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Continuity of the Attractors for a Singularly Perturbed Hyperbolic Problem

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Resumo

Neste artigo consideramos problemas hiperbólicos da forma

$$u_{tt} + \eta A^{\frac{1}{2}} u_t + a u_t + Au = f(u)$$

com $A : D(A) \subset X \to X$ setorial em um espaço de Hilbert X, $a > 0$, $\eta \geq 0$. Para o caso em que f é dissipativa e A é $-\Delta + \frac{\delta}{2} I$ com condições de fronteira de Neumann, $\delta > 0$, provamos que a dinâmica assintótica deste problema é contínua relativamente ao parâmetro η em $\eta = 0$. Para $\eta = 0$, este problema tem estrutura hyperbólica e, para $\eta > 0$, tem estrutura parabólica. Isto permite aproximar problemas hiperbólicos amortecidos por problemas do tipo parabólico.

CONTINUITY OF THE ATTRACTORS FOR A SINGULARLY PERTURBED HYPERBOLIC PROBLEM

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ABSTRACT. We consider damped hyperbolic problems of the form

$$u_{tt} + \eta A^{\frac{1}{2}} u_t + a u_t + Au = f(u)$$

with $A : D(A) \subset X \to X$ sectorial in a Hilbert space X, $a > 0$, $\eta \geq 0$. We prove that asymptotic dynamics of this problem is continuous with respect to η at $\eta = 0$. For $\eta = 0$, this problem has hyperbolic structure and, for $\eta > 0$, has parabolic structure. This allow us to approximate of a damped hyperbolic problem by a parabolic problem.

1. INTRODUCTION

In [7, 8] it is proved that the attractors of the family of hyperbolic problems $\epsilon u_{tt} + u_t = \Delta u + f(u)$, in bounded domains, with Dirichlet boundary conditions and for dissipative functions f, are continuous at $\epsilon = 0$. This shows that parabolic problems are approximated by damped hyperbolic problems. In this paper we consider the approximation of a damped hyperbolic problem by a "parabolic" problem. This approximation will be used in later works to obtain finite dimensional approximations of damped hyperbolic problems.

Consider the following family of one dimensional scalar hyperbolic problems

$$\begin{aligned} &u_{tt} + 2\eta\,(-\Delta + \tfrac{\delta}{2})^{1/2} u_t + 2a\, u_t = \Delta u + f(u), \quad x \in (0,1),\ t > 0 \\ &u_x(0) = u_x(1) = 0,\ t > 0, \end{aligned} \tag{1.1}$$

where $\eta \geq 0$, $a > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a C^2 function satisfying the dissipativeness condition.

$$\limsup_{|u| \to +\infty} \frac{f(u)}{u} \leq -\delta. \tag{1.2}$$

For each $\eta \geq 0$, the equation (1.1) generates C^1-semigroup T_η on $Y^0 = H^1(0,1) \times L^2(0,1)$. The semigroup $T_\eta(t)$, $t \geq 0$ is gradient system and asymptotically smooth. Furthermore, $T_\eta(t)$ admits global attractors $\mathcal{A}_\eta$. Using regularity results we have $\mathcal{A}_\eta \subset Y^1$.

Let $X = L^2(0,1) = X^0$ and denote by A the operator $A : D(A) \subset X^0 \to X^0$ given by $D(A) = \{u \in H^2(0,1); u'(0) = u'(1) = 0\}$, $Au = -u_{xx} + \frac{\delta}{2} u$, $u \in D(A)$. The operator A is positive and self adjoint and therefore a sectorial operator. Let X^α be the fractional power spaces associated to A.

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For convenience we rewrite (1.1) in the following abstract form

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = C_\eta \begin{bmatrix} u \\ v \end{bmatrix} + h(\begin{bmatrix} u \\ v \end{bmatrix}) \tag{1.3}$$

where $D(C_\eta) = X^1 \times X^{\frac{1}{2}} = Y^1$,

$$C_\eta = \begin{bmatrix} 0 & I \\ -A & -2(\eta A^{1/2} + a) \end{bmatrix} \text{ e } h(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} 0 \\ f^e(u) \end{bmatrix}. \tag{1.4}$$

For $\eta > 0$, $-C_\eta$ is a sectorial operator and generates an analytic semigroup of contractions (see [3, 4]).

Define in the space $Y^0 = X^{\frac{1}{2}} \times X^0$ the inner product $\langle (u,v),(w,z)\rangle_{Y^0} = \langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}w\rangle + \langle v,z\rangle$, and in Y^1, $\langle (u,v),(w,z)\rangle_{Y^1} = \langle Au, Aw\rangle + \langle A^{\frac{1}{2}}v, A^{\frac{1}{2}}z\rangle$, where $\langle .,. \rangle$ is the usual inner product in $L^2(0,1)$.

Considering

$$\delta_Y(A,B) = \sup_{x\in A}\inf_{y\in B} d_Y(x,y) \tag{1.5}$$

we can define the continuity of a family of sets $B_\eta \subset Y$ in the following form: a family B_η is continuous in η_0 if it is upper semicontinuous at η_0; that is, $\delta_Y(B_\eta, B_{\eta_0}) \to 0$ as $\eta \to \eta_0$, and lower semicontinuous at η_0; that is, $\delta_Y(B_{\eta_0}, B_\eta) \to 0$ as $\eta \to \eta_0$.

The main result in this paper is

Teorema 1.1. *The family of global attractors $\mathcal{A}_\eta$, $\eta \geq 0$ is continuous at $\eta = 0$.*

This shows that the attractor for the strongly semigroup generated by (1.1) when $\eta = 0$ can be approximated by the attractors of the analytic semigroup generated by the problem (1.1) with $\eta > 0$.

This result will be used in a future work to obtain finite dimensional approximation (using spatial discretization) of the attractor of (1.1) with $\eta = 0$ by approximating the the attractor of (1.1) with $\eta > 0$. In the approximation of the attractor of (1.1) with $\eta > 0$ by attractors of finite dimensional problems it is crucial to be able to exploit the analyticity of the semigroup generated by C_η, $\eta > 0$.

This paper is organized as follows. In Section 2, we prove upper semicontinuity, which is based on [7]. In order to do it we prove the family of attractors $\mathcal{A}_\eta$ is uniform bounded in Y^0 and Y^1. Using compact embeddings and Arzelà-Ascoli Theorem we obtain the upper semicontinuity. In Section 3, we prove lower semicontinuity. In order to do it we use some theorems which appear in [5], [2].

2. Upper semicontinuity

The proof of upper semicontinuity for the family $\{\mathcal{A}_\eta, \eta \geq 0\}$ is organized in three parts. We begin proving that this family is bounded in Y^0-norm and obtaining an estimate on the integrals of the L^2-norm of the u_t. Then, we prove that the attractors are uniformly bounded in Y^1-norm. Finally, with these information and Arzelá-Ascoli Theorem, we prove the upper semicontinuity of the family at $\eta = 0$. We remark that in the proof of the uniform boundedness we use a alternative energy functional similar to the one used in [7].

If λ_1 denotes the first eigenvalue of the operador A we have the following result.

Lema 2.1. *Let b be a real nonnegative number satisfying*

$$b \leq \inf\{\frac{1}{8}, \frac{\lambda_1}{4}, \frac{\sqrt{\lambda_1}}{4}, \frac{9a\lambda_1}{4(3\lambda_1 + 4a^2)}\}.$$

Then

$$\frac{1}{2}\|\psi\|_{L^2}^2 + 2b\langle\phi,\psi\rangle + \frac{1}{2}\|\phi\|_{H^1}^2 \geq \frac{1}{4}(\|\psi\|_{L^2}^2 + \|\phi\|_{H^1}^2) \tag{2.1}$$

$$\frac{1}{2}\|\psi\|_{L^2}^2 + 2b\langle\phi,\psi\rangle + \frac{1}{2}\|\phi\|_{H^1}^2 \leq \frac{3}{4}(\|\psi\|_{L^2}^2 + \|\phi\|_{H^1}^2) \tag{2.2}$$

and

$$(2a - 2b)\|\psi\|_{L^2}^2 + 4ab\langle\phi,\psi\rangle + 2b\|\phi\|_{H^1}^2 \geq \inf\{\frac{a}{2}, \frac{b}{2}\}(\|\psi\|_{L^2}^2 + \|\phi\|_{H^1}^2). \tag{2.3}$$

With this lemma we are in condition to prove the attractors are uniformly bounded in Y^0. Firstly, we prove the semigroups $T_\eta(t)$, $t \geq 0$, are uniformly bounded dissipative; that is, the following result holds.

Teorema 2.1. *Let $0 < \eta < \eta_0$. There exists a constant $C_0 > 0$ and, for any r_0, there exists $t_0 = t(r_0)$ such that*

$$\|T_\eta(t)(u,v)\|_{Y^0} \leq C_0,\ t \geq t_0$$

for all (u,v) with $\|(u,v)\|_{Y^0} \leq r_0$

Proof: We fix a constant b satisfying the hypotheses of Lemma (2.1). For any $\eta > 0$, we define the following energy functional in Y^0,

$$V_b(\phi,\psi) = \frac{1}{2}|\psi\|_{L^2}^2 + \frac{1}{2}\|\phi\|_{H^1}^2 + 2b\langle\psi,\phi\rangle - \int_0^1 F(\phi(x))dx, \tag{2.4}$$

where $F(v) = \int_0^v f(s)ds$. This functional is similar to [7]. Let $u_\eta(t,u_0,v_0)$ be the solution of (1.1), then

$$\begin{aligned} \frac{d}{dt}V_b(u_\eta(t), \frac{du_\eta}{dt}(t)) &= \tfrac{1}{2}\frac{d}{dt}[\|\frac{du_\eta}{dt}(t)\|_{L^2}^2 + \|u_\eta(t)\|_{H^1}^2] + 2b\|\frac{du_\eta}{dt}(t)\|_{L^2}^2 \\ &+2b\langle\frac{d^2u_\eta}{dt^2}(t), u_\eta(t)\rangle - \langle f(u_\eta(t)), \frac{du_\eta}{dt}(t)\rangle \end{aligned}$$

Since $\langle \frac{d^2u}{dt^2}(t) + Au, \frac{du}{dt}\rangle = \frac{1}{2}\frac{d}{dt}(\|\frac{du_\eta}{dt}(t)\|_{L^2}^2 + \|u_\eta(t)\|_{H^1}^2)$, and, by equation (1.1), we obtain

$$\begin{aligned} \frac{d}{dt}\ V_b(u_\eta(t), \frac{du_\eta}{dt}(t)) &= -2\eta\|A^{\frac{1}{4}}\frac{du_\eta}{dt}(t)\|_{L^2}^2 + (2b - 2a)\|\frac{du_\eta}{dt}(t)\|_{L^2}^2 \\ &-4ab\langle\frac{du_\eta}{dt}(t), u_\eta(t)\rangle + 2b\langle f(u_\eta(t)), u_\eta(t)\rangle - 2b\|u_\eta(t)\|_{H^1}^2 - 4b\eta\langle A^{\frac{1}{2}}\frac{du_\eta}{dt}(t), u_\eta(t)\rangle \\ &\leq -4ab\langle\frac{du_\eta}{dt}(t), u_\eta(t)\rangle + (2b - 2a)\|\frac{du_\eta}{dt}(t)\|_{L^2}^2 + 2b\langle f(u_\eta(t)), u_\eta(t)\rangle \\ &-2\eta\|A^{\frac{1}{4}}\frac{du_\eta}{dt}(t)\|_{L^2}^2 - 2b\|u_\eta(t)\|_{H^1}^2 + 4b\eta\|A^{\frac{1}{4}}\frac{du_\eta}{dt}(t)\|_{L^2}\|A^{\frac{1}{4}}u_\eta(t)\|_{L^2} \end{aligned} \tag{2.5}$$

From the hypothesis of Lemma (2.1), we have $b \leq \frac{1}{8}$, and using Youngs Inequality and Interpolation, we get

$$4b\eta\|A^{\frac{1}{4}}\frac{du_\eta}{dt}(t)\|_{L^2}\|A^{\frac{1}{4}}u_\eta(t)\|_{L^2} - 2\eta\|A^{\frac{1}{4}}\frac{du_\eta}{dt}(t)\|_{L^2}^2 \leq \eta K\|u_\eta(t)\|_{H^1}^2 \tag{2.6}$$

Now, using (2.6) and (2.3), in (2.5) we get

$$\begin{aligned}\frac{d}{dt}V_b(u_\eta(t),\frac{du_\eta}{dt}(t)) \leq & -\tfrac{5}{8}k_0(\|\frac{du_\eta}{dt}(t)\|_{L^2}^2+\|u_\eta(t)\|_{H^1}^2)\\ & +2b\langle f(u_\eta(t)),u_\eta(t)\rangle\end{aligned} \tag{2.7}$$

for all $\eta \leq \eta_0$, where $k_0 = \inf\{\frac{a}{2},\frac{b}{2}\}$.
Since (1.2), there exists $c_{\delta/4} > 0$ such that

$$f(v)v \leq -\frac{\delta}{4}v^2 + c_{\delta/4} \tag{2.8}$$

$$F(v) \leq -\frac{\delta}{4}v^2 + c_{\delta/4} \tag{2.9}$$

Thus, by (2.8), we have:

$$\langle f(u_\eta(t)),u_\eta(t)\rangle \leq -\frac{\delta}{4}\|u_\eta(t)\|_{L^2}^2 + c_{\delta/4} \tag{2.10}$$

Therefore, by (2.10), we have from (2.7) that

$$\frac{d}{dt}V_b(u_\eta(t),\frac{du_\eta}{dt}(t)) \leq -\frac{5}{8}k_0(\|\frac{du_\eta}{dt}(t)\|_{L^2}^2+\|u_\eta(t)\|_{H^1}^2) + c_{\delta/4} \tag{2.11}$$

On the other hand, the functional V_b is lower bounded. In fact, from (2.9) and (2.1), we have

$$V_b(u_\eta(t),\frac{du_\eta}{dt}(t)) \geq \frac{1}{4}(\|\frac{du_\eta}{dt}(t)\|_{L^2}^2+\|u_\eta(t)\|_{H^1}^2) - c_{\delta/4} \tag{2.12}$$

Finally, we show that for any $r_0 > 0$ exists $t_0 = t(r_0)$ such that if $(\|\frac{du_\eta}{dt}(0)\|_{L^2}^2 + \|u_\eta(0)\|_{H^1}^2) \leq r_0$ then

$$V_b(u_\eta(t),\frac{du_\eta}{dt}(t)) \leq K_0$$

for all $t \geq t_0$.
In fact, let $\bar{r}$ be large enough such that $-\frac{5}{8}k_0\bar{r}^2 + c_{\delta/4} = -1$. Hence, if $(\|\frac{du_\eta}{dt}(t)\|_{L^2}^2 + \|u_\eta(t)\|_{H^1}^2) \geq \bar{r}$, then $\frac{d}{dt}V_b(u_\eta(t),\frac{du_\eta}{dt}(t)) \leq -1$.
Let $M = \max\{V(u,v);\ \|u\|_{H^1}^2 + \|v\|_{L^2}^2 \leq \bar{r}\}$ and $C_M = \{(u,v); V(u,v) \leq M\}$ which is bounded due to (2.12) and it follows by definition $B_{\bar{r}}(0) \subset C_M$.

We show that for each $(u_0,v_0) \in B_{r_0}$ exists $t_0 = t(u_0,v_0)$ such that $T_\eta(t_0)(u_0,v_0) \in C_M$. Moreover, $\sup\{t(u_0,v_0);\ (u_0,v_0) \in B_{r_0}\} < \infty$. In fact, we have two cases to consider

i) If $(u_0,v_0) \in C_M$ then $V(u_0,v_0) \leq M$.

ii) If $(u_0,v_0) \notin C_M$ then $(u_0,v_0) \notin B_{\bar{r}}(0)$. Consequently, we have that $\frac{d}{dt}V_b(u(t),v(t)) \leq -1$ for all t such that $(u(t),v(t)) \notin B_{\bar{r}}(0)$, thus $V_b(u(t),v(t)) \leq V_b(u_0,v_0) - t$.

Hence, $V_b(u(t),v(t)) \leq K - t$, where $K = \max\{V(u,v); (u,v) \in B_{r_0}\}$. Therefore, consider $t_0 = K - M > 0$. Moreover, t_0 was chosen uniform in $B_{r_0}(0)$.

We verified that C_M is positively invariant under $T_\eta(t)$. In fact, if exists the last time, $\bar{t}$, such that $V(T_\eta(\bar{t})(u_0,v_0)) = M$, $T_\eta(t)(u_0,v_0) \in C_M$ for $t_0 \leq t \leq \bar{t}$ e $T_\eta(t)(u_0,v_0) \notin C_M$ for $\bar{t} < t$, thus $T_\eta(t)(u_0,v_0) \notin B_{\bar{r}}(0)$. Hence, $V(T_\eta(t)(u_0,v_0)) < V(T_\eta(\bar{t})(u_0,v_0)) = M$, what is a contradiction.

Therefore, for any r_0 exists t_0 such that $V(T_\eta(t)(u_0,v_0)) \leq M$ for all $t \geq t_0$ and consequently, from (2.12)

$$\|\frac{du_\eta}{dt}(t)\|_{L^2}^2 + \|u_\eta(t)\|_{H^1}^2 \leq C_0$$

para $t \geq t_0$.

Since the attractors $\mathcal{A}_\eta$ are invariant, it follows from the Theorem 2.1, the following result

Corolário 2.1. *There exists a constant $C_1 > 0$ such that*

$$\|\phi\|_{H^1}^2 + \|\psi\|_{L^2}^2 \leq C_1,$$

for all $(\phi,\psi) \in \mathcal{A}_\eta$ and for all η, $0 \leq \eta \leq \eta_0$.

We also need and estimate for $\displaystyle\int_0^\infty \|\frac{du_\eta}{dt}(s)\|_{L^2}^2 ds$ to be able to prove the bounded of attractors in Y^1. We get this estimate at the following proprosition.

Proposição 2.1. *Let $0 \leq \eta \leq \eta_0$. For any $r_0 > 0$ there exists a constant $C_0^*(r_0)$ such that, for any solution $u_\eta(t)$ of (1.1), with $(\|u_0\|_{H^1}^2 + \|v_0\|_{L^2}^2)^{\frac{1}{2}} \leq r_0$, we have*

$$\int_0^\infty \|\frac{du_\eta}{dt}(s)\|_{L^2}^2 ds \leq C_0^*(r_0)$$

Proof: For any $\eta > 0$, we consider the Lyapunov functional

$$V(\phi,\psi) = V_0(\phi,\psi) = \frac{1}{2}\|\psi\|_{L^2}^2 + \frac{1}{2}\|\phi\|_{H^1}^2 - \int_0^1 F(\phi(x))dx,$$

where $F(v)$ is the same as in the proof of Theorem 2.1.

Consider $u_\eta(t) = u_\eta(t,u_0,v_0)$, a solution of (1.1). From (2.5) with $b = 0$, we have

$$\frac{d}{dt}V(u_\eta(t),\frac{du_\eta}{dt}(t)) \leq -2a\|\frac{du_\eta}{dt}(t)\|_{L^2}^2 - 2\eta\|A^{\frac{1}{4}}\frac{du_\eta}{dt}(t)\|_{L^2}^2 \leq -2a\|\frac{du_\eta}{dt}(t)\|_{L^2}^2.$$

Thus,

$$V(u_\eta(t),\frac{du_\eta}{dt}(t)) - V(u_0,v_0) \leq -2a\int_0^t \|\frac{du_\eta}{dt}(s)\|_{L^2}^2 ds \tag{2.13}$$

or,

$$\int_0^t \|\frac{du_\eta}{dt}(s)\|_{L^2}^2 ds \leq \frac{1}{2a}(V(u_0,v_0) - V(u_\eta(t),\frac{du_\eta}{dt}(t))) \tag{2.14}$$

Under the assumptions on f, we have that

i) $V(\phi,\psi) \geq c(\|\phi\|_{H^1}^2 + \|\psi\|_{L^2}^2) - c'$;
ii) $V(\phi,\psi) \leq c_1(r_0)(\|\phi\|_{H^1}^2 + \|\psi\|_{L^2}^2)$;

Hence,

$$\begin{aligned}\int_0^t \|\frac{du_\eta}{dt}(s)\|_{L^2}^2 ds &\leq \frac{1}{2a}(V(u_0,v_0) - V(u_\eta(t),\frac{du_\eta}{dt}(t))) \\ &\leq \frac{1}{2a}(V(u_0,v_0) - c(\|u_\eta(t)\|_{H^1}^2 + \|\frac{du_\eta}{dt}(t)\|_{L^2}^2) + c') \leq \frac{1}{2a}(V(u_0,v_0) + c')\end{aligned}$$

Therefore,

$$\int_0^\infty \|\frac{du_\eta}{dt}(s)\|_{L^2}^2 ds = \lim_{t\to\infty}\int_0^t \|\frac{du_\eta}{dt}(s)\|_{L^2}^2 ds \le \frac{1}{2a}(V(u_0,v_0)+c') \tag{2.15}$$

Using ii), we get

$$\int_0^\infty \|\frac{du_\eta}{dt}(s)\|_{L^2}^2 ds \le \frac{1}{2a}(V(u_0,v_0)+c') \le c_1(r_0)(\|u_0\|_{H^1}^2 + \|v_0\|_{L^2}^2) + c' = C_0^*(r_0) \tag{2.16}$$

Moreover, (i) implies that

$$\begin{aligned}\|\frac{du_\eta}{dt}(t)\|_{L^2}^2 + \|u_\eta(t)\|_{H^1}^2 &\le \tfrac{1}{c}(V(u_\eta(t),\frac{du_\eta}{dt}(t))+c') \le \tfrac{1}{c}(V(u_0,v_0)+c') \\ &\le \tfrac{1}{c}(c_1(r_0)(\|v_0\|_{L^2}^2 + \|u_0\|_{H^1}^2)+c')\end{aligned}$$

Therefore, $(\|\frac{du_\eta}{dt}(t)\|_{L^2}^2 + \|u_\eta(t)\|_{H^1}^2) \le C_1^*(r_0)$.

Now we prove that the attractors $\mathcal{A}_\eta$ are uniformly bounded in Y^1. This uniform boundedness and a compact imbedding are necessary conditions to obtain that the limit of the orbits in $\mathcal{A}_\eta$ is an orbit in $\mathcal{A}$.

Teorema 2.2. *Let $0<\eta<\eta_0$. There exists a constant $C_2>0$ and for each r_0 and r_1, exist constants $C_2^*(r_0)>0$ and $C_3^*(r_1)>0$, such that for any solution $u_\eta(t,u_0,v_0)$ of (1.1), with $\|(u_0,v_0)\|_{Y^0}^2 \le r_0$ and $\|(u_0,v_0)\|_{Y^1}^2 \le r_1$, then*

$$\|\frac{d^2u_\eta}{dt^2}(t)\|_0^2 + \|\frac{du_\eta}{dt}(t)\|_{H^1}^2 + \|u_\eta(t)\|_{H^2}^2 \le C_2^*(r_0) + C_3^*(r_1)e^{-C_2 t},$$

for all $t \le 0$

Proof: Assume that $\|(u_0,v_0)\|_{Y^1}^2 \le r_1$, then $T_\eta(t)(u_0,v_0) \in C^1([0,\infty),Y^0)\bigcap C([0,\infty),Y^1)$. We now consider the following linear problem:

$$\begin{aligned} &w_{tt} + 2\eta A^{1/2}w_t + 2a\,w_t + Aw = f'(u)\frac{du_\eta}{dt}, \quad 0<x<1,\ t>0 \\ &w_x(0) = w_x(1) = 0,\ t>0, \\ &w_\eta(0) = v_0,\ \frac{dw_\eta}{dt}(0) = -2\eta A^{\frac{1}{2}} - 2av_0 - Au_0 - f(u_0)\end{aligned} \tag{2.17}$$

As $f'(u_\eta)\frac{du_\eta}{dt} \in C([0,\infty),L^2)$ and $(w_\eta(0),\frac{dw_\eta}{dt}(0)) \in Y^0$ then there is only one solution $w_\eta(t,x)$ of (2.17). But $\frac{du_\eta}{dt}(t)$ satisfy (2.17), then $w_\eta(t,x) = \frac{du_\eta}{dt}(t)$.

We fix constant b satisfying the hypotheses of Lemma (2.1) and we define a functional to equation (2.17)

$$V_b(\phi,\psi) = \frac{1}{2}\|\psi\|_{L^2}^2 + \frac{1}{2}\|\phi\|_{H^1}^2 + 2b\langle\psi,\phi\rangle,$$

Using $\langle \frac{d^2w}{dt^2}(t) + Aw, \frac{dw}{dt}\rangle = \frac{1}{2}\frac{d}{dt}(\|\frac{dw_\eta}{dt}(t)\|_{L^2}^2 + \|w_\eta(t)\|_{H^1}^2)$ and (2.17), we get

$$\begin{aligned}\frac{d}{dt}V_b(w_\eta(t), \frac{dw_\eta}{dt}(t)) &= -2\eta\|A^{\frac{1}{4}}\frac{dw_\eta}{dt}(t)\|_{L^2}^2 + (2b-2a)\|\frac{dw_\eta}{dt}(t)\|_{L^2}^2 \\ &- 4ab\langle\frac{dw_\eta}{dt}(t), w_\eta(t)\rangle - 2b\|w_\eta(t)\|_{H^1}^2 \\ &+ 2b\langle f'(u_\eta(t))w_\eta(t), w_\eta(t)\rangle - 4b\eta\langle A^{\frac{1}{2}}\frac{dw_\eta}{dt}(t), w_\eta(t)\rangle \\ &+ \langle f'(u_\eta(t))w_\eta(t), \frac{dw_\eta}{dt}(t)\rangle\end{aligned} \tag{2.18}$$

Using (2.3), Youngs Inequality and Interpolation in (2.18) we obtain, for $\eta \le \eta_0$,

$$\begin{aligned}\frac{d}{dt}V_b\,(w_\eta(t), \frac{dw_\eta}{dt}(t)) &\le -\frac{5k_0}{8}(\|\frac{dw_\eta}{dt}(t)\|_{L^2}^2 + \|w_\eta(t)\|_{H^1}^2) \\ &+\|f'(u_\eta)w_\eta\|_{L^2}\|\frac{dw_\eta}{dt}\|_{L^2} + \frac{2b}{\sqrt{\lambda_1}}\|f'(u_\eta)w_\eta\|_{L^2}\|w_\eta\|_{H^1} \\ &\le -\frac{k_0}{8}(\|\frac{dw_\eta}{dt}(t)\|_{L^2}^2 + \|w_\eta(t)\|_{H^1}^2) + \frac{\lambda_1+4b^2}{2k_0\lambda_1}\|f'(u_\eta(t))w_\eta(t)\|_{L^2}^2\end{aligned} \tag{2.19}$$

We now obtain an estimate for $\|f'(v)\|_{\mathcal{L}(L^2,L^2)}$. Due to $f \in C^2(\mathbb{R},\mathbb{R})$ then $f^e : H^1(0,1) \to L^2(0,1)$ is a C^1 mapping. Moreover, if $v \in H^2(0,1)\cap H^1(0,1)$ then $f'(v)$ is a continuous and linear mapping from L^2 into L^2 and $\|f'(v)w\|_{L^2}^2 \le \|f'(v)\|_\infty^2\|w\|_{L^2}^2$. But, $\|f'(v)\|_\infty^2 \le \sqrt{2}(\|(f'(v))'\|_{L^2}^2 + \|f'(v)\|_{L^2}^2)$ and:

i) $\|(f'(v))'\|_{L^2}^2 \le c^*(\|v\|_{H^1})\|v'\|_{L^2}^2$;
ii) $\|f'(v)\|_{L^2}^2 \le c^{**}(\|v\|_{H^1})$.

Therefore

$$\|f'(v)\|_{\mathcal{L}(L^2,L^2)} \le C^*(\|v\|_{H^1})(1+\|v\|_{H^1}^2). \tag{2.20}$$

From equation (1.1), we have

$$\begin{aligned}\|u_\eta\|_{H^2} = \|Au_\eta\|_{L^2} &= \|f(u_\eta) - \frac{d^2u_\eta}{dt^2} - 2\eta A^{\frac{1}{2}}\frac{du_\eta}{dt} - 2a\frac{du_\eta}{dt}\|_{L^2} \\ &\le \|f(u_\eta)\|_{L^2} + \|\frac{d^2u_\eta}{dt^2}\|_{L^2} + \|2\eta\frac{du_\eta}{dt}\|_{H^1} + \|2a\frac{du_\eta}{dt}\|\end{aligned}$$

Therefore,

$$\begin{aligned}\|u_\eta\|_{H^2}^2 &\le d(\|f(u_\eta)\|_{L^2}^2 + \|\frac{dw_\eta}{dt}\|_{L^2}^2 + \|w_\eta\|_{L^2}^2 + \|w_\eta\|_{H^1}^2) \\ &\le c(\|f(u_\eta)\|_{L^2}^2 + \|\frac{dw_\eta}{dt}\|_{L^2}^2 + \|w_\eta\|_{H^1}^2)\end{aligned} \tag{2.21}$$

From Proposition (2.1), we have $(\|\frac{du_\eta}{dt}\|_{L^2}^2 + \|u_\eta\|_{H^1}^2) \le C_1^*(r_0)$, for $(\|u_0\|_{H^1}^2 + \|v_0\|_{L^2}^2) \le r_0$, then, by (2.20),

$$\|f'(u_\eta(t))w_\eta(t)\|_{L^2}^2 \le C^*(\|u_\eta\|_{H^1})\|(1+\|u_\eta\|_{H^1}^2)\|w_\eta(t)\|_{L^2}^2 \le C_2(r_0)\|w_\eta(t)\|_{L^2}^2 \tag{2.22}$$

Therefore, using (2.22) and (2.2), in (2.19), we get

$$\begin{aligned}\frac{d}{dt}\ V_b(w_\eta(t),\frac{dw_\eta}{dt}(t)) &\leq -\frac{k_0}{8}(\|\frac{dw_\eta}{dt}(t)\|_{L^2}^2+\|w_\eta(t)\|_{H^1}^2)\\ &+(\frac{1}{2k_0}+\frac{2b^2}{k_0\lambda_1})\|w_\eta(t)\|_{L^2}^2C_2(r_0)\\ &\leq -\frac{k_0}{6}(\frac{1}{2}\|\frac{dw_\eta}{dt}(t)\|_{L^2}^2+2b\langle\frac{dw_\eta}{dt}(t),w_\eta(t)\rangle+\frac{1}{2}\|w_\eta(t)\|_{H^1}^2)\\ &+(\frac{1}{2k_0}+\frac{2b^2}{k_0\lambda_1})\|w_\eta(t)\|_{L^2}^2C_2(r_0)\\ &\leq -\frac{k_0}{6}V_b(w_\eta,\frac{dw_\eta}{dt})+C_2^*(r_0)\|w_\eta\|_0^2\end{aligned} \tag{2.23}$$

Solving the differential inequality and using the Proposition (2.1), we get

$$\begin{aligned}V_b(w_\eta(t),\frac{dw_\eta}{dt}(t)) &\leq e^{-\frac{k_0}{6}t}V_b(w_\eta(0),\frac{dw_\eta}{dt}(0))+C_2^*(r_0)\int_0^\infty\|w_\eta(s)\|_{L^2}^2ds\\ &\leq e^{-\frac{k_0}{6}t}V_b(w_\eta(0),\frac{dw_\eta}{dt}(0))+C_2^*(r_0)C_0^*(r_0)\end{aligned} \tag{2.24}$$

Hence, using (2.1) and (2.2), in (2.24) we get

$$\|w_\eta(t)\|_{H^1}^2+\|\frac{dw_\eta}{dt}(t)\|_{L^2}^2\leq 3e^{-\frac{k_0}{6}t}(\|w_\eta(0)\|_{H^1}^2+\|\frac{dw_\eta}{dt}(0)\|_{L^2}^2)+C_3^*(r_0) \tag{2.25}$$

As we saw before in (2.21), we obtain from (2.25)

$$\begin{aligned}&\|w_\eta\ (t)\|_{H^1}^2+\frac{dw_\eta}{dt}(t)\|_{L^2}^2+\|u_\eta(t)\|_{H^2}^2\\ &\leq c\|f(u_\eta)\|_{L^2}^2+(1+c)(\|w_\eta(t)\|_{H^1}^2+\|\frac{dw_\eta}{dt}(t)\|_{L^2}^2)\\ &\leq C_4^*(r_0)+(1+c)[C_3^*(r_0)+e^{-\frac{k}{6}t}(\|w_\eta(0)\|_{H^1}^2+\|\frac{dw_\eta}{dt}(0)\|_{L^2}^2)]\end{aligned} \tag{2.26}$$

Since $\|\frac{dw_\eta}{dt}(0)\|_{L^2}\leq 2\eta\|u_0\|_{H^1}+2a\|v_0\|_{L^2}+\|u_0\|_{H^2}+\|f(u_0)\|_{L^2}$, we get from (2.26),

$$(\|w_\eta(t)\|_{H^1}^2+\|\frac{dw_\eta}{dt}(t)\|_{L^2}^2+\|u_\eta(t)\|_{H^2}^2)\leq C_5^*(r_0)+C_6^*e^{-\frac{k}{6}t}r_1 \tag{2.27}$$

Finally, we deduce the following results:

Corolário 2.2. *Let $0<\eta\leq\eta_0$. The system (1.1) is bounded dissipative in Y^1, uniformly in η, that means, exist a set $B\subset Y^1$, B bounded such that*

$$T_\eta(t)(C)\subset B,\ \textit{para}\ t\leq t_1$$

for any subset $C\subset Y^1$, C bounded and some $t_1=t(r_0,r_1)$.

Corolário 2.3. *Let $0<\eta\leq\eta_0$. There are constants C_3 and C_4 such that:*

i)

$$\|\phi\|_{H^2}^2+\|\psi\|_{H^1}^2\leq C_3,\ \textit{for all}\ (\phi,\psi)\in\mathcal{A}_\eta$$

ii)

$$\|\frac{d^2u_\eta}{dt^2}(t)\|_{L^2}^2\leq C_4\ \textit{for all solution}\ (u_\eta(t),\frac{du_\eta}{dt}(t))\in\mathcal{A}_\eta.$$

We now are in conditions to prove upper semicontinuity.

Teorema 2.3. *The family of global attractors, $\mathcal{A}_\eta$, of (1.1) is upper semicontinuous in $\eta = 0$, $\mathcal{A}_0 := \mathcal{A}$; that means, $\forall \epsilon > 0$, $\exists \eta_0$ such that $\eta \leq \eta_0$,*

$$\delta_{Y^0}(\mathcal{A}_\eta, \mathcal{A}) \leq \epsilon$$

Proof: In order to prove the upper semicontinuity we prove the following statement

Statement 1. *Let η_k be a sequence of positive numbers such that $\lim_{k\to\infty} \eta_k = 0$ and consider $(u_k(t), \frac{du_k}{dt}(t))$ a sequence of solutions of (1.1) with $\eta = \eta_k$ and $(u_k(t), \frac{du_k}{dt}(t)) \in \mathcal{A}_{\eta_k}$ for all $t \geq 0$. Then exists a subsequence η_{k_j} such that $(u_{k_j}(0), \frac{du_{k_j}}{dt}(0))$ converges to $(\bar{u}, \bar{v}) \in \mathcal{A}$ in Y_0.*

Proof of Statement(1):. In fact, by Corollary 2.3, there exists a bounded set B in Y^1 such that

$$(\bigcup_\eta \mathcal{A}_\eta) \cup \mathcal{A} \subset B.$$

Thus, $\cup_t \cup_k u_k(t)$ is a pre-compact (relatively) set in H^1 and the family $u_k(t) \in C(\mathbb{R}, H^1)$, is equicontinuous from $\mathbb{R}$ in H^1.
Consider J_m, $m \geq 0$, a sequence of compact intervals in $\mathbb{R}$ such that $J_m \subset J_{m+1}$, $m \geq 0$ and $\cup_m J_m = \mathbb{R}$. By Arzelà-Ascolis Theorem, exists a subsequence u_{k_0} of u_k such that u_{k_0} converge to $\bar{u}$ in $C(J_0, H^1)$. By induction, using Arzelà-Ascolis Theorem in the subsequence u_{k_m} we extract a subsequence $u_{k_{m+1}}$ that converges to $\bar{u}$ in $C(J_{m+1}, H^1)$. Considering the Cantor diagonal procedure, there exists a subsequence of positive numbers η_{j_k} of η_k and a correspondence subsequence u_{j_k} of u_k which are solutions of (1.1) with $\eta = \eta_{j_k}$ and such that $u_{j_k} \to \bar{u}$ in $C(J, H^1)$ for all compact interval J. Then $\bar{u} \in C(\mathbb{R}, H^1)$ and $\sup_{t\in\mathbb{R}} \|\bar{u}(t)\|_{H^1} \leq K_1$. Also, $\|A^{\frac{1}{2}} \frac{du_{j_k}}{dt}(t)\|_{L^2} = \|\frac{du_{j_k}}{dt}(t)\|_{H^1} \leq \bar{K}$, for all t and for all η_{j_k}. Therefore,

$$\sup_{t\in\mathbb{R}} (2\eta_{j_k} \|A^{\frac{1}{2}} \frac{du_{j_k}}{dt}(t)\|_{L^2}) \to 0$$

as $j_k \to \infty$.

It follows from Corollary 2.3 that $\frac{du_{j_k}}{dt}(t) \in L^\infty(J, H^1)$ and $\frac{d^2 u_{j_k}}{dt^2} \in L^\infty(J, L^2)$. Hence, $\frac{du_{j_k}}{dt}(t) \in W^{1,\infty}(J, L^2) \hookrightarrow C^\mu(J, L^2)$ and the family $\{\frac{du_{j_k}}{dt}(t) \in C^\mu(J, L^2) : k \geq 1\}$ is bounded; that is $\|\frac{du_{j_k}}{dt}\|_{C^\mu(J,L^2)} \leq K$, $k \geq 1$. Therefore, the family $\frac{du_{j_k}}{dt}$ is equicontinuous in $C(J, L^2)$ and consequently $\{\frac{du_{j_k}}{dt} : k = 1, 2, 3 \cdots\}$ is relatively compact in $C(J, L^2)$. Thus, exists a subsequence of $\frac{du_{j_k}}{dt}$, that we continue denote to $\frac{du_{j_k}}{dt}$, and exists $\bar{v} \in C(J, L^2)$ such that $\frac{du_{j_k}}{dt} \to \bar{v}$ in $C(J, L^2)$, for all compact J. Since $\frac{d}{dt}$ is a closed operator and $u_{j_k} \to \bar{u}$, then $\frac{d\bar{u}}{dt} = \bar{v}$ and $\sup_{t\in\mathbb{R}}\{\|\bar{v}(t)\|_{L^2}\} \leq \tilde{K}$. But $(\bar{u}, \bar{v})$ is globally defined in Y^0, it remains only to show that $(\bar{u}, \bar{v})$ is a weak solution of (1.1) with $\eta = 0$. Indeed, since (u_k, v_k) satisfy the integral equation:

$$(u_k, v_k) = e^{Ct}(u_{0k}, v_{0k}) + \int_0^t e^{C(t-s)}(0, f(u_k(s)) - 2\eta A^{\frac{1}{2}}v_k(s))ds, \tag{2.28}$$

$(u_{0k}, v_{0k}) \to (\bar{u}(0), \bar{v}(0))$ in $H^1 \times L^2$ and

$$\|e^{Ct}((u_{0k}, v_{0k}) - (\bar{u}(0), \bar{v}(0)))\|_{Y^0} \leq Me^{\omega t}\|(u_{0k}, v_{0k}) - (\bar{u}(0), \bar{v}(0))\|_{Y^0}$$

then $\|e^{Ct}((u_{0k}, v_{0k}) - (\bar{u}(0), \bar{v}(0)))\|_{Y^0} \to 0$ as $k \to \infty$ uniformly to t in compacts. Also $(0, f(u_k(s))) \to (0, f(\bar{u}(s)))$ in $C(\mathbb{R}, H^1 \times L^2)$ uniformly to s in compacts. And, as we observed before,

$$(2\eta_{j_k}\|A^{\frac{1}{2}}v_{j_k}(s)\|_{L^2}) \to 0$$

as $j_k \to \infty$, uniformly to s in compacts. Thus,

$$e^{Ct}(u_{0k}, v_{0k}) + \int_0^t e^{C(t-s)}[(0, f(u_k(s))) + (0, -2\eta A^{\frac{1}{2}}v_k(s))]ds$$

converges to

$$e^{Ct}(\bar{u}(0), \bar{v}(0)) + \int_0^t e^{C(t-s)}[(0, f(\bar{u}(s)))]ds$$

as $k \to \infty$. But

$$e^{Ct}(u_{0k}, v_{0k}) + \int_0^t e^{C(t-s)}[(0, f(u_k(s))) + (0, -2\eta A^{\frac{1}{2}}v_k(s))]ds = (u_k, v_k).$$

converges to $(\bar{u}, \bar{v})$ as $k \to \infty$. Therefore,

$$(\bar{u}, \bar{v}) = e^{Ct}(\bar{u}(0), \bar{v}(0)) + \int_0^t e^{C(t-s)}[(0, f(\bar{u}(s)))]ds,$$

and it is a weak solution of (1.1) with $\eta = 0$.

To complete the proof, we now show that the statement implies the upper semicontinuity. Indeed, suppose that the family of the attractors is not upper semicontinuity in $\eta = 0$, that means there is $\epsilon_0 > 0$, a sequence of positive numbers η_n converging to 0 and a corresponding sequence (u_{0n}, v_{0n}) with $(u_{0n}, v_{0n}) \in \mathcal{A}_{\eta_n}$ such that

$$\delta_{Y^0}((u_{0n}, v_{0n}), \mathcal{A}) \geq \epsilon_0, \quad \forall n \in \mathbb{N}.$$

Considering $u_n(t)$ solution of (1.1) with initial condition (u_{0n}, v_{0n}), we have $(u_n(t), \frac{d}{dt}u_n(t)) \in \mathcal{A}_{\eta_n}$ and there is no subsequence of (u_{0n}, v_{0n}) converging to some $(\bar{u}, \bar{v}) \in \mathcal{A}$. What contradicts the Statement 1

3. Lower Semicontinuity of the attractors

Considering the theorem proved by Hale in [5], we can see that the proof of lower semicontinuity of attractors needs the verification of several hypotheses. Among them, there is the hypothesis on the lower semicontinuity of the local unstable manifolds. We prove this using the results obtained by Carvalho and Hines in [2].

Then, we begin by proving lower semicontinuity of the local unstable manifolds in the equilibrium points.

First of all, the equilibrium points from (1.1) are the same for all $\eta \geq 0$. Theses equilibrium points are given by: $\Phi_\eta = (\phi, 0)$ where ϕ satisfy

$$\phi'' + f(\phi) = 0.$$

Therefore, the family $\Phi_\eta = (\phi, 0)$ are continuous in η.

We denote the equilibrium points by $\Phi_j = (\phi_j, 0)$. For each $\eta \geq 0$ and Φ_j let $S_{j\eta}$ and $U_{j\eta}$ be a local stable and unstable manifolds respectively, for the (1.1) linearized in Φ_j. Let $P_{j\eta}$ and $Q_{j\eta}$ be the associated spectral projections onto $U_{j\eta}$ and $S_{j\eta}$ respectively. Let $W_{j\eta}$ be the unstable manifold of Φ_j for the (1.1). There is a neighborhood $N_{j\eta}$ of Φ_j and a homeomorphism, $\sigma_{j\eta} : N_{j\eta} \bigcap U_{j\eta} \to N_{j\eta} \bigcap W_{j\eta}$.

Let the linearized equations in Φ_j be,

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A & -2(\eta A^{1/2} + a) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ f'(\phi_j)u \end{bmatrix} \tag{3.1}$$

We need to prove that: given

$$C'_\eta(\Phi_j) = \begin{bmatrix} 0 & I \\ -A & -2(\eta A^{1/2} + a) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ f'(\phi_j) & 0 \end{bmatrix}$$

then there are $\sigma_- < 0$ and $\sigma_+ > 0$ such that spectrum of $C'_\eta(\Phi_j)$, $\sigma(C'_\eta(\Phi_j))$ satisfy

$$\sigma(C'_\eta(\Phi_j)) \bigcap [\sigma_-, \sigma_+] = \emptyset$$

At first, if $\mu_{\eta k}$ is a complex eigenvalue of $C'_\eta(\Phi_j)$ then $\mathrm{Re}(\mu_{\eta k}) \leq -a$. In fact, if λ is a eigenvalue of $C'_\eta(\Phi)$ with associated eigenvector $(u, v) \in H^2 \times H^1$ then λ and (u, v) satisfy:

$$\lambda^2 u(x) + 2a\lambda u(x) + 2\eta\lambda A^{\frac{1}{2}}(x) + Au(x) + f'(\phi(x))u(x) = 0. \tag{3.2}$$

If $\lambda \in \mathbb{C}$ then by multiplying (3.2) for $\overline{u}$ and integrating of 0 to 1 and by considering u normalized in L^2, we get:

$$\lambda^2 + 2a\lambda + 2\eta\lambda \|A^{1/4}u\|^2_{L^2} + \|u\|^2_{H^1} + \int_0^1 f'(\phi)|u(x)|^2 dx = 0, \tag{3.3}$$

Separating the imaginary part we obtain:

$$2\mathrm{Re}(\lambda)\mathrm{Im}(\lambda) + 2a\mathrm{Im}(\lambda) + 2\eta\mathrm{Im}(\lambda)\|A^{1/4}u\|^2_{L^2} = 0.$$

Since $\lambda \in \mathbb{C}$, then

$$\mathrm{Re}(\lambda) = -\eta\|A^{1/4}u\|^2_{L^2} - a \leq -a.$$

Thus, it is enough to show the spectrum separation for all $\eta > 0$ only for real eigenvalues of $C'_\eta(\phi)$ in a neighborhood of zero.

With this goal we prove the following result

Teorema 3.1. *The eigenvalues of $C'_\eta(\Phi)$ converge to the eigenvalues of $C'(\Phi)$ as $\eta \to 0$.*

Proof: We prove that for any $\epsilon > 0$, exists $\eta_0 = \eta_0(\epsilon)$ such that for $\eta \leq \eta_0$ we have $K_\epsilon \subset \rho(C'_\eta(\Phi))$, where $K_\epsilon = B_K \backslash \bigcup_j B_\epsilon(\mu_j)$ e $\mu_j \in \sigma(C'(\Phi))$.

In fact, we rewrite $C'_\eta(\Phi)$ as

$$C'_\eta(\Phi) = C'(\Phi) + B_\eta,$$

where $B_\eta = \begin{bmatrix} 0 & 0 \\ 0 & -2\eta A^{1/2}\end{bmatrix}$. At first, we remark B_η is $C'(\Phi)$-relatively bounded due to

$$\|B_\eta(\begin{bmatrix} u \\ v\end{bmatrix})\|_{H^1\times L^2} = 2\eta\|A^{1/2}v\|_{L^2} = 2\eta\|v\|_{H^1} \leq 2\eta\|C'(\Phi)(\begin{bmatrix} u \\ v\end{bmatrix})\|_{H^1\times L^2}.$$

And, for $\lambda \in K_\epsilon \bigcap \rho(C'(\Phi))$,

$$\begin{aligned}\|B_\eta(\lambda - C'(\Phi))^{-1}\| &\leq 2\eta\|C'(\Phi)(\lambda - C'(\Phi))^{-1}\| \\ &= 2\eta\|\lambda(\lambda - C'(\Phi))^{-1} - I\| \leq 2\eta(1 + \|\lambda(\lambda - C')^{-1}\|) \\ &\leq 2\eta(1 + |\lambda|\|R(\lambda, C'(\Phi))\|) \leq 2\eta(1 + KM) < 1\end{aligned}$$

where $\|R(\lambda, C'(\Phi)\| \leq M$, $|\lambda| \leq K$ and $\eta \leq \eta_0 = \dfrac{1}{4(1+KM)}$.

Therefore, $\lambda \in \rho(C'_\eta(\Phi))$ and $\sigma(C'_\eta(\Phi)) \bigcap B_K \subset \bigcup_j B_\epsilon(\mu_j)$ for $\eta \leq \eta_0$.

Lema 3.1. *Let $\Phi = (\phi, 0)$ be an equilibrium point of (1.1), for all $\eta \geq 0$. And $g^e : H^1 \times L^2 \to H^1 \times L^2$ which is given by $g^e(\begin{bmatrix} u \\ v\end{bmatrix})(x) = g(x, \begin{bmatrix} u \\ v\end{bmatrix})$ and*

$$g(x,(r,s)) = \begin{bmatrix} 0 \\ f(r)\end{bmatrix} - \begin{bmatrix} 0 \\ f(\phi(x))\end{bmatrix} - \begin{bmatrix} 0 & 0 \\ f'(\phi(x)) & 0\end{bmatrix}\begin{bmatrix} r - \phi(x) \\ s\end{bmatrix}.$$

Then, the Lipschitz constant of g^e in $B_{\frac{1}{n}}(\Phi)$ converges to zero as $n \to \infty$.

Proof: Let (u, v), $(w, z) \in H^1 \times L^2 \bigcap B_{\frac{1}{n}}(\Phi)$, then by the Mean Value Theorem we have:

$$\begin{aligned}\|g^e(u,v) - g^e(w,z)\|^2_{H^1\times L^2} &= \|f(u) - f(w) - f'(\phi)(u-w)\|^2_{L^2} \\ &= \int_0^1 [f'(u(x) - t_x(u(x) - w(x))) - (f'(\phi(x))]^2(u(x) - w(x))^2 dx \\ &\leq \int_0^1 [f'(u(x) - t_x(u(x) - w(x))) - f'(\phi(x))]^2 dx\|u - w\|^2_{H^1} \\ &\leq K(\|u - w\|^2_{H^1})\|u - w\|^2_{H^1}\end{aligned}$$

where $K(\|u - w\|^2_{H^1})$ goes to zero as $\|u - w\|_{H^1}$ goes to zero.

From the results in [3] we have the resolvent of C_η compact then $T_\eta(t)$ is a compact semigroup and

$$\begin{aligned}&\|T_\eta(s)(u,v)\| \leq Ke^{\rho s}\|(u,v)\|,\ s \leq 0,\ (u,v) \in W^u_{j\eta}, \\ &\|T_\eta(s)(u,v)\| \leq Ke^{-\beta s}\|(u,v)\|,\ s \geq 0,\ (u,v) \in W^s_{j\eta}.\end{aligned}$$

Thus, from Theorem 4.3 in [2] we have the following result

Proposição 3.1. *The distance between the flow and the unstable manifold is bounded by $Me^{-\beta t}\|Q_{j\eta}(u_0, v_0) - \sigma_{j\eta}(P_{j\eta(u_0,v_0)})\|$ and the Lipschtiz constant $L_{\sigma_{j\eta}}$ of $\sigma_{j\eta}$ can be chosen uniformly in η.*

We have the following result on the continuity of semigroups $T_\eta(t)$ on $\mathcal{A}$ in $\eta = 0$

Teorema 3.2. *For any $\epsilon > 0$, $\tau > 0$*
i. There is $\eta_0 = \eta_0(\epsilon, \tau)$ such that

$$\|T_\eta(t)(\begin{bmatrix} u \\ v\end{bmatrix}) - T(t)(\begin{bmatrix} u \\ v\end{bmatrix})\|_Y \leq \epsilon,$$

for all $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{A}$ *and* $0 < t < \tau$;

ii. There are a real number $\delta^* = \delta(\epsilon, \tau) > 0$ *and a* $\eta_0 > 0$ *such that*

$$\|T_\eta(t)(u,v)_\eta - T_0(t)(u,v)\|_Y \leq \epsilon$$

for $0 \leq t \leq \tau$, $(u,v) \in \mathcal{A}$, $(u,v)_\eta \in \mathcal{A}_\eta$, $\|(u,v) - (u,v)_\eta\|_Y \leq \delta^*$ *e* $\eta \leq \eta_0$.

Proof: i) We rewrite $C_\eta(u,v) + h(u,v)$,

$$C_\eta \begin{bmatrix} u \\ v \end{bmatrix} + h(\begin{bmatrix} u \\ v \end{bmatrix}) = C \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -2\eta A^{\frac{1}{2}} v + f(u) \end{bmatrix} = C \begin{bmatrix} u \\ v \end{bmatrix} + g(\begin{bmatrix} u \\ v \end{bmatrix}) + h(\begin{bmatrix} u \\ v \end{bmatrix}),$$

where $g(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} 0 \\ -2\eta A^{\frac{1}{2}} v \end{bmatrix}$ e $h(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} 0 \\ f(u) \end{bmatrix}$. Using Variation Constant Formula we have:

$$T_\eta(t) \begin{bmatrix} u \\ v \end{bmatrix} = e^{Ct} \begin{bmatrix} u \\ v \end{bmatrix} + \int_0^t e^{C(t-s)} (g(T_\eta(s)(\begin{bmatrix} u \\ v \end{bmatrix})) + h(T_\eta(s) \begin{bmatrix} u \\ v \end{bmatrix}))ds$$

and

$$T(t) \begin{bmatrix} u \\ v \end{bmatrix} = e^{Ct} \begin{bmatrix} u \\ v \end{bmatrix} + \int_0^t e^{C(t-s)} h(T(s)(\begin{bmatrix} u \\ v \end{bmatrix}))ds.$$

Thus, for $0 \leq t \leq \tau$, we have:

$$\begin{aligned} \|T_\eta(t)(\begin{bmatrix} u \\ v \end{bmatrix}) - T(t)(\begin{bmatrix} u \\ v \end{bmatrix})\|_Y \leq & \int_0^t \|e^{C(t-s)}[h(T_\eta(s)(\begin{bmatrix} u \\ v \end{bmatrix})) - h(T(s)(\begin{bmatrix} u \\ v \end{bmatrix}))]\|ds \\ & + \int_0^t \|e^{C(t-s)} g(T_\eta(s)(\begin{bmatrix} u \\ v \end{bmatrix}))\|ds \end{aligned} \tag{3.4}$$

Using e^{Ct} is a contraction semigroup, $h : H^1 \times L^2 \to H^1 \times L^2$ Lipschitz in bounded and for $(u,v) \in \mathcal{A}$, $\|(u,v)\|_{Y^1} \leq r_1$ and Theorem (2.2), we obtain $\|T_\eta(s)(\begin{bmatrix} u \\ v \end{bmatrix})\|_{Y^1} \leq K$ for all $s \geq 0$ and $(u,v) \in \mathcal{A}$ then, by (3.4), we get

$$\|T_\eta(t)(\begin{bmatrix} u \\ v \end{bmatrix}) - T(t)(\begin{bmatrix} u \\ v \end{bmatrix})\|_Y \leq \int_0^t L\|T_\eta(s)(\begin{bmatrix} u \\ v \end{bmatrix}) - T(s)(\begin{bmatrix} u \\ v \end{bmatrix}))\|ds + 2\eta M K \tau$$

Finally, by using Gronwall, we obtain

$$\|T_\eta(t)(\begin{bmatrix} u \\ v \end{bmatrix}) - T(t)(\begin{bmatrix} u \\ v \end{bmatrix})\|_Y \leq 2\eta M K \tau e^{MLt} \leq 2\eta M K \tau e^{ML\tau}.$$

ii) Using i) and $\|T_\eta(t)((u,v)_\eta) - T_\eta(t)((u,v))\| \leq e^{Lt}\|(u,v) - (u,v)_\eta\|_Y$ we have

$$\|T_\eta(t)(u,v)_\eta - T(t)(u,v)\|_Y \leq e^{Lt}\|(u,v)_\eta - (u,v)\|_Y + 2\eta M K \tau e^{ML\tau}$$

Thus, we consider δ^* such that $e^{L\tau}\|(u,v)_\eta - (u,v)\|_Y \leq \frac{\epsilon}{2}$ and η_0 such that $2\eta M K \tau e^{ML\tau} \leq \frac{\epsilon}{2}$ for $\eta \leq \eta_0$ then

$$\|T_\eta(t)(u,v)_\eta - T(t)(u,v)\|_Y \leq \epsilon.$$

Observação 3.1. *It follows from the convergence of the eigenvalues in η that:*

i. If the equilibrium points of (1.1) for $\eta = 0$ are hyperbolic then the equilibrium points of (1.1) are also hyperbolic.

ii. The projections $P_{j\eta}$ and $Q_{j\eta}$ are uniform bounded.

iii. From the results in [9]*, the equilibrium points of the parabolic problem $u_t = u_{xx} + f(u)$ are generically hyperbolic then, by using a perturbation arguments in f, we get that the equilibrium points of (1.1), which are in finite number, are also generically hyperbolic.*

Using remarks i. and ii. and Proposition(3.1), by theorem 2.1 in [2], we get that the local unstable manifolds are lower semicontinuity.

The lower semicontinuity of local unstable manifolds together with continuity of $T_\eta(t)$ for $\eta \geq 0$ and the fact of $T_\eta(t)$ is a gradient system, we get the main result.

Teorema 3.3. *The family of the global attractors, $\mathcal{A}_\eta$, of (1.1) is lower semicontinuous in $\eta = 0$, $\mathcal{A}_0 := \mathcal{A}$.*

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