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Patterns in parabolic problems with nonlinear boundary conditions

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Resumo

Neste artigo estudamos a existência de padrões para problemas parabólicos com condição de fronteira não linear em domínios pequenos, conectados por canais finos. Analisamos a convergência de auto-valores e auto-vetores do operador de Laplace em nesses domínios e usamos esta informação para mostrar que a dinâmica assintótica da equação do calor nesses domínios é equivalente a dinâmica assintótica de um sistema de duas equações diferenciais ordinárias (fracamente) acopladas por difusão. A principal ferramenta empregada é a Teoria de Variedades Invariantes.

PATTERNS IN PARABOLIC PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

ALEXANDRE N. CARVALHO* AND GERMAN LOZADA-CRUZ†

ABSTRACT. In this paper we study the existence patterns in parabolic problems with nonlinear boundary conditions on small domains connected by thins channels. We Study the convergence of eigenvalues and eigenfunctions of the Laplace operator in such domains. This information is used to show that the asymptotic dynamics of the heat equation in this domain is equivalent to the asymptotic dynamics of a system of two differential equations diffusively (weakly) coupled. The main tool employed here is the Invariant Manifold Theory.

1. INTRODUCTION

Our aim is to show the existence of patterns for the following problem

$$\begin{cases} u_t = \frac{\delta}{\lambda_2^\epsilon}\Delta u + f(u) \text{ in } \Omega_\epsilon \\ \frac{\delta}{\lambda_2^\epsilon}\frac{\partial u}{\partial n} = g(u) \qquad \text{on } \partial\Omega_\epsilon, \end{cases} \tag{1.1}$$

where $\Omega_\epsilon = \Omega \cup R_\epsilon$ (see Figure 1), $\Omega = \Omega_0^L \cup \Omega_0^R$ with Ω_0^L, $\Omega_0^R \subset \mathbb{R}^N$ domains, R_ϵ is a channel that connects Ω_0^L and Ω_0^R, $\frac{\delta}{\lambda_2^\epsilon}$ is the diffusion coefficient, f and g are nonlinear functions that satisfy certain dissipativeness properties and λ_2^ϵ is the second eigenvalue for the problem

$$\begin{cases} -\Delta u &= \lambda u \text{ in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \partial\Omega_\epsilon. \end{cases} \tag{1.2}$$

H. Matano [10] in 1979 shows an important result on existence of patterns for reaction-diffusion problems with homogeneous Neumann boundary conditions

$$\begin{cases} u_t &= \Delta u + k\ f(u) \text{ in } D \\ \frac{\partial u}{\partial n} &= 0 \ \text{ on } \partial D, \end{cases} \tag{1.3}$$

where D is a nonconvex domain. In this direction other results were obtained later by the others authors like J. K. Hale and J. Vegas [7], S. Jimbo [9], etc.

N. Consul and J. Sòla-Morales [6] in 1995 extended Matano's results proving the existence of stable nonconstant stable equilibria for diffusion equations with nonlinear boundary conditions

$$\begin{cases} u_t &= \Delta u \quad \text{in } D \\ \frac{\partial u}{\partial n} &= k\ f(u) \text{ on } \partial D. \end{cases} \tag{1.4}$$

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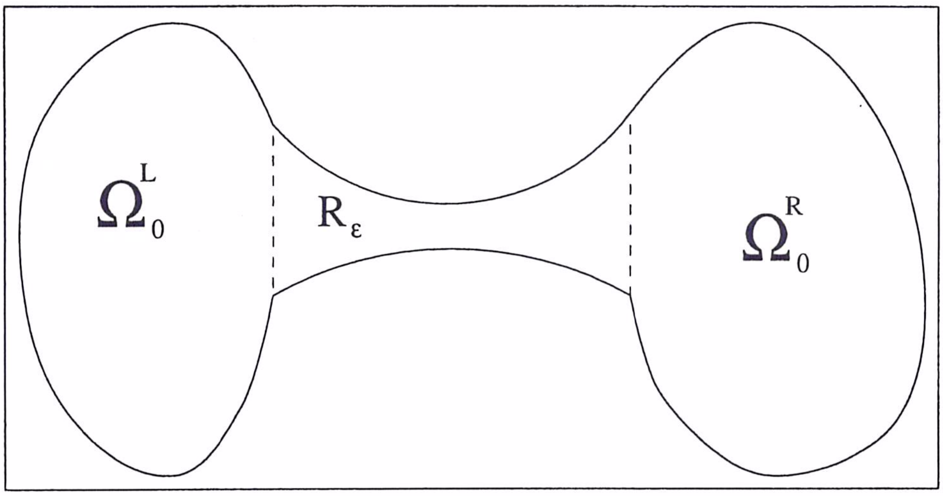

FIGURE 1. Domain Ω_ϵ

The problem (1.4) is also a model of reaction-diffusion like the problem (1.3), but the reaction happens only in the boundary of the domain, for example the presence of solid catalyzer.

Denoting by λ_k^ϵ, $k = 1, 2, ...$ the eigenvalues of (1.2), the first eigenvalue $\lambda_1^\epsilon = 0$, the second eigenvalue λ_2^ϵ go to zero when $\epsilon \longrightarrow 0$ and third eigenvalue is bounded away from zero for ϵ sufficient small. Thus we conclude that the first eigenvalue μ_1^ϵ of

$$\begin{cases} \frac{\delta}{\lambda_2^\epsilon}\Delta u & = \mu\ u \text{ in } \Omega_\epsilon \\ \frac{\delta}{\lambda_2^\epsilon}\frac{\partial u}{\partial n} & = 0 \text{ on } \partial\Omega_\epsilon, \end{cases} \tag{1.5}$$

is zero, the second eigenvalue $\mu_2^\epsilon = \delta$ and $\mu_k^\epsilon \longrightarrow +\infty$ as $\epsilon \longrightarrow 0$, $k \in \mathbb{N}$, $k \geq 3$.

With this we intuitively guess that the system (1.1) should approach a system of two couple ordinary differential equations.

We show that if f and g satisfy $f(u)u < 0$, $g(u)(u) < 0$ when $|u|$ is large then, the problem (1.1) has a stable nonconstant equilibrium for ϵ and δ sufficient small. This will be shown using invariant manifold theorem.

This paper is organized as follows. In section 2 we study the eigenvalue problem (1.2), showing the convergence of eigenvalues and eigenfunctions. In section 3 we present problem (1.1) and determine the system couple of ordinary differential equations that approach (1.1). Also, we show that the dynamics of this system is equivalent to dynamics of the system (1.1), here we use invariant manifold theorem and the results of the section 2. In section 4 we present the patterns for specific functions f and g with δ sufficient small.

2. EIGENVALUE PROBLEM

There are several works where eigenvalue problems for the Laplacian on varying domains are studied, for example: [1, 3, 7, 9, 11] etc.

Consider Ω_0^L and Ω_0^R domains in $\mathbb{R}^N$, $N \geq 2$ with $\overline{\Omega_0^L} \cap \overline{\Omega_0^R} = \emptyset$. Put $R_\epsilon = \{(\epsilon x, \epsilon^\eta y) : (x,y) \in R_1\} \subset \mathbb{R} \times \mathbb{R}^{N-1}$, where $\eta > 0$, R_1 is a channel that connects Ω_0^L and Ω_0^R. Let $\Omega_\epsilon = \Omega_0^L \cup R_\epsilon \cup \Omega_0^R$ (see Figure 1) where $0 < \epsilon < \epsilon_0$ with $\epsilon_0 > 0$. Denote $\Omega = \Omega_0^L \cup \Omega_0^R$.

Consider the eigenvalue problem (1.2).

Teorema 2.1. (Convergence of Eigenvalues)

(*i*) $\lambda_1^\epsilon \to 0$, *when* $\epsilon \to 0$

(*ii*) $\lambda_2^\epsilon \to 0$, *when* $\epsilon \to 0$

(*iii*) 0 *is not accumulation point of* $\{\lambda_3^\epsilon : 0 < \epsilon < \epsilon_0\}$.

Proof.

(*i*) It is easy see that

$$\begin{aligned}\lambda_1^0 &= \lambda_1^\epsilon = 0 \\ \varphi_1^0 &= |\Omega|^{-\frac{1}{2}},\ \varphi_1^\epsilon = |\Omega_\epsilon|^{-\frac{1}{2}}.\end{aligned}$$

(*ii*) Now, we study the second eigenvalue λ_2^ϵ of (1.2). Here we use the idea of Carvalho and Pereira in [2]. First we exhibit a test function $\varphi^\epsilon \in H^1(\Omega_\epsilon) \cap C(\Omega_\epsilon)$ that only depend of x,

$$\varphi^\epsilon = \begin{cases} c_1^\epsilon & \text{in } \Omega_0^L \\ \text{linear} & \text{in } R_\epsilon \\ c_2^\epsilon & \text{in } \Omega_0^R, \end{cases} \tag{2.1}$$

where $c_1^\epsilon, c_1^\epsilon$ are solutions of

$$\begin{cases} \int_{\Omega_\epsilon} \varphi^\epsilon = 0 \\ \|\varphi^\epsilon\|_{L^2(\Omega_\epsilon)} = 1. \end{cases} \tag{2.2}$$

The system (2.2) is equivalent to the following system

$$\begin{cases} c_1^\epsilon|\Omega_0^L| + \int_{R_\epsilon} \varphi^\epsilon + c_2^\epsilon|\Omega_0^R| & = 0 \\ (c_1^\epsilon)^2|\Omega_0^L| + \int_{R_\epsilon} (\varphi^\epsilon)^2 + (c_2^\epsilon)^2|\Omega_0^R| & = 1. \end{cases} \tag{2.3}$$

Observe that $\int_{R_\epsilon} \varphi^\epsilon \to 0$ when $\epsilon \to 0$, also $\int_{R_\epsilon} (\varphi^\epsilon)^2 \to 0$ when $\epsilon \to 0$. Since $|\Omega_0^L|$ and $|\Omega_0^L|$ are real nonnegative real numbers, the second equation of (2.3) follow that c_1^ϵ and c_2^ϵ are bounded in $\mathbb{R}$. Thus, there is a subsequence, denoted by c_1^ϵ and c_2^ϵ, that converges for c_1^0 and c_2^0 respectively. The first equation of (2.3) we see that c_1^ϵ and c_2^ϵ have opposite sign, for uniqueness of the limit we consider c_1^ϵ with negative signal. Thus we have that the system (2.3) has a limit system given by

$$\begin{cases} c_1^0|\Omega_0^L| + c_2^0|\Omega_0^R| & = 0 \\ (c_1^0)^2|\Omega_0^L| + (c_2^0)^2|\Omega_0^R| & = 1. \end{cases} \tag{2.4}$$

Solving (2.4) we have

$$c_1^0 = -\sqrt{\frac{|\Omega_0^R|}{|\Omega_0^L||\Omega|}},\ c_2^0 = \sqrt{\frac{|\Omega_0^L|}{|\Omega_0^R||\Omega|}}, \tag{2.5}$$

where $|\Omega| = |\Omega_0^L| + |\Omega_0^R|$. With this we have the following function

$$\varphi_2^0 = \begin{cases} c_1^0, & \text{in } \Omega_0^L \\ c_2^0, & \text{in } \Omega_0^R. \end{cases} \tag{2.6}$$

Obviously

$$\int_\Omega \varphi_2^0 = 0 \text{ and } \|\varphi_2^0\|_{L^2(\Omega)} = 1.$$

In fact φ_2^0 is the eigenfunction associated to λ_2^0.

Now by the min-max principle we have

$$\begin{aligned} \lambda_2^\epsilon &\le \int_{\Omega_\epsilon} |\nabla \varphi_2^\epsilon|^2 = \int_{R_\epsilon} |\nabla \varphi_2^\epsilon|^2 \\ &\le \left(\frac{c_2^\epsilon - c_1^\epsilon}{c\epsilon}\right)^2 \int_{R_\epsilon} 1 \\ &\le C_0 \epsilon^{\eta(N-1)-1}, \end{aligned} \tag{2.7}$$

where $C_0 = \frac{4M^2|R_1|}{c}$, $M = \max\{C_1, C_2\}$ with C_1, C_2 upper bounds for c_1^ϵ and c_2^ϵ respectively.

From (2.7) follows that $\lambda_2^\epsilon \to 0$ where $\epsilon \to 0$ for $\eta > \frac{1}{N-1}$.

(*iii*) Now, let λ_3^ϵ the third eigenvalue of (1.2) and φ_3^ϵ an associated normalized eigenfunction. The eigenfunction φ_3^ϵ satisfy

$$\begin{cases} \int_{\Omega_\epsilon} \varphi_3^\epsilon = 0 \\ \int_{\Omega_\epsilon} \varphi_3^\epsilon \varphi_2^\epsilon = 0 \\ \|\varphi_3^\epsilon\|_{L^2(\Omega_\epsilon)} = 1, \end{cases} \tag{2.8}$$

where φ_2^ϵ is an eigenfunction associated to λ_2^ϵ.

Now we show that 0 is not accumulation point of $\{\lambda_3^\epsilon : 0 < \epsilon < \epsilon_0\}$, that is

$$\liminf_{\epsilon \to 0} \lambda_3^\epsilon > 0. \tag{2.9}$$

In fact, suppose that (2.9) is not satisfy, thus there is a sequence $\{\epsilon_j\}_{j=1}^\infty$ with $\epsilon_j \to 0$ as $j \to \infty$ such that

$$\lim_{j \to \infty} \lambda_3^{\epsilon_j} = 0,$$

then

$$\int_{\Omega_0^L} |\nabla \varphi_3^{\epsilon_j}|^2 \le \int_{\Omega_{\epsilon_j}} |\nabla \varphi_3^{\epsilon_j}|^2 = \lambda_3^{\epsilon_j} \|\varphi_3^{\epsilon_j}\|^2 = \lambda_3^{\epsilon_j}$$

The last inequality follows that

$$\int_{\Omega_0^L} |\nabla \varphi_3^{\epsilon_j}|^2 \to 0, \text{ when } \epsilon \to 0. \tag{2.10}$$

By the same reasoning

$$\int_{\Omega_0^R} |\nabla \varphi_3^{\epsilon_j}|^2 \to 0, \text{ when } \epsilon \to 0. \tag{2.11}$$

From (2.10) and (2.11) it follows that

$$\varphi_3^{\epsilon_j} \longrightarrow \begin{cases} c_1, & \text{in } \Omega_0^L \\ c_2, & \text{in } \Omega_0^R, \end{cases} \tag{2.12}$$

where c_1 and c_2 satisfy the follow system

$$\begin{cases} c_1|\Omega_0^L| + c_2|\Omega_0^R| & = 0 \\ c_1 c_1^0|\Omega_0^L| + c_2 c_2^0|\Omega_0^R| & = 0 \\ (c_1)^2|\Omega_0^L| + (c_2)^2|\Omega_0^R| & = 1. \end{cases} \tag{2.13}$$

But the system is incompatible, e.g., do not exist c_1, c_2 that satisfy (2.13). Thus we have a contradiction. Therefore (2.9) is valid. ∎

Teorema 2.2. (Convergence of Eigenfunctions)

(i) $\varphi_1^\epsilon \longrightarrow \varphi_1^0$ *in* $H^k(\Omega)$ *when* $\epsilon \longrightarrow 0$, $k \geq 1$

(ii) $\varphi_2^\epsilon \longrightarrow \varphi_2^0$ *in* $L^2(\Omega)$ *when* $\epsilon \longrightarrow 0$

(iii) $\varphi_2^\epsilon \longrightarrow \varphi_2^0$ *in* $H^1(\Omega)$, $H^2(\Omega)$ *when* $\epsilon \longrightarrow 0$

Proof.

(i) Observe that $\varphi_1^\epsilon = \frac{1}{|\Omega_\epsilon|^{\frac{1}{2}}}$ and $\varphi_1^0 = \frac{1}{|\Omega|^{\frac{1}{2}}}$ are the corresponding eigenfunctions of $\lambda_1^\epsilon = \lambda_1^0 = 0$.
It is clear that $\varphi_1^\epsilon \longrightarrow \varphi_1^0$ in $H^k(\Omega)$ for all integer $k \geq 0$.

(ii) Let λ_2^ϵ be the second eigenvalue of (1.2) and φ_2^ϵ a corresponding normalized eigenfunction. The eigenfunction φ_2^ϵ satisfy the following conditions

$$\begin{cases} \int_{\Omega_\epsilon} \varphi_2^\epsilon & = 0 \\ \|\varphi_2^\epsilon\|_{L^2(\Omega_\epsilon)} & = 1. \end{cases} \tag{2.14}$$

From (2.14)

$$\begin{aligned} \left|\int_\Omega \varphi_2^\epsilon\right| &= \left|\int_{R_\epsilon} \varphi_2^\epsilon\right| \leq \int_{R_\epsilon} |\varphi_2^\epsilon| \\ &\leq \Big(\int_{R_\epsilon} |\varphi_2^\epsilon|^2\Big)^{\frac{1}{2}}\Big(\int_{R_\epsilon} 1^2\Big)^{\frac{1}{2}} \\ &\leq (|R_\epsilon|)^{\frac{1}{2}}\|\varphi_2^\epsilon\|_{L^2(\Omega_\epsilon)}. \end{aligned} \tag{2.15}$$

From (2.15) it follows that

$$\int_\Omega \varphi_2^\epsilon \longrightarrow 0 \text{ e } \int_{R_\epsilon} \varphi_2^\epsilon \longrightarrow 0 \text{ when } \epsilon \longrightarrow 0. \tag{2.16}$$

Also we have that

$$\int_{R_\epsilon} |\varphi_2^\epsilon|^2 \longrightarrow 0 \text{ when } \epsilon \longrightarrow 0. \tag{2.17}$$

Using (2.17) in the second equation of (2.14) follow that

$$\int_\Omega |\varphi_2^\epsilon|^2 \longrightarrow 1 \text{ when } \epsilon \longrightarrow 0. \tag{2.18}$$

Now we show that

$$\int_\Omega \varphi_2^\epsilon \varphi_2^0 \to 1, \text{ when } \epsilon \to 0. \tag{2.19}$$

In fact, we know that

$$\int_\Omega |\nabla \varphi_2^\epsilon|^2 \leq \int_{\Omega_\epsilon} |\nabla \varphi_2^\epsilon|^2 = \lambda_2^\epsilon. \tag{2.20}$$

Thus we have

$$\int_\Omega |\nabla \varphi_2^\epsilon|^2 \to 0 \text{ when } \epsilon \to 0. \tag{2.21}$$

Therefore

$$\nabla \varphi_2^\epsilon \to 0, \text{ when } \epsilon \to 0. \tag{2.22}$$

Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, φ_2^ϵ has a convergent subsequence, that we write again φ_2^ϵ, such that this subsequence converges for φ. Thus we have

$$\begin{cases} \varphi_2^\epsilon \to \varphi, & \text{strongly in } L^2(\Omega) \\ \nabla \varphi_2^\epsilon \to 0, & \text{strongly in } L^2(\Omega). \end{cases} \tag{2.23}$$

Since ∇ is a closed operator we have that

$$\begin{cases} \varphi \in H^1(\Omega) \\ \nabla \varphi = 0. \end{cases} \tag{2.24}$$

From (2.24) and the uniqueness of limit follow that

$$\varphi = \varphi_2^0 = \begin{cases} c_1, & \text{in } \Omega_0^L \\ c_2, & \text{in } \Omega_0^R, \end{cases} \tag{2.25}$$

where c_1 and c_2 satisfy

$$\begin{cases} c_1|\Omega_0^L| + c_2|\Omega_0^R| & = 0 \\ (c_1)^2|\Omega_0^L| + (c_2)^2|\Omega_0^R| & = 1. \end{cases} \tag{2.26}$$

From (2.26) it follows that

$$c_1 = \pm\sqrt{\frac{|\Omega_0^R|}{|\Omega_0^L||\Omega|}} \text{ and } c_2 = \pm\sqrt{\frac{|\Omega_0^L|}{|\Omega_0^R||\Omega|}}.$$

Returning to (2.19) we have

$$\begin{aligned} \int_\Omega \varphi_2^\epsilon \varphi_2^0 &= \int_{\Omega_0^L} \varphi_2^\epsilon \varphi_2^0 + \int_{\Omega_0^R} \varphi_2^\epsilon \varphi_2^0 \\ &\to \int_{\Omega_0^L} c_1 c_1^0 + \int_{\Omega_0^R} c_2 c_2^0, \text{ when } \epsilon \to 0. \end{aligned}$$

The first equation of (2.26) we see that c_1 and c_2 have opposite signs. We consider c_1 with negative sign and c_2 with positive sign. In this case $c_1 = c_1^0$ and $c_2 = c_2^0$. Thus we have

$$\int_\Omega \varphi_2^\epsilon \varphi_2^0 \longrightarrow (c_1^0)^2|\Omega_0^L| + (c_2^0)^2|\Omega_0^R| = 1, \text{ when } \epsilon \to 0.$$

If $c_1 > 0$ and $c_2 < 0$ the constants c_1^0 and c_2^0 have the same signs. Also the last equation is valid. Therefore (2.19) is proved.

Now we show $\varphi_2^\epsilon \longrightarrow \varphi_2^0$, when $\epsilon \to 0$. In fact,

$$\begin{aligned} \|\varphi_2^\epsilon - \varphi_2^0\|^2_{L^2(\Omega)} &= \|\varphi_2^\epsilon\|^2_{L^2(\Omega)} + \|\varphi_2^0\|^2_{L^2(\Omega)} - 2\int_{\Omega_0} \varphi_2^\epsilon\varphi_2^0 \\ &\leq \|\varphi_2^\epsilon\|^2_{L^2(\Omega_\epsilon)} + 1 - 2\int_\Omega \varphi_2^\epsilon\varphi_2^0 \\ &\leq 2 - 2\int_\Omega \varphi_2^\epsilon\varphi_2^0. \end{aligned} \tag{2.27}$$

Using (2.19) and (2.27) it follows that

$$\|\varphi_2^\epsilon - \varphi_2^0\|_{L^2(\Omega)} \longrightarrow 0, \text{ when } \epsilon \to 0. \tag{2.28}$$

(*iii*) From (2.28) and (2.21) we conclude that

$$\|\varphi_2^\epsilon - \varphi_2^0\|_{H^1(\Omega)} \longrightarrow 0, \text{ when } \epsilon \to 0. \tag{2.29}$$

Consider the norm

$$\|u\|_{H^2(\Omega)} = \|\Delta u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}, \; u \in H^2(\Omega), \tag{2.30}$$

that is equivalent to the norm in $H^2(\Omega)$. Since $\Delta\varphi_2^\epsilon = \lambda_2^\epsilon\varphi_2^\epsilon$ it follows that

$$\|\Delta\varphi_2^\epsilon\|_{L^2(\Omega)} = \|\lambda_2^\epsilon\varphi_2^\epsilon\|_{L^2(\Omega)} \leq \lambda_2^\epsilon \longrightarrow 0 \text{ when } \epsilon \to 0. \tag{2.31}$$

From (2.31) and (2.18) there exists a constant C such that

$$\|\varphi_2^\epsilon\|_{H^2(\Omega)} \leq C < \infty. \tag{2.32}$$

Since the embedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$ is compact, φ_2^ϵ has a subsequence we denoted again by φ_2^ϵ, such that it is converge for φ in $H^1(\Omega)$. We also know that the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact and since this limit is unique it follows that $\varphi = \varphi_2^\epsilon$. Thus we have that $\varphi_2^\epsilon \longrightarrow \varphi_2^0$ in $H^2(\Omega)$ in the norm given by (2.30) when $\epsilon \to 0$. ■

3. Reaction-diffusion problem

In this section we consider the reaction-diffusion problem (1.1) with $\Omega_\epsilon = \Omega \cup R_\epsilon$ where R_ϵ is given in the section 2.

Without loss generality we will suppose that $|\Omega| = 1$.

By theorems 2.1 and 2.2, we intuitively guess that the system (1.1) can be approximated by a system of two coupled ordinary differential equations.

Now we determine the limit system for the problem (1.1). This can be seen in Carvalho and Cuminato [5] and Carvalho [4].

First we need understand the eigenvalue problem (1.5).

Denote by $\mu_1^\epsilon < \mu_2^\epsilon \leq \mu_3^\epsilon \leq ...$ the eigenvalues of problem (1.5) arranged in increasing order, counting multiplicity and $\varphi_1^\epsilon, \varphi_2^\epsilon, \varphi_3^\epsilon, ...$ a sequence of orthonormal eigenfunctions corresponding to the sequence of eigenvalues. Then we have

Proposição 3.1.

$$\begin{aligned} &\mu_1^\epsilon \equiv 0, \quad \varphi_1^\epsilon \to 1, \ \textit{when } \epsilon \to 0 \\ &\mu_2^\epsilon \equiv \delta, \quad \varphi_2^\epsilon \to \varphi_2^0 \ \textit{when } \epsilon \to 0. \end{aligned} \tag{3.1}$$

Moreover $\mu_3^\epsilon \to +\infty$ *when* $\epsilon \to 0$.

Proof. The proof is an immediate consequence of $\mu_k^\epsilon = \frac{\delta}{\lambda_2^\epsilon}\lambda_k^\epsilon$, $k = 1, 2, ...,$ where λ_k^ϵ are eigenvalues of (1.2). ∎

Now we consider the following decomposition of u

$$u = u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \omega,$$

where $\omega \in V^\perp$, $V = \text{span}[\varphi_1^\epsilon, \varphi_2^\epsilon]$,

$$u_1 = \int_{\Omega_\epsilon} u\varphi_1^\epsilon, \tag{3.2}$$

$$u_2 = \int_{\Omega_\epsilon} u\varphi_2^\epsilon, \tag{3.3}$$

$$\omega = u - u_1\varphi_1^\epsilon - u_2\varphi_2^\epsilon. \tag{3.4}$$

This decomposition induces, in the problem (1.1), the following decomposition

$$\begin{aligned} \dot{u}_1 &= \int_{\Omega_\epsilon} f(u)\varphi_1^\epsilon + \int_{\partial\Omega_\epsilon} g(u)\varphi_1^\epsilon \\ \dot{u}_2 &= -\delta u_2 + \int_{\Omega_\epsilon} f(u)\varphi_2^\epsilon + \int_{\partial\Omega_\epsilon} g(u)\varphi_2^\epsilon \\ \dot{\omega} &= \frac{\delta}{\lambda_2^\epsilon}\Delta\omega + f(u) - [\int_{\Omega_\epsilon} f(u)\varphi_1^\epsilon + \int_{\partial\Omega_\epsilon} g(u)\varphi_1^\epsilon]\varphi_1^\epsilon \\ &\quad -[\int_{\Omega_\epsilon} f(u)\varphi_2^\epsilon + \int_{\partial\Omega_\epsilon} g(u)\varphi_2^\epsilon]\varphi_2^\epsilon \\ \frac{\delta}{\lambda_2^\epsilon}\frac{\partial\omega}{\partial n} &= g(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \omega). \end{aligned} \tag{3.5}$$

Since the third eigenvalue μ_3^ϵ goes to infinity when $\epsilon \to 0$, we guess that ω is not important in the asymptotic behavior (1.1) and we have

$$\begin{cases} \dot{u}_1 \sim \int_{\Omega_\epsilon} f(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon)\varphi_1^\epsilon + \int_{\partial\Omega_\epsilon} g(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon)\varphi_1^\epsilon \\ \dot{u}_2 \sim -\delta u_2 + \int_{\Omega_\epsilon} f(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon)\varphi_2^\epsilon + \int_{\partial\Omega_\epsilon} g(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon)\varphi_2^\epsilon. \end{cases} \tag{3.6}$$

Using the convergence of eigenvalues and eigenfunctions we have that limit system should be

$$\begin{cases} \dot{u}_1 = \int_\Omega f(u_1 + u_2\varphi_2^0) + \int_{\partial\Omega} g(u_1 + u_2\varphi_2^0) \\ \dot{u}_2 = -\delta u_2 + \int_\Omega f(u_1 + u_2\varphi_2^0)\varphi_2^0 + \int_{\partial\Omega} g(u_1 + u_2\varphi_2^0)\varphi_2^0. \end{cases} \tag{3.7}$$

which is equivalent to

$$\begin{cases} \dot{u}_1 & = |\Omega_0^L| f(u_1 + c_1^0 u_2) + |\Omega_0^R| f(u_1 + c_2^0 u_2) \\ & + |\partial\Omega_0^L| g(u_1 + c_1^0 u_2) + |\partial\Omega_0^R| g(u_1 + c_2^0 u_2) \\ \dot{u}_2 & = -\delta u_2 + |\Omega_0^L| c_1^0 f(u_1 + c_1^0 u_2) + |\Omega_0^R| c_2^0 f(u_1 + c_2^0 u_2) \\ & + |\partial\Omega_0^L| c_1^0 g(u_1 + c_1^0 u_2) + |\partial\Omega_0^R| c_2^0 g(u_1 + c_2^0 u_2), \end{cases} \tag{3.8}$$

where

$$c_1^0 = -\Big[\frac{|\Omega_0^R|}{|\Omega_0^L|}\Big]^{\frac{1}{2}}, \; c_2^0 = \Big[\frac{|\Omega_0^L|}{|\Omega_0^R|}\Big]^{\frac{1}{2}}.$$

The variables u_1 and u_2 may not be the best choice of variables to study this problem. A better choice is probably a set of variables that reflected the average over Ω_0^L and Ω_0^R. To relate u_1 and u_2 with these averages we consider

$$v_1 = \big(|\Omega_0^L|\big)^{-1} \int_{\Omega_0^L} u(x)dx \text{ and } v_2 = \big(|\Omega_0^R|\big)^{-1} \int_{\Omega_0^R} u(x)dx$$

thus,

$$\begin{cases} u_1 & = |\Omega_0^L| v_1 + |\Omega_0^R| v_2 \\ u_2 & = -\big(|\Omega_0^L||\Omega_0^R|\big)^{\frac{1}{2}}(v_1 - v_2) \end{cases} \tag{3.9}$$

and

$$\begin{cases} v_1 = & u_1 + c_1^0 u_2 \\ v_2 = & u_1 + c_2^0 u_2. \end{cases} \tag{3.10}$$

With the variables (3.10) the system (3.8) becomes

$$\begin{cases} \dot{v}_1 & = -\delta|\Omega_0^R|(v_1 - v_2) + f(v_1) + \frac{|\partial\Omega_0^L|}{|\Omega_0^L|} g(v_1) \\ \dot{v}_2 & = \delta|\Omega_0^L|(v_1 - v_2) + f(v_2) + \frac{|\partial\Omega_0^R|}{|\Omega_0^R|} g(v_2). \end{cases} \tag{3.11}$$

Now our aim is to show that the dynamics of (1.1) can be described by the dynamics of (3.11), for this we employ the Invariant Manifold Theory.

INVARIANT MANIFOLDS

Let X and Y be Banach spaces and $-A : \mathcal{D}(A) \subset X \to X$ be a sectorial operator such that $\operatorname{Re} \sigma(-A) > 0$. Let $\{T(t) : t \geq 0\}$ be an analytic of semigroup of bounded linear operators generated by A and denote by $(-A)^\alpha$ the α fractional power of $-A$ and $X^\alpha := \mathcal{D}((-A)^\alpha)$ be a Banach space endowed with the graphic norm (see [8]).

Definição 3.1. *Let $F : X^\alpha \times Y \to X$, $G : X^\alpha \times Y \to Y$ be locally Lipschitz continuous functions and $B \in L(Y)$. A set $S \subset X^\alpha \times Y$ is an invariant manifold for the differential equation*

$$\begin{cases} \dot{x} = Ax + F(x,y) \\ \dot{y} = By + G(x,y), \end{cases} \tag{3.12}$$

if there exists $\sigma : Y \to X^\alpha$ such that $S = \{(x,y) \in X^\alpha \times Y : x = \sigma(y)\}$ and for any $(x_0, y_0) \in S$, there exists a solution $(x(\cdot), y(\cdot))$ of (3.12) on $\mathbb{R}$ such that $(x(t), y(t)) \in S$, $\forall t \in \mathbb{R}$. An invariant manifold S is said to be exponentially attracting if there are positive constants γ and K such that

$$\|x(t) - \sigma(y(t))\|_{X^\alpha} \leq Ke^{-\gamma t}\|x_0 - \sigma(y_0)\|_{X^\alpha},$$

whenever $(x(t), y(t))$ is a solution of (3.12).

Observe that the system (3.7) can be rewritten as follows

$$\begin{cases} \dot{u}_1 &= G_0^\epsilon(u_1, u_2, \omega) \\ \dot{u}_2 &= -\delta u_2 + G_1^\epsilon(u_1, u_2, \omega) \\ \dot{\omega} &= A_\epsilon \omega + F_\epsilon(u_1, u_2, \omega), \end{cases} \tag{3.13}$$

where

$$\begin{aligned} A_\epsilon &= \frac{\delta}{\lambda_2^\epsilon}\Delta \\ B(u_1, u_2)^\top &= (0, -\delta u_2)^\top \\ G_0^\epsilon(u_1, u_2, \omega) &= \int_\Omega f(u)\varphi_1^\epsilon + \int_{\partial\Omega_\epsilon} g(u)\varphi_1^\epsilon \\ G_1^\epsilon(u_1, u_2, \omega) &= \int_\Omega f(u)\varphi_2^\epsilon + \int_{\partial\Omega_\epsilon} g(u)\varphi_2^\epsilon \\ F_\epsilon(u_1, u_2, \omega) &= f(u) - G_0^\epsilon(u_1, u_2, \omega)\varphi_1^\epsilon - G_1^\epsilon(u_1, u_2, \omega)\varphi_2^\epsilon. \end{aligned} \tag{3.14}$$

Putting $y = (u_1, u_2)$ we have

$$\begin{cases} \dot{\omega} &= A_\epsilon \omega + F_\epsilon(y, \omega), \\ \dot{y} &= By + G_\epsilon(y, \omega), \end{cases} \tag{3.15}$$

where $G_\epsilon(y, \omega) = (G_0^\epsilon(u_1, u_2, \omega), G_1^\epsilon(u_1, u_2, \omega))$. Let $\epsilon > 0$ be a positive parameter and X_ϵ be a Banach space and $-A_\epsilon : \mathcal{D}(A_\epsilon) \subset X_\epsilon \to X_\epsilon$ be a sectorial operator. Denote by $(-A_\epsilon)^\alpha$, the α fractional powers of $-A_\epsilon$ and $X_\epsilon^\alpha := \mathcal{D}((-A_\epsilon)^\alpha)$, the associated fractional power spaces.

Lema 3.1. (Gronwall's Inequality) *Let ψ, α, β : $[a,b] \to \mathbb{R}^+$ be continuous functions with $\beta(t)$ integrable, $L > 0$ a constant. Assume that*

$$\psi(t) \leq \alpha(t) + L\int_t^b \beta(s)\psi(s)ds, \ a \leq t \leq b. \tag{3.16}$$

Then,

$$\psi(t) \leq \alpha(t) + L\int_t^b \beta(s)\alpha(s)e^{\int_t^s L\beta(u)du}ds. \tag{3.17}$$

Furthermore, if α is decreasing

$$\psi(t) \leq \alpha(b)e^{\int_t^b L\beta(u)du} - \int_t^b e^{\int_t^s L\beta(u)du}\dot{\alpha}(s)ds \leq \alpha(t)e^{\int_t^s L\beta(u)du}. \tag{3.18}$$

In particular if $\beta(t) \equiv 1$ we have that

$$\psi(t) \leq \alpha(t) + L\int_t^b \alpha(s)e^{L(s-t)}ds \tag{3.19}$$

and if in addition α is decreasing we have that

$$\psi(t) \leq \alpha(b)e^{L(b-t)} - \int_t^b e^{L(s-t)}\dot{\alpha}(s)ds \leq \alpha(t)e^{L(b-t)}. \tag{3.20}$$

Proof. Put $R(t) := \int_t^b \beta(s)\psi(s)ds$, differentiate and use (3.16). ■

Teorema 3.1. *Let X_ϵ and Y be a Banach spaces, A_ϵ be a sectorial operator on X_ϵ. Assume that $F_\epsilon : X_\epsilon^\alpha \times Y \to X_\epsilon$ and $G_\epsilon : X_\epsilon^\alpha \times Y \to Y$ are functions satisfying*

(1) $\|F_\epsilon(x,y) - F_\epsilon(z,w)\|_{X_\epsilon} \leq L_F\big(\|x-z\|_{X_\epsilon^\alpha} + \|y-w\|_Y\big)$,
(2) $\|F_\epsilon(x,y)\|_{X_\epsilon} \leq N_F$,
(3) $\|G_\epsilon(x,y) - G_\epsilon(z,w)\|_Y \leq L_G\big(\|x-z\|_{X_\epsilon^\alpha} + \|y-w\|_Y\big)$,
(4) $\|G_\epsilon(x,y)\|_Y \leq N_G$,

for each (x,y), (z,w) in $X_\epsilon^\alpha \times Y$. Assume also that

(5) $\|e^{-A_\epsilon t}w\|_{X_\epsilon^\alpha} \leq Me^{-\beta(\epsilon)t}\|w\|_{X_\epsilon^\alpha}$, $t \geq 0$,
(6) $\|e^{-A_\epsilon t}w\|_{X_\epsilon^\alpha} \leq Mt^{-\alpha}e^{-\beta(\epsilon)t}\|w\|_{X_\epsilon}$, $t > 0$, *for any $w \in X_\epsilon^\alpha$ where $\beta(\epsilon) \to \infty$ when $\epsilon \to 0$.*

and that $B \in L(Y)$. Consider the weakly couple system

$$\begin{cases} \dot{x} &= A_\epsilon x + F_\epsilon(x,y), \\ \dot{y} &= By + G_\epsilon(x,y). \end{cases} \tag{3.21}$$

Then, for small enough ϵ, there is an exponentially attracting invariant manifold

$$S_\epsilon = \{(x,y) : x = \sigma_\epsilon(y), y \in Y\}$$

for (3.21), where $\sigma_\epsilon : Y \to X_\epsilon^\alpha$ satisfies

$$s(\epsilon) = \sup_{y\in Y}\|\sigma_\epsilon(y)\|_{X_\epsilon^\alpha},$$

$$\|\sigma_\epsilon(y) - \sigma_\epsilon(z)\|_{X_\epsilon^\alpha} \leq l(\epsilon)\|y-z\|_Y,$$

with $s(\epsilon)$, $l(\epsilon) \to$ when $\epsilon \to 0$. If F_ϵ, G_ϵ are smooth; then σ_ϵ is smooth and its derivative $D\sigma_\epsilon$ satisfy

$$\sup_{y\in Y}\|D\sigma_\epsilon(y)\|_{L(Y,X_\epsilon^\alpha)} \leq l(\epsilon).$$

Corolário 3.1. *Let $f, g : \mathbb{R} \to \mathbb{R}$ are functions global Lipschitz, bounded with bounded derivative. Assume that*

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} < 0 \quad \textit{and} \quad \limsup_{|u|\to\infty} \frac{g(u)}{u} < 0.$$

Then, there exists a exponentially attracting invariant manifold

$$S = \{(u_1, u_2, \omega) : \omega = \sigma_\epsilon(u_1, u_2), (u_1, u_2) \in \mathbb{R}^2\}$$

for (3.5). *The flow on S is given by $u(t,x) = u_1\varphi_1^\epsilon(x) + u_2\varphi_2^\epsilon(x) + \sigma_\epsilon(u_1, u_2)$, where (u_1, u_2) is the solution of*

$$\begin{cases} \dot{u}_1 & = G_0^\epsilon(u_1, u_2, \sigma(u_1, u_2)) \\ \dot{u}_2 & = -\delta u_2 + G_1^\epsilon(u_1, u_2, \sigma(u_1, u_2)). \end{cases} \tag{3.22}$$

Furthermore, $\sigma_\epsilon \to 0$ in $C^1(V, V^\perp)$, as $\epsilon \to 0$, where $V = span\ [\varphi_1^\epsilon, \varphi_2^\epsilon]$.

Proof. First let us define

$$\begin{aligned} M_1 &= \sup_{x\in\mathbb{R}} f(x),\ L_1 = \sup_{x\in\mathbb{R}} f'(x), \\ M_2 &= \sup_{x\in\mathbb{R}} g(x),\ L_2 = \sup_{x\in\mathbb{R}} g'(x). \end{aligned}$$

We will show that G_0^ϵ and G_1^ϵ are globally Lipschitz continuous bounded functions with bounded derivative. In fact, if $a = u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \omega$ then,

$$\begin{aligned} |G_0^\epsilon(u_1,u_2,\omega)| &= |\int_{\Omega_\epsilon} f(a)\varphi_1^\epsilon + \int_{\partial\Omega_\epsilon} g(a)\varphi_1^\epsilon| \le \int_{\Omega_\epsilon} |f(a)|\ |\varphi_1^\epsilon| + \int_{\partial\Omega_\epsilon} |g(a)|\ |\varphi_1^\epsilon| \\ &\le M_1 \int_{\Omega_\epsilon} |\varphi_1^\epsilon| + M_2 \int_{\partial\Omega_\epsilon} |\varphi_1^\epsilon| \le M_1(|\Omega_\epsilon|)^{\frac{1}{2}} + M_2 \frac{|\partial\Omega_\epsilon|}{|\Omega_\epsilon|^{\frac{1}{2}}} \\ &\le M_1^{f,g}, \end{aligned}$$

where $M_1^{f,g} = M_1|\Omega_\epsilon|^{\frac{1}{2}} + M_2\frac{|\partial\Omega_\epsilon|}{|\Omega_\epsilon|^{\frac{1}{2}}}$, $|\Omega_\epsilon| = 1 + \epsilon^{\eta+1}|R_1| \le l_1$, $m_1 = \max|\varphi_1|$, $m_2 = \max|\varphi_2|$ and $|\partial\Omega_\epsilon| \le |\partial\Omega| + |\partial R_\epsilon| \le |\partial\Omega| + \epsilon + \epsilon^{\eta+1} \le l_2$. Also,

$$\begin{aligned} |G_1^\epsilon(u_1,u_2,\omega)| &= |\int_{\Omega_\epsilon} f(a)\varphi_2^\epsilon + \int_{\partial\Omega_\epsilon} g(a)\varphi_2^\epsilon| \le \int_{\Omega_\epsilon} |f(a)|\ |\varphi_1^\epsilon| + \int_{\partial\Omega_\epsilon} |g(a)|\ |\varphi_2^\epsilon| \\ &\le M_1 \int_{\Omega_\epsilon} |\varphi_2^\epsilon| + M_2 \int_{\partial\Omega_\epsilon} |\varphi_1^\epsilon| \le M_1 m_2 |\Omega_\epsilon| + M_2\ m_2|\partial\Omega_\epsilon| \\ &\le M_1 m_2 l_1 + M_2 m_2 l_2 = M_2^{f,g}. \end{aligned}$$

To obtain the global Lipschitz continuity, let $y_1 = (u_1, u_2)$, $y_2 = (v_1, v_2)$, $a = u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \omega_1$, $b = v_1\varphi_1^\epsilon + v_2\varphi_2^\epsilon + \omega_2$, then we have

$$\begin{aligned} |G_0^\epsilon(y_1,\omega_1) - G_0^\epsilon(y_2,\omega_2)| &= |\int_{\Omega_\epsilon} [f(a) - f(b)]\varphi_1^\epsilon + \int_{\partial\Omega_\epsilon} [g(a) - g(b)]\varphi_1^\epsilon| \\ &\le \int_{\Omega_\epsilon} |f(a) - f(b)|\ |\varphi_1^\epsilon| + \int_{\partial\Omega_\epsilon} |g(a) - g(b)|\ |\varphi_1^\epsilon| \\ &\le \int_{\Omega_\epsilon} |f'(\tau_1)|\ |a-b|\ |\varphi_1^\epsilon| + \int_{\partial\Omega_\epsilon} |g'(\tau_2)|\ |a-b|\ |\varphi_1^\epsilon| \\ &\le L_1 \int_{\Omega_\epsilon} |a-b|\ |\varphi_1^\epsilon| + L_2 \int_{\partial\Omega_\epsilon} |a-b|\ |\varphi_1^\epsilon|, \end{aligned}$$

$\tau_1, \tau_2 \in [a, b]$. Hence

$$\begin{aligned}
|G_0^\epsilon(y_1, \omega_1) \quad - \quad & G_0^\epsilon(y_2, \omega_2)| \leq L_1\big[\ |u_1 - v_1| \int_{\Omega_\epsilon} |\varphi_1^\epsilon|^2 + |u_2 - v_2| \int_{\Omega_\epsilon} |\varphi_1^\epsilon|\ |\varphi_2^\epsilon| \\
& + \int_{\Omega_\epsilon} |\omega_1 - \omega_2|\ |\varphi_2^\epsilon|\ \big] + L_2\big[|u_1 - v_1| \int_{\partial\Omega_\epsilon} |\varphi_1^\epsilon|^2 \\
& + |u_2 - v_2| \int_{\partial\Omega_\epsilon} |\varphi_1^\epsilon|\ |\varphi_2^\epsilon| + \int_{\partial\Omega_\epsilon} |\omega_1 - \omega_2|\ |\varphi_2^\epsilon|\ \big] \\
\leq \quad & L_1\big[\ \|y_1 - y_1\|_{\mathbb{R}^2} + \|\omega_1 - \omega_2\|_{L^2(\Omega_\epsilon)}\ \big] + L_2\big[\ C_0(|\Omega|)|u_1 - v_1| \\
& + C_1(|\Omega|)|u_2 - v_2| + C_2(|\Omega|)\ \epsilon^{\frac{1-\eta}{2}} \|\omega_1 - \omega_2\|_{H^1(\Omega_\epsilon)}\ \big] \\
\leq \quad & L_1\big[\ \|y_1 - y_1\|_{\mathbb{R}} + \|\omega_1 - \omega_2\|_{L^2(\Omega_\epsilon)}\ \big] \\
& + L_2\big[C_3(|\Omega|)\|y_1 - y_2\|_{\mathbb{R}^2} + C_2(|\Omega|)\ \epsilon^{\frac{1-\eta}{2}} \|\omega_1 - \omega_2\|_{H^1(\Omega_\epsilon)}\ \big] \\
\leq \quad & L_3(|\Omega|)\ \|y_1 - y_2\|_{\mathbb{R}^2} + L_4(|\Omega|)\epsilon^{\frac{1-\eta}{2}}\ \|\omega_1 - \omega_2\|_{H^1(\Omega_\epsilon)},
\end{aligned}$$

where $L_3(|\Omega|) = L_1 + L_2C_3(|\Omega|)$, $C_3(|\Omega|) = \max\{C_0(|\Omega|), C_1(|\Omega|)\}$, $L_4(|\Omega|) = L_1 + L_2C_2(|\Omega|)$.

$$\begin{aligned}
|G_1^\epsilon(y_1, \omega_1) - G_1^\epsilon(y_2, \omega_2)| \quad &= \quad |\int_{\Omega_\epsilon} [f(a) - f(b)]\varphi_2^\epsilon + \int_{\partial\Omega_\epsilon} [g(a) - g(b)]\varphi_2^\epsilon| \\
&\leq \quad \int_{\Omega_\epsilon} |f(a) - f(b)|\ |\varphi_2^\epsilon| + \int_{\partial\Omega_\epsilon} |g(a) - g(b)|\ |\varphi_2^\epsilon| \\
&\leq \quad \int_{\Omega_\epsilon} |f'(\tau_1)|\ |a - b|\ |\varphi_2^\epsilon| + \int_{\partial\Omega_\epsilon} |g'(\tau_2)|\ |a - b|\ |\varphi_2^\epsilon| \\
&\leq \quad L_1 \int_{\Omega_\epsilon} |a - b|\ |\varphi_2^\epsilon| + L_2 \int_{\partial\Omega_\epsilon} |a - b|\ |\varphi_2^\epsilon|.
\end{aligned}$$

and

$$\begin{aligned}
|G_1^\epsilon(y_1, \omega_1) \quad - \quad & G_1^\epsilon(y_2, \omega_2)| \leq L_1\big[|u_1 - v_1| + |u_2 - v_2| + \|\omega_1 - \omega_2\|_{L^2(\Omega_\epsilon)}\big] \\
+ \quad & L_2\big[\ |u_1 - v_1| + C_3(|\Omega|)\epsilon^{\frac{1-\eta}{2}} |u_2 - v_2| + C_4(|\Omega|)\epsilon^{\frac{1-\eta}{2}} \|\omega_1 - \omega_2\|_{H^1(\Omega_\epsilon)}\big] \\
\leq \quad & L_1\big[\ \|y_1 - y_1\|_{\mathbb{R}^2} + \|\omega_1 - \omega_2\|_{H^1(\Omega_\epsilon)}\big] \\
+ \quad & L_2\big[\ |u_1 - v_1| + C_4(|\Omega|)|u_2 - v_2| + C_5(|\Omega|)\epsilon^{\frac{1-\eta}{2}} \|\omega_1 - \omega_2\|_{H^1(\Omega_\epsilon)}\big] \\
\leq \quad & L_5(|\Omega|)\|y_1 - y_1\|_{\mathbb{R}^2} + L_6(|\Omega|)\epsilon^{\frac{1-\eta}{2}} \|\omega_1 - \omega_2\|_{H^1(\Omega_\epsilon)},
\end{aligned}$$

where $L_5(|\Omega|) = L_1 + C_6(|\Omega|)L_2$, $C_6 = \max\{1, C_4(|\Omega|)\}$, $L_6(|\Omega|) = L_1 + C_5(|\Omega|)L_2$. Thus

$$\begin{aligned}
\|G_\epsilon(\omega_1, y_1) - G_\epsilon(\omega_2, y_2)\|_{\mathbb{R}^2} &= \|(G_0^\epsilon(\omega_1, y_1) - G_0^\epsilon(\omega_2, y_2), G_1^\epsilon(\omega_1, y_1) - G_1^\epsilon(\omega_2, y_2))\|_{\mathbb{R}^2} \\
&= \quad |G_0^\epsilon(\omega_1, y_1) - G_0^\epsilon(\omega_2, y_2)| + |G_1^\epsilon(\omega_1, y_1) - G_1^\epsilon(\omega_2, y_2)|
\end{aligned}$$

and

$$
\begin{aligned}
\|G_\epsilon(\omega_1, y_1)-&G_\epsilon(\omega_2, y_2)\|_{\mathbb{R}^2} \\
&\leq L_3(|\Omega|)\ \|y_1 - y_2\|_{\mathbb{R}^2} + L_4(|\Omega|)\ \epsilon^{\frac{1-\eta}{2}}\ \|\omega_1 - \omega_2\|_{H^1(\Omega_\epsilon)} \\
&+ L_5(|\Omega|)\|y_1 - y_1\|_{\mathbb{R}^2} + L_6(|\Omega|)\ \epsilon^{\frac{1-\eta}{2}}\|\omega_1 - \omega_2\|_{H^1(\Omega_\epsilon)} \\
&\leq L_7(|\Omega|)\ \|y_1 - y_2\|_{\mathbb{R}^2} + L_8(|\Omega|)\epsilon^{\frac{1-\eta}{2}}\ \|\omega_1 - \omega_2\|_{H^1(\Omega_\epsilon)} \\
&\leq L_7(|\Omega|)\ \|y_1 - y_2\|_{\mathbb{R}^2} + L_8(|\Omega|)\epsilon^{\frac{1-\eta}{2}} \left(\frac{\lambda_2^\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{\delta}{\lambda_2^\epsilon}\right)^{\frac{1}{2}} \|\nabla(\omega_1 - \omega_2)\|_{L^2(\Omega_\epsilon)} \\
&\leq L_7(|\Omega|)\ \|y_1 - y_2\|_{\mathbb{R}^2} + \frac{L_8(|\Omega|)}{\delta}\ \epsilon^{\frac{1-\eta}{2}}(\lambda_2^\epsilon)^{\frac{1}{2}}\|\omega_1 - \omega_2\|_{X_\epsilon^{\frac{1}{2}}(\Omega_\epsilon)},
\end{aligned}
$$

where $L_7 = L_3(|\Omega|)+L_3(|\Omega|)$, $L_8 = L_4(|\Omega|)+L_6(|\Omega|)$. Using (2.7) we see that the expression $\epsilon^{\frac{1-\eta}{2}}(\lambda_2^\epsilon)^{\frac{1}{2}}$ is bounded uniformly for ϵ in a neighborhood of zero; that is,

$$\epsilon^{\frac{1-\eta}{2}}\ (\lambda_2^\epsilon)^{\frac{1}{2}} \leq C_0,\ \epsilon \to 0$$

$$
\begin{aligned}
\|G_\epsilon(\omega, y)\|_{\mathbb{R}^2} &= \|(G_0^\epsilon(\omega, y), -\delta u_2 + G_1^\epsilon(\omega, y))\|_{\mathbb{R}^2} = |G_0^\epsilon(\omega, y)| + |G_1^\epsilon(\omega, y)| \\
&\leq M_1^{f,g} + M_2^{f,g} = N_G.
\end{aligned}
$$

Also

$$
\begin{aligned}
|\frac{\partial}{\partial u_1}G_0^\epsilon(u_1, u_2, \omega)| &\leq \int_{\Omega_\epsilon} |f'(a)|\ |\varphi_1^\epsilon|^2 + \int_{\partial\Omega_\epsilon} |g'(b)\ |\varphi_1^\epsilon|^2 \\
&\leq L_1 + L_2 m_1^2|\partial\Omega_\epsilon| \leq L_1 + L_2 m_1^2 l_2
\end{aligned}
$$

$$
\begin{aligned}
|\frac{\partial}{\partial u_2}G_0^\epsilon(u_1, u_2, \omega)| &\leq \int_{\Omega_\epsilon} |f'(a)|\ |\varphi_1^\epsilon|\ |\varphi_2^\epsilon| + \int_{\partial\Omega_\epsilon} |g'(b)\ |\varphi_1^\epsilon|\ |\varphi_2^\epsilon| \\
&\leq L_1 + L_2\ m_1 m_2|\partial\Omega_\epsilon| \leq L_1 + L_2 m_1 m_2 l_2
\end{aligned}
$$

$$
\begin{aligned}
|\frac{\partial}{\partial \omega}G_0^\epsilon(u_1, u_2, \omega)| &\leq \int_{\Omega_\epsilon} |f'(a)|\ |\varphi_1^\epsilon| + \int_{\partial\Omega_\epsilon} |g'(b)\ |\varphi_1^\epsilon| \\
&\leq L_1 \int_{\Omega_\epsilon} |\varphi_1^\epsilon| + L_2 \int_{\partial\Omega_\epsilon} |\varphi_1^\epsilon| \\
&\leq L_1(|\Omega_\epsilon|)^{\frac{1}{2}} + L_2\frac{|\partial\Omega_\epsilon|}{|\Omega_\epsilon|^{\frac{1}{2}}}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
|\frac{\partial}{\partial u_1} G_1^\epsilon(u_1,u_2,\omega)| &\leq L_1 + L_2 m_1^2 l_2 \\
|\frac{\partial}{\partial u_2} G_1^\epsilon(u_1,u_2,\omega)| &\leq L_1 + L_2 m_2^2 l_2 \\
|\frac{\partial}{\partial \omega} G_0^\epsilon(u_1,u_2,\omega)| &\leq L_1 m_1 l_1 + L_2 m_1 l_2.
\end{aligned}
$$

Now for the function F_ϵ we have

$$\|F_\epsilon(\omega_1,y_1) - F_\epsilon(\omega_2,y_2)\|_{X_\epsilon} = \|\int_0^1 \frac{d}{ds} F_\epsilon((\omega_1,y_1)s + (\omega_2,y_2)(1-s))ds\|_{L^2(\Omega_\epsilon)}. \tag{3.23}$$

Now

$$
\begin{aligned}
\frac{d}{ds} F_\epsilon((\omega_1,y_1)s + (\omega_2,y_2)(1-s)) &= \frac{d}{ds}[\ f((\omega_1,y_1)s + (\omega_2,y_2)(1-s)) \\
&- G_0^\epsilon((\omega_1,y_1)s + (\omega_2,y_2)(1-s))\varphi_1^\epsilon \\
&- G_0^\epsilon((\omega_1,y_1)s + (\omega_2,y_2)(1-s))\varphi_1^\epsilon\ ] \\
&= f'(*)\ [(\omega_1,y_1) - (\omega_2,y_2)] \\
&- G_0^\epsilon)'(*)\ [(\omega_1,y_1) - (\omega_2,y_2)]\varphi_1^\epsilon \\
&- (G_1^\epsilon)'(*)\ [(\omega_1,y_1) - (\omega_2,y_2)]\varphi_2^\epsilon.
\end{aligned}
$$

Replacing this last expression in (3.23) we have

$$
\begin{aligned}
\|F_\epsilon(\omega_1,y_1) &- F_\epsilon(\omega_2,y_2)\ \|_{X_\epsilon} \leq \int_0^1 \|f'(*)\ [(\omega_1,y_1) - (\omega_2,y_2)] \\
&- G_0^\epsilon)'(*)\ [(\omega_1,y_1) - (\omega_2,y_2)]\varphi_1^\epsilon \\
&- (G_1^\epsilon)'(*)\ [(\omega_1,y_1) - (\omega_2,y_2)]\varphi_2^\epsilon\|_{L^2(\Omega_\epsilon)} \\
&\leq |f'(*)|\ \|(\omega_1,y_1) - (\omega_2,y_2)\|_{L^2(\Omega_\epsilon)} \\
&+ |(G_0^\epsilon)'(*)|\ \|(\omega_1,y_1) - (\omega_2,y_2)\|_{L^2(\Omega_\epsilon)} \\
&+ |(G_1^\epsilon)'(*)|\ \|(\omega_1,y_1) - (\omega_2,y_2)\|_{L^2(\Omega_\epsilon)} \\
&\leq L_1[\ \|\omega_1 - \omega_2\|_{L^2(\Omega_\epsilon)} + \|y_1 - y_2\|_{\mathbb{R}^2}\ ] \\
&+ \tilde{L}[\ \|\omega_1 - \omega_2\|_{L^2(\Omega_\epsilon)} + \|y_1 - y_2\|_{\mathbb{R}^2}\ ] \\
&+ \tilde{\tilde{L}}[\ \|\omega_1 - \omega_2\|_{L^2(\Omega_\epsilon)} + \|y_1 - y_2\|_{\mathbb{R}^2}\ ]
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\|F_\epsilon(\omega_1,y_1) - F_\epsilon(\omega_2,y_2)\|_{X_\epsilon} &\leq \hat{L}\|y_1 - y_2\|_{\mathbb{R}^2} + \hat{L}\|\omega_1 - \omega_2\|_{H^1(\Omega_\epsilon)} \\
&\leq \hat{L}\|y_1 - y_2\|_{\mathbb{R}^2} + \hat{L}\|\nabla\omega_1 - \nabla\omega_2\|_{L^2(\Omega_\epsilon)} \\
&\leq \hat{L}\|y_1 - y_2\|_{\mathbb{R}^2} + \hat{L}\ (\frac{\lambda_2^\epsilon}{\delta})^{\frac{1}{2}}\ (\frac{\delta}{\lambda_2^\epsilon})^{\frac{1}{2}}\ \|\nabla\omega_1 - \nabla\omega_2\|_{L^2(\Omega_\epsilon)} \\
&\leq \hat{L}\|y_1 - y_2\|_{\mathbb{R}^2} + \frac{\hat{L}}{\delta}\ (\lambda_2^\epsilon)^{\frac{1}{2}}\ \|\omega_1 - \omega_2\|_{X_\epsilon^{\frac{1}{2}}(\Omega_\epsilon)}
\end{aligned}
$$

where $\hat{L} = L_1 + \tilde{L} + \tilde{\tilde{L}}$, $\tilde{L} = \max(G_0^\epsilon)'$, $\tilde{\tilde{L}} = \max(G_1^\epsilon)'$. Note that, for $\eta > \frac{1}{N-1}$, (2.7) implies

$$\hat{L}(\lambda_2^\epsilon)^{\frac{1}{2}} \leq \hat{L} C_0 \epsilon^{\frac{\eta(N-1)-1}{2}} \to 0, \text{ quando } \epsilon \to 0.$$

Finally, we have that

$$\begin{aligned} |F_\epsilon(u_1,u_2,\omega)| &\leq |f(\cdot)| + |G_0^\epsilon(u_1,u_2,\omega)|\, |\varphi_1^\epsilon| + |G_1^\epsilon(u_1,u_2,\omega)|\, |\varphi_2^\epsilon| \\ &\leq M_1 + M_1^{f,g} m_1 + M_2^{f,g} m_2 = M_{(f,g)}. \end{aligned}$$

$$\int_{\Omega_\epsilon} |F(u_1,u_2,\omega)|^2 \leq M_{(f,g)}^2 \int_{\Omega_\epsilon} \leq M_{(f,g)}^2 |\Omega_\epsilon| \leq M_{(f,g)}^2 l_1$$

and $\|F(u_1,u_2,\omega)\|_{L^2(\Omega_\epsilon)} \leq N_F$, where $N_F = M_{(f,g)}\sqrt{l_1}$.

With this we see that F_ϵ and G_ϵ satisfy the hipotheses of Theorem 3.1. ∎

Teorema 3.2. *Assume that the system* (3.11) *is structurally stable. Then, for small enough ϵ, the flow on the invariant manifold given by* (3.22) *is topologically equivalent to the flow* (3.11).

Proof. For each ϵ enough small the flow in the two-dimensional subspace V given by $v(t,x) = u_1\varphi_1^0 + u_2\varphi_2^0$, where (u_1,u_2) is solution of

$$\begin{cases} \dot{u}_1 &= G_0^0(u_1,u_2,\sigma_\epsilon(u_1,u_2)) \\ \dot{u}_2 &= -\delta u_2 + G_1^0(u_1,u_2,\sigma_\epsilon(u_1,u_2)). \end{cases} \tag{3.24}$$

Since the application $v \to u = v + \sigma_\epsilon(v)$ is a conjugation between orbits of this flow (3.24) and the orbits of (3.22), we only have to prove that the system (3.24) is topologically conjugate to (3.11). We see that what remains to be proved is that the vector fields

$$\begin{aligned} X_\epsilon(u_1,u_2) &= (X_0(\epsilon), X_1(\epsilon)) \text{ and} \\ X_0(u_1,u_2) &= (X_0(0), X_1(0)), \end{aligned}$$

where

$$\begin{aligned} X_0(\epsilon) &= \int_{\Omega_\epsilon} f(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \sigma(u_1,u_2))\varphi_1^\epsilon \\ &\quad + \int_{\partial\Omega_\epsilon} g(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \sigma(u_1,u_2))\varphi_1^\epsilon \\ X_1(\epsilon) &= -\delta u_2 + \int_{\Omega_\epsilon} f(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \sigma(u_1,u_2))\varphi_2^\epsilon \\ &\quad + \int_{\partial\Omega_\epsilon} g(u_1\varphi_1^\epsilon + u_2\varphi_2^\epsilon + \sigma(u_1,u_2))\varphi_2^\epsilon \\ X_0(0) &= \int_{\Omega_\epsilon} f(u_1 + u_2\varphi_2^0) + \int_{\partial\Omega_\epsilon} g(u_1 + u_2\varphi_2^0) \\ X_1(0) &= -\delta u_2 + \int_{\Omega_\epsilon} f(u_1 + u_2\varphi_2^0)\varphi_2^0 + \int_{\partial\Omega_\epsilon} g(u_1 + u_2\varphi_2^0)\varphi_2^0 \end{aligned}$$

are C^1 closed. This follows easily from the fact that σ approaches 0 in the C^1 topology and the asymptotic properties of the eigenvalues and eigenfunctions of $\frac{\delta}{\lambda_2^\epsilon}\Delta$ as $\epsilon \to 0$. ∎

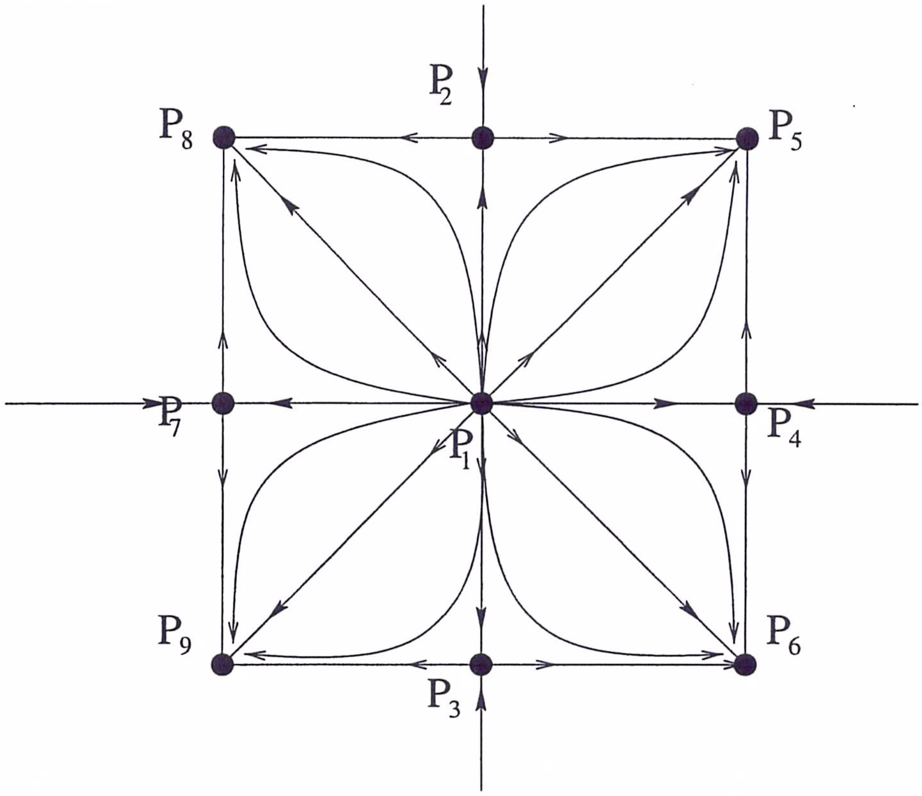

FIGURE 2. Approximate Phase Portrait: $\delta << 1$, $g(u) = u - u^3$, $f \equiv 0$

4. PATTERNS

In this section we to return to the system (3.11). Now we want obtain the patterns for (1.1). For this purpose we consider δ sufficiently small, $g(u) = u - u^3$ and $f(u) = 0$. Thus the system (3.11) have nine equilibrium points (see Figure 2).

$$\begin{aligned} P_1 &= (0,0),\ P_2 = (0,1),\ P_3 = (0,-1), \\ P_4 &= (1,0),\ P_5 = (1,1),\ P_6 = (1,-1), \\ P_7 &= (-1,0),\ P_8 = (-1,1),\ P_9 = (-1,-1). \end{aligned}$$

All theses equilibrium points are hyperbolic. The equilibrium P_5, P_6, P_8, P_9 are stable. We have seen that the dynamics of (3.11) is equivalent to dynamics of (1.1), the equilibrium points of the form $P = (v_1, v_2)$ with $v_1 \neq v_2$ correspond to stable nonconstant equilibrium (patterns) for (1.1). Thus we have two patterns for the system (1.1). Observe that the equilibrium stable in the form $P = (v_1, v_2)$ with $v_1 = v_2$ not correspond to patterns for (1.1).

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