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C^ℓ-Determinacy of Weighted Homogeneous Germs

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A C^∞ map germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ is C^ℓ-G- determined, $0 \leq \ell < \infty$, ($G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$) if for each germ g such that $j^\ell g(0) = j^\ell f(0)$, g is C^ℓ-G-equivalent to f.

We give estimates for the degree of C^ℓ-G-determinacy of f, $0 \leq \ell < \infty$,($G = \mathcal{R}$,$\mathcal{C}$ or $\mathcal{K}$), of weighted homogeneous map-germs satisfying a convenient Lojasiewicz condition. Our work extends the results given in [R], where the first author obtains estimates for the degree of C^ℓ-G-determinacy of homogeneous map-germs.

The question of determining the degree of C^0-G-determinacy of f has been considered by several authors (e.g. [DG], [D], [P]), but those results do not include the C^ℓ case, $0 < \ell < \infty$. As an application, we use the degree of C^1-determinacy and the Newton diagram to obtain equisingular deformations in the Briançon-Speder example.

1 Basic definitons

The basic notation is the same as in [R] or [W].

The groups C^ℓ-G, ($G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$; $0 \leq \ell < \infty$) are defined as the groups $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$, taking diffeomorphisms of class C^ℓ, $\ell \geq 1$, or homeomorphisms, when $\ell = 0$.

The ring of C^∞ map-germs $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ is denoted by C_n and m_n denotes its maximal ideal.

As in [W], given $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ we denote by $I_{\mathcal{R}}f$ the ideal of C_n generated by the $p \times p$ minors of the jacobean matrix of f, by $I_{\mathcal{C}}f$ the ideal generated by the coordinate functions of f, and by $I_{\mathcal{K}}f$ the ideal $I_{\mathcal{R}}f + I_{\mathcal{C}}f$.

Let $N_{\mathcal{C}}f(x) = |f(x)|^2$, $N_{\mathcal{R}}f(x) = |df(x)|^2 = \sum_j M_j^2$, where the M_j are the generators of $I_{\mathcal{R}}f$ and $N_{\mathcal{K}}f = N_{\mathcal{R}}f + N_{\mathcal{C}}f$.

We say that $N_G f$ satisfies a Lojasiewicz condition if there exist constants $c > 0$ and $\alpha > 0$ such that $N_G f(x) \geq c|x|^\alpha$.

Proposition 1.1 ([W]) *$N_G f(x)$ satisfies a Lojasiewicz condition if and only if f is finitely C^ℓ-G-determined for any ℓ, $0 \leq \ell < \infty$.*

Definition 1.2 *$f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ is weighted homogeneous of type $(r_1, r_2, \ldots, r_n : d_1, d_2, \ldots, d_p)$ if for all $\lambda \in \mathbb{R} - \{0\}$ and r_i, $d_j \in \mathbb{Q}^+$:*

$$f\left(\lambda^{r_1}x_1, \lambda^{r_2}x_2, \ldots, \lambda^{r_n}x_n\right) = \left(\lambda^{d_1}f_1(x), \lambda^{d_2}f_2(x), \ldots, \lambda^{d_p}f_p(x)\right) .$$

Definition 1.3 *The filtration of a monomial $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$, with respect to a fixed $(r_1, r_2, \ldots, r_n)$ is $fil(x^\alpha) = \Sigma\alpha_i r_i$, $i = 1, \ldots, n$.*

The filtration of a germ f in C_n is defined by $\text{fil}(f) = \min \text{fil}(x^\alpha)$ where $x^\alpha \in j^\infty(f)$ (the Taylor series of f). This definition can be extended to C_{n+r}, the ring of r-parameter families of germs in n-variables, by defining $\text{fil}(x^\alpha t^\beta) = \text{fil}(x^\alpha)$.

The filtration of a map germ $f = (f_1, f_2, \ldots, f_p)$ is defined by $\text{fil}(f) = (d_1, d_2, \ldots, d_p)$, where d_i is the filtration of each f_i.

2 Estimates for the degree of C^ℓ-determinacy

Definition 2.1 *Let $(r_1, r_2, \ldots, r_n; 2K)$ be fixed. We define the standard control function $\rho_K(x)$ by $\rho_K(x) = x_1^{2\alpha_1} + x_2^{2\alpha_2} + \cdots + x_n^{2\alpha_n}$, where the a_i are chosen in such a way that the function ρ_K is weighted homogeneous of type $(r_1, r_2, \ldots, r_n; 2K)$.*

We observe that $\rho_K(x)$ satisfies a Lojasiewicz condition $\rho_K(x) \geq c|x|^{2\alpha}$ for some constants c and α.

Lemma 1 *Let $h(x)$ be a weighted homogeneous polynomial of type $(r_1, \ldots, r_n; 2K)$ and $h_t(x)$, $t \in [0,1]$, a weighted homogeneous deformation of h. Then:*

a. *There exists a constant c_1 such that $|h_t(x)| \leq c_1 \rho_K(x)$.*

b. *If there exist constants c and α such that $|h_t(x)| \geq c|x|^\alpha$,*

   then $|h_t(x)| \geq c_2 \rho_K(x)$ for some constant c_2.

Proof. Let $M = \{(y,t) \in \mathbb{R}^n \times [0,1] / \rho_K(y) = 1\}$.

To prove (a), we first observe that for each pair (x,t) fixed, there is a pair $(y,t) \in M$, and a real number $\lambda \neq 0$, such that $(x,t) = (\lambda^{r_1} y_1, \ldots, \lambda^{r_n} y_n, t)$.

Now, let $c_1 = \sup\{h_t(y) / (y,t) \in M\}$. Then,

$$h_t(x) = h_t(\lambda y) = \lambda^{2k} h_t(y) \leq \lambda^{2k} c_1 \rho_K(y) = c_1 \rho_K(x).$$

To prove (b), let $c_2 = \inf\{h_t(y) \text{ such that } (y,t) \in M\}$.

From the hypothesis, $c_2 > 0$, hence $c_2 \rho_K(x) = c_2 \lambda^{2k} \rho_K(y) \leq \lambda^{2k} h_t(y) = h_t(x)$.

Lemma 2 *Let $\rho(x)$ the standard control of type $(r_1, \ldots, r_n; 2K)$, $r_1 \leq r_2 \leq \cdots \leq r_n$, $h_t(x)$ a deformation of h such that:*

$$\mathit{fil}(h_t) \geq 2k + \ell r_n + 1, \quad t \in [0,1], \quad \ell \geq 1 \ .$$

Then the function $\nu(x) = \frac{h_t(x)}{\rho(x)}$ is differentiable of class C^ℓ.

Proof. We will proceed by induction on the class of differentiability.

First we consider $\ell = 1$.

The gradient of $\nu(x)$ is given by $\nabla\nu(x) = \frac{\nabla h_t(x)}{\rho(x)} - \frac{\nabla\rho(x)}{\rho(x)} \frac{h_t(x)}{\rho(x)}$, with $\text{fil}(\nabla\rho(x)) \geq 2k - r_n$ and $\text{fil}(h_t(x)) \geq 2k - r_n + 1$. Hence $\text{fil}|(\nabla\rho(x)) \cdot h_t(x)| \geq 4k + 1$.

Each term of $\nabla\nu(x)$ is of form $\frac{g(x) \cdot m(x)}{\rho(x)}$, where $m(x)$ is weighted homogeneous of type $(r_1, \ldots, r_n; 2K)$ and $\lim_{x \to 0} g(x) = 0$. It follows from Lemma 1 that $\frac{m(x)}{\rho(x)}$ is bounded, hence $\nabla\nu(x)$ is continuous.

Let us assume by induction that for all $\nu = \frac{h}{\rho}$ with $\text{fil}(h) \geq 2k + (\ell - 1)r_n + 1$, ν is of class $C^{\ell-1}$.

Given $\nu = \frac{h}{\rho}$ with $\mathrm{fil}(h) \geq 2k + \ell r_n + 1$, then $\nabla\nu(x) = \frac{H(x)}{\rho(x)}$ with $\mathrm{fil}(H) \geq 2k + (\ell - 1)r_n + 1$, and ν is of class C^ℓ.

Case 1: $G = \mathcal{R}$.

Proposition 2.2 *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a weighted homogeneous polynomial map germ of type $(r_1, \ldots, r_n\,; d_1, \ldots, d_p)$ with $r_1 \leq \cdots \leq r_n$, $d_1 \leq \cdots \leq d_p$, satisfying a Lojasiewicz condition $N_{\mathcal{R}} f(x)| \geq c|x|^\alpha$, for constants c and α. Then:*

(a) Deformations of f defined by $f_t(x) = f(x) + t\Theta(x)$, $\Theta = (\Theta_1, \ldots, \Theta_p)$ with $fil(\Theta_i) \geq d_i - r_1 + \ell r_n + 1$, for all i, $\ell \geq 1$ and $t \in [0,1]$ are C^ℓ-$\mathcal{R}$-trivial.

(b) Weighted homogeneous deformations f_t of f of type $(r_1, \ldots, r_n, d_1, \ldots, d_p)$ are C^0-$\mathcal{R}$-trivial, for small t.

We observe that for each $p \times p$ minor M_I of df, there is an s_I such that M_I is weighted homogeneous of type $(r_1, \ldots, r_n; s_I)$.

Let $N^*_{\mathcal{R}} f$ be defined by $N^*_{\mathcal{R}} f = \sum_I M_I^{2\alpha_I}$, where $\alpha_I = \frac{l.c.m.(s_I)}{s_I}$. Then, $N^*_{\mathcal{R}} f$ is a weighted homogeneous control function of type $(r_1, \ldots, r_n; 2K)$.

For deformations f_t of f, we define the control $N^*_{\mathcal{R}} f_t$ by $N^*_{\mathcal{R}} f_t = \sum_I M_{t_I}^{2\alpha_I}$, where M_{t_I} are the $p \times p$ minors of J_{f_t}, and the α_I are the same as above. If f_t is weighted homogeneous of same type than f, then $N^*_{\mathcal{R}} f_t$ is weighted homogeneous of type $(r_1, \ldots, r_n; 2K)$ for all t. If $f_t(x) = f(x) + t\Theta(x)$ and $\mathrm{fil}(\Theta_i) \geq d_i$, it follows that $\mathrm{fil}(N^*_{\mathcal{R}} f_t) \geq \mathrm{fil}(N^*_{\mathcal{R}} f)$.

Lemma 3 *There exist constants c_1 and c_2 such that:*

$$c_2 \rho_K(x) \leq N^*_{\mathcal{R}} f_t \leq c_1 \rho_K(x) \; .$$

Proof. When f_t is weighted homogeneous of same type than f, the result follows from Lemma 1.

If $\mathrm{fil}(f_t) > \mathrm{fil}(f)$, we write $N^*_{\mathcal{R}} f_t = N^*_{\mathcal{R}} f + tR(x,t)$ where $R(x,t)$ is polynomial with $\mathrm{fil}(R(x,t)) > \mathrm{fil}(N^*_{\mathcal{R}} f)$. Then $N^*_{\mathcal{R}} f \leq N^*_{\mathcal{R}} f_t + |R_t(x)|$, for $0 \leq t \leq 1$. By Lemma 1, there exists a constant c_2 such that: $c_2 \rho_K(x) \leq N^*_{\mathcal{R}} f \leq N^*_{\mathcal{R}} f_t + |R_t(x)|$.

Since $\mathrm{fil}(R(x,t)) > \mathrm{fil}(N^*_{\mathcal{R}} f)$, it follows that $\frac{|R_t(x)|}{\rho_k(x)} \to 0$ when $x \to 0$ (Lemma 2).

Thus $c_2\rho_k(x) \leq N^*_{\mathcal{R}} f_t$.

It is easy to see that there exists a constant c_1 such that $N^*_{\mathcal{R}} f_t \leq c_1\rho(x)$ for small t.

Proof of the Proposition 2.2.

(a) Let M_{t_I} a $p \times p$ minor of J_{f_t}, $I = (i_1, i_2, \ldots, i_p) \subset (1, 2, \ldots, n)$. Then, there exists a vector field W_I associated to M_{t_I}, such that:

$$(1.2.1) \qquad \tfrac{\partial f_t}{\partial t} M_{t_I} = df(W_I), \text{ where } W_I = \sum_1^n w_i \frac{\partial}{\partial x_i}, \text{ with:}$$

$$\begin{cases} w_i = 0; \text{ if } i \notin I \\ w_{i_k} = \sum_{j=1}^p N_{ji_k} \left(\frac{\partial f_t}{\partial t}\right)_j \end{cases} ; \quad i_k \in I \text{ and } N_{ji_k} \text{ is the } (p-1) \times (p-1) \text{ minor}$$

cofactor of $\frac{\partial f_j}{\partial x_{i_k}}$ in df. (See [G] or [R] for more details).

Then $\mathrm{fil}(W_{i_k}) = \min\left(\mathrm{fil}(N_{ji_k}) + \mathrm{fil}\left(\frac{\partial f_t}{\partial t}\right)_j\right)$, $j = 1, \ldots, p$, that is:

$$\mathrm{fil}(w_{i_k}) = d - r_I + r_{i_k} - r_1 + \ell r_n + 1$$

where $d = d_1 + d_2 + \cdots + d_p$ and $r_I = r_{i_1} + r_{i_2} + \cdots + r_{i_p}$.

The least possible filtration of w_i for $i = 1, \ldots, n$ is $\mathrm{fil}(w_1)$, and $\mathrm{fil}(w_1) = d - r + \ell r_n + 1$, where $r = r_1 + r_2 + \cdots + r_p$.

From (1.2.1), $\frac{\partial f_t}{\partial t} N^*_{\mathcal{R}} f_t = df_t(W_R)$, where $W_R = \sum_I M_I^{2\alpha_I - 1} w_i$. Then $\mathrm{fil}(W_R) = 2k + \ell r_n + 1$.

Let $\nu : \mathbb{R}^n \times \mathbb{R}, 0 \to \mathbb{R}^n \times \mathbb{R}, 0$ be the vector field defined by $\nu(x) = \frac{W_R}{N^*_{\mathcal{R}} f_t}$. By Lemma 2, ν is of class C^ℓ. The equation $\frac{\partial f_t}{\partial t}(x,t) = (df_t)_x(x,t)(\nu(x,t))$ implies the C^ℓ-$\mathcal{R}$-triviality of the family $f_t(x)$ in a neighbourhood of $t = 0$. Since the same argument is true in a neighbourhood of $t = \bar{t}$, $\forall t \in [0,1]$, the proof is complete.

(b) The vector field is constructed as in case (a). Here, $\mathrm{fil}(W_R) \geq 2k + r_1$, and $\mathrm{fil}(W_{R_i}) \geq 2k + r_i$, where the W_{R_i} are the components of W_R. Then, the vector field $\nu(x) = \frac{W_R}{N^*_{\mathcal{R}} f_t}$ is continuous.

Furthermore, $\nu(x,t) \leq c|x|$, and this condition implies the integrability of the vector field $\nu \cdot$ ([K])

Case 2: $G = \mathcal{C}$.

Let $N_C^* f = \sum_{i=1}^p f_i^{2\beta_i}$, where $\beta_i = k/d_i$, and $k = l.c.m.(d_i)$.

Given a deformation f_t of f, $f_t = f + t\Theta$, let $N_C^* f_t = \sum_i f_{ti}^{2\beta_i}$, where each β_i is as above.

Proposition 2.3 *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a weighted homogeneous polynomial map germ of type $(r_1, \ldots, r_n; d_1, \ldots, d_p)$ with $r_1 \leq \cdots \leq r_n$, $d_1 \leq \cdots \leq d_p$, satisfying a Lojasiewicz condition $N_C(f(x)) \geq c|x|^\alpha$, for constants c and α. Then:*

(a) Deformations $f_t = f + t\Theta$ of f, with $fil(\Theta_i) \geq d_p + \ell r_n + 1$, for all $i, t \in [0,1]$ and $\ell \geq 1$ are C^ℓ-$\mathcal{C}$-trivial.

(b) Small deformations of f, with $fil(\Theta_i) = d_p + 1$, $i = 1, \ldots, p$ are C^0-$\mathcal{C}$-trivial.

Proof. **(a)** C^ℓ-$\mathcal{C}$-triviality of the family f_t is obtained by constructing C^ℓ-map germs V_i, $V_i : \mathbb{R}^n \times \mathbb{R}, 0 \to \mathbb{R}^p, 0$; $i = 1, \ldots, p$, $V_i = \left(V_{i_1}, V_{i_2}, \ldots, V_{i_p}\right)$, with $V_{i_j}(x, 0) = \delta_{i_j}(x)$ in such a way that: $\frac{\partial f_t}{\partial t} = \sum_{i=1}^p V_i(x,t)(f_{ti})$.

Since $\frac{\partial f_t}{\partial t} = \frac{\sum_{i=1}^p (f_t)_i^{2\beta_i - 1}\left(\frac{\partial f_t}{\partial t}\right)}{N_C^* f_t}(f_{ti})$, we define $W_i(x,t) = (f_{ti})^{2\beta_i - 1}\left(\frac{\partial f_t}{\partial t}\right)$. Then $\frac{\partial f_t}{\partial t} = \frac{\sum_{i=1}^p W_i(x,t)}{N_C^* f_t}(f_{ti})$ with fil $(W_i(x,t)) \geq 2k + \ell r_n + 1$ for all i.

Let $V : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}, 0 \to \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}, 0$ be the vector field defined by: $(0, V_p, 0)$, where $V_p(x,y,t) = \sum_{i=1}^p \frac{W_i(x,t)}{N_C^* f_t} y_i$. As V is of class C^ℓ, the result follows by integrating V.

Case (b) is analogous to (a).

In order to obtain a better estimate, as in [R] we prove the following lemma:

Let c be a constant such that $|f_{ti}(x)|^2 \leq c\rho(x)$, and V and U be neighbourhoods of the region $|y| < c\rho(x)^{1/2}$ in $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} - \{0, 0, t\}$,

$$V = \left\{(x,y,t) \big/ |y| \leq c_1 \rho(x)^{1/2}, \text{ with } c_1 > c\right\}$$

and U is chosen in such a way that $U \subset \overline{U} \subset V$.

Lemma 4 *There exists a conic bump function* $p : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}$ *such that:* $p_{|\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} - \{0,0,t\}}$ *is smooth, and*

$$\begin{cases} p(x,y,t) = 1, \text{ for all } (x,y,t) \in \overline{U} \\ p(x,y,t) = 0, \text{ outside of } V \\ 0 \leq p(x,y,t) \leq 1, \text{ in } V - \overline{U} \\ p(0,0,t) = 0, \text{ for all } t \end{cases}$$

Proof. We define the function $p(x,y_i) = h(\Theta_i)$, with $\Theta_i = \frac{y_i}{\rho(x)^{1/2}}$, where for each i, the set $|y_i| \leq c\rho(x)^{1/2}$, is in $\mathbb{R}^n \times \mathbb{R}^p$ and $h : \mathbb{R} \to \mathbb{R}$ is the usual bump function,

$$\begin{cases} h(\Theta) = 1 & , \text{ if } 0 \leq \Theta \leq \Theta_1 ; \\ h(\Theta) = 0 & , \text{ if } \Theta \geq \Theta_2 ; \\ 0 \leq h(\Theta) \leq 1 & , \text{ if } \Theta_1 < \Theta < \Theta_2 . \end{cases}$$

Since $|f_{t_i}| \leq c\rho(x)^{1/2}$, for a constant c, we have:

$$\begin{cases} h(\Theta_i) \leq 1 & \text{if } |y_i| \leq c\rho(x)^{1/2} \\ 0 \leq h(\Theta_i) \leq 1 & \text{if } c\rho(x)^{1/2} \leq |y_i| \leq c_1\rho(x)^{1/2} \\ h(\Theta_i) = 0 & \text{if } c_1\rho(x)^{1/2} \leq |y_i| . \end{cases}$$

The desired conic bump function is defined by:

$$p(x,y,t) = p(x,y_1)p(x,y_2)\cdots p(x,y_p).$$

Proposition 2.4 *Let* $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ *be a weighted homogeneous polynomial map germ of type* $(r_1, \ldots, r_n; d_1, \ldots, d_p)$ *with* $r_1 \leq \cdots \leq r_n$, $d_1 \leq \cdots \leq d_p$, *satisfying a Lojasiewicz condition:* $N_C|f(x)| \geq c|x|^\alpha$, *for constants* c *and* α. *Then:*

(a) *Deformations* $f_t = f + t\Theta$ *of* f , *with* $fil(\Theta_i) \geq d_p + \ell r_n$, $i = 1, \ldots, p$, $t \in [0,1]$ *and* $\ell \geq 1$ *are* C^ℓ-$\mathcal{C}$-*trivial.*

(b) *Small deformations of* f , *with* $fil(\Theta_i) = d_p$, $i = 1, \ldots, p$ *are* C^0-$\mathcal{C}$-*trivial.*

Proof. **(a)** Let V be the vector field defined by $V = \frac{W}{N_C^s f_t}$, where W is defined in a neighbourhood of $\{0,0,t\}$ in $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}$, with $W_j(x,y,t) = p(x,y,t)w_j(x,y,t)$ for $i \leq j \leq p$, and the w_j are defined as in Proposition 1.3.

We just have to check the class of differentiability of $W_j = (w_{j1}, \ldots, w_{jp})$ where: $w_{ji} = \frac{h_{ji}}{\rho_k(x)} p(x,y,t) y_i$, and $h_{ji} = \Theta_i f_{tj}^{2\beta_j - 1}$. Then, $\mathrm{fil}(h_{ji}) \geq 2k + \ell r_n$, and

$$|W_{ji}| = \left|\frac{h_{ji}}{\rho(x)}\right| |p y_j| \leq \left|\frac{h_{ji}}{\rho(x)}\right| (\rho(x))^{1/2} .$$

Applying Lemmas 4 and 2, we see that each W_{ji} is of class C^ℓ and W is of class C^ℓ.

As in proposition 2.3, case (b) is analogous to (a).

Case 3: $G = \mathcal{K}$.

Proposition 2.5 *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a weighted homogeneous polynomial map germ of type $(r_1, \ldots, r_n; d_1, \ldots, d_p)$ with $r_1 \leq \cdots \leq r_n$, $d_1 \leq \cdots \leq d_p$, satisfying a Lojasiewicz condition $N_{\mathcal{K}} f(x)| \geq c|x|^\alpha$, for constants c and α. Then:*

(a) Deformations $f_t = f + t\Theta$ of f , with fil$(\Theta_i) \geq d_p + \ell r_n$, $i = 1, \ldots, p$, $t \in [0,1]$ and $\ell \geq 1$ are C^ℓ-$\mathcal{K}$-trivial.

(b) Small deformations off , with fil$(\Theta_i) = d_p$, $i = 1, \ldots, p$ are C^0-$\mathcal{K}$-trivial.

Proof. Since the group C^ℓ-$\mathcal{K}$ is the semi-direct product of the groups C^ℓ-$\mathcal{R}$ and C^ℓ-$\mathcal{C}$, the vector fields are defined as in cases $G = \mathcal{R}$ and $\mathcal{C}$, and the control function $N^*_{\mathcal{K}} f$ is defined by: $N^*_{\mathcal{K}} f = N^*_{\mathcal{R}} f^\alpha + N^*_{\mathcal{C}} f^\beta$ where α and β are constants such that $N^*_{\mathcal{K}} f$ is weighted homogeneous.

As a consequence of the above results, we obtain a general estimate for the degree of C^ℓ-G-determinacy ($G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$).

Proposition 2.6 *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a weighted homogeneous polynomial map germ of type $(r_1, \ldots, r_n; d_1, \ldots, d_p)$, $r_1 \leq \cdots \leq r_n$, $d_1 \leq \cdots \leq d_p$, satisfying a Lojasiewicz condition: $N_G f(x)| \geq c|x|^\alpha$, for constants c and α.*

(a) f is k-C^ℓ-G-determined, where k is the least integer bigger than or equal to:

$\left(\frac{d_p + \ell r_n + 1}{r_1} - 2\right)$, $0 < \ell < \infty$,

(b) *f is k-C^0-G-determined,* $k = \frac{d_p}{r_1}$,

(c) *Small deformations of f, of degree* $\frac{d_p}{r_1}$ *are* C^0*-G-trivial.*

Example 2.7 Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (x^a - y^b; xy)$, with $a \geq b$ even integers. f is a weighted homogeneous map germ of type $(r, s; d, r+s)$, where $d =$ "l.c.m."(a,b) and $r = d/a$, $s = d/b$.

Let $f_t = f + t\Theta$,with $\Theta = (\Theta_1, \Theta_2)$ a deformation of f.

If $\text{fil}(\Theta_1) = d$ and $\text{fil}(\Theta_2) = s + r$, the family f_t is C^0-$\mathcal{R}$-trivial for small t.

The family f_t is C^ℓ-$\mathcal{R}$-trivial for any t if $\text{fil}(\Theta_1) \geq d - r + \ell s + 1$ and $\text{fil}(\Theta_2) \geq (\ell + 1)s + 1$.

For any $G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$, f is k-C^ℓ-G-determined, where $k = \frac{sup(d,r+s)}{r}$ if $\ell = 0$, and k is the least integer bigger than or equal to: $\left(\frac{sup(d,r+s)+\ell s+1}{r} - 2\right)$ if $\ell \geq 1$.

The Briançon-Speder example

Let $f(x,y,z) = z^5 + y^7x + x^{15}$, f is weighted homogeneous of type (1,2,3;15).

Briançon and Speder showed in [BS] that the family $F(x,y,z,t) = z^5 + y^7x + x^{15} + ty^6z$ is topologically trivial, but the variety $F^{-1}(0)$ is not equisingular along the parameter space.

From Proposition 2.2, it follows that deformations F of f, by terms with fil $\geq 15 + 3\ell$,($\ell \geq 1$) are C^ℓ-$\mathcal{R}$-trivial, hence the hipersurface $F^{-1}(0)$ is Whitney equisingular along the parameter space.

Since $\text{fil}(y^9) = 18$, the family $F(x,y,z,t) = z^5 + y^7x + x^{15} + ty^9$ is C^1-$\mathcal{R}$-trivial, hence $F^{-1}(0)$ is Whitney-equisingular along the parameter space. It is easy to see that the monomials x^{15} and z^5 belong to the integral closure of $< x_i \frac{\partial F}{\partial x_j} >$, and then equisingularity of deformations along these directions follows from Teissier's condition (c), ([T]). Then if we consider the new filtration $(r_1, r_2, r_3) = (3, 5, 9)$ associated to the monomials x^{15}, y^9 and z^5, it follows from condition (c) that for any deformation F of f by terms with filtration ≥ 45, the variety $F^{-1}(0)$ is Whitney equisingular at 0.

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