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A Lusternik-Schnirelmann Theorem for General Spaces

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RESUMO

Neste trabalho nós generalizamos um conhecido teorema de Lusternik- Schnirelmann sobre coberturas por fechados de n-esferas e funções antipodais, para espaços e funções mais gerais. Esta generalização nos permite obter um nova generalização do teorema de Borsuk-Ulam.

A LUSTERNIK-SCHNIRELMANN THEOREM FOR GENERAL SPACES

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Abstract

In this note we generalize a well known theorem of Lusternik and Schnirelmann about closed coverings of n-spheres and antipodal maps for more general spaces and maps. This generalization provides a new generalization of the Borsuk-Ulam Theorem

1 Introduction

A famous theorem due to Lusternik and Schnirelmann says the following

Theorem 1 *: Let $a : S^n \longrightarrow S^n$ be the antipodal map of the n-dimensional sphere S^n. In each family of n+1 closed sets covering S^n at least one set contains a pair of antipodal points $\{x, a(x)\}$*

In [1] we encounter the following

Theorem 2 *Let X be a paracompact Hausdorff space with topological dimension, $\dim X \leq n$. Suppose that ι is a fixed point free involution of X. Then there exists a closed cover $\{C_1, ..., C_k\}$ of X with $k \leq n+2$ such that no C_i contains a pair $\{x, \iota(x)\}$, $i = 1, ..., k$.*

This Theorem and others of the same paper was studied by the third author,under supervision of the first author, in his Master Thesis [6] and it inspired the present work which generalize Theorem 1 by changing the n-sphere for more general "nice" spaces with some homotopic restrictions. Based on our result we give a new generalization of the Borsuk-Ulam Theorem.

The antipodal map a in Theorem 1 generates a free $\mathbb{Z}_2$-action on S^n and provides the principal $\mathbb{Z}_2$-bundle $\xi_n = \xi(S^n, a) : \mathbb{Z}_2 \cdots \mathsf{S}^n \to \mathsf{S}^n/a = \mathbb{RP}^n$. Denoting the Universal principal $\mathbb{Z}_2$-bundle by $\gamma_{\mathbb{Z}_2} : \mathbb{Z}_2 \cdots \mathsf{E}\mathbb{Z}_2 \to \mathsf{B}\mathbb{Z}_2$, then $\mathsf{B}\mathbb{Z}_2$ is an Eilenberg-Mac Lane space $K(\mathbb{Z}_2, 1) \cong \mathbb{RP}^\infty$, so there exists a characteristic

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class $c_1(\gamma_{\mathbb{Z}_2}) = c_1 \in H^1(\mathsf{B}\mathbb{Z}_2; \mathbb{Z}_2)$ corresponding to the inverse of the Hurewicz isomorphism $\mathcal{H}_{\mathsf{B}\mathbb{Z}_2} : \pi_1(\mathsf{B}\mathbb{Z}_2) \to H_1(\mathsf{B}\mathbb{Z}_2)$. Thus for every principal $\mathbb{Z}_2$-bundle over a paracompact Hausdorff space $B(\xi)$, $\xi = f_\xi^!(\gamma_{\mathbb{Z}_2}) : \mathbb{Z}_2 \cdots E(\xi) \to B(\xi)$, where $f_\xi^!(\gamma_{\mathbb{Z}_2})$ is the pull-back of the bundle $\gamma_{\mathbb{Z}_2}$ by the classifying map $f_\xi : B(\xi) \to \mathsf{B}\mathbb{Z}_2$, we have the characteristic class $c_1(\xi) = f_\xi^*(c_1) \in H^1(B(\xi); \mathbb{Z}_2)$. This characteristic class satisfies the following well known

Lemma 1 *If ξ and η are principal $\mathbb{Z}_2$-bundles over the same paracompact, Hausdorff, path connected and locally path connected base space B then,*

$$\xi \equiv \eta \iff c_1(\xi) = c_1(\eta)$$

In particular $c_1(\xi) = 0$ if and only if ξ is trivial.

We can generalize the bundles ξ_n, changing the pair (S^n, a) by the pair (X, ι) where X is a paracompact, Hausdorff, path connected and locally path connected space and ι is a continuous involution without fixed points, and Theorem 1 can be generalized in this direction with the aid of Lemma 1. To be more precise, we prove the following

Theorem 3 *: Given a paracompact, Hausdorff, path connected and locally path connected space X and a continuous involution $\iota : X \longrightarrow X$ without fixed points such that*

i) There exists $\bar{e} \in \pi_1(X/\iota)$ with order$(\bar{e}) = 2$;

ii) $\langle c_1(\xi(X, \iota)), \mathcal{H}_{X/\iota}(\bar{e})\rangle \neq 0$, where $\mathcal{H}_{X/\iota}$ is the Hurewicz homomorphism.
then for each family of 3 closed sets, $\mathcal{F} = \{C_0, C_1, C_2\}$ covering X there exists at least one $C_j \in \mathcal{F}$ such that $\iota(C_j) \cap C_j \neq \emptyset$.

Theorem 4 *: Given an n-simple space X and a continuous involution ι, both satisfying the same hypotheses of Theorem 3 and in addition*

iii) For $2 \leq k < n$, $\quad 2\pi_k(X) \cong \begin{cases} \pi_k(X), & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even} \end{cases}$

then for each family of n+1 closed sets, $\mathcal{F} = \{C_0, C_1, ..., C_n\}$ covering X there exists at least one $C_j \in \mathcal{F}$ such that $\iota(C_j) \cap C_j \neq \emptyset$.

Remark 1 *The condition i) can not be dropped as can be seen by the example $(S^1 \times S^1, \iota)$ where $\iota(x, y) = (x, -y)$. The torus T^2 has a covering by 3 closed sets, C_0, C_1 and C_2 such that $C_j \cap \iota(C_j) = \emptyset$ for all $j = 0, 1, 2$ where $C_j = \{(x, (\cos\theta, \sin\theta)) \in T^2 : \frac{2\pi j}{3} \leq \theta \leq \frac{2\pi(j+1)}{3}\}$.*

Remark 2 *Theorem 1 is a consequence of the Borsuk-Ulam Theorem which says that any map $f : S^n \to \mathbb{R}^n$ must identify a pair of antipodal points of S^n. We say that a pair (X, ι) of a space X with a free continuous involution ι is of type LS(n) if (X, ι) satisfies the conclusion of Theorem 4. In this way every pair (X, ι) satisfying i) and ii) is of type LS(2) and if in addition it satisfies iii) then (X, ι) is of type LS(n). Note that, if (X, ι) is of type LS(n) then for any map $f : X \to \mathbb{R}^n$ there exists $x_0 \in X$ such that $f(x_0) = f \circ \iota(x_0)$. To see this suppose* *that there is a map $f : X \to \mathbb{R}^n$ such that $f(x) \neq f \circ \iota(x)$, $\forall\, x \in X$, then the map $h : (X, \iota) \to (S^{n-1}, a)$ given by $h(x) = \frac{f(x) - f\circ\iota(x)}{\|f(x) - f\circ\iota(x)\|}$ is an equivariant map ($a \circ h = h \circ \iota$), and by Theorem 2 we can find n+1 closed sets $C_0, C_1, ..., C_n$ of S^{n-1} such that $C_j \cap a(C_j) = \emptyset$ for $j = 0, 1, ..., n$ and from this it follows easily that there are closed sets $F_j = h^{-1}(C_j)$ of X, $j = 0, 1, ..., n$ such that $F_j \cap \iota(F_j) = \emptyset$ for all $j = 0, ...n$ which contradicts the LS type of X. Thus The pairs (X, ι) satisfying i), ii) and iii) also satisfies the Borsuk-Ulam Theorem. We will see in paragraph 3 a more general version of the Borsuk-Ulam Theorem for our spaces.*

2 Proof of Theorems 3 and 4

We need here the following lemma due to P. Olum [5].

Lemma 2 *: Let Y be a CW-complex, $A \subseteq Y$ be a path connected subspace, $Y_2 = A \cup Y^{(2)}$ (where $Y^{(2)}$ is the 2-skeleton of Y) and Z any topological space. If $f_0 : A \to Z$ is a continuous map and $\psi : \pi_1(Y, y_0) \to \pi_1(Z, z_0)$ is a homomorphism then, there exists a continuous map $f_2 : Y_2 \to Z$ such that $f_2|_A = f_0$ and $(f_2)_* = \psi$.*

Let (X, ι) be a pair formed by a space together with a continuous involution satisfying the hypotheses of Theorem 3, then we can construct the principal $\mathbb{Z}_2$-bundle $\xi(X, \iota) : \mathbb{Z}_2 \cdots X \to X/\iota$

Applying Lemma 2 for $Z = X/\iota$, $Y = \mathbb{RP}^\infty$ and $A = \{\text{point}\}$ the base point of $\mathbb{RP}^\infty$, then by the hypothesis i) of Theorem 3 we have that there exists a homomorphism $\psi : \pi_1(\mathbb{RP}^\infty) \to \pi_1(X/\iota)$ sending the non zero element $v \in \pi_1(\mathbb{RP}^\infty) \cong \mathbb{Z}_2$ to $\bar{e} \in \pi_1(X/\iota)$, we find a continuous map $f_2 : \mathbb{RP}^2 \to X/\iota$ such that $\psi = (f_2)_*$, the induced homomorphism on π_1 of f_2.

Using now ii) if $u = \mathcal{H}_{\mathbb{RP}^2}(v) \in H_1(\mathbb{RP}^2)$ we obtain that

$$\langle (f_2)^*(c_1(\xi(X, \iota))), u\rangle = \langle c_1(\xi(X, \iota)), (f_2)_*(u)\rangle = \langle c_1(\xi(X, \iota)), \mathcal{H}_{X/\iota}(\bar{e})\rangle \neq 0$$

and this implies that $(f_2)^*(c_1(\xi(X, \iota))) \neq 0$ in $H^1(\mathbb{RP}^2; \mathbb{Z}_2) \cong \mathbb{Z}_2$, then

$$(f_2)^*(c_1(\xi(X, \iota))) = j_2^*(c_1) = c_1(\xi_2)$$

because ξ_2 is not trivial, where $j_2 : \mathbb{RP}^2 \to \mathbb{RP}^\infty$ is the inclusion. Hence by Lemma 1 we conclude that $(f_2)^!(\xi(X, \iota)) = (f_\xi \circ f_2)^!(\gamma_{\mathbb{Z}_2}) \equiv j_2^!(\gamma_{\mathbb{Z}_2}) = \xi_2$.

We have then the following situation

$$\begin{array}{ccccc}
\mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_2 \\
\vdots & & \vdots & & \vdots \\
\mathsf{S}^2 & \xrightarrow{F_2} & \mathsf{X} & \xrightarrow{F_\xi} & \mathsf{E}\mathbb{Z}_2 \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{RP}^2 & \xrightarrow{f_2} & \mathsf{X}/\iota & \xrightarrow{f_\xi} & \mathsf{B}\mathbb{Z}_2
\end{array}$$

diagram 1

where F_2 and F_ξ are bundle maps.

Now if $X = C_0 \cup C_1 \cup C_2$ is a closed covering of X then $S^2 = H_0 \cup H_1 \cup H_2$ is a closed covering of S^2, where $H_j = F_2^{-1}(C_j)$, $j = 0, 1, 2$. Thus by Theorem 1 for at least one H_{j_0} we have $H_{j_0} \cap a(H_{j_0}) \neq \emptyset$. Since F_2 is a bundle map, for each $x \in S^2$ $F_2(a(x)) = \iota(F_2(x))$, hence

$$\begin{aligned} H_{j_0} \cap a(H_{j_0}) \neq \emptyset \quad &\Rightarrow \quad F_2^{-1}(C_{j_0}) \cap a^{-1} F_2^{-1}(C_{j_0}) \neq \emptyset \\ &\Rightarrow \quad F_2^{-1}(C_{j_0}) \cap (F_2 \circ a)^{-1}(C_{j_0}) \neq \emptyset \\ &\Rightarrow \quad F_2^{-1}(C_{j_0}) \cap (\iota \circ F_2)^{-1}(C_{j_0}) \neq \emptyset \\ &\Rightarrow \quad F_2^{-1}(C_{j_0}) \cap F_2^{-1}\iota^{-1}(C_{j_0}) \neq \emptyset \\ &\Rightarrow \quad F_2^{-1}\left(C_{j_0} \cap \iota(C_{j_0})\right) \neq \emptyset \\ &\Rightarrow \quad C_{j_0} \cap \iota(C_{j_0}) \neq \emptyset \end{aligned}$$

which concludes the proof of Theorem 3.

For Theorem 4 we note that if a diagram

$$\begin{array}{ccccc} \mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_2 \\ \vdots & & \vdots & & \vdots \\ \mathsf{S}^n & \xrightarrow{F_n} & \mathsf{X} & \xrightarrow{F_\xi} & \mathsf{E}\mathbb{Z}_2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{RP}^n & \xrightarrow{f_n} & \mathsf{X}/\iota & \xrightarrow{f_\xi} & \mathsf{B}\mathbb{Z}_2 \end{array}$$

diagram 2

similar to diagram 1 can be constructed, in other words, if there exists a bundle map F_n from ξ_n to $\xi(X, \iota)$ covering a map $f_n : \mathbb{RP}^n \to X/\iota$ then following the same steps of the proof of Theorem 3 we can conclude Theorem 4. By Lemma 1 if

$$(f_n)^*(c_1(\xi(X, \iota))) = j_n^*(c_1) = c_1(\xi_n) \tag{1}$$

where $j_n : \mathbb{RP}^n \to \mathbb{RP}^\infty$ is the inclusion, then f_n is covered by a bundle map F_n. Since the characteristic classes $c_1(\xi)$ lives in the first cohomology groups and the formula (1) is valid for f_2 we need only to extend the map f_2 to $\mathbb{RP}^n$ to obtain such an f_n with the required property (1).

To do this let (Y, A) be a CW pair and Z an n-simple space, $n \geq 1$. For an abelian group G, let $K = K(G, n)$ be an Eilenberg-MacLane space and $i_n \in H^n(K; G)$ an n-characteristic element.

Given $u_n \in H^n(Z; G)$ there exists a map $\varphi : Z \to K$ such that $\varphi^*(i_n) = u_n$. The induced homomorphism $\varphi_* : \pi_n(Z) \to \pi_n(K)$ gives

$$\varphi_{u_n}^p : H^p(Y, A; \pi_n(Z)) \to H^p(Y, A; \pi_n(K)), \quad \forall\, p \in \mathbb{N}$$

Let $f_A : A \to Z$ be a map, $Y^{(p)}$ the p-skeleton of Y and $Y_p = Y^{(p)} \cup A$.

With these notations we have the following (see [2])

Proposition 1 *If $\varphi^{n+1}_{u_n}$ is a monomorphism and $g_n : Y_n \to Z$ is an extension of f_A, then there exists $g : Y_{n+1} \to Z$ such that $g|_{Y_{n-1}} = g_n|_{Y_{n-1}}$ if and only if $\delta f^*_A(u) = 0$, where δ is the coboundary operator.*

Using proposition 1 with $Z = X/\iota$, $(Y, A) = (\mathbb{R}P^\infty, \emptyset)$ and starting with $g_2 = f_2 : \mathbb{R}P^2 \to X/\iota$, since $\pi_k(X/\iota) \cong \pi_k(X)$ for all $k \geq 2$ we have that if

$$\varphi^{k+1}_{u_k} : H^{k+1}(\mathbb{R}P^\infty; \pi_k(X)) \to H^{k+1}(\mathbb{R}P^\infty; \pi_k(K(\mathbb{Z}_2, k))$$

is injective for all $2 \leq k < n$ then we can conclude that there exists a map $f_n : \mathbb{R}P^n \to X/\iota$ such that $f_n|_{\mathbb{R}P^2} = f_2$ and therefore satisfying the formula (1). By the universal- coefficient theorem for cohomology we have

$$H^{k+1}(\mathbb{R}P^\infty; \pi_k(X)) \cong \begin{cases} \mathrm{H}om_{\mathbb{Z}}(\mathbb{Z}_2, \pi_k(X)) & \text{if k is even} \\ \mathrm{E}xt(\mathbb{Z}_2, \pi_k(X)) & \text{if k is odd} \end{cases}$$

but from the definitions of Hom and Ext we have $\mathrm{H}om_{\mathbb{Z}}(\mathbb{Z}_2, \pi_k(X)) \cong \{\alpha \in \pi_k(X) : 2\alpha = 0\}$ and $\mathrm{E}xt(\mathbb{Z}_2, \pi_k(X)) \cong \pi_k(X)/2\pi_k(X)$. Therefore if X satisfies iii) then $H^{k+1}(\mathbb{R}P^\infty; \pi_k(X)) = 0$ thus $\varphi^{k+1}_{u_k}$ is injective for all $2 \leq k < n$ and the Theorem 4 follows.

3 Examples

(1) If X is a simply connected space and ι is any continuous involution of X without fixed points, then the conclusion of Theorem 3 works for (X, ι), once we have $\pi_1(X/\iota) \cong \mathbb{Z}_2$ which implies i) and using the same notation as that in the proof of Theorem 3 then by the naturality of the Hurewicz homomorphism the diagram

$$\begin{array}{ccc} \pi_1(\mathbb{R}P^2) & \xrightarrow{(f_2)_*} & \pi_1(X/\iota) \\ \mathcal{H}_{\mathbb{R}P^2} \downarrow & & \downarrow \mathcal{H}_{X/\iota} \\ H_1(\mathbb{R}P^2) & \xrightarrow{(f_2)_*} & H_1(X/\iota) \end{array}$$

is commutative, so it follows that $(f_2)_* : H_1(\mathbb{R}P^2) \to H_1(X/\iota)$ is a non zero homomorphism, in other words, $(f_2)_*(u) \neq 0$ where $u = \mathcal{H}_{\mathbb{R}P^2}(v)$. Thus since X is path connected it follows that the bundle $\xi(X, \iota)$ is not trivial, hence from Lemma 1 $c_1(\xi(X, \iota)) \neq 0$ and in this way $\langle c_1(\xi(X, \iota)), (f_2)_*(u) \rangle \neq 0$. But $(f_2)_*(u) = (f_2)_*\mathcal{H}_{\mathbb{R}P^2}(v) = \mathcal{H}_{X/\iota}(f_2)_*(v) = \mathcal{H}_{X/\iota}(\bar{e})$.

(2) The conclusion of Theorem 4 is valid for all n-connected spaces X, $n \geq 2$ and all continuous involution of X without fixed points, because from the above example X satisfies i) and ii) and the condition iii) is obviously true for X.

In remark 2 we saw that the Borsuk-Ulam Theorem remains valid if one changes the sphere S^n by a pair (X, ι) of type LS(n). We observe in the following result that with the aid of Theorem 2, the codomain also can be changed by a more general space.

Theorem 5 *If* (X, ι) *is a pair of type* $LS(n)$ *and* Y *is a separable metric space with topological dimension,* $\dim(Y) \leq \frac{n-1}{2}$ *then, for any map* $f : X \to Y$ *there is an* $x \in X$ *such that* $f(x) = f(\iota(x))$.

Proof: Suppose that $f(x) \neq f(\iota(x))$ for all $x \in X$ and let (W, τ) be the space $W = Y \times Y - \Delta$ where Δ is the diagonal, with the involution $\tau(x, y) = (y, x)$, then W is paracompact, Hausdorff, $\dim(W) \leq n - 1$ and τ is a free continuous involution of W. Thus the map $g : (X, \iota) \to (W, \tau)$ given by $g(x) = (f(x), f(\iota(x)))$ is an equivariant map, and by Theorem 2 there exists a closed cover $\{C_0, C_1, ...C_n\}$ of W such that $C_j \cap \tau(C_j) = \emptyset$ for all $j = 0, 1, ..., n$, this provides a closed cover $\mathcal{D} = \{D_j = g^{-1}(C_j)\}_{0 \leq j \leq n}$ of X with $D_j \cap \tau(D_j) = \emptyset$ which contradicts the LS type of X.

Theorem 4 provides examples of pairs (X, ι) satisfying the hypothesis of the above Theorem.

In [3] M. Izydorek and J. Jaworowski constructed for each k and $n \leq 2k-1$ a map f from the n-sphere S^n into a specific contractible k-dimensional complex Y such that $f(x) \neq f(-x)$, for all $x \in S^n$. Thus the upper bound for dimension in the above Theorem is sharp for general cases.

Another generalization of the same nature of the Borsuk-Ulam Theorem with "nice" topological spaces X and Y satisfying some homological conditions can be found in [4].

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References

[1] Aarts, J. M., Fokkink, R. J., Vermeer, H. *Variations on a Theorem of Lusternik and Schnirelmann*, Topology, 35 (4) (1996), 1051-1056.

[2] Biasi, C. and Libardi, A. K. M., *Some applications of obstruction theory*, Notas do Instituto de Ciências Exatas de São Carlos - USP (1988).

[3] Izydorek, M. and Jaworowski, J., *Antipodal coincidence for maps of spheres into complexes*, Proc. Amer. Math. Soc. 123 (1995), 1947-1950.

[4] Mattos, D., Santos, E. L. and Pergher, P. L. Q., *A Borsuk-Ulam theorem for general spaces* (preprint).

[5] Olum, P. *Obstructions to extensions and homotopies*, Ann. of Math. 52 (2), (1950), 1-50.

[6] Santos, E. E. F. *Coloração de funções em espaços de dimensão topológica finita*, Master Thesis, (2001) (in Portuguese).

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