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LOCAL ZETA FUNCTION FOR CURVES, NON DEGENERACY CONDITIONS AND NEWTON POLYGONS

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Resumo

Neste trabalho é feita uma descrição dos polos de funções locais zeta de Igusa $Z(s,f,v)$ quando $f(x,y)$ satisfaz uma nova condição de não degeneração, chamada *não degeneração aritmética*. Mais precisamente, para cada polinômio $f(x,y)$, é considerado um conjunto finito de polígonos convexos $\Gamma^A(f)=\{\Gamma_{f,1},..,\Gamma_{f,l_0}\}$ chamado de *polígono de Newton aritmético* associado a $f(x,y)$, e é introduzida a noção de *não degeneração aritmética* com respeito a $\Gamma^A(f)$. O conjunto de polinômios degenerados com respeito a um polígono de Newton fixado, no sentido de Kouchnirenko, contem um subconjunto aberto, com relação à topologia de Zariski, formado por polinômios não degenerados com respeito a algum polígono aritmético. Se L é um corpo, o principal resultado deste trabalho mostra que para quase todas a valorizações não-arquimedianas v de L, os polos de $Z(s,f,v)$, com $f(x,y)\in L[x,y]$, podem ser descritos explicitamente em termos das equaçẽs dos segmentos que determinam os polígonos $\Gamma_{f,1}$,.., Γ_{f,l_0}. Alem disto, é mostrado um procedimento para se obter explicitamente a função $Z(s,f,v)$.

Abstract

This paper is dedicated to the description of the poles of the Igusa local zeta functions $Z(s,f,v)$ when $f(x,y)$ satisfies a new non degeneracy condition, that we have called *arithmetic non degeneracy*. More precisely, we attach to each polynomial $f(x,y)$, a collection of convex sets $\Gamma^A(f)=\{\Gamma_{f,1},..,\Gamma_{f,l_0}\}$ called the *arithmetic Newton polygon* of $f(x,y)$, and introduce the notion of *arithmetic non degeneracy with respect to* $\Gamma^A(f)$. The set of degenerate polynomials, with respect to a fixed geometric Newton polygon, contains an open subset, for the Zariski topology, formed by non degenerate polynomials with respect to some arithmetic Newton polygon.If L is a number field, our main result asserts that for almost all non-archimedean valuations v of L, the poles of $Z(s,f,v)$, with $f(x,y)\in L[x,y]$, can be described explicitly in terms of the equations of the straight segments that conform the boundaries of the convex sets $\Gamma_{f,1}$,.., Γ_{f,l_0}. Moreover, our proof gives an effective procedure to compute $Z(s,f,v)$.

1. Introduction

Let L be a number field, v a non-archimedean valuation of L, and L_v the completion of L with respect to v. Let O_v the ring of integers of L_v, P_v the maximal ideal of O_v, and $\overline{L_v}=O_v/\ P_v$ the residue field of L_v. The cardinality of $\overline{L_v}$ is denoted by q, thus $\overline{L_v}=\mathbb{F}_q$. We fix a local parameter π for O_v. For $z\in L_v$, $\mid z\mid_v:=q^{-v(z)}$ denotes its normalized absolute value. Given a non-constant polynomial $g(x,y)\in L[x,y]$, and a non-archimedean valuation v of L, the Igusa local zeta function $Z(s,g,v)$ attached to $g(x,y)$ and v is defined as follows:

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$$Z(s,g,v) := \int_{O_v^2} \mid g(x,y) \mid_v^s \mid dxdy \mid,$$

where $s \in \mathbb{C}$ satisfying Re(s) > 0 and $\mid dxdy \mid$ denotes a Haar measure of L_v^2 normalized so that the measure of O_v^2 is one.

Igusa showed that the local zeta function $Z(s,g,v)$ is a rational function of q^{-s}, for general polynomials with coefficients in a $p-$adic field [5]. Igusa's local zeta functions are related to the number of solutions of congruences modulo p^m and to exponential modulo p^m. There are several conjectures and intriguing connections between Igusa's local zeta functions with topology and singularity theory [2].

This paper is dedicated to the description of the poles of the Igusa local zeta functions $Z(s,f,v)$ when $f(x,y)$ satisfies a new non degeneracy condition, that we have called *arithmetic non degeneracy.* More precisely, we attach to each polynomial $f(x,y)$, a collection of convex sets $\Gamma^A(f) = \{\Gamma_{f,1},..,\Gamma_{f,l_0}\}$ called the *arithmetic Newton polygon* of $f(x,y)$, and introduce the notion of *arithmetic non degeneracy with respect to* $\Gamma^A(f)$ (see section 4).

The set of degenerate polynomials, with respect to a fixed geometric Newton polygon, contains an open subset, for the Zariski topology, formed by non degenerate polynomials with respect to some arithmetic Newton polygon.

Our main result asserts that for almost all non-archimedean valuations v of L, the poles of $Z(s,f,v)$, with $f(x,y) \in L[x,y]$, can be described explicitly in terms of the equations of the straight segments that conform the boundaries of the convex sets $\Gamma_{f,1},.., \Gamma_{f,l_0}$ (see theorems 6.1, 6.2). Moreover, our proof gives an effective procedure to compute $Z(s,f,v)$.

The description of the poles of the local zeta functions in terms of Newton polygons is a well known fact for non degenerate polynomials in the sense of Kouchnirenko [13], [10], [3], [4], [16], [17]. However, in the degenerate case there are no previous results showing a relation between the poles of local zeta functions and Newton polygons.

A second approach to the description of the poles of local zeta functions is based on resolution of singularities. Currently, as a consequence of the work of many people, among them, Veys [14], [15], Meuser [11], Igusa [7], Strauss [12], there is a completely description of the poles in terms of resolution of singularities for arbitrary polynomials in two variables. As far as the authors understand the main result of this paper is complementary to those mentioned results. Indeed, our approach does not require the computation of a resolution of singularities, and it gives an effective procedure to compute the corresponding local zeta function. However, our approach is only valid for a certain class of polynomials, and not for all polynomials.

The notion of non degeneracy due Kouchnirenko is useful in the computation of Milnor numbers, number of isolated solutions of polynomial equations, etc., [9]. The authors hope that the notion of arithmetic non degeneracy may be useful for other purposes, like those above mentioned, different from the description of the poles of local zeta functions.

In this paper, we work with $p-$adic fields of characteristic zero, however all results are valid in positive characteristic.

The plan of this paper is as follows. In the section 2, we review some well-known results about geometric Newton polygons, non degeneracy in the sense of Kouchnirenko, and computation of certain $p-$adic integrals attached to cones and non degenerate polynomials, in the sense of Kouchnirenko. In section 3, we compute explicitly the local zeta function of some degenerate polynomials, in the sense of Kouchnirenko. These examples constitute the basic models for the effective computation of the local zeta functions of arithmetically non degenerate polynomials. In section 4, we introduce the notion of arithmetic Newton polygon and arithmetic non-degeneracy condition. In section 5, we compute the several $p-$adic integrals that appears in the effective computation of the local zeta function of an arithmetically non degenerate polynomial. In section 6, we prove the main result.

2. Geometric Newton polygons

In this section we review some well-known results and notions about Newton polygons, Kouchnirenko's non degeneracy condition, and the computation of certain p−adic integrals attached to cones and non degenerate (in the sense of Kouchnirenko) polynomials. All these notions will be used along this paper.

We set $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geqq 0\}$. Let $f(x,y) = \sum_{(i,j)} a_{i,j}x^iy^j \in K[x,y]$, be a non-constant polynomial in two variables with coefficients in a field K, satisfying $f(0,0)=0$. The set

$$supp(f) = \{(i,j) \in \mathbb{N}^2 \mid a_{i,j} \neq 0\}$$

is called the *support* of f. *The "geometric" Newton polygon* $\Gamma^{geom}(f)$ of f is defined as the convex hull in $\mathbb{R}_+^2$ of the set

$$\bigcup_{(i,j)\in supp(f)} \left((i,j) + \mathbb{R}_+^2\right).$$

By a *proper face* γ of $\Gamma^{geom}(f)$, we mean a non-empty convex set γ obtained by intersecting $\Gamma^{geom}(f)$ with an straight line H, such that $\Gamma^{geom}(f)$ is contained in one of two half-spaces determined by H. The straight line H is named *the supporting line* of γ. The faces are points (zero-dimensional faces) or straight segments (one-dimensional faces). The last ones are also called *facets.*

Given a face $\gamma \subseteq \Gamma^{geom}(f)$ we set

$$f_\gamma(x,y) := \sum_{(i,j)\in\gamma} a_{i,j}x^iy^j.$$

Definition 2.1. *A non-constant polynomial $f(x,y)$ satisfying $f(0,0)=0$ is named non degenerate with respect to $\Gamma^{geom}(f)$, in the sense Kouchnirenko, if it satisfies:*

1. *the origin of K^2 is a singular point of $f(x,y)$;*
2. *for each proper face $\gamma \subseteq \Gamma^{geom}(f)$, including $\Gamma^{geom}(f)$ itself, the system of equations*

$$f_\gamma(x,y) = \frac{\partial f_\gamma}{\partial x}(x,y) = \frac{\partial f_\gamma}{\partial y}(x,y) = 0,$$

does not have solutions on $K^{\times 2}$.

We set $<,>$ for the usual inner product in $\mathbb{R}^2$, and identify the dual vector space with $\mathbb{R}^2$. For $a \in \mathbb{R}_+^2$, we define

$$m(a) := \inf_{x\in\Gamma(f)}\{<a,x>\}.$$

The first meet locus of $a \in \mathbb{R}_+^2 \smallsetminus \{0\}$ is defined by

$$F(a) := \{x \in \Gamma^{geom}(f) \mid <a,x>= m(a)\}.$$

The first meet locus $F(a)$ of a is a proper face of $\Gamma^{geom}(f)$.

Given a face γ of $\Gamma^{geom}(f)$, the *cone associated* to γ is defined to be the set

$$\Delta_\gamma := \left\{a \in (\mathbb{R}_+)^2 \smallsetminus \{0\} \mid F(a) = \gamma\right\}.$$

We define an equivalence relation on $\mathbb{R}_+^2 \smallsetminus \{0\}$ by

$$a \simeq a' \text{ if and only if } F(a) = F(a').$$

The following two propositions describe the geometry of the equivalences classes of $\simeq$ (see e.g. [13], [3]).

Proposition 2.1. *Let γ be a proper face of $\Gamma^{geom}(f)$. If γ is a one-dimensional face and a is perpendicular vector on γ, then*

$$\Delta_\gamma = \{\alpha a \mid \alpha \in \mathbb{R},\ \alpha > 0\}. \tag{2.1}$$

If γ is a zero-dimensional face and w_1, w_2 are the facets of $\Gamma^{geom}(f)$ which contain γ, and a_1, a_2 are vectors which are perpendicular on respectively w_1, w_2, then

$$\Delta_\gamma = \{\alpha_1 a_1 + \alpha_2 a_2 \mid \alpha_i \in \mathbb{R},\ \alpha_i > 0\}. \tag{2.2}$$

A set of the form (2.2) (respectively (2.1)) is called a *cone strictly positive spanned* by the vectors $\{a_1, a_2\}$ (respectively by $\{a\}$). Let Δ be a strictly positive spanned by a vector set A, with $A = \{a_1, a_2\}$, or $A = \{a\}$. If A is linearly independent over $\mathbb{R}$, the cone Δ is called *a simplicial cone.* In this last case, if the elements of A are in $\mathbb{Z}^2$, the cone Δ is called a *rational simplicial* cone. If A can be completed to be a basis of $\mathbb{Z}-$module $\mathbb{Z}^2$, the cone Δ is named *a simple cone.*

A vector $a \in \mathbb{R}^2$ is called *primitive* if the components of a are positive integers whose greatest common divisor is one.

For every facet of $\Gamma^{geom}(f)$ there is a unique primitive vector in $\mathbb{R}^2$ which is perpendicular to this facet. Let D be the set of all these vectors. Thus each equivalence class under $\simeq$ is a cone strictly positively spanned by vectors of D.

From all above, we have that there exists a partition of $\mathbb{R}^2$ of the following form:

$$\mathbb{R}^2 = \{(0,0)\} \bigcup \bigcup_{\gamma \subset \Gamma^{geom}(f)} \Delta_\gamma, \tag{2.3}$$

where γ runs through all proper faces of $\Gamma^{geom}(f)$, and Δ_γ denotes the rational simplicial cone associated to the face γ.

2.1. Local zeta functions of non degenerate polynomials.

Let $f(x,y) \in L_v[x,y]$ be a non-constant, non degenerate polynomial with respect to its geometric Newton polygon $\Gamma^{geom}(f)$. Along this subsection we shall suppose that $\Gamma^{geom}(f)$ has only a compact facet, whose supporting line is

$$\{(x,y) \in \mathbb{R}^2 \mid ax + by = d_0\}.$$

Then the polynomial $f(x,y)$ admits a decomposition of the form $f(x,y) = f_0(x,y) + f_1(x,y)$, where $f_0(x,y)$ is weighted homogeneous with weighted degree d_0, with respect to the weight (a,b), a, b coprime; and all monomials of $f_1(x,y)$ have weighted degree higher than d_0.

In this particular case, $\Gamma^{geom}(f)$ induces on $\mathbb{R}^2$ the following partition:

$$\mathbb{R}^2 = \{(0,0)\} \bigcup \bigcup_{i=1}^{5} \Delta_i, \tag{2.4}$$

with

$$\begin{array}{ll} \Delta_1 := (0,1)\mathbb{R}_+, & \Delta_2 := (0,1)\mathbb{R}_+ + (a,b)\mathbb{R}_+, \\ \Delta_3 := (a,b)\mathbb{R}_+, & \Delta_4 := (a,b)\mathbb{R}_+ + (1,0)\mathbb{R}_+, \\ \Delta_5 := (1,0)\mathbb{R}_+. & \end{array}$$

The polynomial $f(x,y)$ is non degenerate with respect to $\Delta_i, i = 1,2,4,5$, i.e. if $\gamma_i, i = 1,2,4,5$, are the corresponding first meet locus, then the polynomials $f_{\gamma_i}, i = 1,2,4,5$, do not have singular points on $L_v^{\times 2}$, for $i = 1,2,4,5$.

We define

$$E_i := \{(x,y) \in O_v^2 \mid (v(x), v(y)) \in \Delta_i\},$$

$$Z(s,f,v,\Delta_i) := \int\limits_{E_i} \mid f(x,y) \mid_v^s \mid dxdy \mid, \ i = 1,2,3,4,5,$$

and

$$Z(s,f,v,O_v^{\times 2}) := \int\limits_{O_v^{\times 2}} \mid f(x,y) \mid_v^s \mid dxdy \mid .$$

With all the above notation, and partition (2.4), it follows that

$$Z(s,f,v) = Z(s,f,v,O_v^{\times 2}) + \sum_{i=1}^{5} Z(s,f,v,\Delta_i). \tag{2.5}$$

The computation of the integral $Z(s,f,v,O_v^{\times 2})$ is well-known. It can be computed by using the $p-$adic stationary phase formula, abbreviated SPF ([5], theorem 10.2.1).We formally summarize, in the case of two variables, this fact as follows.

Lemma 2.1. *Let $\overline{E}$ be a subset of $\mathbb{F}_q^2$ and $\overline{S}(f) \subseteq \overline{E}$ a subset consisting of all $\overline{z} \in \overline{E}$ such that $\overline{f}(\overline{z}) = \frac{\partial \overline{f}}{\partial x}(\overline{z}) = \frac{\partial \overline{f}}{\partial y}(\overline{z}) = 0$. Let $E, S(f)$ be the preimages of $\overline{E}$, $\overline{S}(f)$ under $O_v \to O_v/\pi O_v \cong \mathbb{F}_q$ and $N(f,E)$ the number of zeros of $\overline{f}(\overline{x})$ in $\overline{E}$. Then it holds that*

$$\int\limits_{E} \mid f(x,y) \mid_v^s \mid dxdy \mid = q^{-2}\left(Card\{\overline{E}\} - N(f,E)\right)$$

$$+q^{-2}\left(N(f,E) - Card\{\overline{S}(f)\}\frac{(1-q^{-1})q^{-s}}{1-q^{-1-s}}\right) + \int\limits_{S} \mid f(x,y) \mid_v^s \mid dxdy \mid .$$

By using an approach based on SPF, it is possible to compute explicitly several classes of local zeta functions ([5], [16], [17] and the references therein).

In the case in which the polynomial $f(x,y)$ does not have singularities on L_v^2, by applying SPF repeatedly, it is possible to compute effectively the corresponding local zeta function (see lemma 3.1 in [16], or lemma 2.4 in [17]). We summarize formally this result as follows.

Lemma 2.2. *Let $f(x,y) \in L_v[x,y]$ be a non-constant polynomial such that it does not have singular points on L_v^2. Then*

$$Z(s,f,v) = \frac{U(q^{-s})}{1-q^{-1-s}},$$

with $U(q^{-s}) \in \mathbb{Q}[q^{-s}]$.

The computation of the integrals $Z(s,f,v,\Delta_i)$, $i = 1,2,4,5$, is also well-known. There are basically three different forms to study the poles of $Z(s,f,v,\Delta)$, when f is non degenerate with respect Δ. The first approach due to Lichtin and Meuser [10] uses toroidal resolution of singularities; this approach is based on Varchenko's approach to estimating oscillatory integrals with non degenerate phase [13]. The second approach due to Denef uses monomial changes of variables [3] (see also [4]) and the last one approach is due to second author and it is based on SPF formula [16], [17]. We formally summarize, in the case of two variables, the mentioned result.

Lemma 2.3. *Let $f(x,y) \in L_v[x,y]$ be a non-constant polynomial, with $f(0,0) = 0$, and $\Gamma^{geom}(f)$ its geometric Newton polygon. $\Gamma^{geom}(f)$ has three facets γ_1, γ_2, γ_3. We set $y = d_{\gamma_1}$ for the equation of the supporting line of γ_1; $ax + by = d_{\gamma_2}$, with a,b coprime, for the equation of the supporting line of γ_2; and $x = d_{\gamma_3}$ for the equation of the supporting line of γ_3.*

1. *If $f_{\gamma_1}(x,y)$ does not have singularities on the torus $L_v^{\times 2}$, then*

$$Z(s,f,v,\Delta_1)=\frac{L_1(q^{-s})}{(1-q^{-1-s})(1-q^{-1-d_{\gamma_1}s})},\ L_1(q^{-s})\in\mathbb{Q}[q^{-s}].$$

2.

$$Z(s,f,v,\Delta_2)=\frac{L_2(q^{-s})}{(1-q^{-1-s})(1-q^{-1-d_{\gamma_1}s})(1-q^{-(a+b)-d_{\gamma_2}s})},$$
$$L_3(q^{-s})\in\mathbb{Q}[q^{-s}].$$

3.

$$Z(s,f,v,\Delta_4)=\frac{L_4(q^{-s})}{(1-q^{-1-s})(1-q^{-1-d_{\gamma_3}s})(1-q^{-(a+b)-d_{\gamma_2}s})},$$
$$L_4(q^{-s})\in\mathbb{Q}[q^{-s}].$$

4. *If $f_{\gamma_3}(x,y)$ does not have singularities on the torus $L_v^{\times 2}$, then*

$$Z(s,f,v,\Delta_5)=\frac{L_1(q^{-s})}{(1-q^{-1-s})(1-q^{-1-d_{\gamma_3}s})},\ L_5(q^{-s})\in\mathbb{Q}[q^{-s}].$$

The core of this paper is the explicit computation of integrals of same type that $Z(s,f,v,\Delta_3)$, when $f(x,y)$ is degenerate in the sense of Kouchnirenko, but satisfies a new non degeneracy condition. This will be done in sections 4, 5.

Remark 2.1. Given a Δ be a rational simplicial cone, there exists a partition of Δ into simple cones ([8], p. 32-33). This fact can be used to obtain a partition of $\mathbb{R}_+^2$ in simple cones. In this case, it is possible to compute the integral $Z(s,f,v,\Delta)$ in a simple form,when f is non degenerate, in the sense of Kouchnirenko. However this procedure produces a larger list of candidates to poles of $Z(s,f,v,\Delta)$. We shall illustrate this procedure in the next section when we shall compute explicitly the local zeta function of $(y^3-x^2)^2+x^4y^4$, and $(y^3-x^2)^2(y^3-ax^2)+x^4y^4$, $a\neq 1$.

3. Model cases

In this section we shall compute explicitly the local zeta functions for some degenerate polynomials, in the sense of Kouchnirenko. These examples constitute the basic models for the effective computation of the local zeta functions of arithmetically non degenerate polynomials.

3.1. **The local zeta function of $(y^3-x^2)^2+x^4y^4$.** We assume that the characteristic of the residue field of L_v is different of 2. The origin of L_v^2 is the only singular point of $f(x,y)$.This polynomial is degenerate with respect to its geometric Newton polygon.

By using a decomposition of $\mathbb{R}_+^2$ into rational simple cones, it is possible to reduce the computation of $Z(s,f,v)$ to the computation of $p-$adic integrals on rational simple cones, as it was explained in subsection 2.1. The following is the conic decomposition of $\mathbb{R}_+^2$ into rational simple cones, obtained from the geometric Newton polygon of $f(x,y)$:

$$\mathbb{R}^2=\{(0,0)\}\bigcup\bigcup_{j=1}^{9}\Delta_j, \tag{3.1}$$

with

(3.2)

$\Delta_1 := (0,1)\mathbb{R}_+,$	$E_1 = \{(0,1)n, n \in \mathbb{N}\setminus\{0\}\},$
$\Delta_2 := ((0,1)\mathbb{R}_+ + (1,1)\mathbb{R}_+),$	$E_2 = \{(0,1)n + (1,1)m,\ \ n,m \in \mathbb{N}\setminus\{0\}\},$
$\Delta_3 := (1,1)\mathbb{R}_+,$	$E_3 = \{(1,1)n,\ n \in \mathbb{N}\setminus\{0\}\},$
$\Delta_4 := ((1,1)\mathbb{R}_+ + (3,2)\mathbb{R}_+),$	$E_4 = \{(1,1)n + (3,2)m,\ \ n,m \in \mathbb{N}\setminus\{0\}\},$
$\Delta_5 =: (3,2)\mathbb{R}_+,$	$E_5 = \{(3,2)n,\ n \in \mathbb{N}\setminus\{0\}\},$
$\Delta_6 := ((3,2)\mathbb{R}_+ + (2,1)\mathbb{R}_+),$	$E_6 = \{(3,2)n + (2,1)m,\ \ n,m \in \mathbb{N}\setminus\{0\}\},$
$\Delta_7 := (2,1)\mathbb{R}_+,$	$E_7 = \{(2,1)n,\ n \in \mathbb{N}\setminus\{0\}\},$
$\Delta_8 := ((2,1)\mathbb{R}_+ + (1,0)\mathbb{R}_+),$	$E_8 = \{(2,1)n + (1,0)m,\ \ n,m \in \mathbb{N}\setminus\{0\}\}.$
$\Delta_9 := (1,0)\mathbb{R}_+,$	$E_9 = \{(1,0)n,\ \ n \in \mathbb{N}\setminus\{0\}\}.$

By the considerations done in subsection 2.1, and with the notation introduced there; it is sufficient to compute the integrals

$$Z(s,f,v,O_v^{\times 2}),\quad Z(s,f,v,\Delta_i), i = 1,..,9.$$

3.2. **Computation of $Z(s,f,v,O_v^{\times 2})$.** Since $f(x,y)$ does not have singularities on $L_v^{\times 2}$, the integral can be computed by using SPF.

3.3. **Computation of $Z(s,f,v,\Delta_i)$, $i = 1,2,3,4,6,7,8,9$.** These integrals correspond to the case in which f is non degenerate, in the sense of Kouchnirenko, with respect Δ_i, $i = 1, 2, 3, 4, 6, 7, 8, 9$. We compute explicitly the integrals corresponding to Δ_i, $i = 1,2$, as follows.

3.3.1. *Case* $Z(s,f,v,\Delta_1)$.

$$\begin{aligned} Z(s,f,v,\Delta_1) &= \sum_{n=1}^{\infty} \int_{O_v^\times \times \pi^n O_v^\times} \mid f(x,y) \mid_v^s \mid dxdy \mid \\ &= \sum_{n=1}^{\infty} q^{-n} \int_{O_v^{\times 2}} \mid (\pi^{3n}y^3 - x^2)^2 + \pi^{4n}x^4y^4 \mid_v^s \mid dxdy \mid \\ &= (1-q^{-1})^2 \sum_{n=1}^{\infty} q^{-n} = q^{-1}(1-q^{-1}). \end{aligned}$$

3.3.2. *Case* $Z(s,f,v,\Delta_2)$.

$$Z(s,f,v,\Delta_2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \int_{\pi^m O_v^\times \times \pi^{n+m} O_v^\times} \mid f(x,y) \mid_v^s \mid dxdy \mid =$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{-2m-n-4ms} \int_{O_v^{\times 2}} \mid (\pi^{3n+m}y^3 - x^2)^2 + \pi^{4m+4n}x^4y^4 \mid_v^s \mid dxdy \mid$$

$$= (1-q^{-1})q^{-1}\frac{q^{-2-4s}}{(1-q^{-2-4s})}.$$

The other integrals can be computed in the same form. The following table summarizes these computations.

(3.3)

Cone	$Z(s,f,v,\Delta)$
Δ_1	$q^{-1}(1-q^{-1})$
Δ_2	$(1-q^{-1})q^{-1}\frac{q^{-2-4s}}{(1-q^{-2-4s})}$
Δ_3	$(1-q^{-1})^2\frac{q^{-2-4s}}{(1-q^{-2-4s})}$
Δ_4	$(1-q^{-1})^2\frac{q^{-7-16s}}{(1-q^{-2-4s})(1-q^{-5-12s})}$
Δ_6	$\frac{(1-q^{-1})^2 q^{-8-18s}}{(1-q^{-5-12s})(1-q^{-3-6s})}$
Δ_7	$\frac{(1-q^{-1})^2 q^{-3-6s}}{(1-q^{-3-6s})}$
Δ_8	$\frac{(1-q^{-1}) q^{-4-6s}}{(1-q^{-3-6s})}$
Δ_9	$q^{-1}(1-q^{-1})$

3.4. **Computation of $Z(s,f,v,\Delta_5)$ (an integral on a degenerate face in the sense of Kouchnirenko).**

$$\begin{aligned} Z(s,f,v,\Delta_5) &= \sum_{n=1}^{\infty} \int_{\pi^{3n}O_v^\times \times \pi^{2n}O_v^\times} \mid f(x,y) \mid_v^s \mid dxdy \mid \\ &= \sum_{n=1}^{\infty} q^{-5n-12ns} \int_{O_v^{\times 2}} \mid (y^3-x^2)^2 + \pi^{8n}x^4y^4 \mid_v^s \mid dxdy \mid . \end{aligned} \tag{3.4}$$

In order to compute (3.4), we first compute the integral

$$I(s,f^{(n)},v) := \int_{O_v^{\times 2}} \mid (y^3-x^2)^2 + \pi^{8n}x^4y^4 \mid_v^s \mid dxdy \mid , \; n \geqslant 1. \tag{3.5}$$

For that, we use the following change of variables

$$\begin{array}{cccc} \Phi : & O_v^{\times 2} & \longrightarrow & O_v^{\times 2} \\ & (x,y) & \longrightarrow & (x^3y, x^2y). \end{array} \tag{3.6}$$

The map Φ gives an analytic bijection of $O_v^{\times 2}$ onto itself, and preserves the Haar measure because its Jacobian $J_\Phi(x,y) = x^4y$ satisfies $\mid J_\Phi(x,y) \mid_v = 1$, for every $x, y \in O_v^\times$. Thus, if we set

$$f^{(n)} \circ \Phi(x,y) = x^{12}y^4 \widetilde{f^{(n)}}(x,y), \tag{3.7}$$

with

$$\widetilde{f^{(n)}}(x,y) := (y-1)^2 + \pi^{8n}x^8y^4, \tag{3.8}$$

then from (3.6), and (3.7), it follows that

$$I(s,f^{(n)},v) = \int_{O_v^{\times 2}} \mid \widetilde{f^{(n)}}(x,y) \mid_v^s \mid dxdy \mid . \tag{3.9}$$

In order to compute the integral $I(s,f^{(n)},v)$, $n \geqslant 1$, we decompose $O_v^{\times 2}$ as follows

$$O_v^{\times 2} = \bigcup_{y_0 \not\equiv 1 \bmod \pi} O_v^\times \times \{y_0 + \pi O_v\} \bigcup O_v^\times \times \{1 + \pi O_v\}, \tag{3.10}$$

where y_0 runs through a set of representatives in $O_v^\times$ of $\mathbb{F}_q^\times$. From the partition (3.10), and (3.8), it follows that

$$I(s,f^{(n)},v)=(q-2)q^{-1}(1-q^{-1})+\int\limits_{O_v^\times\times\{1+\pi O_v\}} |\widetilde{f^{(n)}}(x,y)|_v^s| \,dxdy\,| . \tag{3.11}$$

The integral in the right side of (3.11), admits the following expansion:

$$I(s,f^{(n)},v)=(q-2)q^{-1}(1-q^{-1})+\sum_{j=1}^{\infty}q^{-j}\int\limits_{O_v^{\times 2}} |\widetilde{f^{(n)}}(x,1+\pi^j z)|_v^s| \,dxdz\,|, \tag{3.12}$$

where $\widetilde{f^{(n)}}(x,1+\pi^j z)=\pi^{2j}z^2+\pi^{8n}x^8(1+\pi^j z)^4$. In the next step, we compute explicitly $|\widetilde{f^{(n)}}(x,1+\pi^j z)|_v^s$, for a fixed $n\geqq 1$. In this case, this can be done by a simple calculation,

$$|\widetilde{f^{(n)}}(x,1+\pi^j z)|_v^s=\begin{cases} q^{-2js}, & \text{if } 1\leqq j\leqq 4n-1,\\ q^{-8ns}\,|z^2+x^8(1+\pi^j z)^4|_v^s, & \text{if } j=4n,\\ q^{-8ns}(1-q^{-1})^2, & \text{if } j\geqq 4n+1,\end{cases} \tag{3.13}$$

for any $(x,z)\in O_v^{\times 2}$.

We note that the explicit computation of the absolute value of $|\widetilde{f^{(n)}}(x,1+\pi^j z)|_v$ depends on an explicit description of the set $\{(w,z)\in\mathbb{R}^2 \mid w\leqq \min\{2z,8n\}\}$, for a fixed n.

We shall see in section 4 that this kind of set is the basis to construct the arithmetic Newton polygon associated to $f(x)$.

Now from (3.12), (3.13) , and the identity $\sum_{k=A}^{B} z^k=\frac{z^A-z^{B+1}}{1-z}$, we obtain an explicit expression for $I(s,f_n,v)$:

$$\begin{aligned} I(s,f^{(n)},v) &= (q-2)q^{-1}(1-q^{-1})+(1-q^{-1})^2\frac{q^{-1-2s}}{(1-q^{-1-2s})}\\ &\quad -(1-q^{-1})^2\frac{q^{-4n-8ns}}{(1-q^{-1-2s})}+q^{-4n-8ns}I_1(s)\\ &\quad +q^{-1}(1-q^{-1})q^{-4n-8ns}, \end{aligned} \tag{3.14}$$

where

$$I_1(s):=\int\limits_{O_v^{\times 2}} |z^2+x^8(1+\pi^j z)^4|_v^s| \,dxdz\,| . \tag{3.15}$$

Since the reduction modulo π of the polynomial $z^2+x^8(1+\pi^j z)^4$ does not have singular points on $\mathbb{F}_q^{\times 2}$ (the characteristic of the residue field is different of 2), SPF implies that $I_1(s)=\frac{U_1(q^{-s})}{(1-q^{-1-s})}$, where $U_1(q^{-s})$ is a polynomial in q^{-s} independent of j and n.

Finally, from (3.4), (3.9) and (3.14), we obtain the following explicit expression for $Z(s,f,v,\Delta_5)$:

$$\begin{aligned} Z(s,f,v,\Delta_5) &= \frac{(q-2)(1-q^{-1})q^{-6-12s}}{(1-q^{-5-12s})}+\frac{(1-q^1)^2q^{-6-14s}}{(1-q^{-1-2s})(1-q^{-5-12s})}\\ &\quad -\frac{(1-q^1)^2q^{-9-20s}}{(1-q^{-1-2s})(1-q^{-9-20s})}+\frac{q^{-9-20s}U_1(q^{-s})}{(1-q^{-1-s})(1-q^{-9-20s})}\\ &\quad +\frac{(1-q^{-1})q^{-10-20s}}{(1-q^{-9-20s})}. \end{aligned} \tag{3.16}$$

From the explicit formulas for $Z(s,f,v,\Delta_i)$, $i=1,2,..,8$, and $Z(s,f,v,O_v^{\times 2})$ (see (3.3), (3.16)), it follows that the real parts of the poles of $Z(s,f,v)$ belong to the set

$$\{-1\}\bigcup\{-\frac{5}{12}\}\bigcup\{-\frac{1}{2},-\frac{9}{20}\}. \tag{3.17}$$

3.5. The local zeta function of $(y^3-x^2)^2(y^3-ax^2)+x^4y^4$. We set $g(x,y)=(y^3-x^2)^2(y^3-ax^2)+x^4y^4 \in L_v[x,y]$, with $a \in O_v^\times$, and $a \not\equiv 1 \bmod \pi$. We assume that the characteristic of the residue field of L_v is different from 2. The polynomial $g(x,y)$ is degenerate with respect to its geometric Newton polygon. The origin of L_v^2 is the only singular point of $g(x,y)$.

By using the same decomposition of $\mathbb{R}_+^2$ into rational simple cones as in the above example, it is possible to reduce the computation of $Z(s,g,v)$ to the computation of the $p-$adic integrals $Z(s,g,v,O_v^{\times 2})$, $Z(s,g,v,\Delta_i), i=1,.., 8, 9$.

3.6. Computation of $Z(s,g,v,O_v^{\times 2})$. Since $g(x,y)$ does not have singularities on $O_v^{\times 2}$, the integral can be computed by using SPF.

3.7. Computation of $Z(s,g,v,\Delta_i)$, $i=1, 2, 3, 4, 6, 7, 8, 9$. These integrals can be computed as the corresponding integrals in the above example. The following table summarizes these computations.

(3.18)

Cone	$Z(s,g,v,\Delta)$
Δ_1	$q^{-1}(1-q^{-1})$
Δ_2	$(1-q^{-1})\frac{q^{-3-6s}}{(1-q^{-2-6s})}$
Δ_3	$(1-q^{-1})^2\frac{q^{-2-6s}}{(1-q^{-2-6s})}$
Δ_4	$(1-q^{-1})^2\frac{q^{-7-24s}}{(1-q^{-2-6s})(1-q^{-5-18s})}$
Δ_6	$\frac{(1-q^{-1})^2q^{-8-27s}}{(1-q^{-5-18s})(1-q^{-3-9s})}$
Δ_7	$\frac{(1-q^{-1})^2q^{-3-9s}}{(1-q^{-3-9s})}$
Δ_8	$\frac{(1-q^{-1})q^{-4-9s}}{(1-q^{-3-9s})}$
Δ_9	$q^{-1}(1-q^{-1})$

3.8. Computation of $Z(s,f,v,\Delta_5)$ (an integral on a degenerate face in the sense of Kouchnirenko)).

$$\begin{aligned} Z(s,g,v,\Delta_5) &= \sum_{n=1}^{\infty} \int\limits_{\pi^{3n}O_v^\times \times \pi^{2n}O_v^\times} \mid g(x,y) \mid_v^s \mid dxdy \mid \\ &= \sum_{n=1}^{\infty} q^{-5n-18ns} \int\limits_{O_v^{\times 2}} \mid (y^3-x^2)^2(y^3-ax^2)+\pi^{2n}x^4y^4 \mid_v^s \mid dxdy \mid . \end{aligned} \tag{3.19}$$

In order to compute (3.19), we first compute the integral

$$I(s,g^{(n)},v) := \int\limits_{O_v^{\times 2}} \mid (y^3-x^2)^2(y^3-ax^2)+\pi^{2n}x^4y^4 \mid_v^s \mid dxdy \mid, \, n \geqslant 1. \tag{3.20}$$

By using the change of variables $\Phi(x,y)$ defined in (3.6), we have that

$$g^{(n)} \circ \Phi(x,y) = x^{18}y^6\widetilde{g^{(n)}}(x,y), \tag{3.21}$$

with

$$\widetilde{g^{(n)}}(x,y) = (y-1)^2(y-a)+\pi^{2n}x^2y^2. \tag{3.22}$$

Since $\Phi(x,y)$ gives a bijection of $O_v^{\times 2}$ onto itself preserving the Haar measure, it follows from (3.20), and (3.21), that

$$I(s,g^{(n)},v) = \int_{O_v^{\times 2}} \mid \widetilde{g^{(n)}}(x,y) \mid_v^s \mid dxdy \mid . \tag{3.23}$$

In order to compute the integral $I(s,g^{(n)},v)$, we decompose $O_v^{\times 2}$ as follows:

$$\begin{aligned} O_v^{\times 2} &= (O_v^{\times} \times \{y_0+\pi O_v \mid y_0 \not\equiv 1, a \bmod \pi\}) \bigcup (O_v^{\times} \times \{1+\pi O_v\}) \\ & \bigcup (O_v^{\times} \times \{a+\pi O_v\}) . \end{aligned} \tag{3.24}$$

From the above partition follows that

$$\begin{aligned} I(s,g^{(n)},v) &= (1-q^{-1})q^{-1}(q-3) \\ &+\sum_{j=1}^{\infty} q^{-j} \int_{O_v^{\times}\times O_v^{\times}} \mid \widetilde{g^{(n)}}(x,1+\pi^j y) \mid_v^s \mid dxdy \mid \\ &+\sum_{j=1}^{\infty} q^{-j} \int_{O_v^{\times}\times O_v^{\times}} \mid \widetilde{g^{(n)}}(x,a+\pi^j y) \mid_v^s \mid dxdy \mid, \end{aligned} \tag{3.25}$$

with

$$\widetilde{g^{(n)}}(x,1+\pi^j y) = \pi^{2j}\gamma_1(x,y)y^2+\pi^{2n}\gamma_2(x,y)x^2, \tag{3.26}$$

$$\gamma_1(x,y)=1-a+\pi^j y,\ \gamma_2(x,y)=(1+\pi^j y)^2, \tag{3.27}$$

and

$$\widetilde{g^{(n)}}(x,a+\pi^j y) = \pi^{j}\delta_1(x,y)y+\pi^{2n}\delta_2(x,y)x^2, \tag{3.28}$$

$$\delta_1(x,y)=(a-1+\pi^j y)^2,\ \delta_2(x,y)=(a+\pi^j y)^2.$$

In the next step, we compute explicitly $\mid \widetilde{g^{(n)}}(x,1+\pi^j y) \mid_v^s$, and $\mid \widetilde{g^{(n)}}(x,a+\pi^j y) \mid_v^s$, for a fixed $n \geqq 1$. By some simple calculations, it holds that

$$\mid \widetilde{g^{(n)}}(x,1+\pi^j y) \mid_v^s = \begin{cases} q^{-2js}, & \text{if } 1 \leqq j \leqq n-1, \\ q^{-2ns} \mid \gamma_1(x,y)y^2+\gamma_2(x,y)x^2 \mid_v^s, & \text{if } j=n, \\ q^{-2ns}, & \text{if } j \geqq n+1, \end{cases} \tag{3.29}$$

for any $(x,z) \in O_v^{\times 2}$, and

$$\mid \widetilde{g^{(n)}}(x,a+\pi^j y) \mid_v^s = \begin{cases} q^{-js}, & \text{if } 1 \leqq j \leqq 2n-1, \\ q^{-2ns} \mid \delta_1(x,y)y+\delta_2(x,y)x^2 \mid_v^s, & \text{if } j=2n, \\ q^{-2ns}, & \text{if } j \geqq 2n+1, \end{cases} \tag{3.30}$$

We note that the explicit computation of the absolute values of $\mid \widetilde{g^{(n)}}(x,1+\pi^j y) \mid_v$, respectively, $\mid \widetilde{g^{(n)}}(x,a+\pi^j y) \mid_v$ depends on an explicit description of the set

$$\{(w,z) \in \mathbb{R}^2 \mid w \leqq \min\{2z,2n\}\},$$

respectively, of the set

$$\{(w,z) \in \mathbb{R}^2 \mid w \leqq \min\{z,2n\}\}, \quad \text{for a fixed } n.$$

We shall see in section 4 that this kind of set is the basis to construct the arithmetic Newton polygon associated to $f(x)$.

Now, from (3.25),(3.29), (3.30) and the identity $\sum_{k=A}^{B} z^k = \frac{z^A - z^{B+1}}{1-z}$, we obtain the following explicit expression for $I(s, g^{(n)}, v)$:

$$
\begin{aligned}
I(s,g^{(n)},v) &= q^{-1}(1-q^{-1})(q-3)+\left(1-q^{-1}\right)^2 \frac{q^{-1-2s}}{1-q^{-1-2s}} \\
&\quad -\left(1-q^{-1}\right)^2 \frac{q^{-n-2ns}}{1-q^{-1-2s}} + q^{-n-2ns} I_1(s) + q^{-1}\left(1-q^{-1}\right) q^{-n-2ns} \\
&\quad +\left(1-q^{-1}\right)^2 \frac{q^{-1-s}}{1-q^{-1-s}} - \left(1-q^{-1}\right)^2 \frac{q^{-2n-2ns}}{1-q^{-1-s}} + q^{-2n-2ns} I_2(s) \\
&\quad + q^{-1}\left(1-q^{-1}\right) q^{-2n-2ns},
\end{aligned} \tag{3.31}
$$

where

$$
\begin{aligned}
I_1(s) &:= \int_{O_v^{\times 2}} \mid \gamma_1(x,y)y^2 + \gamma_2(x,y)x^2 \mid_v^s dxdz \mid, \\
I_2(s) &:= \int_{O_v^{\times 2}} \mid \delta_1(x,y)y + \delta_2(x,y)x^2 \mid_v^s dxdz \mid .
\end{aligned} \tag{3.32}
$$

Since the reduction modulo π of each of the polynomials $\gamma_1(x,y)y^2+\gamma_2(x,y)x^2$, $\delta_1(x,y)y+\delta_2(x,y)x^2$ does not have singular points on $\mathbb{F}_q^{\times 2}$ (the characteristic of the residue field is different of 2), SPF implies that

$$
\begin{aligned}
I_1(s) &= \frac{U_1(q^{-s})}{1-q^{-1-s}}, \\
I_2(s) &= \frac{U_2(q^{-s})}{1-q^{-1-s}},
\end{aligned} \tag{3.33}
$$

where $U_1(q^{-s})$, and $U_1(q^{-s})$ are polynomials in q^{-s}, independent of j, and n.

Finally, from (3.19), and (3.31), we obtain the following explicit expression for $Z(s,f,v,\Delta_5)$:

$$
\begin{aligned}
Z(s,f,v,\Delta_5) &= \frac{(q-3)(1-q^{-1})q^{-6-18s}}{(1-q^{-5-18s})} + \frac{(1-q^{1})^2 q^{-6-20s}}{(1-q^{-1-2s})(1-q^{-5-18s})} \\
&\quad - \frac{(1-q^{1})^2 q^{-6-20s}}{(1-q^{-1-2s})(1-q^{-6-20s})} + \frac{q^{-6-20s} U_1(q^{-s})}{(1-q^{-1-s})(1-q^{-6-20s})} \\
&\quad + \frac{(1-q^{-1})q^{-7-20s}}{(1-q^{-6-20s})} + \frac{(1-q^{1})^2 q^{-6-19s}}{(1-q^{-1-s})(1-q^{-5-18s})} \\
&\quad - \frac{(1-q^{1})^2 q^{-7-20s}}{(1-q^{-1-s})(1-q^{-7-20s})} + \frac{q^{-7-20s} U_2(q^{-s})}{(1-q^{-1-s})(1-q^{-7-20s})} \\
&\quad + \frac{(1-q^{-1})q^{-8-20s}}{(1-q^{-7-20s})}.
\end{aligned} \tag{3.34}
$$

From the explicit formulas for $Z(s,f,v,\Delta_i)$, $i=1,2,..,8$, and $Z(s,f,v,O_v^{\times 2})$ (see (3.18), (3.34)), it follows that the real parts of the poles of $Z(s,f,v)$ belong to the set

$$
\{-1\} \bigcup \{-\frac{5}{18}\} \bigcup \{-\frac{1}{2}, -\frac{3}{10}\} \bigcup \{-\frac{7}{20}\}. \tag{3.35}
$$

4. Arithmetic Newton polygons and non degeneracy conditions

In this section we introduce the notions of arithmetic Newton polygon and arithmetic non degeneracy. With these notions we construct an open subset of polynomials, for the Zariski topology, contained in the set of degenerate polynomials with a fixed geometric newton polygon $\Gamma^{geom}(f_0)$. We have called this set of polynomials arithmetically non degenerate. The relevance of this fact is that the poles of the local zeta function of an arithmetically non degenerate polynomial can be described in terms of the corresponding arithmetic Newton polygon (see section 6).

4.1. **Arithmetically non degenerate semiquasihomogeneous polynomials.** Let K be a field, and a, b two coprime positive integers. Along this paper, a quasihomogeneous polynomial $g(x,y) \in K[x,y]$ with respect to the weight (a,b) will be a polynomial of type

$$g(x,y) = cx^u y^v \prod_{i=1}^{l} \left(y^a - \alpha_i x^b\right)^{e_i}, \tag{4.1}$$

satisfying

$$g(t^a x, t^b y) = t^d g(x,y),$$

for every $t \in K^\times$. The integer d is the weighted degree of $g(x,y)$ with respect to (a,b). The above definition of quasihomogeneity assumes implicitly that the polynomial $g(x^a, y^b)$ factorizes as a product of linear polynomials with coefficients in K.

We put

$$f_j(x,y) := c_j x^{u_j} y^{v_j} \prod_{i=1}^{l_j} \left(y^a - \alpha_{i,j} x^b\right)^{e_{i,j}}, \tag{4.2}$$

and d_j for the weighted degree of $f_j(x,y)$ with respect to (a,b), i.e.

$$d_j := ab \left(\sum_{i=1}^{l_j} e_{i,j} \right) + au_j + bv_j.$$

A semiquasihomogeneous polynomial $f(x,y)$ with coefficients in K, in two variables, is a polynomial of the form

$$f(x,y) = \sum_{j=0}^{l_f} f_j(x,y), \tag{4.3}$$

with $d_0 < d_1 < .. < d_{l_f}$.

The polynomial $f_0(x,y)$ is called the quasihomogeneous part of $f(x,y)$. If $f_0(x,y)$ has singular points on the torus $K^{\times 2}$, i.e., if some $e_{i,0} > 1$, $i = 1, 2, .., l_0$, the polynomial $f(x,y)$ is degenerate, in the sense of Kouchnirenko, with respect to its geometric Newton polygon $\Gamma^{geom}(f)$. From now on, *we shall suppose that $f(x,y)$ is a degenerate polynomial in the sense of Kouchnirenko.*

Definition 4.1. *Let $f(x,y) \in K[x,y]$ be a semiquasihomogeneous polynomial, $\theta \in K^\times$ a fixed root of $f_0(1,\theta^a) = 0$, and $e_{j,\theta}$ the multiplicity of θ as root of the polynomial function $f_j(1,y^a)$. To each term $f_j(x,y)$ in (4.3), we associate an straight line of the form*

$$w_{j,\theta}(z) := (d_j - d_0) + e_{j,\theta} z, \; j = 0, 1, .., l_f,$$

and define the arithmetic Newton diagram $\mathcal{N}_{f,\theta}$ of $f(x,y)$ at θ as

$$\mathcal{N}_{f,\theta} := \{(z,w) \in \mathbb{R}_+^2 \mid w \leqq \min_{0 \le j \le l_f} \{w_{j,\theta}(z)\}. \tag{4.4}$$

Let $\partial\mathcal{N}_{f,\theta}$ *denotes the topological boundary of* $\mathcal{N}_{f,\theta}$. *The set*

$$\Gamma_{f,\theta} := \partial\mathcal{N}_{f,\theta} \setminus \{(z,w) \in \mathbb{R}_+^2 \mid w = 0\},$$

will be called the arithmetic Newton polygon of $f(x,y)$ *at* θ.

Definition 4.2. *Let* $f(x,y) \in K[x,y]$ *be a non-constant semiquasihomogeneous polynomial. The arithmetic Newton diagram* $\mathcal{N}^A(f)$ *of* $f(x,y)$ *is defined to be the set*

$$\mathcal{N}^A(f) := \{\mathcal{N}_{f,\theta} \mid \theta \in K^\times, \ f_0(1,\theta^a) = 0\},$$

and the arithmetic Newton polygon $\Gamma^A(f)$ *of* $f(x,y)$ *is defined to be the set*

$$\Gamma^A(f) := \{\Gamma_{f,\theta} \mid \theta \in K^\times, \ f_0(1,\theta^a) = 0\}.$$

We highlight that $\Gamma^A(f)$ (and also $\mathcal{N}^A(f)$) is a finite collection of convex sets; and no just a convex set as occurs in the case of $\Gamma^{geom}(f)$.

If the polynomial $f_0(x,y)$ is interpreted as the weighted tangent cone of $f(x,y)$, then for each tangent direction of the form $(\theta : 1)$, $\theta \in K^\times$, there exists an arithmetic Newton polygon $\Gamma_{f,\theta}$ at θ, and all these Newton polygons constitute the arithmetic Newton polygon $\Gamma^A(f)$ of $f(x,y)$.

Let θ be a fixed root of $f_0(1,\theta^a) = 0$. Each arithmetic Newton polygon $\Gamma_{f,\theta}$ is conformed by a finite number of straight segments. A point $Q \in \Gamma_{f,\theta}$ will be called a *vertex*, if $Q = (0,0)$ or if it is the intersection point of two straight segments with different slopes. Let Q_k, $k = 0, 1, .., r+1$, denote the vertices of $\Gamma_{f,\theta}$, with $Q_0 := (0,0)$, and $Q_{r+1}$is a point at infinity. In order to fix the notation, we shall suppose from now on that the straight segment between Q_{k-1} and Q_k has an equation of the form

$$w_{k,\theta}(z) = (\mathcal{D}_k - d_0) + \mathcal{E}_k z, \quad k = 1, 2, .., r+1. \tag{4.5}$$

Then from the above notation, it follows that

$$Q_k = (\tau_k, (\mathcal{D}_k - d_0) + \mathcal{E}_k\tau_k), \ k = 1, 2, .., r, \tag{4.6}$$

with

$$\tau_k := \frac{(\mathcal{D}_{k+1} - \mathcal{D}_k)}{\mathcal{E}_k - \mathcal{E}_{k+1}} > 0, \ k = 1, 2, .., r, \tag{4.7}$$

(see in the examples of the next section how to obtain these numbers for specific cases). We note that $\mathcal{D}_k = d_{j_k}$, and $\mathcal{E}_k = e_{j_k,\theta}$, for some index j_k. In particular, $\mathcal{D}_1 = d_0$, $\mathcal{E}_1 = e_{0,\theta}$, and the first equation is $w_{1,\theta}(z) = \mathcal{E}_1 z$. We also remark that the last segment that determines $\Gamma_{f,\theta}$ is a half-line with origin at Q_r, i.e. a infinite segment from Q_r to Q_{r+1}. This half-line may be horizontal, i.e. $\mathcal{E}_{r+1} = 0$.

Given a vertex Q of $\Gamma_{f,\theta}$ *the face function* $f_Q(x,y)$ is defined to be the polynomial

$$f_Q(x,y) := \sum_{w_{j,\theta}(Q)=0} f_j(x,y), \tag{4.8}$$

where $w_{j,\theta}(z)$ is the straight line corresponding to the term $f_j(x,y)$.

Now we introduce a notion of non degeneracy with respect to an arithmetic Newton polygon.

Definition 4.3. *A semiquasihomogeneous polynomial* $f(x,y)$ *is named arithmetically non degenerate with respect to* $\Gamma_{f,\theta}$ *at* θ, *if it satisfies:*

1. *the origin of* K^2 *is a singular point of* $f(x,y)$;
2. *the polynomial* $f(x,y)$ *does not have singular points on* $K^{\times 2}$;
3. *the system of equations*

$$f_Q(x,y) = \frac{\partial f_Q}{\partial x}(x,y) = \frac{\partial f_Q}{\partial y}(x,y) = 0,$$

does not have solutions on $K^{\times 2}$, *for each vertex* $Q \neq (0,0)$ *of* $\Gamma_{f,\theta}$.

Definition 4.4. *A semiquasihomogeneous polynomial $f(x, y)$ with coefficients in a field K, will be called arithmetically non degenerate with respect to $\Gamma^A(f)$, if it satisfies:*

1. *the origin of K^2 is a singular point of $f(x, y)$;*
2. *the polynomial $f(x, y)$ does not have singular points on $K^{\times 2}$;*
3. *the polynomial $f(x, y)$ is arithmetically non degenerate with respect to $\Gamma_{f,\theta}$, for each $\theta \in K^\times$ satisfying $f_0(1, \theta^a) = 0$.*

4.2. **Arithmetically non degenerate polynomials.** Let $f(x, y) \in K[x, y]$ be a non-constant polynomial, and $\Gamma^{geom}(f)$ its geometric Newton polygon. Each facet γ of $\Gamma^{geom}(f)$ has a perpendicular primitive vector $a_\gamma = (a_1(\gamma), a_2(\gamma))$ and the corresponding supporting line has an equation of the form $< a, x >= d_a(\gamma)$. The polynomial $f(x, y)$ is a semiquasihomogeneous polynomial with respect to the weight a_γ, for each facet γ of $\Gamma^{geom}(f)$.

Definition 4.5. *Let $f(x, y) \in K[x, y]$ be a non-constant polynomial. The arithmetic Newton diagram $\mathcal{N}^A(f)$ of $f(x, y)$ is defined to be the set*

$$\mathcal{N}^A(f) := \{\mathcal{N}^A(f_\gamma) \mid \gamma \text{ a facet of } \Gamma^{geom}(f)\},$$

and the arithmetic Newton polygon $\Gamma^A(f)$ of $f(x, y)$ is defined to be the set

$$\Gamma^A(f) := \{\Gamma^A(f_\gamma) \mid \gamma \text{ a facet of } \Gamma^{geom}(f)\}.$$

Definition 4.6. *A polynomial $f(x, y) \in K[x, y]$ is named arithmetically non degenerate with respect to its arithmetic Newton polygon, if it satisfies:*

1. *the origin of K^2 is a singular point of $f(x, y)$;*
2. *the polynomial $f(x, y)$ does not have singular points on $K^{\times 2}$;*
3. *for each facet γ of $\Gamma^{geom}(f)$, the semiquasihomogeneous polynomial $f(x, y)$ with respect to the weight a_γ, is arithmetically non degenerate with respect to $\Gamma^A(f_\gamma)$.*

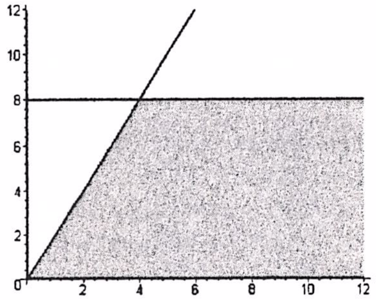
As an immediate consequence of the definition of arithmetically non degenerate polynomial, we have that for a given fixed geometric Newton polygon $\Gamma^{geom}(f_0)$, the set of degenerate polynomials, in the sense of Kouchnirenko with respect to $\Gamma^{geom}(f_0)$, contains an open set consisting of arithmetically non degenerate polynomials.

4.3. **Examples.** The following three examples illustrate the above definitions for specific polynomials, we assume that K is a field of characteristic zero.

Example 4.1. We set $f(x, y) = (y^3 - x^2)^2 + x^4y^4 \in K[x, y]$. The polynomial $f(x, y)$ is a degenerate, in the sense of Kouchnirenko, semiquasihomogeneous polynomial with respect to the weight $(3, 2)$.The origin of K^2 is the only singular point of $f(x, y)$. In this case the arithmetic Newton polygon of $f(x, y)$ is equal to $\Gamma_{f,1}$. The Newton polygon $\Gamma_{f,1}$ is conformed by the following two straight segments:

$$\begin{aligned} w_{1,1}(z) &= 2z, \quad 0 \leq z \leq 4, \\ w_{2,1}(z) &= 8, \quad z \geqslant 4. \end{aligned}$$

Thus $\mathcal{D}_1 = d_0 = 12$, $\mathcal{E}_1 = 2$, $\mathcal{D}_2 = 20$, $\tau_0 = 0$, $\tau_1 = 4$. The arithmetic Newton diagram $\mathcal{N}_{f,1}$ is showed in the next figure.



Arithmetic Newton Diagram of $f(x, y)$.

The face functions are

$$f_{(0,0)}(x,y) = (y^3 - x^2)^2, \ f_{(4,8)}(x,y) = (y^3 - x^2)^2 + x^4y^4.$$

Since $f_{(4,8)}(x, y)$ does not have singular points on $K^{\times 2}$, $f(x, y)$ is arithmetically non degenerate.

Example 4.2. We set

$$g(x,y) = (y^3 - x^2)^5 + (y^3 - x^2)^3x^6y^3 + (y^3 - x^2)^2x^{12} + x^{24} \in K[x, y].$$

The polynomial $g(x, y)$ is a degenerate, in the sense of Kouchnirenko, semiquasihomogeneous polynomial with respect to the weight $(3, 2)$. The origin of K^2 is the only singular point of $g(x, y)$. Following the definition 4.1 we obtain $g(x, y) = g_0(x, y) + g_1(x, y) + g_2(x, y) + g_3(x, y)$, where

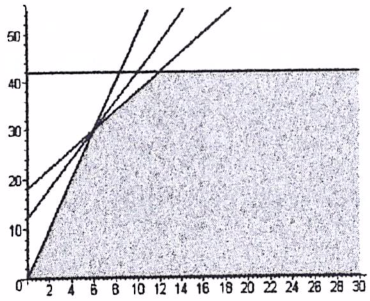
$$\begin{aligned} g_0(x,y) &= (y^3 - x^2)^5, \ \textit{hence } d_0 = 30, \ e_{0,1} = 5 \textit{ and } w_{0,1} = 5z; \\ g_1(x,y) &= (y^3 - x^2)^3x^6y^3, \ \textit{hence } d_1 = 42, \ e_{1,1} = 3 \textit{ and } w_{1,1} = 12 + 3z; \\ g_2(x,y) &= (y^3 - x^2)^2x^{12}, \ \textit{hence } d_2 = 48, \ e_{2,1} = 2 \textit{ and } w_{2,1} = 18 + 2z; \\ g_3(x,y) &= x^{24}, \ \textit{hence } d_3 = 72, \ e_{3,1} = 0 \textit{ and } w_{3,1} = 42. \end{aligned}$$

Since there is only one root $\theta = 1$ of $g_0(1, \theta^a) = 0$, the arithmetic Newton polygon of $g(x, y)$ is equal to $\Gamma_{g,1}$, which is conformed by the following three straight segments:

$$\begin{aligned} w_{1,1}(z) &= 5z, \ \ 0 \leq z \leq 6, \\ w_{2,1}(z) &= 18 + 2z, \ \ 6 \leq z \leq 12, \\ w_{3,1}(z) &= 42, \ \ z \geqslant 12. \end{aligned}$$

Thus $\mathcal{D}_1 = d_0 = 30$, $\mathcal{E}_1 = 5$, $\mathcal{D}_2 = 48$, $\mathcal{E}_2 = 2$, $\mathcal{D}_3 = 72$, $\mathcal{E}_3 = 0$, $\tau_0 = 0$, $\tau_1 = 6$, $\tau_2 = 12$.

We observe that the segment $w(z) = 12 + 3z$ has only the point $(6, 30)$ in the arithmetic Newton diagram of $\mathcal{N}_{g,1}$, which is showed in the next figure.



Arithmetic Newton Diagram of $g(x,y)$.

The face functions are

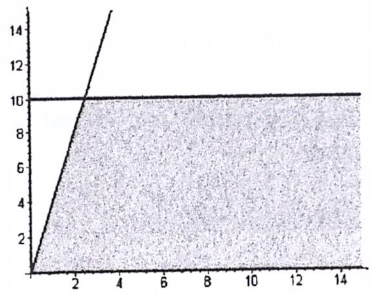
$$\begin{aligned} g_{(0,0)}(x,y) &= (y^3-x^2)^5, \\ g_{(6,30)}(x,y) &= (y^3-x^2)^5+(y^3-x^2)^3x^6y^3+(y^3-x^2)^2x^{12}, \\ g_{(12,40)}(x,y) &= (y^3-x^2)^2x^{12}+x^{24}. \end{aligned}$$

Since $g_{(6,30)}(x,y)=(y^3-x^2)^2\left((y^3-x^2)^3+(y^3-x^2)+x^{12}\right)$ has singular points on $K^{\times 2}$, $g(x,y)$ is arithmetically degenerate.

Example 4.3. We set $h(x,y)=(y^5-x^3)^4(y^5-ax^3)(y^5-bx^3)+x^{20}\in K[x,y]$, with $1\neq a\neq b$.The polynomial $h(x,y)$ is a degenerate, in the sense of Kouchnirenko, semiquasihomogeneous polynomial with respect the weight $(5,3)$. The origin of K^2 is a singular point of $h(x,y)$, and there is no singularities on the torus $K^{\times 2}$. In this case the arithmetic Newton polygon of $h(x,y)$ is equal to $\{\Gamma_{h,1},\Gamma_{h,a},\Gamma_{h,b}\}$. The Newton polygon $\Gamma_{h,1}$ is conformed by the following two straight segments:

$$\begin{aligned} w_{1,1}(z) &= 4z, \quad 0\leq z\leq \frac{5}{2}, \\ w_{2,1}(z) &= 10, \quad z\geqslant \frac{5}{2}. \end{aligned}$$

The arithmetic Newton diagram $\mathcal{N}_{h,1}$ is showed in the next figure



Arithmetic Newton Diagram of $h(x,y)$ at 1.

The face functions are

$$\begin{aligned} h_{(0,0)}(x,y) &= (y^5-x^3)^4(y^5-ax^3)(y^5-bx^3) \\ h_{(\frac{5}{2},10)}(x,y) &= (y^5-x^3)^4(y^5-ax^3)(y^5-bx^3)+x^{20}. \end{aligned}$$

Since

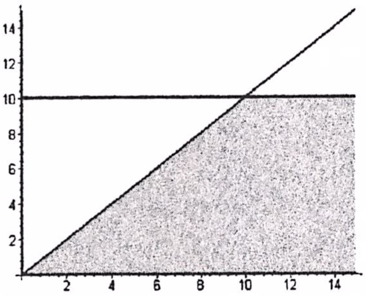
$$h_{(\frac{5}{2},10)}(x,y) = (y^5 - x^3)^4(y^5 - ax^3)(y^5 - bx^3) + x^{20}$$

has no singular points on $K^{\times 2}$, $f(x,y)$ is arithmetically non degenerate with respect to $\Gamma_{h,1}$.

The Newton polygon $\Gamma_{h,a}$ is conformed by the following two straight segments:

$$\begin{aligned} w_{1,a}(z) &= z, \quad 0 \leq z \leq 10, \\ w_{2,a}(z) &= 10, \quad z \geqslant 10. \end{aligned}$$

The arithmetic Newton diagram $\mathcal{N}_{h,a}$ is showed in the next figure



Arithmetic Newton Diagram of $h(x,y)$ at a.

The face functions are

$$\begin{aligned} h_{(0,0)}(x,y) &= (y^5 - x^3)^4(y^5 - ax^3)(y^5 - bx^3), \\ h_{(10,10)}(x,y) &= (y^5 - x^3)^4(y^5 - ax^3)(y^5 - bx^3) + x^{20}. \end{aligned}$$

Since

$$h_{(10,10)}(x,y) = (y^5 - x^3)^4(y^5 - ax^3)(y^5 - bx^3) + x^{20}$$

has no singular points on $K^{\times 2}$, $f(x,y)$ is arithmetically non degenerate with respect to $\Gamma_{h,a}$.

The Newton polygon $\Gamma_{h,b}$ is equal to $\Gamma_{h,a}$. Thus the polynomial $h(x,y)$ is arithmetically non degenerate with respect Γ_h^A.

5. The integrals $Z(s,f,v,\Delta_3)$

In this section, by using the notion of arithmetic non degeneracy introduced in the above section, we shall compute explicitly the "degenerate" integrals

$$Z(s,f,v,\Delta_3) := \int_{E_3} \mid f(x,y) \mid_v^s \mid dxdy \mid,$$

where $f(x,y) \in L_v[x,y]$ is a semiquasihomogeneous polynomial with respect to the weight (a,b), a, b coprime, and $f(x,y)$ is a degenerate polynomial, in the sense of Kouchnirenko, but it is arithmetically non degenerate; and the cone Δ_3 is given by partition (2.4) of $\mathbb{R}^2$, i.e.,

$$\Delta_3 := (a,b)\mathbb{R}_+, \text{ and } E_3 := \{(x,y) \in O_v^2 \mid (v(x), v(y)) \in \Delta_3\}.$$

Definition 5.1. *Given a fixed semiquasihomogeneous polynomial $f(x,y)$ non degenerate with respect to $\Gamma^A(f)$, we define for each $\Gamma_{f,\theta}$, with $\theta \in L_v^\times$, $f_0(1,\theta^a)=0$,*

$$\mathcal{P}(\Gamma_{f,\theta}) := \bigcup_{i=1}^{r_\theta}\{-\frac{1}{\mathcal{E}_i}, -\frac{(a+b)+\tau_i}{\mathcal{D}_{i+1}+\mathcal{E}_{i+1}\tau_i}, -\frac{(a+b)+\tau_i}{\mathcal{D}_i+\mathcal{E}_i\tau_i}\}\bigcup \bigcup_{\{\mathcal{E}_{r+1}\neq o\}}\{-\frac{1}{\mathcal{E}_{r+1}}\}, \tag{5.1}$$

and

$$\mathcal{P}(\Gamma^A(f)) := \bigcup_{\{\theta \mid f_0(1,\theta^a)=0\}} \mathcal{P}(\Gamma_{f,\theta}). \tag{5.2}$$

The data $\mathcal{D}_i$, $\mathcal{E}_i$, τ_i, for each i, are obtained from the equations of the straight segments conforming $\Gamma_{f,\theta}$.

The main result of this section is the following.

Theorem 5.1. *Let $f(x,y) = \sum_{j=0}^{l_f} f_j(x,y) \in O_v[x,y]$ be a semiquasihomogeneous polynomial, with respect to the weight (a,b), a, b coprime, with $f_j(x,y)$ as in (4.2), $\alpha_{i,j}$, $c_j \in O_v^\times$, for every i,j, and $\alpha_{i,j} \not\equiv \alpha_{r,t} \bmod \pi$, when $(i,j)\neq(r,t)$. If $f(x,y)$ is arithmetically non degenerate with respect to $\Gamma^A(f)$, then the real parts of the poles of $Z(s,f,v,\Delta_3)$ belong to the set*

$$\{-1\}\bigcup\{-\frac{a+b}{d_0}\}\bigcup\mathcal{P}(\Gamma^A(f)).$$

The proof will be given at the end of this section, and it will be accomplished by computing explicitly several $p-$adic integrals. Moreover the proof provides an effective method to compute these integrals.

The following two examples illustrate the theorem.

Example 5.1. We set $f(x,y) = (y^3-x^2)^2 + x^4y^4 \in L_v[x,y]$. We suppose that the characteristic of the residue field is different from 2. The polynomial $f(x,y)$ is arithmetically non degenerate, semiquasihomogeneous polynomial with respect to the weight $(3,2)$ (see example (4.1). In this case, we have the following data attached the arithmetic Newton polygon $\Gamma^A(f) = \Gamma_{f,1}$: $a=3$, $b=2$, $\mathcal{D}_1 = d_0 = 12$, $\tau_1 = 4, \mathcal{E}_1 = 2$, $\mathcal{D}_2 = 20$. Thus according the above theorem, the real parts of the poles of the integral $Z(s,f,v)$ belong to the set

$$\{-1\}\bigcup\{-\frac{5}{12}\}\bigcup\{-\frac{1}{2},-\frac{9}{20}\}.$$

In subsection 3.1, we computed explicitly the local zeta function $Z(s,f,v)$ (cf. (3.17)).

Example 5.2. We set $g(x,y) = (y^3-x^2)^2(y^3-ax^2) + x^4y^4 \in L_v[x,y]$, with $a \not\equiv 1 \bmod \pi$. We assume that the characteristic of the residue field of L_v is different from 2. The polynomial $g(x,y)$ is quasihomogeneous with respect to the weight $(3,2)$. The origin of L_v^2 is the only singular point of $g(x,y)$.

The arithmetic Newton polygon $\Gamma^A(g) = \{\Gamma_{g,1}, \Gamma_{g,a}\}$. The Newton polygon $\Gamma_{g,1}$ is conformed by the following two straight segments:

$$\begin{aligned} w_{1,1}(z) &= 2z, \ \ 0\leq z\leq 1,\\ w_{2,1}(z) &= 2, \ \ z\geqslant 1. \end{aligned}$$

Thus $\mathcal{D}_1 = d_0 = 18$, $\mathcal{E}_1 = 2$, $\mathcal{D}_2 = 20$, $\mathcal{E}_2 = 0$, $\tau_0 = 0$, $\tau_1 = 1$.The polynomial is arithmetically non degenerate with respect $\Gamma_{g,1}$. According to the previous theorem the contribution of the arithmetic Newton polygon $\Gamma_{g,1}$ to the set of real parts of $Z(s,g,v)$ is $\{-\frac{1}{2}, -\frac{9}{20}\}$.

The Newton polygon $\Gamma_{g,a}$ is conformed by the following two straight segments:

$$\begin{aligned} w_{1,1}(z) &= z, \ \ 0\leq z\leq 2,\\ w_{2,1}(z) &= 2, \ \ z\geqslant 2. \end{aligned}$$

Thus $\mathcal{D}_1 = d_0 = 18$, $\mathcal{E}_1 = 1$, $\mathcal{D}_2 = 20$, $\mathcal{E}_2 = 0$, $\tau_0 = 0$, $\tau_1 = 2$. The polynomial is arithmetically non degenerate with respect $\Gamma_{g,a}$. According to the previous theorem the contribution of the arithmetic Newton polygon $\Gamma_{g,a}$ to the set of real parts of $Z(s,g,v)$ is $\{-1, -\frac{7}{20}\}$. Then according to the previous theorem the real parts of the poles of $Z(s,g,v)$ belong to the set $\{-1\}\bigcup\{-\frac{5}{18}\}\bigcup\{-\frac{1}{2}, -\frac{9}{20}\}\bigcup\{-\frac{7}{20}\}$. In subsection 3.2, we computed explicitly the local zeta $Z(s,g,v)$ (cf. (3.35)).

5.1. **Computation of the integrals** $Z(s,f,v,\Delta_3)$**.** The explicit computation of the integrals $Z(s,f,v,\Delta_3)$ is based on the explicit computation of several simple p-adic integrals. In this subsection we present all these computations.

Proposition 5.1. *Let $f(x,y) \in O_v[x,y]$ be a semiquasihomogeneous polynomial, with respect to the weight (a,b), a,b coprime, and*

$$f^{(m)}(x,y) := \pi^{-d_0} f(\pi^a x, \pi^b y) = \sum_{j=0}^{l_f} \pi^{(d_j - d_0)m} f_j(x,y), \tag{5.3}$$

with $f_j(x,y)$ as in (4.2). Then there exists a measure-preserving bijection

$$\begin{aligned} \Phi \quad &: \quad O_v^{\times\, 2} \to O_v^{\times\, 2} \\ (u,w) \quad &\to \quad (\Phi_1(x,y), \Phi_2(x,y)), \end{aligned}$$

such that $f^{(m)} \circ \Phi(x,y) = u^{N_i} w^{M_i} \widetilde{f^{(m)}}(u,w)$, with $\widetilde{f^{(m)}}(u,w)$ not vanishing identically on the hypersurface $uw = 0$, and

$$\widetilde{f^{(m)}}(u,w) = \sum_{j=0}^{l_f} \pi^{(d_j - d_0)m} \widetilde{f}_j(u,w), \tag{5.4}$$

with

$$\widetilde{f}_j(u,w) = c_j u^{A_j} w^{B_j} \prod_{i=1}^{l_j} (w - \alpha_{i,j})^{e_{i,j}}, \text{ or } \widetilde{f}_j(u,w) = c_j u^{L_j} w^{F_j} \prod_{i=1}^{l_j} (1 - \alpha_{i,j} u)^{e_{i,j}}. \tag{5.5}$$

Proof. We denote by Φ_1 the map

$$\begin{aligned} \Phi_1(u,w) \quad &: \quad O_v^{\times\, 2} \to O_v^{\times\, 2} \\ (u,w) \quad &\to \quad (x,y), \end{aligned}$$

with $x = u$, $y = uw$, and by Φ_2 the map

$$\begin{aligned} \Phi_2(u,w) \quad &: \quad O_v^{\times\, 2} \to O_v^{\times\, 2} \\ (u,w) \quad &\to \quad (x,y), \end{aligned}$$

with $x = uw$, $y = w$.

Since $\mid \det \Phi_1'(u,w) \mid_v = 1$, $\mid \det \Phi_2'(u,w) \mid_v = 1$, and the maps Φ_1, Φ_2 give a measure-preserving bijection of $O_v^{\times\, 2}$ to itself. We shall show that Φ is a finite composition of maps of the form Φ_1 or Φ_2. The proof will be accomplished by induction on $\min\{a,b\}$.

Case $\min\{a,b\} = 1$

In this case, it is sufficient to take Φ as follows:

$$\Phi = \begin{cases} \Phi_1 \circ .. \circ \Phi_1, \; b - \text{times, if } a = 1, \\ \\ \Phi_2 \circ .. \circ \Phi_2, \; a - \text{times, if } b = 1. \end{cases}$$

Induction hypothesis

Suppose by induction hypothesis that the proposition is valid for all polynomials $f^{(m)}(x,y)$ of the form (5.3) satisfying $1 \leqq \min\{a,b\} \leqq k$, with $k \geqq 2$.

Case $\min\{a,b\} = k+1$, $k \geqq 2$

Let $f^{(m)}(x,y)$ be a polynomial of the form (5.3) satisfying $\min\{a,b\} = k+1$, $k \geqq 2$, and $a > b$. By applying the Euclidean algorithm to a, b, we have that

$$a = q_1 b + r_1, \qquad 0 \leqq r_1 < b,$$

for some q_1, $r_1 \in \mathbb{N}$. Because a, b are coprime, necessarily $1 \leqq r_1 < b$.

We set $\Psi = \underbrace{\Phi_2 \circ .. \circ \Phi_2}_{q_1 - \text{times}}$, i.e. $x = uw^{q_1}$, $y = w$, thus

$$f^{(m)} \circ \Psi(x,y) = u^{A_i} w^{B_i} f^{*(m)}(u,w)$$

with $f^{*(m)}(u,w)$ not vanishing identically on $uw = 0$, and

$$f^{*(m)}(u,w) = \sum_{j=0}^{l_f} \pi^{(d_j - d_0)m} f_j^*(u,w), \tag{5.6}$$

with

$$f_j^*(u,w) = c_j u^{C_j} w^{D_j} \prod_{i=1}^{l_j} \left(w^{r_1} - \alpha_{i,j} u^b\right)^{e_{i,j}}.$$

Since $\min\{r_1, b\} = r_1$, and $1 \leqq r_1 \leqq b - 1 = k$, by applying the induction hypothesis applied to the polynomial $f^{*(m)}(u,v)$ in (5.6), there exists a map Θ, that is a finite composition of maps of the form Φ_1 or Φ_2, such that $\left(f^{(m)} \circ \Psi\right) \circ \Theta = f^{(m)} \circ (\Psi \circ \Theta) = u^{N_i} w^{M_i} \widetilde{f^{(m)}}(u,v)$, and $\widetilde{f^{(m)}}(u,v)$ has all announced properties in the proposition. Therefore, it is sufficient to take $\Phi = \Psi \circ \Theta$.

The case $b > a$ is proved in an analogous form. □

Remark 5.1. We shall use the above proposition, only in the case in which $\alpha_{i,j} \in O_v^\times$, for every i, j. In this case, we can suppose that

$$\widetilde{f^{(m)}}(u,w) = \sum_{j=0}^{l_f} \pi^{(d_j - d_0)m} \widetilde{f}_j(u,w), \tag{5.7}$$

with

$$\widetilde{f}_j(u,w) = c_j u^{A_j} w^{B_j} \prod_{i=1}^{l_j} (w - \alpha_{i,j})^{e_{i,j}}. \tag{5.8}$$

Because $\widetilde{f}_j(u,w) = c_j u^{L_j} w^{F_j} \prod_{i=1}^{l_j} \left(1 - \alpha_{i,j} u\right)^{e_{i,j}}$ can be transformed into the above case by a simple algebraic manipulation.

We define

$$I(s, f^{(m)}, v, O_v^{\times 2}) := \int_{O_v^{\times 2}} \mid f^{(m)}(x,y) \mid_v^s \mid dxdy \mid .$$

Proposition 5.2. *Let $f(x,y) \in O_v[x,y]$ be a semiquasihomogeneous polynomial, with respect to the weight (a,b), a, b coprime, and*

$$f^{(m)}(x,y) = \pi^{-d_0} f(\pi^a x, \pi^b y) = \sum_{j=0}^{l_f} \pi^{(d_j - d_0)m} f_j(x,y), \tag{5.9}$$

with $f_j(x,y)$ *as in (4.2).If* $\alpha_{i,j}$, $c_j \in O_v^\times$, *for every* i, j, *and* $\alpha_{i,j} \not\equiv \alpha_{r,t}$ *mod* π, *when* $(i,j) \neq (r,t)$, *then*

$$(5.10) \qquad \begin{aligned} I(s,f^{(m)},v,O_v^{\times 2}) &= q^{-2}(q-1-l_0)(q-1) \\ &+ \sum_{\{\theta\in O_v^\times \mid f_0(1,\theta^a)=0\}} \sum_{k=1}^{\infty} q^{-k} \int_{O_v^\times \times O_v^\times} \mid \widetilde{f^{(m)}}(x,\theta+\pi^k y) \mid_v^s \mid dxdy \mid, \end{aligned}$$

with $l_0 :=$Card$(\{\theta \in O_v^\times \mid f_0(1,\theta^a)=0\}$,

$$(5.11) \qquad \widetilde{f^{(m)}}(x,\theta+\pi^k y) = \sum_{j=0}^{l_f} \pi^{(d_j-d_0)m+ke_{j,\theta}} \gamma_j(x,y) y^{e_{j,\theta}},$$

and $\overline{\gamma_j(x,y)} \in \mathbb{F}_q^\times$, for every $(x,y) \in O_v^\times \times O_v^\times$.

Proof. By proposition (5.1), it holds that

$$(5.12) \qquad I(s,f^{(m)},v,O_v^{\times 2}) = \int_{O_v^{\times 2}} \mid f^{(m)}(x,y) \mid_v^s \mid dxdy \mid = \int_{O_v^{\times 2}} \mid \widetilde{f^{(m)}}(x,y) \mid_v^s \mid dxdy \mid .$$

Now, we take the partition $O_v^{\times 2} = A \cup B$, with

$$A := O_v^\times \times \{z \in O_v^\times \mid y \not\equiv \theta \text{ mod } \pi, \text{ for any } \theta \in O_v^\times,\ f_0(1,\theta^a) = 0\ \},$$

$$B := O_v^\times \times \bigcup_{\{\theta\in O_v^\times \mid f_0(1,\theta^a)=0\}} \{z \in O_v^\times \mid y \equiv \theta \text{ mod } \pi\}\ .$$

Thus

$$(5.13) \qquad I(s,f^{(m)},v,O_v^{\times 2}) = \int_A \mid \widetilde{f^{(m)}}(x,y) \mid_v^s \mid dxdy \mid + \int_B \mid \widetilde{f^{(m)}}(x,y) \mid_v^s \mid dxdy \mid .$$

Since $\mid \widetilde{f^{(m)}}(x,\ z+\pi y) \mid_v = 1$, when $z \not\equiv \theta$ mod π (cf. remark 5.1, and $\alpha_{i,j} \not\equiv \alpha_{r,t}$ mod π, when $(i,j) \neq (r,t)$), it holds that

$$(5.14) \qquad \int_A \mid \widetilde{f^{(m)}}(x,y) \mid_v^s \mid dxdy \mid = q^{-2}(q-1-\text{Card}(\{\theta \in O_v^\times \mid f_0(1,\theta^a)=0\}))(q-1).$$

Now, the results follows from (5.13) and (5.14). □

Given a real number x, $[x]$ denotes the greatest integer less or equal to x.

Proposition 5.3. *With the hypothesis of proposition 5.2. If* $f(x,y)$ *is arithmetically non degenerate with respect to* $\Gamma_{f,\theta}$, *then*

$$
\begin{aligned}
J(s,m,\theta) \quad : \quad &= \sum_{k=1}^{\infty} q^{-k} \int\limits_{O_v^\times \times O_v^\times} \mid \widetilde{f^{(m)}}(x,\theta+\pi^k y) \mid_v^s \mid dxdy \mid = \\
&(1-q^{-1})^2 \sum_{i=0}^{r-1} \frac{q^{-(1+s\mathcal{E}_{i+1})}}{1-q^{-(1+s\mathcal{E}_{i+1})}} q^{-(\mathcal{D}_{i+1}-d_0)ms-(1+s\mathcal{E}_{i+1})[m\tau_i]} \\
&-(1-q^{-1})^2 \sum_{i=0}^{r-1} \frac{1}{1-q^{-(1+s\mathcal{E}_{i+1})}} q^{-(\mathcal{D}_{i+1}-d_0)ms-(1+s\mathcal{E}_{i+1})[m\tau_{i+1}]} \\
&+(1-q^{-1})^2 \frac{q^{-(1+s\mathcal{E}_{r+1})}}{1-q^{-(1+s\mathcal{E}_{r+1})}} q^{-(\mathcal{D}_{r+1}-d_0)ms-(1+s\mathcal{E}_{r+1})[m\tau_r]} \\
&+\sum_{i=1}^{r} q^{-(\mathcal{D}_i-d_0)ms-(1+s\mathcal{E}_i)[m\tau_i]} I_i(s),
\end{aligned} \tag{5.15}
$$

where τ_i, $i=0,1,..,r$, are the abscissas of the vertices of $\Gamma_{f,\alpha_{l,0}}$, and $I_i(s) = \frac{L_j(q^{-s})}{1-q^{-1-s}}$, $L_i(q^{-s}) \in \mathbb{Q}[q^{-s}], i=1,2,..,r$. Moreover the $I_i(s)$ do not dependent on k.

Proof. The proof is based on an explicit computation of the v−adic absolute value of $\widetilde{f^{(m)}}(x,\theta+\pi^k y)$, when $x,\ y \in O_v^\times$. In order to accomplish it, we attach to $\widetilde{f^{(m)}}(x,\theta+\pi^k y)$ the convex set $\mathcal{N}_{\widetilde{f^{(m)}}(x,\theta+\pi^k y)}$ defined as follows. We associate to each term

$$\pi^{(d_j-d_0)m+ke_{j,\theta}}\gamma_j(x,y)y^{e_{j,\theta}} \tag{5.16}$$

of $\widetilde{f^{(m)}}(x,\theta+\pi^k y)$ (see (5.11)) an straight line of the form

$$\widetilde{w}_{j,\theta}(\widetilde{z}) := (d_j-d_0)m + e_{j,\theta}\widetilde{z}, \quad j=0,1,..,l_f,$$

and associate to $\widetilde{f^{(m)}}(x,\theta+\pi^k y)$ the convex set

$$\mathcal{N}_{\widetilde{f^{(m)}}(x,\theta+\pi^k y)} = \{(\widetilde{z},\widetilde{w}) \in \mathbb{R}_+^2 \mid \widetilde{w} \leqq \min_{0\leq j\leq l_f} \{\widetilde{w}_{j,\theta}(\widetilde{z})\}\}. \tag{5.17}$$

Now, we set

$$
\begin{array}{rccc}
F_m: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\
 & (u,w) & \longrightarrow & (\widetilde{u},\widetilde{w}),
\end{array}
$$

with $u=\frac{\widetilde{u}}{m}$, $z=\frac{\widetilde{u}}{m}$. Then $F_m(\mathcal{N}_{f,\theta}) = \mathcal{N}_{\widetilde{f^{(m)}}(x,\theta+\pi^k y)}$ (see (4.4)). The homomorphism F_m sends the topological boundary of $\mathcal{N}_{f,\theta}$ into the topological boundary of $\mathcal{N}_{\widetilde{f^{(m)}}(x,\theta+\pi^j y)}$. Thus the vertices of $\partial\mathcal{N}_{\widetilde{f^{(m)}}(x,\theta+\pi^k y)}$ are

$$F_m(Q_i) := \{.(m\tau_i,(\mathcal{D}_i-d_0)m+m\mathcal{E}_i\tau_i),\ i=1,2,..,r,(0,0),\ \ i=0; \tag{5.18}$$

where the τ_i are the abscissas of the vertices of $\Gamma_{f^{(m)},\theta}$ (cf. (4.6), (4.7)).

Now$\mid \widetilde{f^{(m)}}(x,\theta+\pi^k y) \mid_v$ can be computed from $\mathcal{N}_{\widetilde{f^{(m)}}(x,\theta+\pi^k y)}$, as follows (we use the notation introduced at (4.5), (4.6), (4.7)):

If $m\tau_i < j < m\tau_{i+1}$, $i=0,1,..,r-1$, then

$$\mid \widetilde{f^{(m)}}(x,\theta+\pi^k y) \mid_v = q^{-(\mathcal{D}_{i+1}-d_0)m-\mathcal{E}_{i+1}j}\ ,\ (x,y) \in O_v^\times \times O_v^\times\ . \tag{5.19}$$

If $j > m\tau_r$, then

$$\mid \widetilde{f^{(m)}}(x,\theta+\pi^k y) \mid_v = q^{-(\mathcal{D}_{r+1}-d_0)m-\mathcal{E}_{r+1}j}\ ,\ (x,y) \in O_v^\times \times O_v^\times\ . \tag{5.20}$$

If $j = m\tau_i$, $i = 1, 2, .., r$, then

$$| \widetilde{f^{(m)}}(x, \theta + \pi^k y) |_v = q^{-(\mathcal{D}_i - d_0)m - \mathcal{E}_i j} \, | \widetilde{f^{(m)}_{F_m(Q_i)}}(x, y) |_v \,, \quad (x, y) \in O_v^\times \times O_v^\times \,, \tag{5.21}$$

with

$$\widetilde{f^{(m)}_{F_m(Q_i)}}(x, y) = \sum_{\widetilde{w}_{i,\theta}(F_m(Q_i))=0} \gamma_i(x, y) y^{e_{i,\theta}};$$

here $\widetilde{w}_{i,\theta}(\widetilde{z})$ is the straight line corresponding to the term

$$\pi^{(d_i - d_0)m + ke_{i,\theta}} \gamma_i(x, y) y^{e_{i,\theta}}.$$

Then by using (5.19),(5.20), and (5.21), it follows that

$$\begin{aligned} J(s, m, \theta) &= \sum_{k=1}^{\infty} q^{-k} \int_{O_v^\times \times O_v^\times} | \widetilde{f^{(m)}}(x, \theta + \pi^k y) |_v^s | \, dxdy \, |= \\ & (1 - q^{-1})^2 \sum_{i=0}^{r-1} \left(q^{-(\mathcal{D}_{i+1} - d_0)ms} \sum_{j=[m\tau_i]+1}^{[m\tau_{i+1}]-1} q^{-(1 + s\mathcal{E}_{i+1})j} \right) \\ & + (1 - q^{-1})^2 \sum_{j=[m\tau_r]+1}^{\infty} q^{-(\mathcal{D}_{r+1} - d_0)ms - (1 + s\mathcal{E}_{r+1})j} \\ & + \sum_{i=1}^{r} q^{-(\mathcal{D}_i - d_0)ms - (1 + s\mathcal{E}_i)[m\tau_i]} I_i(s), \end{aligned} \tag{5.22}$$

with

$$I_i(s) := \int_{O_v^{\times 2}} | \widetilde{f^{(m)}_{F_m(Q_i)}}(x, y) |_v | \, dxdy \, | \,.$$

Since

$$f_{Q_i} \circ \Phi(\pi^{ma} x, \pi^{mb}(\theta + \pi^k y)) = \pi^{md_0} x^M y^N \widetilde{f^{(m)}_{F_m(Q_i)}}(x, \theta + \pi^k y),$$

and f_{Q_i} does not have singularities on $L_v^{\times 2}$, and Φ is a K-analytic isomorphism on $L_v^{\times 2}$(cf. proposition 5.1); it follows that $\widetilde{f^{(m)}_{F_m(Q_i)}}(x, \theta + \pi^k y)$ does not have singularities on $L_v^{\times 2}$. Therefore by lemma 2.2, it follows that

$$I_i(s) = \frac{L_i(q^{-s})}{1 - q^{-1-s}}, \quad L_i(q^{-s}) \in \mathbb{Q}[q^{-s}], \; i = 1, 2, .., r. \tag{5.23}$$

Finally, the result follows from (5.22), by using the algebraic identity

$$\sum_{k=A}^{B} z^k = \frac{z^A - z^{B+1}}{1 - z}. \tag{5.24}$$

□

Lemma 5.1. *Let $f(x, y) \in O_v[x, y]$ be a semiquasihomogeneous polynomial, with respect to the weight (a, b), a, b coprime, and*

$$f^{(m)}(x, y) = \pi^{-d_0} f(\pi^a x, \pi^b y) = \sum_{j=0}^{l_f} \pi^{(d_j - d_0)m} f_j(x, y), \tag{5.25}$$

with $f_j(x,y)$ as in (4.2), and $\alpha_{i,j}$, $c_j \in O_v^\times$, for every i,j, and $\alpha_{i,j} \not\equiv \alpha_{r,t} \bmod \pi$, when $(i,j) \neq (r,t)$. If $f(x,y)$ is arithmetically non degenerate with respect to $\Gamma^A(f)$, then

$$I(s,f^{(m)},v,O_v^{\times 2}) = q^{-2}(q-1-l_0)(q-1) + \sum_{\{\theta\in O_v^\times \mid f_0(1,\theta^a)=0\}} J(s,m,\theta), \tag{5.26}$$

where the functions $J(s,m,\theta)$ were explicitly computed in proposition 5.3.

Proof. The result follows immediately from propositions 5.2, and 5.3, by using the fact that $f(x,y)$ is arithmetically non degenerate with respect to $\Gamma^A(f)$. □

5.2. **Some algebraic identities.** The following algebraic identities follows easily from (5.24).

These identities will be used later on, for the explicit computation of certain $p-$adic integrals.

$$\begin{aligned} S_1(s,i) &:= \sum_{m=1}^{\infty} q^{-(a+b)m-d_0ms-(\mathcal{D}_{i+1}-d_0)ms-(1+s\mathcal{E}_{i+1})[m\tau_i]} \\ &= \begin{cases} \frac{q^{-(a+b)-d_0ms}}{1-q^{-(a+b)-d_0ms}}, & \text{if } i=0, \\ \sum_{l=0}^{\mathcal{E}_i-\mathcal{E}_{i+1}-1}\left(\frac{q^{-B_{i,l}-A_l(\gamma_i+\delta_i s)}}{1-q^{-\gamma_i-\delta_i s}}\right), & \text{if } i=1,..,r, \end{cases} \end{aligned} \tag{5.27}$$

with

(5.28)

$\tau_i = \frac{\mathcal{D}_{i+1}-\mathcal{D}_i}{\mathcal{E}_i-\mathcal{E}_{i+1}}$,
$A_l := \begin{cases} 0, \text{ if } l\neq 0, \\ 1, \text{ if } l=0. \end{cases}$
$B_{i,l} := (a+b)l + [l\tau_i] + s([l\tau_i]\mathcal{E}_{i+1} + l\mathcal{D}_{i+1})$,
$\gamma_i := (a+b)(\mathcal{E}_i-\mathcal{E}_{i+1}) + (\mathcal{D}_{i+1}-\mathcal{D}_i)$,
$\delta_i := \mathcal{D}_{i+1}(\mathcal{E}_i-\mathcal{E}_{i+1}) + (\mathcal{D}_{i+1}-\mathcal{D}_i)\mathcal{E}_{i+1}$.

$$\begin{aligned} S_2(s,i) \;&:= \sum_{m=1}^{\infty} q^{-(a+b)m-d_0ms-(\mathcal{D}_{i+1}-d_0)ms-(1+s\mathcal{E}_{i+1})[m\tau_{i+1}]} \\ &= \sum_{l=0}^{\mathcal{E}_{i+1}-\mathcal{E}_{i+2}-1}\left(\frac{q^{-G_{i,l}-A_l(\rho_i+\sigma_i s)}}{1-q^{-\rho_i-\sigma_i s}}\right), \quad i=0,..,r-1, \end{aligned} \tag{5.29}$$

with

(5.30)

$\tau_{i+1} = \frac{\mathcal{D}_{i+2}-\mathcal{D}_{i+1}}{\mathcal{E}_{i+1}-\mathcal{E}_{i+2}}$,
$G_{i,l} := (a+b)l + [l\tau_{i+1}] + s([l\tau_{i+1}]\mathcal{E}_{i+1} + l\mathcal{D}_{i+1})$,
$\rho_i := (a+b)(\mathcal{E}_{i+1}-\mathcal{E}_{i+2}) + (\mathcal{D}_{i+2}-\mathcal{D}_{i+1})$,
$\delta_i := \mathcal{D}_{i+1}(\mathcal{E}_{i+1}-\mathcal{E}_{i+2}) + (\mathcal{D}_{i+2}-\mathcal{D}_{i+1})\mathcal{E}_{i+1}$.

(iii)

$$S_3(s,r) := \sum_{m=1}^{\infty} q^{-(a+b)m-d_0ms-(\mathcal{D}_{r+1}-d_0)ms-(1+s\mathcal{E}_{r+1})[m\tau_r]} = S_1(s,r).$$

Remark 5.2. With the above notation, it holds that the real parts of the poles of $S_1(s,i)$, $S_2(s,i)$, belong to the set $\mathcal{P}(\Gamma_{f,\theta})$.

5.3. **Proof of Theorem 5.1.** The integral $Z(s,f,v,\Delta_3)$ admits the following expansion:

$$
\begin{aligned}
Z(s,f,v,\Delta_3) &= \sum_{m=1}^{\infty} \int\limits_{\pi^{am}O_v^\times \times \pi^{bm}O_v^\times} \mid f(x,y) \mid_v^s \mid dxdy \mid \\
&= \sum_{m=1}^{\infty} q^{-(a+b)m\ -d_0\ ms} \int\limits_{O_v^{\times 2}} \mid f^{(m)}(x,y) \mid_v^s \mid dxdy \mid, \qquad (5.31)
\end{aligned}
$$

with

$$f^{(m)}(x,y) = \pi^{-md_0} f(\pi^{am}x, \pi^{bm}y) = \sum_{j=0}^{l_f} \pi^{(d_j-d_0)m} f_j(x,y).$$

By lemma 5.1, the integral

$$I(f^{(m)},s,v) = \int\limits_{O_v^{\times 2}} \mid f^{(m)}(x,y) \mid_v^s \mid dxdy \mid$$

satisfies

$$I(s,f^{(m)},v,O_v^{\times 2}) = q^{-2}(q-1-l_0)(q-1) + \sum_{\{\theta\in O_v^\times \mid f_0(1,\theta^a)=0\}} J(s,m,\theta), \qquad (5.32)$$

with l_0 :=Card$(\{\theta \in O_v^\times \mid f_0(1,\theta^a)=0\}$. The functions $J(s,m,\theta)$ were computed in proposition 5.3, for a fixed θ. Thus from (5.31), and (5.32), it follows that

$$
\begin{aligned}
Z(s,f,v,\Delta_3) &= q^{-2}(q-1-l_0)(q-1)\frac{q^{-(a+b)-d_0 s}}{1-q^{-(a+b)-d_0 s}} \qquad (5.33)\\
&+ \sum_{\{\theta\in O_v^\times \mid f_0(1,\theta^a)=0\}} \left(\sum_{m=1}^{\infty} q^{-(a+b)m\ -d_0\ ms} J(s,m,\theta) \right).
\end{aligned}
$$

By using the explicit formula for $J(s,m,\theta)$ given in proposition 5.3 and remark 5.2, we have that

$$
\begin{aligned}
\sum_{m=1}^{\infty} q^{-(a+b)m\ -d_0\ ms} J(s,m,\theta) &= (1-q^{-1})^2 \sum_{i=0}^{r-1} \frac{q^{-(s\mathcal{E}_{i+1}+1)}}{1-q^{-(s\mathcal{E}_{i+1}+1)}} S_1(s,i) \\
&-(1-q^{-1})^2 \sum_{i=0}^{r-1} \frac{1}{1-q^{-(1+s\mathcal{E}_{i+1})}} S_2(s,i) \\
&+(1-q^{-1})^2 \frac{q^{-(1+s\mathcal{E}_{r+1})}}{1-q^{-(1+s\mathcal{E}_{r+1})}} S_1(s,r) \\
&+\sum_{i=1}^{r} I_i(s) S_2(s,i-1), \qquad (5.34)
\end{aligned}
$$

where the data $\mathcal{E}_i$, $\mathcal{D}_i$, $i=1,2,..,r$, depend on the arithmetic Newton polygon $\Gamma_{f,\theta}$. Now by remark 5.2, the real parts of the poles of $S_1(s,i)$, $S_2(s,i)$, belong to the set $\mathcal{P}(\Gamma_{f,\theta})$. Therefore the real parts of the poles of

$$\sum_{\{\theta\in O_v^\times \mid f_0(1,\theta^a)=0\}} \left(\sum_{m=1}^{\infty} q^{-(a+b)m\ -d_0\ ms} J(s,m,\theta) \right)$$

belong to the set $\bigcup_{\{\theta\in O_v^\times \mid f_0(1,\theta^a)=0\}} \mathcal{P}(\Gamma_{f,\theta})$; and from (5.33), it follows that the real parts of the poles of $Z(s,f,v,\Delta_3)$ belong to the set

$$\{-1\}\bigcup\{-\frac{a+b}{d_0}\}\bigcup \bigcup_{\{\theta\in O_v^\times \mid f_0(1,\theta^a)=0\}} \mathcal{P}(\Gamma_{f,\theta}).$$

6. Main results

In this section we prove the main result of this paper, that is the description of the real parts of the poles of the local zeta functions attached to arithmetically non degenerate polynomials. The proof will be accomplished by reducing it to the case of semiquasihomogeneous polynomials.

6.1. **Local zeta functions for arithmetically non degenerate semiquasihomogeneous curves.** Let $f(x,y)$ be an arithmetically non degenerate semiquasihomogeneous polynomial with coefficients in a number field L. In this subsection we shall describe the poles of $Z(s,f,v)$ for almost all non archimedean valuations v of L. We denote by L_v the corresponding local field.

Theorem 6.1. *Let $f(x,y)\in L[x,y]$ be a semiquasihomogeneous polynomial with respect to the weight (a,b), a, b, coprime. If $f(x,y)$ is arithmetically non degenerate with respect to its arithmetic Newton polygon $\Gamma^A(f)$, then for almost all non-archimedean valuations v of L, the real parts of the poles of $Z(s,f,v)$ belong to the set*

$$\{-1\}\bigcup\{-\frac{a+b}{d_0}\}\bigcup\mathcal{P}(\Gamma^A(f)), \tag{6.1}$$

where d_0 is the weighted degree of the quasihomogeneous part of $f(x,y)$.

Proof. We consider the conic subdivision $\mathbb{R}_+^2=\{(0,0)\}\bigcup\bigcup_{i=1}^5\Delta_i$, induced by the geometric Newton polygon $\Gamma^{geom}(f)$ of $f(x,y)$ (see subsection 2.1). From the above subdivision, it follows that the computation of $Z(s,f,v)$ is reduced to the computation of the integrals $Z(s,f,v,O_v^{\times 2})$, $Z(s,f,v,\Delta_i)$, $i=1,2,3,4,5$, such as it was explained at subsection 2.1. By lemmas 2.2,2.3, the real parts of the poles of the integrals $Z(s,f,v,O_v^{\times 2})$, $Z(s,f,v,\Delta_i)$, $i=1,2,4,5$, belong to the set

$$\{-1\}\bigcup\{-\frac{a+b}{d_0}\}. \tag{6.2}$$

Thus the problem is reduced to give a list of the real parts of the poles of the integral $Z(s,f,v,\Delta_3)$, for almost all v. Because the polynomial $f(x,y)$ satisfies all the conditions of theorem 5.1 for almost all v, it holds that the real parts of the poles of $Z(s,f,v,\Delta_3)$ belong to the set

$$\{-1\}\bigcup\{-\frac{a+b}{d_0}\}\bigcup \bigcup_{\{\theta\in O_v^\times \mid f_0(1,\theta^a)=0\}} \mathcal{P}(\Gamma_{f,\theta}). \tag{6.3}$$

The result follows by doing the union of the sets (6.2) and (6.3). □

6.2. **Local zeta functions for arithmetically non degenerate curves.** The notion of arithmetic non degeneracy, for polynomials in two variables, appears naturally when it tries of computing local zeta functions of polynomials in two variables by using Newton polygons. Let $f(x,y)\in L_v[x,y]$ be a polynomial such that the origin of L_v^2 is a singular point and such that $f(x,y)$ does not have singular points on $L_v^{\times 2}$. As it was explained in section 2, there is a partition of $\mathbb{R}^2$, induced by the geometric polygon $\Gamma^{geom}(f)$, of the form

$$\mathbb{R}^2=\{(0,0)\}\bigcup \bigcup_{\gamma\subset\Gamma^{geom}(f)} \Delta_\gamma, \tag{6.4}$$

where Δ_γ is the corresponding cone to the proper face γ. We denote by $a_\gamma = (a_1(\gamma), a_2(\gamma))$ a perpendicular and primitive vector to facet γ of $\Gamma^{geom}(f)$; and by $< a, x >= d_a(\gamma)$, the equation of the corresponding supporting line.

The computation of $Z(s, f, v)$ is reduced to the computation of $Z(s, f, v, O_v^{\times 2})$, $Z(s, f, v, \Delta_\gamma)$, for γ a proper face of $\Gamma^{geom}(f)$ (see subsection 2.1).

Since $f(x, y)$ does not have singular points on the torus $L_v^{\times 2}$, the integral $Z(s, f, v, O_v^{\times 2})$ can be computed by using SPF repeatedly (cf. lemma 2.2).

The computation of the integrals $Z(s, f, v, \Delta_\gamma)$, for γ a proper face of $\Gamma^{geom}(f)$, involves two cases, according if the semiquasihomogeneous polynomial $f(x, y)$, with respect to $a_\gamma = (a_1(\gamma), a_2(\gamma))$, is an arithmetically non degenerate or not (see theorem 5.1).

Our definition of arithmetic non degeneracy for curves were introduced in a such a way that the computation of the corresponding local zeta functions can be done as was described above.

Given a polynomial $f(x, y)$ with coefficients in a field L, we put

$$\mathcal{P}(\Gamma^{geom}(f)) := \{-\frac{a_1(\gamma) + a_2(\gamma)}{d_a(\gamma)} \mid \gamma \text{ a facet of } \Gamma^{geom}(f)\},$$

and

$$\mathcal{P}(\Gamma^A(f)) := \bigcup_{\gamma \text{ a facet of } \Gamma^{geom}(f)} \mathcal{P}(\Gamma^A(f_\gamma));$$

here the polynomial $f(x, y)$ is considered as a semiquasihomogeneous polynomial with respect to the weight a_γ.

The following is the main result of the present paper.

Theorem 6.2 (Main Theorem). *Let $f(x, y)$ be polynomial with coefficients in a number field L. If $f(x, y)$ is arithmetically non degenerate with respect to its arithmetic newton polygon $\Gamma^A(f)$, then for almost all non-archimedean valuations v of L, the real parts of the poles of $Z(s, f, v)$ belong to the set*

$$\{-1\} \bigcup \mathcal{P}(\Gamma^{geom}(f)) \bigcup \mathcal{P}(\Gamma^A(f)). \tag{6.5}$$

Proof. The theorem follows from theorem 5.1, by the previous considerations. □

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