(t_1), \lambda_2\big) + \tfrac{k_1}{\rho} H_1(\|\lambda_2 - \lambda_1\|)\big(e^{\rho t} - e^{\rho t_1}\big) \\ &\leq e^{\rho t_1} V\big(t_1, y(t_1) - x(t_1), \lambda_2\big) + \tfrac{k_1}{\rho} H_1(\|\lambda_2 - \lambda_1\|) e^{\rho t}. \end{aligned}$$

Therefore

$$V\big(t, y(t) - x(t), \lambda_2\big) \leq e^{-\rho(t-t_1)} V\big(t_1, y(t_1) - x(t_1), \lambda_2\big) + \frac{k_1}{\rho} H_1(\|\lambda_2 - \lambda_1\|).$$

From (2.4) it follows that

$$\begin{aligned} \alpha_1 \|y(t) - x(t)\|^\beta &\leq V\big(t, y(t) - x(t), \lambda_2\big) \\ &\leq e^{-\rho(t-t_1)} V\big(t_1, y(t_1) - x(t_1), \lambda_2\big) + \tfrac{k_1}{\rho} H_1(\|\lambda_2 - \lambda_1\|) \\ &\leq \alpha_2 e^{-\rho(t-t_1)} \|y(t_1) - x(t_1)\|^\beta + \tfrac{k_1}{\rho} H_1(\|\lambda_2 - \lambda_1\|). \end{aligned}$$

Therefore we conclude that

$$\begin{aligned} \|y(t) - x(t)\| &\leq \left[\tfrac{\alpha_2}{\alpha_1} e^{-\rho(t-t_1)} \|y(t_1) - x(t_1)\|^\beta + \tfrac{k_1}{\rho\alpha_1} H_1(\|\lambda_2 - \lambda_1\|)\right]^{1/\beta} \\ &\leq \left[\tfrac{\alpha_2}{\alpha_1} e^{-\rho(t-t_1)} \|y(t_1) - x(t_1)\|^\beta\right]^{1/\beta} + \left[\tfrac{k_1}{\rho\alpha_1} H_1(\|\lambda_2 - \lambda_1\|)\right]^{1/\beta} \\ &= M \|y(t_1) - x(t_1)\| e^{-\alpha(t-t_1)} + k H(\|\lambda_2 - \lambda_1\|), \end{aligned}$$

for $t \geq t_1$, where we have used that $(a+b)^{1/p} \leq a^{1/p} + b^{1/p}$ for $a, b \geq 0$ and $p \geq 1$.

■

In a natural way one can prove that Theorem 2.3 and Theorem 2.4 imply synchronization. Thus, to obtain synchronization we adopt the following strategy. We first use Theorem 2.1 to prove the uniform dissipativeness and then we use Theorem 2.3 or Theorem 2.4 to show the synchronization. When we use Theorem 2.3 we need to prove that the evolution operator is uniformly exponentially bounded. For this purpose the following theorem is useful.

Theorem 2.5. *Let X be a Banach space, $\Gamma \subset X$ be a compact subset, and for each $\mu \in \Gamma$ let $A(\mu) \in \mathcal{L}(\mathbb{R}^n)$. Suppose that $\mu \mapsto A(\mu)$ is a continuous map from Γ to $\mathcal{L}(\mathbb{R}^n)$. Suppose also that for each $\mu \in \Gamma$ all eigenvalues of $A(\mu)$ have negative real part. Then there exist positive constants β and M such that*

$$\|e^{A(\mu)t}\| \leq M e^{-\beta t}, \qquad \forall t \geq 0 \quad \textit{and} \quad \forall \mu \in \Gamma.$$

Proof: Let $\lambda_1(\mu) \leq \lambda_2(\mu) \leq \cdots \leq \lambda_n(\mu)$, be the eigenvalues of $A(\mu)$, and $\alpha(\mu) := -\max\left\{ \mathrm{Re}\left(\lambda_i(\mu)\right) : 1 \leq i \leq n \right\}$. The function $\mu \in \Gamma \mapsto \alpha(\mu) \in \mathbb{R}$ is continuous (See Kato[25, p. 213]). Let $\alpha := \max_{\mu\in\Gamma} \alpha(\mu)$ and $\beta > \alpha$. From Kato[25] it follows that one can choose smooth simple positively oriented closed curve $\gamma \subset \{z : \mathrm{Re}(z) < \beta\}$, such that the spectrum $\sigma(A(\mu))$ of $A(\mu)$ is contained in the interior of γ for every $\mu \in \Gamma$. From Coppel[11, p. 56] or Daleckiĭ-Kreĭn[14, p. 25] it follows that, for $t \geq 0$ and $\mu \in \Gamma$,

$$e^{A(\mu)t} = \frac{1}{2\pi i}\int_\gamma \left(zI - A(\mu)\right)^{-1} e^{tz} dz.$$

Therefore,

$$\|e^{A(\mu)t}\| \leq \frac{1}{2\pi}\int_\gamma \left\|\left(zI - A(\mu)\right)^{-1}\right\| |e^{tz}|\, |dz| \leq \left(\frac{1}{2\pi}\int_\gamma \left\|\left(zI - A(\mu)\right)^{-1}\right\| |dz|\right) e^{-\beta t}.$$

Since $(\mu, z) \mapsto zI - A(\mu)$ is a continuous function, we have that

$$(\mu, z) \in \Gamma \times \gamma \mapsto \left\|\left(zI - A(\mu)\right)^{-1}\right\|$$

is continuous. Since $\Gamma \times \gamma$ is compact we have that there exists a positive constant K such that $\left\|\left(zI - A(\mu)\right)^{-1}\right\| \leq K$ for every $(\mu, z) \in \Gamma \times \gamma$. Therefore we have that

$$\|e^{A(\mu)t}\| \leq Me^{-\beta t},$$

where $M := \dfrac{1}{2\pi}\int_\gamma K|dz| = \dfrac{K}{2\pi}\mathrm{length}(\gamma)$.

■

3. Applications

Example 3.1 (Lorenz System). Consider the system

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases} \tag{3.1}$$

Let $m, \bar{m} \in C_K(\mathbb{R}, \mathbb{R})$. From the system above we construct the systems

$$\begin{cases} \dot{x} = -\sigma_1 x + \sigma_1 y \\ \dot{y} = r_1\big(x + m(t)\big) - y - z\big(x + m(t)\big) \\ \dot{z} = -b_1 z + y\big(x + m(t)\big) \end{cases} \tag{3.2}$$

$$\begin{cases} \dot{u} = -\sigma_2 u + \sigma_2 v \\ \dot{v} = r_2\big(x + \bar{m}(t)\big) - v - w\big(x + \bar{m}(t)\big) \\ \dot{w} = -b_2 w + v\big(x + \bar{m}(t)\big) \end{cases} \tag{3.3}$$

where $(\sigma_1, r_1, b_1), (\sigma_2, r_2, b_2) \in \Gamma$ and

$$\Gamma := \Big\{(\sigma, r, b) \in \mathbb{R}^3 : \sigma_m \leq \sigma \leq \sigma_M, \quad r_m \leq r \leq r_M \quad \text{e} \quad b_m \leq b \leq b_M\Big\},$$

with $0 < \sigma_m \leq \sigma_M$, $0 < r_m \leq r_M$ and $0 < b_m \leq b_M$. Let $\Lambda = \Gamma \times C_K(\mathbb{R}, \mathbb{R})$ and $\lambda = (\gamma, m(t)) \in \Lambda$, where $\gamma = (\sigma, r, b) \in \Gamma$.

Now we will show that systems (3.2) and (3.3) satisfy conditions of Theorem 2.1. For system (3.2), consider the following Liapunov function

$$V(x, y, z, \lambda_1) = x^2 + \sigma_1 y^2 + \sigma_1 (z - r_1)^2,$$

where $\lambda_1 = (\sigma_1, r_1, b_1, m(t))$. Then we have

$$V(x, y, z, \lambda_1) \geq x^2 + \sigma_m y^2 + \sigma_m (|z| - r_M)^2 + \sigma_m ({r_m}^2 - {r_M}^2) := a(x, y, z).$$

$$V(x, y, z, \lambda_1) \leq x^2 + \sigma_M y^2 + \sigma_M (|z| + r_M)^2 := b(x, y, z).$$

We also have

$$\dot{V}(x, y, z, \lambda_1) = -2\sigma_1 x^2 + 2\sigma_1 xy - 2\sigma_1 y^2 - 2\sigma_1 b_1 z^2 + 2\sigma_1 b_1 r_1 z.$$

Hence

$$-\dot{V}(x, y, z, \lambda_1) \geq \sigma_m x^2 + \sigma_m y^2 + 2\sigma_m b_m z^2 - 2\sigma_M b_M r_M |z| =: c(x, y, z).$$

The functions a, b and c obviously satisfy the conditions of Theorem 2.1, and so there exists a bounded set $\mathcal{B}_1 \subset \mathbb{R}^3$ such that, if $\big(x(t), y(t), z(t)\big)$ is a solution of system (3.2), then there exists t_1 such that $\big(x(t), y(t), z(t)\big) \in \mathcal{B}_1$ for $t \geq t_1$.

Using the above idea, for system (3.3) we consider $V(u, v, w, \lambda_2)$ as a Liapunov function, where $\lambda_2 = (\sigma_2, r_2, b_2, \bar{m}(t))$. Then

$$a(u,v,w) \leq V(u,v,w,\lambda_2) \leq b(u,v,w) \quad \text{and} \quad -\dot{V}(u,v,w,\lambda_2) \geq c(u,v,w).$$

Therefore we conclude that there exists a bounded set $\mathcal{B}_2 \subset \mathbb{R}^3$ such that, if $\big(u(t), v(t), w(t)\big)$ is a solution of system (3.3), then there exists t_2 such that $\big(u(t), v(t), w(t)\big) \in \mathcal{B}_2$ for $t \geq t_2$.

Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ and $t_3 = \max\{t_1, t_2\}$. Then for every solution $\big(x(t), y(t), z(t)\big)$ of (3.2) and $\big(u(t), v(t), w(t)\big)$ of (3.3), we have

$$\Big(\big(x(t), y(t), z(t)\big), \big(u(t), v(t), w(t)\big)\Big) \in \mathcal{B} \times \mathcal{B}$$

for $t \geq t_3$.

Therefore systems (3.2) and (3.3) satisfy condition (i) of Theorem 2.4. Let us show that these systems satisfy the remaining conditions of Theorem 2.4. Let

$$f(x,y,z,u,v,w,\lambda) = \begin{pmatrix} -\sigma x + \sigma y \\ r\big(x + m(t)\big) - y - z\big(x + m(t)\big) \\ -bz + y\big(x + m(t)\big) \end{pmatrix}$$

and

$$g(x,y,z,u,v,w,\lambda) = \begin{pmatrix} -\sigma u + \sigma v \\ r\big(x + m(t)\big) - v - w\big(x + m(t)\big) \\ -bw + v\big(x + m(t)\big) \end{pmatrix}.$$

If we let $X := u - x$, $Y := v - y$ and $Z := w - z$, we obtain

$$g(x,y,z,u,v,w,\lambda) - f(x,y,z,u,v,w,\lambda) = \begin{pmatrix} -\sigma X + \sigma Y \\ -Y - Z\big(x + m(t)\big) \\ -bZ + Y\big(x + m(t)\big) \end{pmatrix}.$$

Then we have the following system

$$\begin{cases} \dot{X} = -\sigma X + \sigma Y \\ \dot{Y} = -Y - Z\big(x + m(t)\big) \\ \dot{Z} = -bZ + Y\big(x + m(t)\big) \end{cases} \tag{3.4}$$

For this system we consider the Liapunov function

$$V(X,Y,Z,\lambda) = 2X^2 + \sigma Y^2 + \sigma Z^2.$$

Then we have

$$\dot{V}(X,Y,Z,\lambda) = -4\sigma X^2 - 2\sigma Y^2 - 2\sigma b Z^2 + 4\sigma XY.$$

Let $\rho > 0$ such that $\rho < 2\sigma_m$, $\rho < 2b_m$ and $\rho < \rho_-$, where $0 < \rho_- < \rho_+$ are the roots of $\xi^2 - (2+2\sigma_M)\xi + 2\sigma_m = 0$. Then

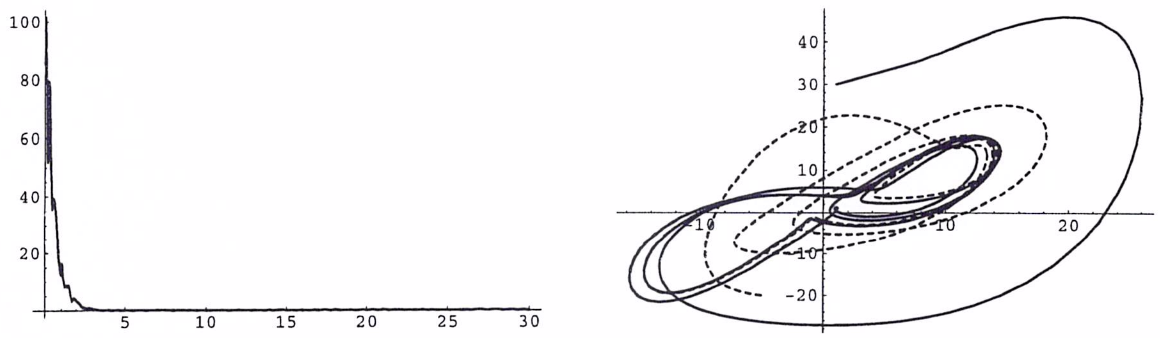
$$-\Big(\dot{V}(X,Y,Z,\lambda) + \rho V(X,Y,Z,\lambda)\Big) = 2(2\sigma-\rho)X^2 + \sigma(2-\rho)Y^2 + \sigma(2b-\rho)Z^2 - 4\sigma XY.$$

Using Sylvester's criteria and the previous conditions on ρ, we show that $-\Big(\dot{V}(X,Y,Z,\lambda)+ \rho V(X,Y,Z,\lambda)\Big)$ is positive defined. Therefore we have

$$\dot{V}(X,Y,Z,\lambda) \leq -\rho V(X,Y,Z,\lambda).$$

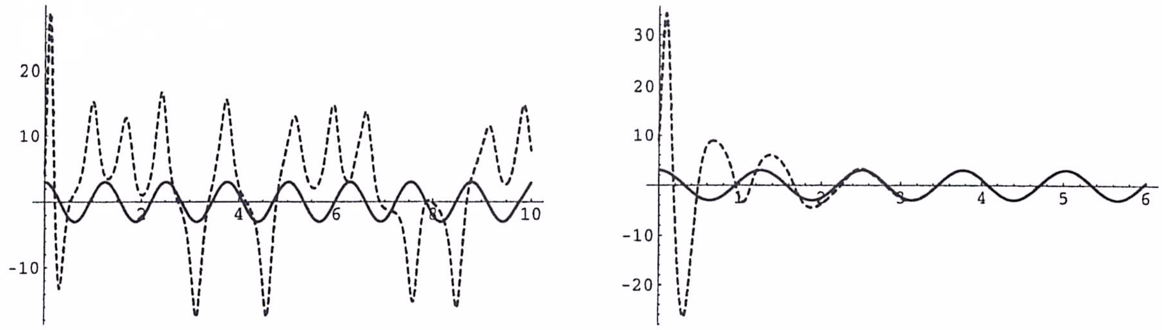
The remaining conditions of Theorem 2.4 are easy to verify. Therefore we conclude, from Theorem 2.4, that systems (3.2) and (3.3) synchronize.

In Figure 1 and Figure 2 we present some simulations that show the synchronization and the codification of messages with these systems. In these simulations we used the following parameters and initial conditions values: $\sigma_1 = 10$, $r_1 = 28$, $b_1 = 8/3$, $\sigma_2 = 10 + 1/100$, $r_2 = 28 + 1/100$, $b_2 = 8/3 + 1/100$, $m(t) = 3\cos(5t)$, $\bar{m}(t) = 3\cos(5t) + (1/100)\sin(t)$, $x(0) = 1$, $y(0) = 30$, $z(0) = -10$, $u(0) = -5$, $v(0) = -20$ and $w(0) = 20$.



(a) $|x(t)-u(t)|+|y(t)-v(t)|+|z(t)-w(t)|$ (b) $(x(t),y(t)) = —$, $(u(t),v(t)) = ---$

FIGURE 1. Synchronization of Lorenz System



(a) $m(t) = —$, $x(t) + m(t) = - - -$ (b) $m(t) = —$, $m_r(t) = - - -$

FIGURE 2. **Codification of Messages with Lorenz System**

Example 3.2 (Chua's Like System). Consider the system

$$\begin{cases} \dot{x} = -\alpha x + \alpha y + \alpha b x + \alpha \dfrac{a-b}{2} h(x) \\ \dot{y} = -x - y + z \\ \dot{z} = -\beta y - \sigma z \end{cases} \tag{3.5}$$

where $h(x) = |x+1| - |x-1|$, with $a > b > 0$ and $b < 1$.

Let $m, \bar{m} \in C_K(\mathbb{R}, \mathbb{R})$. From system (3.5) we construct the systems:

$$\begin{cases} \dot{x} = -\alpha_1 x + \alpha_1 y + \alpha_1 b_1 \big(x + m(t)\big) + \alpha_1 \dfrac{a_1 - b_1}{2} h\big(x + m(t)\big) \\ \dot{y} = -\big(x + m(t)\big) - y + z \\ \dot{z} = -\beta_1 y - \sigma_1 z \end{cases} \tag{3.6}$$

$$\begin{cases} \dot{u} = -\alpha_2 u + \alpha_2 v + \alpha_2 b_2 \big(x + \bar{m}(t)\big) + \alpha_2 \dfrac{a_2 - b_2}{2} h\big(x + \bar{m}(t)\big) \\ \dot{v} = -\big(x + \bar{m}(t)\big) - v + w \\ \dot{w} = -\beta_2 v - \sigma_2 w \end{cases} \tag{3.7}$$

where $(\alpha_1, \beta_1, \sigma_1, a_1, b_1), (\alpha_2, \beta_2, \sigma_2, a_2, b_2) \in \Gamma$, where Γ is the set of $\gamma = (\alpha, \beta, \sigma, a, b) \in \mathbb{R}^5$ such that $0 < \alpha_m \leq \alpha \leq \alpha_M$, $0 < \beta_m \leq \beta \leq \beta_M$, $0 < \sigma_m \leq \sigma \leq \sigma_M$, $0 < b_m \leq b \leq b_M < 1$ and $b_M < a_m \leq a \leq a_M$. Let $\Lambda = \Gamma \times C_K(\mathbb{R}, \mathbb{R})$ and $\lambda = \big(\gamma, m(t)\big) \in \Lambda$, where $\gamma = (\alpha, \beta, \sigma, a, b) \in \Gamma$.

Now we will show that systems (3.6) and (3.7) satisfy the conditions of Theorem 2.1. For system (3.6), consider the following Liapunov function

$$V(x,y,z,\lambda_1) = \beta_1 x^2 + \alpha_1\beta_1 y^2 + \alpha_1 z^2.$$

where $\lambda_1 = \big(\alpha_1, \beta_1, \sigma_1, a_1, b_1, m(t)\big)$. Then we have

$$V(x,y,z,\lambda_1) \geq \beta_m x^2 + \alpha_m\beta_m y^2 + \alpha_m z^2 =: a(x,y,z).$$

$$V(x,y,z,\lambda_1) \leq \beta_M x^2 + \alpha_M\beta_M y^2 + \alpha_M z^2 =: b(x,y,z).$$

We also have

$$\begin{gathered}
-\dot{V}(x,y,z,\lambda_1) = 2\alpha_1\beta_1(1-b_1)x^2 + 2\alpha_1\beta_1 y^2 + 2\alpha_1\beta_1\sigma_1 z^2 \\
-\alpha_1\beta_1\Big(2b_1 m(t) + (a_1 - b_1)h\big(x+m(t)\big)\Big)x + 2\alpha_1\beta_1 y m(t) \geq \\
2\alpha_m\beta_m(1-b_M)x^2 + 2\alpha_m\beta_m y^2 + 2\alpha_m\sigma_m z^2 \\
-2\alpha_M\beta_M\big(b_M K + (a_M - b_m)\big)|x| - 2\alpha_M\beta_M K|y| =: c(x,y,z).
\end{gathered}$$

The functions a, b and c obviously satisfy the conditions of Theorem 2.1, and so there exists a bounded set $\mathcal{B}_1 \subset \mathbb{R}^3$ such that, if $\big(x(t), y(t), z(t)\big)$ is a solution of system (3.6), then there exists t_1 such that $\big(x(t), y(t), z(t)\big) \in \mathcal{B}_1$ for $t \geq t_1$.

Using the above idea, for system (3.7) we consider $V(u,v,w,\lambda_2)$ as a Liapunov function, where $\lambda_2 = \big(\alpha_2, \beta_2, \sigma_2, a_2, b_2, \bar{m}(t)\big)$. Let $M \geq 0$ such that $\|x + m(t)\| \leq M$ for $(x,y,z) \in \mathcal{B}_1$. Then we have

$$a(u,v,w) \leq V(u,v,w,\lambda_2) \leq b(u,v,w)$$

and

$$\begin{gathered}
-\dot{V}(u,v,w,\lambda_2) = 2\alpha_2\beta_2 u^2 + 2\alpha_2\beta_2 v^2 - 2\alpha_2\beta_2 uv + 2\alpha_2\sigma_2 w^2 \\
-2\alpha_2\beta_2 b_1\big(x+m(t)\big)u - \alpha_2\beta_2(a_2 - b_2)h\big(x+m(t)\big)u + 2\alpha_2\beta_2\big(x+m(t)\big)v \geq \\
2\alpha_2\beta_2 u^2 + 2\alpha_2\beta_2 v^2 + -\alpha_2\beta_2(u^2+v^2) + 2\alpha_2\sigma_2 w^2 \\
-2\alpha_2\beta_2 b_1\big(x+m(t)\big)u - \alpha_2\beta_2(a_2 - b_2)h\big(x+m(t)\big)u + 2\alpha_2\beta_2\big(x+m(t)\big)v \geq \\
\alpha_m\beta_m u^2 + \alpha_m\beta_m v^2 + 2\alpha_m\sigma_m w^2 \\
-\alpha_M\beta_M\big(2b_M M + (a_M - b_m)\big)|u| - 2\alpha_M\beta_M M|y| =: c_2(x,y,z).
\end{gathered}$$

Therefore we conclude that there exists a bounded set $\mathcal{B}_2 \subset \mathbb{R}^3$ such that, if $\big(u(t), v(t), w(t)\big)$ is a solution of system (3.7), then there exists t_2 such that $\big(u(t), v(t), w(t)\big) \in \mathcal{B}_2$ for $t \geq t_2$.

Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ and $t_3 = \max\{t_1, t_2\}$. Then for every solution $\big(x(t), y(t), z(t)\big)$ of (3.6) and $\big(u(t), v(t), w(t)\big)$ of (3.7), we have

$$\Big(\big(x(t), y(t), z(t)\big), \big(u(t), v(t), w(t)\big)\Big) \in \mathcal{B} \times \mathcal{B}$$

for $t \geq t_3$.

Therefore systems (3.6) and (3.7) satisfies condition (i) of Theorem 2.3. Let us show that these systems satisfy the remaining conditions of Theorem 2.3. Let

$$f(x,y,z,u,v,w,\lambda) = \begin{pmatrix} -\alpha x + \alpha y + \alpha b\big(x + m(t)\big) + \alpha\frac{a-b}{2}h\big(x+m(t)\big) \\ -\big(x + m(t)\big) - y + z \\ -\beta y - \sigma z \end{pmatrix}$$

and

$$g(x,y,z,u,v,w,\lambda) = \begin{pmatrix} -\alpha u + \alpha v + \alpha b\big(x + m(t)\big) + \alpha\frac{a-b}{2}h\big(x+\bar{m}(t)\big) \\ -\big(x + m(t)\big) - v + w \\ -\beta v - \sigma w \end{pmatrix}.$$

Then we have

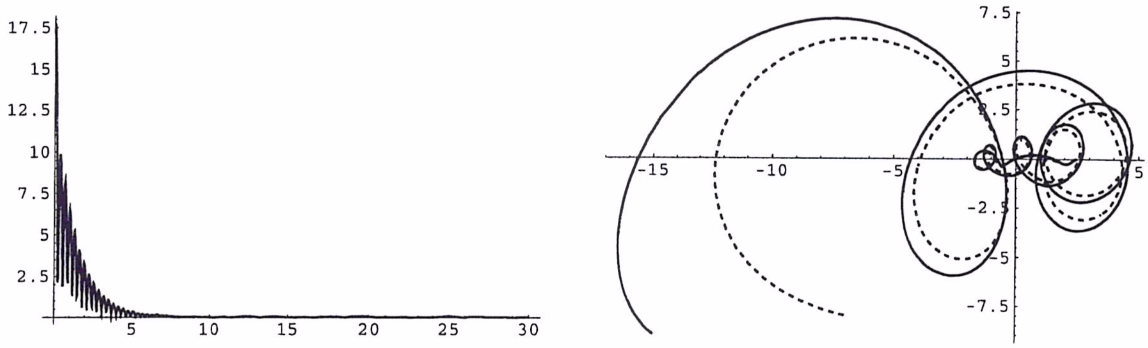
$$g(x,y,z,u,v,w,\lambda) - f(x,y,z,u,v,w,\lambda) = \begin{pmatrix} -\alpha & \alpha & 0 \\ 0 & -1 & 1 \\ 0 & -\beta & -\sigma \end{pmatrix} \begin{pmatrix} u - x \\ v - y \\ w - z \end{pmatrix}$$

Therefore systems (3.6) e (3.7) satisfy condition (ii) of Theorem 2.3 with

$$F(t,x,y,z,u,v,w,\lambda) = \begin{pmatrix} -\alpha & \alpha & 0 \\ 0 & -1 & 1 \\ 0 & -\beta & -\sigma \end{pmatrix} =: A(\gamma).$$

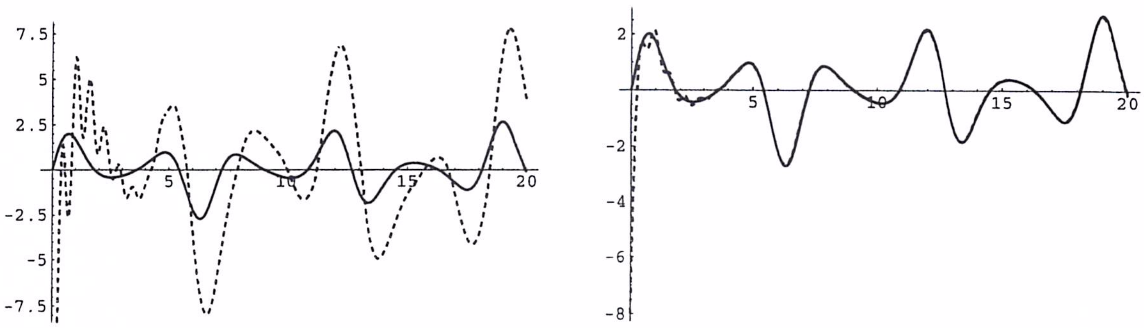
Let $\mu_1 = -\alpha$, $\mu_2 = \frac{-1-\sigma-\sqrt{(\sigma+1)^2-4\beta}}{2}$ and $\mu_3 = \frac{-1-\sigma+\sqrt{(\sigma+1)^2-4\beta}}{2}$ the eigenvalues of $A(\gamma)$. We see that they have negative real parts. Then, by Theorem 2.5, we obtain condition (iii) of Theorem 2.3. Condition (iv) is verified similarly to the Example 3.3. Therefore systems (3.6) and (3.7) satisfy conditions of Theorem 2.3, and then we conclude that these systems synchronize.

In Figure 3 and Figure 4 we present some simulations that show the synchronization and the codification of messages with these systems. In these simulations we used the following parameters and initial conditions values: $\alpha_1 = 7$, $\beta_1 = 100$, $\sigma_1 = 1/2$, $a_1 = 8/7$, $b_1 = 5/7$, $\alpha_2 = 7 + 1/100$, $\beta_2 = 100 + 1/100$, $\sigma_2 = 1/2 + 1/100$, $a_2 = 8/7 + 1/100$, $b_2 = 5/7 + 1/100$, $m(t) = e^{\cos(t)} \sin(\sqrt{3}t)$, $\bar{m}(t) = e^{\cos(t)} \sin(\sqrt{3}t) + (1/100)\cos(2t)$, $x(0) = -15$, $y(0) = -9$, $z(0) = 8$, $u(0) = -7$, $v(0) = -8$ and $w(0) = 0$.



(a) $|x(t)-u(t)|+|y(t)-v(t)|+|z(t)-w(t)|$ (b) $(x(t), y(t))$ = —, $(u(t), v(t))$ = - - -

FIGURE 3. **Synchronization of Chua's Like System**



(a) $m(t)$ = —, $x(t) + m(t)$ = - - - (b) $m(t)$ = —, $m_r(t)$ = - - -

FIGURE 4. **Codification of Messages with Chua's Like System**

Example 3.3 (Excited Duffing's Like System). Consider the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega x - cy - q\big(h(x)\big)^3 - rh(x)\cos(t) \end{cases} \tag{3.8}$$

where $h : \mathbb{R} \to \mathbb{R}$ is bounded and Lipschitz on bounded sets. Let $m, \bar{m} \in C_K(\mathbb{R}, \mathbb{R})$, then from the system above we construct the systems

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega_1 x - c_1 y - q_1 \Big(h\big(x + m(t)\big)\Big)^3 - r_1 h\big(x + m(t)\big)\cos(t) \end{cases} \tag{3.9}$$

$$\begin{cases} \dot{u} = v \\ \dot{v} = -\omega_2 u - c_2 v - q_2 \Big(h\big(x + \bar{m}(t)\big)\Big)^3 - r_2 h\big(x + \bar{m}(t)\big)\cos(t) \end{cases} \tag{3.10}$$

where $(\omega_1, c_1, q_1, r_1), (\omega_2, c_2, q_2, r_2) \in \Gamma$ and

$$\Gamma := \Big\{(\omega, c, q, r) \in \mathbb{R}^4 : \omega_m \le \omega \le \omega_M, \quad c_m \le c \le c_M, \quad q_m \le q \le q_M \ \text{ e } \ r_m \le r \le r_M\Big\},$$

with $0 < \omega_m \le \omega_M$, $0 < c_m \le c_M$, $0 < q_m \le q_M$ e $0 < r_m \le r_M$. Let $\Lambda = \Gamma \times C_K(\mathbb{R}, \mathbb{R})$ and $\lambda = (\gamma, m(t)) \in \Lambda$, where $\gamma = (\omega, c, q, r) \in \Gamma$.

Now we will show that systems (3.9) and (3.10) satisfy conditions of Theorem 2.1. Let $L \ge 0$ such that $|h(x)| \le L$. To system (3.9) we consider the following Liapunov function

$$V(x, y, \lambda_1) = (2\omega_1 + c_1^2)x^2 + 2y^2 + 2c_1 xy,$$

where $\lambda_1 = (\omega_1, c_1, q_1, r_1, m(t))$.

Then we have

$$V(x, y, \lambda_1) \ge 2\omega_m x^2 + y^2 =: a(x, y).$$

$$V(x, y, \lambda_1) \le (2\omega_M + c_M^2)x^2 + 2y^2 + 2c_M|x||y| =: b(x, y).$$

We also have

$$\begin{aligned} -\dot{V}(x, y, \lambda_1) = 2c_1\omega_1 x^2 + 2c_1 y^2 + 4q_1 y\Big(h\big(x + m(t)\big)\Big)^3 + 4r_1 y h\big(x + m(t)\big)\cos(t) + \\ 2c_1 q_1 x\Big(h\big(x + m(t)\big)\Big)^3 + 2c_1 r_1 x h\big(x + m(t)\big)\cos(t) \ge \\ 2c_m\omega_m x^2 + 2c_m y^2 - 2c_M L(q_M L^2 + r_M)|x| - 4L(q_M L^2 + r_M)|y| =: c(x, y). \end{aligned}$$

It is clear that the functions a, b e c satisfy conditions of Theorem 2.1. Therefore there exists a bounded set $\mathcal{B}_1 \subset \mathbb{R}^2$ such that, if $\big(x(t), y(t)\big)$ is a solution of system (3.9), then there exists t_1 such that $\big(x(t), y(t)\big) \in \mathcal{B}_1$ for every $t \ge t_1$.

In the same way, for system (3.10) we consider $V(u,v,\lambda_2)$ as a Liapunov where $\lambda_2 = (\omega_2, c_2, q_2, r_2, \bar{m}(t))$. Then

$$a(u,v,w) \le V(u,v,w,\lambda_2) \le b(u,v,w) \quad \text{and} \quad -\dot{V}(u,v,w,\lambda_2) \ge c(u,v,w).$$

Therefore there exists a bounded set $\mathcal{B}_2 \subset \mathbb{R}^2$ such that, if $\big(u(t),v(t)\big)$ is a solution of system (3.10), then there exists t_2 such that $\big(u(t),v(t)\big) \in \mathcal{B}_2$ for every $t \ge t_2$.

Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ and $t_3 = \max\{t_1,t_2\}$. Then for every solution $\big(x(t),y(t)\big)$ of (3.9) and every solution of $\big(u(t),v(t)\big)$ of (3.10), we have

$$\Big(\big(x(t),y(t)\big),\big(u(t),v(t)\big)\Big) \in \mathcal{B}\times\mathcal{B}$$

for $t \ge t_3$.

Therefore systems (3.9) and (3.10) satisfy condition (i) of Theorem 2.3. Let us show that these systems satisfy the remaining conditions of Theorem 2.3. Let

$$f(x,y,u,v,\lambda) = \begin{pmatrix} y \\ -\omega x - cy - q\Big(h\big(x+m(t)\big)\Big)^3 - rh\big(x+m(t)\big)\cos(t) \end{pmatrix}$$

and

$$g(x,y,u,v,\lambda) = \begin{pmatrix} v \\ -\omega u - cv - q\Big(h\big(x+m(t)\big)\Big)^3 - rh\big(x+m(t)\big)\cos(t) \end{pmatrix}.$$

Then we have

$$g(x,y,u,v,\lambda) - f(x,y,u,v,\lambda) = \begin{pmatrix} 0 & 1 \\ -\omega & -c \end{pmatrix}\begin{pmatrix} u-x \\ v-y \end{pmatrix}.$$

Therefore systems (3.9) and (3.10) satisfy condition (ii) of Theorem 2.3 with

$$F(t,x,y,u,v,\lambda) = \begin{pmatrix} 0 & 1 \\ -\omega & -c \end{pmatrix} =: A(\gamma).$$

Let $\mu_1(\gamma) := \frac{-c+\sqrt{c^2-4\omega}}{2}$ and $\mu_2(\gamma) := \frac{-c-\sqrt{c^2-4\omega}}{2}$ the eigenvalues of $A(\gamma)$. We see that have negative real part. Then, by Theorem 2.5, we obtain the condition (iii) of Theorem 2.3, with

$$0 < \beta < \min_{\gamma\in\Gamma}\Big\{-\operatorname{Re}\big(\mu_1(\gamma)\big), -\operatorname{Re}\big(\mu_2(\gamma)\big)\Big\} = \frac{c_m}{2}.$$

Let us now verify the condition (iv). We have

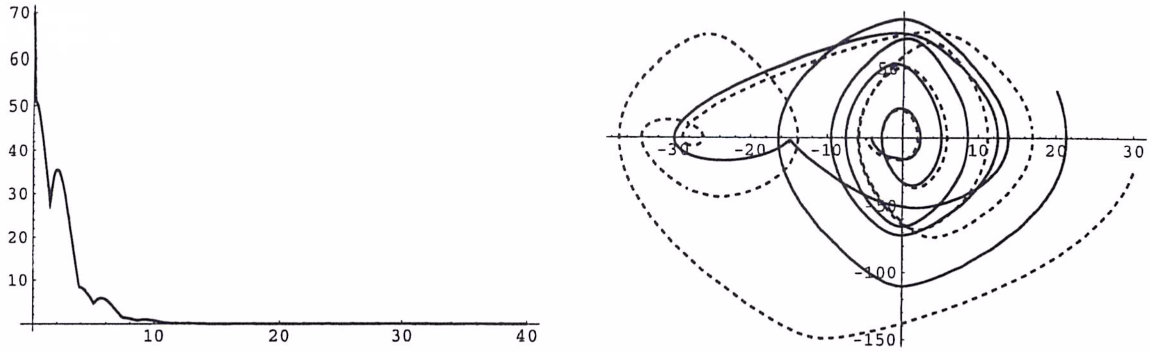
$$f(x,y,u,v,\lambda_2) - f(x,y,u,v,\lambda_1) =$$
$$\begin{pmatrix} 0 \\ -\Big(h\big(x+\bar{m}(t)\big) - h\big(x+m(t)\big)\Big)\left[\Big(h\big(x+\bar{m}(t)\big)\Big)^2 + \Big(h\big(x+m(t)\big)\Big)^2\right] q_1 \end{pmatrix} +$$
$$\begin{pmatrix} 0 \\ -\Big(h\big(x+\bar{m}(t)\big) - h\big(x+m(t)\big)\Big) h\big(x+\bar{m}(t)\big) h\big(x+m(t)\big) q_1 \end{pmatrix} +$$
$$\begin{pmatrix} 0 \\ -(\omega_2-\omega_1)x - (c_2-c_1)y - (q_2-q_1)\Big(h\big(x+\bar{m}(t)\big)\Big)^3 \end{pmatrix} +$$
$$\begin{pmatrix} 0 \\ -(r_2-r_1)h\big(x+\bar{m}(t)\big)\cos(t) - \Big(h\big(x+\bar{m}(t)\big) - h\big(x+m(t)\big)\Big) r_1 \cos(t) \end{pmatrix}.$$

Since the set $\mathcal{C} = \Big\{x + \alpha(t) \,:\, (x,y) \in \mathcal{B} \text{ and } \alpha \in C_K(\mathbb{R},\mathbb{R})\Big\}$ is bounded and h is Lipschitz in bounded sets, let τ be the Lipschitz constant of h in $\mathcal{C}$. Also, let $B \geq 0$ such that $\|(x,y)\| \leq B$ for $(x,y) \in \mathcal{B}$. Then we have

$$\|f(x,y,u,v,\lambda_2) - f(x,y,u,v,\lambda_1)\| \leq$$
$$\tau(3L^2 q_M + r_M)\|\bar{m} - m\| + |\omega_2 - \omega_1|B + |c_2 - c_1|B + |q_2 - q_1)|L^3 + |r_2 - r_1|L.$$

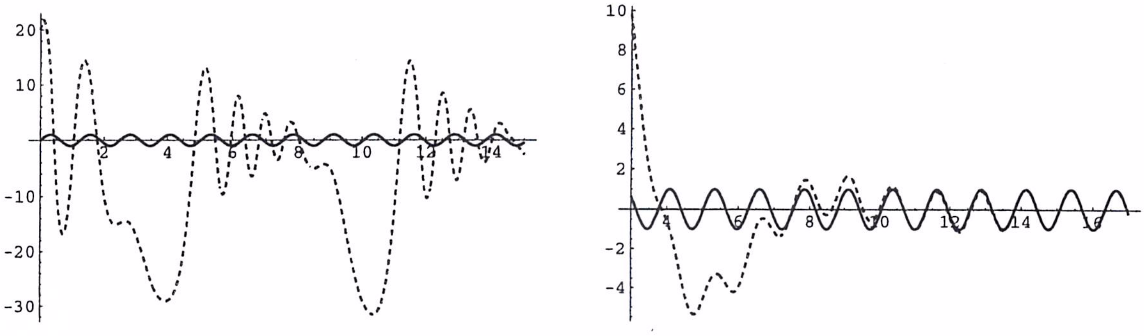
Therefore systems (3.9) and (3.10) satisfy conditions of Theorem 2.3, and then we conclude that these systems synchronize.

In Figure 5 and Figure 6 we present some simulations that show the synchronization and the codification of messages with these systems. In these simulations we used the following parameters and initial conditions values: $h(x) = \arctan(x)$, $\omega_1 = 1$, $c_1 = 1$, $q_1 = 50$, $r_1 = 150$, $\omega_2 = 1 + 1/1000$, $c_2 = 1 + 1/1000$, $q_2 = 50 + 1/1000$, $r_2 = 150 + 1/1000$, $m(t) = \sin(5t)$, $\bar{m}(t) = \sin(5t) + (1/1000)\cos(2t)$, $x(0) = 20$, $y(0) = 35$, $u(0) = 30$ and $v(0) = -25$.



(a) $|x(t) - u(t)| + |y(t) - v(t)|$ (b) $(x(t), y(t)) = —$, $(u(t), v(t)) = - - -$

FIGURE 5. **Synchronization of Excited Duffing's Like System**



(a) $m(t) = —$, $x(t) + m(t) = - - -$ (b) $m(t) = —$, $m_r(t) = - - -$

FIGURE 6. **Codification of Messages with Excited Duffing's Like System**

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Resumo

Neste trabalho apresentamos aplicações de resultados de sincronização ao estudo de codificação e decodificação de sinais, utilizando sistemas caóticos. Em um primeiro resultado abstrato, estabelecemos condições para que um sistema de equações diferenciais não autônomo seja uniformemente dissipativo, utilizando funcionais do tipo Liapunov. Estabelecemos a seguir dois teoremas que fornecem técnicas para o estudo de sincronização de um sistema não autônomo. Demonstramos que a sincronização é robusta com relação à variação de parâmetros. Aplicações a sistemas de comunicação utilizando equações de Lorenz, Duffing e Chua são discutidas.

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