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Existence Results for Partial Neutral Functional Integrodifferential Equations with Unbounded Delay

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Resultados de Existência para Equações Integrodiferenciais do Tipo Neutro com Retardamento não Limitado.

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Resumo

Neste artigo estudamos a existência de soluções fracas em intervalos compactos para uma equação integrodiferencial do tipo neutro com retardamento não limitado modelada na forma $\frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F(t, x_t, \int_0^t h(t, s, x_s)ds)$, onde A é o gerador de um semigrupo fortemente continuo de operadores lineares num espaço de Banach X e F, G, h são funções apropriadas.

Existence Results for Partial Neutral Functional Integrodifferential Equations with Unbounded Delay.

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Abstract

We prove the existence of mild solutions for a partial neutral integrodifferential equation with unbounded delay using the Leray Schauder Alternative. The ideas and techniques in this paper permit deduce a new and correct version of the existence result given recently by Balachandran in [1] for the case of partial neutral integrodifferential equations with finite delay.

Keywords: Functional differential equations, semigroup of bounded linear operators, fractional powers of closed operators.

1 Introduction

The purpose of this paper is to prove the existence of mild solutions for a class of partial neutral integrodifferential equations with unbounded described in the form

$$\begin{aligned} \frac{d}{dt}(x(t) + G(t, x_t)) &= Ax(t) + F(t, x_t, \int_0^t h(t, s, x_s)ds), \qquad t \in [\sigma, T], \\ x_\sigma &= \varphi \in \mathcal{B}, \end{aligned} \tag{1}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on a Banach space X, the history $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically, $0 \leq \sigma < T$ and $G : [\sigma, T] \times \mathcal{B} \to X$, $F : [\sigma, T] \times \mathcal{B} \times X \to X$ and $h : [\sigma, T] \times [\sigma, T] \times \mathcal{B} \to X$ are appropriate functions.

Recently Balachandran [1, 2] studied the existence of mild solutions for a neutral integrodifferential equations with finite delay modeled in the form

$$\begin{aligned} \frac{d}{dt}(x(t) + G(t, x_t)) &= Ax(t) + F(t, x_t, \int_0^t h(t, s, x_s)ds), \qquad t \in [0, T], \\ x_\sigma &= \varphi \in C([-r, 0] : X), \end{aligned} \tag{2}$$

where A is the infinitesimal generator of a compact semigroup of bounded linear operator on X and $x_t \in C([-r, 0] : X)$. In the paper [1], Balachandran proved the existence of global solutions for the neutral problem (2) using the Leray Schauder Alternative and considered some examples and applications. Lamentably the proof of the Balachandran existence result, see Theorem 3.1 in [1], is in error. In fact, the author showed the existence of solution assuming that the operator family $AT(\cdot)$ is bounded in the uniform operator topology on $(0,T]$ which implies that A is bounded!, in opposition with the central idea of the paper and of the considered example.

In this work we prove the existence of mild solution for the neutral integrodifferential problem (1) using the Leray Schauder Alternative. Our principals assumptions are that $\alpha(T(t)(-A)^{\beta}G([\sigma,T]\times B_r(\varphi,\mathcal{B}))=0$ and that $\alpha(T(t)F([\sigma,T]\times B_r(\varphi,\mathcal{B}))=0$ for every $t\in(0,a]$, where $\alpha(\cdot)$ is the Kuratowski's measure of noncompactness in X. We remark that the existence of mild solutions for the problem studied by Balachandran can be obtained using the ideas and techniques in the proof of the main result of this work, see Theorem (2).

Neutral differential equations arises in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature for neutral functional differential equations, is the the Hale book [5] and the references therein. The works in partial neutral functional differential equations with delay, in what follows ANFDE, was initiated by Hernández & Henríquez in [6, 7]. In the cited papers Hernández & Henríquez proved the existence of mild, strong and periodic solutions for a ANFDE modeled in the form

$$\begin{aligned} \frac{d}{dt}(x(t)+G(t,x_t)) &= Ax(t)+F(t,x_t), \qquad t\geq\sigma, \\ x_\sigma &= \varphi\in\mathcal{B}, \end{aligned} \tag{3}$$

where A is the infinitesimal generator of an analytic semigroup of linear operators on X and $\mathcal{B}$ is a phase space defined axiomatically. In general, the results was obtained using the semigroup theory and the Sadovskii fixed point theorem (see [14]). Another related papers are Hernández [9] and Henriquez [8] respect to classical and periodical solutions.

Throughout this paper, X will be a Banach space provided with norm $\|\cdot\|$ and $A:D(A)\to X$ will be the infinitesimal generator of an analytic semigroup $T=(T(t))_{t\geq 0}$ of bounded linear operators on X. For the theory of strongly continuous semigroup, refer to Pazy [13]. We will point out here some notations and properties that will be used in this work. It is well know that there exist constants $\tilde{M}$ and $w\in\mathbb{R}$ such that

$$\| T(t)\|\leq\tilde{M}e^{wt}, \qquad t\geq 0.$$

If T is a uniformly bounded and analytic semigroup such that $0\in\rho(A)$, then it is possible to define the fractional power $(-A)^{\alpha}$, for $0<\alpha\leq 1$, as a closed linear operator on its domain $D(-A)^{\alpha}$. Furthermore, the subspace $D(-A)^{\alpha}$ is dense in X, and the expression

$$\| x\|_{\alpha}=\|(-A)^{\alpha}x\|$$

defines a norm in $D(-A)^{\alpha}$. If X_{α} represents the space $D(-A)^{\alpha}$ endowed with the norm $\|\cdot\|_{\alpha}$, then the following properties are well known ([13], pp. 74):

Lemma 1 *If the previous conditions hold:*

1. *Let* $0 < \alpha \le 1$. *Then* X_α *is a Banach space.*

2. *If* $0 < \beta \le \alpha$ *then* $X_\alpha \to X_\beta$ *is continuous.*

3. *For every* $0 < \alpha \le 1$ *there exists* $C_\alpha > 0$ *such that*
$$\| (-A)^\alpha T(t) \| \le \frac{C_\alpha}{t^\alpha}, \qquad 0 < t \le T.$$

4. *For every* $0 < \alpha \le 1$ *there exists a positive constant* C'_α *such that*
$$\| (T(t) - I)(-A)^{-\alpha} \| \le C'_\alpha t^\alpha, \qquad 0 < t \le T.$$

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato [4]. To establish the axioms of the space $\mathcal{B}$ we follows the terminology used in Hino-Murakami-Naito [11], and thus, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X, endowed with a seminorm $\| \cdot \|_\mathcal{B}$. We will assume that $\mathcal{B}$ satisfies the following axioms:

(A) If $x : (-\infty, \sigma + a) \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$ the following conditions hold:

i) x_t is in $\mathcal{B}$.

ii) $\| x(t) \| \le H \| x_t \|_\mathcal{B}$.

iii) $\| x_t \| \le K(t - \sigma) \sup\{\| x(s) \| : \sigma \le s \le t\} + M(t - \sigma) \| x_\sigma \|_\mathcal{B}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \to [0, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in **(A)**, x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + a)$.

(B) The space $\mathcal{B}$ is complete.

The paper is organized as follows. In section 2, we define the different concepts used in this work and discuss the existence of global solutions for the initial value problem (1). Our results are based on the properties of analytic semigroups and the ideas and techniques in [6, 7], [10] and [1].

Throughout this paper we assume that X is an abstract Banach space with norm $\| \cdot \|$. The terminology and notations are those generally used in operator theory. In particular, if X and Y are Banach spaces, we indicate by $\mathcal{L}(X : Y)$ the Banach space of the bounded linear operators of X in Y and we abbreviate to $\mathcal{L}(X)$ whenever $X = Y$. In addition, $B_r[x : Z]$ will denote the closed ball in a metric space Z, with center at x and radius r.

Finally, we remark that for the proof of our existence theorem we shall make use of the following result which is referred to as the **Leray Schauder Alternative**.

Theorem 1 (**Leray Schauder Alternative**) *Let* D *a convex subset of a Banach space* X *and assume that* $0 \in S$. *Let* $F : D \to D$ *be a completely continuous map. Then the set* $\{x \in D : x = \lambda F(x),\ 0 < \lambda < 1\}$ *is unbounded or the map* F *has a fix point in* D.

2 Existence Results

In this section we study the existence of mild solutions for the abstract Cauchy problem (1). Henceforth we will assume that A is the infinitesimal generator of an analytic semigroup, $T(\cdot) = (T(t))_{t\geq 0}$, of linear operator on X. Further, to avoid unnecessary notations, we suppose that $0 \in \rho(A)$ and that the semigroup $T(\cdot)$ is uniformly bounded, that is to say, $\|T(t)\| \leq \tilde{M}$, for some constant $\tilde{M} \geq 1$ and every $t \geq 0$.

Remark 1 *In the rest of this work, to simplify notations, we only consider the case $\sigma = 0$.*

In order to define a concept of mild solution for (1), by comparison with the abstract Cauchy problem

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(t), \\ x(0) &= x_0. \end{aligned}$$

whose properties are well known (see ([13]), we consider the following definition

Definition 1 *We will say that a function $x : (-\infty, T) \to X$, is a mild solution of the abstract Cauchy problem (1) if: $x_0 = \varphi$; the restriction of $x(\cdot)$ to the interval $I = [0, T]$ is continuous; for each $0 \leq t < T$ the function $AT(t-s)G(s, x_s)$, $s \in [0, t)$, is integrable and*

$$\begin{aligned} x(t) &= T(t)(\varphi(0) + G(0, \varphi)) - G(t, x_t) - \int_0^t AT(t-s)G(s, x_s)ds \\ &\quad + \int_0^t T(t-s)F(s, x_s, \int_0^s h(s, \tau, x_\tau)d\tau)ds, \qquad t \in I. \end{aligned} \tag{4}$$

Along of this paper we assume that the following assumptions hold.

Assumptions :

H_1 The function $F : I \times \mathcal{B} \times X \to X$ satisfies the following Caratheodory conditions:

(i) For every $t \in I$, the function $F(t, \cdot) : \mathcal{B} \times X \to X$ is continuous.

(ii) For each $(\psi, y) \in \mathcal{B} \times X$, the function $F(\cdot, \psi, y) : I \to X$ is strongly measurable.

H_2 There exist constants $0 < \beta < 1$, c_1, c_2 such that the function G is X_β-valued, $(-A)^\beta G : [0, T] \times \mathcal{B} \to X$ is continuous and

$$\| (-A)^\beta G(t, \psi) \| \leq c_1 \| \psi \|_{\mathcal{B}} + c_2$$

for every $(t, \psi) \in [0, T] \times \mathcal{B}$.

H_3 The function $h : I \times I \times \mathcal{B} \to X$ satisfies the following Caratheodory conditions:

(i) For every $(t, s) \in I \times I$ the function $h(t, s, \cdot) : \mathcal{B} \to X$ is continuous.

(ii) For each $\psi \in \mathcal{B}$ the function $h(\cdot, \cdot, \psi) : I \times I \to X$ is strongly measurable.

H$_4$ There exist continuous functions $m_f, m_h : I \to [0, \infty)$ such that for every $\psi \in \mathcal{B}$, $x \in X$ and $0 \le s \le t \le T$

$$\begin{aligned} \| F(t, \psi, x) \| &\le m_f(t) \Omega_f(\| \psi \|_{\mathcal{B}} + \| x \|), \\ \| h(t, s, \psi) \| &\le m_h(s) \Omega_h(\| \psi \|_{\mathcal{B}}), \end{aligned}$$

where $\Omega_f, \Omega_h : [0, \infty) \to (0, \infty)$ are continuous nondecreasing functions.

Let $\varphi \in \mathcal{B}$, $y : (-\infty, T] \to X$ be the function defined by

$$y(t) = \begin{cases} T(t)\varphi(0) & \text{for } t \ge 0, \\ \varphi(t) & \text{for } t \le 0, \end{cases}$$

and $S(T)$ be the space

$$S(T) = \{u : (-\infty, T] \to X : u_0 = 0; \ u \in C([0, T] : X)\}$$

endowed with the norm of uniform convergence on $[0, T]$.

For $\varphi \in \mathcal{B}$, we referred as condition $\mathbf{G}_\varphi$ to

G$_\varphi$ Let $\varphi \in \mathcal{B}$. We said that the system verifies condition $\mathbf{G}_\varphi$, if for every bounded set Q in $S(T)$ the set of functions $\{t \to G(t, x_t + y_t) : x \in Q\}$ is equicontinuous on $[0, T]$.

The proof of the next result follows using the steeps in the proof of Lemma 5.6.7 in Pazy [13]. In the next Lemma $\Gamma(\cdot)$ is the classical gamma function.

Lemma 2 *Let $v(\cdot), w(\cdot) : [0, T] \to [0, \infty)$ continuous functions and $0 < \alpha < 1$. If $v(\cdot)$ is nondecreasing and*

$$v(t) \le w(t) + \vartheta \int_0^t \frac{v(s)}{(t-s)^\alpha} ds, \qquad t \in [0, T],$$

then

$$v(t) \le e^{\frac{\vartheta \Gamma(\alpha)^n T^{n\alpha}}{\Gamma(n\alpha)}} \cdot \sum_{j=0}^{n-1} (\frac{\vartheta T^\alpha}{\alpha})^j \cdot w(t)$$

for every $t \in [0, T]$ and every $n \in N$ such that $n\alpha > 1$.

Remark 2 *In the next result, for a bounded function $\xi : [0, T] \to [0, \infty)$ and $0 \le t \le T$ we employ the notation*

$$\xi_t = \sup\{\xi(\theta) : \theta \in [0, t]\}. \tag{5}$$

Theorem 2 *Let $\varphi \in \mathcal{B}$ and assume that assumptions $\mathbf{H_1} - \mathbf{H_4}$ and condition $\mathbf{G}_\varphi$ hold. Suppose, furthermore, that the following conditions hold:*

(a) *For every $r > 0$ and every $\epsilon > 0$ there are compact sets $W^i_{\epsilon,r} \subset X$ such that $T(\epsilon)(-A)^\beta G(s,\psi) \in W^1_{\epsilon,r}$ and $T(\epsilon)F(s,\psi,x) \in W^2_{\epsilon,r}$ for every $s \in [0,T]$, $\psi \in B_r[0,\mathcal{B}]$ and $x \in B_r[0,X]$.*

(b) $\mu = \| (-A)^{-\beta} \| K_T c_1 < 1$ *and*

$$\int_0^T \left(\frac{bK_T\tilde{M}}{1-\mu} m_f(s) + m_h(s)\right)ds < \int_C^\infty \frac{ds}{\Omega_f(s)+\Omega_h(s)} < \infty,$$

where $C = ab$,

$$b = e^{\frac{b_1\Gamma(1-\beta)^n T^{n(1-\beta)-1}}{\Gamma(n(1-\beta))}} \sum_{j=0}^{n-1} \left(\frac{b_1 T^{1-\beta}}{1-\beta}\right)^j; \quad b_1 = \frac{K_T C_{1-\beta} c_1}{1-\mu}; \quad a = M_T \| \varphi \|_{\mathcal{B}} + \frac{K_T}{1-\mu} a_1;$$

$$a_1 = \| \varphi \|_{\mathcal{B}} (\tilde{M}H + c_1 \| (-A)^{-\beta} \| (\tilde{M} + M_T)) + \| (-A)^{-\beta} \| c_2(\tilde{M}+1) + \frac{C_{1-\beta}c_2T^\beta}{\beta},$$

$C_{1-\beta}$ is the constant in Lemma (1) and n is the first natural number such that $n\alpha > 1$.

Then there exist a mild solution of the integrodifferential problem (1).

Proof: In order to use the Leray Schauder Alternative we obtain *a priori* estimates for the solutions of the integral equation

$$\begin{aligned} x(t) &= \lambda T(t)(\varphi(0) + G(0,\varphi)) - \lambda G(t,x_t) - \lambda \int_0^t AT(t-s)G(s,x_s)ds \\ &\quad + \lambda \int_0^t T(t-s)F\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\right)ds, \qquad t \in [0,T]. \end{aligned} \tag{6}$$

Let $x(\cdot)$ be a solution of (6). Using the assumptions $\mathbf{H_1} - \mathbf{H_4}$, the notation introduced in (5) and the inequality

$$\| x_t \|_{\mathcal{B}} \le M_T \| \varphi \|_{\mathcal{B}} + K_T \sup\{\| x(s) \| : 0 \le s \le t\} = M_T \| \varphi \|_{\mathcal{B}} + K_T \| x \|_t = v(t)$$

we find that

$$\begin{aligned} \| x(t) \| &\le \tilde{M}(H \| \varphi \|_{\mathcal{B}} + \| (-A)^{-\beta} \| (c_1 \| \varphi \|_{\mathcal{B}} + c_2)) + \| (-A)^{-\beta} \| (c_1 \| x_t \|_{\mathcal{B}} + c_2) \\ &\quad \int_0^t \| (-A)^{1-\beta} T(t-s) \| \| (-A)^\beta G(s,x_s) \| ds \\ &\quad + \tilde{M} \int_0^t m_f(s)\Omega_f(v(s) + \int_0^s m_h(\tau)\Omega_h(v(\tau))d\tau)ds \\ &\le \tilde{M}(H \| \varphi \|_{\mathcal{B}} + \| (-A)^{-\beta} \| (c_1 \| \varphi \|_{\mathcal{B}} + c_2)) + \| (-A)^{-\beta} \| c_1 M_T \| \varphi \|_{\mathcal{B}} \\ &\quad + \| (-A)^{-\beta} \| c_1 K_T \| x \|_t + \| (-A)^{-\beta} \| c_2 + \int_0^t \frac{C_{1-\beta}c_1}{(t-s)^{1-\beta}} \| x_s \|_{\mathcal{B}} ds \\ &\quad + \frac{C_{1-\beta}c_2T^\beta}{\beta} + \tilde{M} \int_0^t m_f(s)\Omega_f(v(s) + \int_0^s m_h(\tau)\Omega_h(v(\tau))d\tau)ds \end{aligned}$$

where $C_{1-\beta}$ is the constant in Lemma 1. Since $v(\cdot)$ is nondecreasing follow that

$$\begin{aligned}\| x \|_t &\leq a_1 + \| (-A)^{-\beta} \| c_1 K_T \| x \|_t + C_{1-\beta} c_1 \int_0^t \frac{v(s)}{(t-s)^{1-\beta}} ds \\ &+ \tilde{M} \int_0^t m_f(s) \Omega_f(v(s) + \int_0^s m_h(\tau) \Omega_h(v(\tau)) d\tau) ds \end{aligned}$$

and hence

$$\begin{aligned}\| x \|_t &\leq \frac{a_1}{1-\mu} + \frac{C_{1-\beta} c_1}{1-\mu} \int_0^t \frac{v(s)}{(t-s)^{1-\beta}} ds \\ &+ \frac{\tilde{M}}{1-\mu} \int_0^t m_f(s) \Omega_f(v(s) + \int_0^s m_h(\tau) \Omega_h(v(\tau)) d\tau) ds \end{aligned}$$

where $a_1 = \| \varphi \|_{\mathcal{B}} (\tilde{M} H + c_1 \| (-A)^{-\beta} \| (\tilde{M} + M_T)) + \| (-A)^{-\beta} \| c_2 (\tilde{M} + 1) + \frac{C_{1-\beta} c_2 T^\beta}{\beta}$ and the assumption $\mu = K_T c_1 \| (-A)^{-\beta} \| < 1$ has been used. Employing the last inequality and the definition of $v(\cdot)$ we obtain that

$$\begin{aligned} v(t) &= M_T \| \varphi \|_{\mathcal{B}} + K_T \| x \|_t \leq M_T \| \varphi \|_{\mathcal{B}} + \frac{K_T a_1}{1-\mu} + \frac{K_T C_{1-\beta} c_1}{1-\mu} \int_0^t \frac{v(s)}{(t-s)^{1-\beta}} ds \\ &+ \frac{K_T \tilde{M}}{1-\mu} \int_0^t m_f(s) \Omega_f(v(s) + \int_0^s m_h(\tau) \Omega_h(v(\tau)) d\tau) ds \end{aligned}$$

and from lemma 2 that

$$v(t) \leq b \cdot \left(a + \frac{K_T \tilde{M}}{1-\mu} \int_0^t m_f(s) \Omega_f \left(v(s) + \int_0^s m_h(\tau) \Omega_h(v(\tau)) d\tau \right) ds \right) := w(t)$$

where $a = M_T \| \varphi \|_{\mathcal{B}} + \frac{K_T a_1}{1-\mu}$, $b = e^{\frac{b_1 \Gamma(1-\beta)^n T^{n(1-\beta)-1}}{\Gamma(n(1-\beta))}} \cdot \sum_{j=0}^{n-1} (\frac{b_1 T^{(1-\beta)}}{1-\beta})^j$, $b_1 = \frac{K_T C_{1-\beta} c_1}{1-\mu}$ and n is the first natural number such that $n(1-\beta) > 1$. From the definition of $w(\cdot)$ we see

$$\dot{w}(t) \leq \frac{b K_T \tilde{M}}{1-\mu} m_f(t) \Omega_f \left(v(t) + \int_0^t m_h(\tau) \Omega_h(v(\tau)) d\tau \right)$$

thus

$$\dot{w}(t) \leq \frac{b K_T \tilde{M}}{1-\mu} m_f(t) \Omega_f \left(w(t) + \int_0^t m_h(\tau) \Omega_h(w(\tau)) d\tau \right) \tag{7}$$

since $\Omega_f(\cdot)$ and $\Omega_h(\cdot)$ are nondecreasing. Let $z(t) = w(t) + \int_0^t m_h(\tau) \Omega_h(w(\tau)) d\tau$. Then $z(0) = w(0)$ and from (7) we find

$$\begin{aligned} \dot{z}(t) &\leq \dot{w}(t) + m_h(t) \Omega_h(w(t)) \\ &\leq \frac{b K_T \tilde{M}}{1-\mu} m_f(t) \Omega_f(z(t)) + m_h(t) \Omega_h(z(t)) \\ &\leq \left(\frac{b K_T \tilde{M}}{1-\mu} m_f(t) + m_h(t) \right) (\Omega_f(z(t)) + \Omega_h(z(t))) \end{aligned}$$

and hence,

$$\int_0^t \frac{\dot{z}(s)}{\Omega_f(z(s))+\Omega_h(z(s))}ds \;\leq\; \int_0^t (\frac{bK_T\tilde{M}}{1-\mu}m_f(s)+m_h(s))ds < \int_{z(0)}^{\infty} \frac{1}{\Omega_f(s)+\Omega_h(s)}ds.$$

Then

$$\int_{z(0)}^{z(t)} \frac{ds}{\Omega_h(s)+\Omega_f(s)} < \int_{z(0)}^{\infty} \frac{1}{\Omega_h(s)+\Omega_f(s)}ds,$$

which implies that $z(\cdot)$ is bounded in $[0,T]$, $C=z(0)$, and therefore that the set

$$\{x \in C([0,T]:X) : x(\cdot) \text{ é solução de } (6)\ , \lambda \in (0,1)\}$$

is bounded in $C([0,T]:X)$.

Next, we rewrite equation (4). Let $y:(-\infty,T]\to X$ the function defined by

$$y(t)=\begin{cases} T(t)\varphi(0) & \text{for } t\geq 0,\\ \varphi(t) & \text{for } t\leq 0.\end{cases}$$

If $u(\cdot)$ is a mild solution of (1), we can to decompose it as $u(t)=y(t)+x(t)$, which implies that $u_t=y_t+x_t$, $x_0\equiv 0$ and that $x(\cdot)$ is solution of the integral equation

$$\begin{aligned} x(t) &= T(t)G(0,\varphi)-G(t,x_t+y_t)-\int_0^t AT(t-s)G(s,x_s+y_s)ds \\ &\quad +\int_0^t T(t-s)F(s,x_s+y_s,\int_0^s h(s,\tau,x_\tau+y_\tau)d\tau)ds, \end{aligned} \tag{8}$$

moreover, using the relation $T(t)x-x=A\int_0^t T(s)x,\ x\in X$, follow that

$$\begin{aligned} x(t) &= T(t)(G(0,\varphi)-G(t,x_t+y_t))-\int_0^t AT(t-s)G(s,x_s+y_s)ds \\ &\quad +\int_0^t AT(t-s)G(t,x_t+y_t)ds \\ &\quad +\int_0^t T(t-s)F(s,x_s+y_s,\int_0^s h(s,\tau,x_\tau+y_\tau)d\tau)ds. \end{aligned}$$

On the space

$$S(T)=\{x:(-\infty,0]\to X : x_0=0,\ x\in C([0,T]:X)\}$$

provided with the uniform convergence topology we define the map $\Gamma:S(T)\to S(T)$ by

$$\begin{aligned} \Gamma x(t) &= 0, \qquad t\leq 0,\\ \Gamma x(t) &= T(t)(G(0,\varphi)-G(t,x_t+y_t))-\int_0^t AT(t-s)G(s,x_s+y_s)ds \\ &\quad +\int_0^t AT(t-s)G(t,x_t+y_t)ds \\ &\quad +\int_0^t T(t-s)F(s,x_s+y_s,\int_0^s h(s,\tau,x_\tau+y_\tau)d\tau)ds \qquad t\geq 0. \end{aligned}$$

This expressions leads us to define the maps $\Gamma_i : S(T) \to S(T), i = 1,2,3,4$, by $\Gamma_i x(t) = 0$, for $t \leq 0$ and

$$\begin{aligned}
\Gamma_1 x(t) &= T(t)(G(0,\varphi) - G(t, x_t + y_t)),\\
\Gamma_2 x(t) &= -\int_0^t AT(t-s)G(s, x_s + y_s)ds,\\
\Gamma_3 x(t) &= \int_0^t AT(t-s)G(t, x_t + y_t)ds,\\
\Gamma_4 x(t) &= \int_0^t T(t-s)F(s, x_s + y_s, \int_0^s h(s,\tau, x_\tau + y_\tau)d\tau)ds,
\end{aligned}$$

for $0 \leq t \leq T$ and $i = 1,2,3,4$. From $\mathbf{H_1} - \mathbf{H_3}$, it is clear that Γ_1, Γ_3 and Γ_4 are well defined. On the other hand, since G is X_β-valued and $(-A)^\beta G$ is continuous then both $(-A)^\beta G(s, x_s + y_s)$ and $G(s, x_s + y_s)$ are continuous. In addition, in view of the fact that $T(\cdot)$ is an analytic semigroups (see [13]), the operator function $s \to AT(t-s)$ is continuous in the uniform operator topology on $[0,t)$ and thus $AT(t-s)G(s, x_s + y_s)$ is continuous on $[0,t)$. Applying the estimations established in Lemma (1) we obtain that

$$\begin{aligned}
\| (-A)T(t-s)G(s, x_s + y_s) \| &= \| (-A)^{1-\beta}T(t-s)(-A)^\beta G(s, x_s + y_s) \| \\
&\leq \frac{C_{1-\beta}Cte}{(t-s)^{1-\beta}}
\end{aligned} \tag{9}$$

which, by Bochner's Theorem (see [12]), implies that $\| AT(t-s)G(s, x_s + y_s) \|$ is integrable on $[0,t)$. This concludes the proof that Γ_2 is also a well defined map with values in $S(T)$.

On the other hand, from Axiom (**A**) the map $[0,T] \times S(T) \to \mathcal{B}$; $(s,x) \to x_s$ is continuous. If $(x^n)_{n\in N}$ is a convergent sequence in $S(T)$ with limit x, then from $\mathbf{H_2}$ $(-A)^\beta G(s, x_s^n + y_s)$ converge to $(-A)^\beta G(s, x_s + y_s)$ for $s \in [0,T]$ and from (5) and (9) we infer that $\Gamma_2(x^n) \to \Gamma_2(x)$ in $C([0,T] : X)$, which proves that Γ_2 is continuous. Using $\mathbf{H_1}$,$\mathbf{H_3}$, $\mathbf{H_4}$ and similar arguments to those in the proof of Theorem 3.1 in [1], we can to prove that $\Gamma_1, \Gamma_3, \Gamma_4$ are continuous, we omit details. Thus Γ is continuous.

Next, we will prove that each map Γ_i is completely continuous. Let $B_r[0, S(T)]$ the closed ball in $S(T)$, with center at 0 and radius r. By Ascoli's theorem is sufficient to show that the set $\Gamma_i(B_r[0, S(T)])$ is equicontinuous on $[0,T]$ and that $\Gamma_i(B_r[0, S(T)])(t) = \{\Gamma_i x(t) : x \in B_r[0, S(T)]\}$ is relatively compact in X for every $t \in [0,T]$.

To begin, we observe that from axiom **A**,

$$\| x_t + y_t \|_{\mathcal{B}} \leq K_T r + (M_T + K_T \tilde{M} H) \| \varphi \|_{\mathcal{B}} := r^*,$$

for every $x \in B_r[0, S(T)]$ and every $t \in [0,T]$. Now, we study the property for each Γ_i separately. In what follows we use the notation $B_r = B_r[0, S(T)]$.

$\mathbf{\Gamma_1}$ From $\mathbf{G}_\varphi$ the set $\Gamma_1 B_r$ is equicontinuous on $[0,T]$. Moreover, for $t > 0$ and $0 < \epsilon < t$ follow that

$$\begin{aligned}
\Gamma_1 B_r(t) &= \{T(t)G(0,\varphi) + T(t-\epsilon)(-A^{-\beta})T(\epsilon)(-A^\beta)G(t, x_t + y_t) : x \in B_r\}\\
&\subset \{T(t)G(0,\varphi) + T(t-\epsilon)(-A^{-\beta})W^1_{\epsilon,r^*}\},
\end{aligned}$$

where W^1_{ϵ,r^*} is the compact set in (**a**), thus $\Gamma_1 B_r(t)$ is relatively compact. Since $\Gamma_1 B_r(0) = \{0\}$, the previous remark prove that Γ_1 is a compact operator.

Γ_2 In first time we prove that $\Gamma_2 B_r$ is equicontinuous. Let $0 < t_0 < t \leq T$ and $0 < \epsilon < t_0$. Directly of the definition of Γ_2 we have that for $x \in B_r$

$$\begin{aligned}
&\| \Gamma_2 x(t) - \Gamma_2 x(t_0) \| \\
\leq & \int_0^{t_0-\epsilon} \| (-A)^{1-\beta}(T(t-s) - T(t_0-s))(-A)^{\beta} G(s, x_s + y_s) \| ds \\
&+ \int_{t_0-\epsilon}^{t_0} \| T(t-t_0) - I \| \| (-A)^{1-\beta} T(t_0 - s)(-A)^{\beta} G(s, x_s + y_s) \| ds \\
&+ \int_{t_0}^{t} \| (-A)^{1-\beta} T(t-s)(-A)^{\beta} G(s, x_s + y_s) \| ds \\
\leq & \int_0^{t_0-\epsilon} \| (-A)^{1-\beta}[T(t-s-\frac{\epsilon}{2}) - T(t_0 - s - \frac{\epsilon}{2})] T(\frac{\epsilon}{2})(-A)^{\beta} G(s, x_s + y_s) \| ds \\
&+ 2\tilde{M} C_{1-\beta}(c_1 r^* + c_2)\frac{\epsilon^{\beta}}{\beta} + C_{1-\beta}(c_1 r^* + c_2)\frac{(t-t_0)^{\beta}}{\beta},
\end{aligned}$$

thus

$$\begin{aligned}
&\| \Gamma_2 x(t) - \Gamma_2 x(t_0) \| \\
\leq & \int_0^{t_0-\epsilon} \| (-A)^{1-\beta}[T(t-s-\frac{\epsilon}{2}) - T(t_0 - s - \frac{\epsilon}{2})] T(\frac{\epsilon}{2})(-A)^{\beta} G(s, x_s + y_s) \| ds \\
&+ 2\tilde{M} C_{1-\beta}(c_1 r^* + c_2)(\frac{\epsilon^{\beta}}{\beta} + \frac{(t-t_0)^{\beta}}{\beta}). \qquad (10)
\end{aligned}$$

Let $W^1_{\frac{\epsilon}{2},r^*}$ be the compact set in (a). Since $T(\cdot)$ is analytic, the function $s \to (-A)^{1-\beta}T(s)$ is continuous in the uniform operator topology on $(0, T]$ and consequently the set of functions $\{ s \to (-A)^{1-\beta}T(s)x : x \in W^1_{\frac{\epsilon}{2},r^*} \}$ is equicontinuous on $[\frac{\epsilon}{2}, T]$. Let $0 < \delta < \epsilon$ such that

$$\| (-A)^{1-\beta}T(s)x - (-A)^{1-\beta}T(s')x \| < \epsilon \qquad (11)$$

for every $x \in W^1_{\frac{\epsilon}{2},r^*}$ and every $\frac{\epsilon}{2} \leq s, s' \leq T$ with $| s - s' | < \delta < \epsilon$. Using (11) in (10), follows that for $| t - t_0 | < \delta < \epsilon$ and $x \in B_r$

$$\| \Gamma_2 x(t) - \Gamma_2 x(t_0) \| \leq (t_0 - \epsilon)\epsilon + 4\tilde{M} C_{1-\beta}(c_1 r^* + c_2)\frac{\epsilon^{\beta}}{\beta}.$$

The last inequality prove that $\Gamma(B_r)$ is equicontinuous from the right side at t_0. In the same manner we can to prove that $\Gamma_2(B_r)$ is equicontinuous at any $t_0 \geq 0$. We omit details.

Now, we prove that $\Gamma_2 B_r(t)$ is relatively compact for every $t \in [0, T]$. The case $t = 0$ is trivial. Let $0 < \epsilon < t \leq T$. For $x \in B_r$ we find,

$$\begin{aligned}
\Gamma_2 x(t) &= \int_0^{t-\epsilon} (-A)^{1-\beta} T(t - s - \frac{\epsilon}{2}) T(\frac{\epsilon}{2})(-A)^{\beta} G(s, x_s + y_s) ds \\
&+ \int_{t-\epsilon}^{t} (-A)^{1-\beta} T(t-s)(-A)^{\beta} G(s, x_s + y_s) ds. \qquad (12)
\end{aligned}$$

Let $W^1_{\frac{\epsilon}{2},r^*}$ be the compact set in (a) and $B_{\epsilon} = \{(-A)^{1-\beta}T(s)x : s \in [\frac{\epsilon}{2}, T], x \in W^1_{\frac{\epsilon}{2},r^*} \}$. Using that $(-A)^{\beta}T(\cdot)$ is strongly continuous on $[\frac{\epsilon}{2}, T]$, we infer that B_{ϵ} is relatively compact in X and from the mean value theorem for the Bochner integral and (12) we find that

$$\Gamma_2 x(t) \in (t-\epsilon)\overline{Conv(B_\epsilon)} + C_\epsilon \tag{13}$$

where $Conv(B_\epsilon)$ denote the convex hull of B_ϵ and $Diam(C_\epsilon) \le 2C_{1-\beta}(c_1 r^* + c_2)\frac{\epsilon^\beta}{\beta}$. Since $Conv(B_\epsilon)$ is relatively compact, we conclude from (13) that $\Gamma_2 B_r(t)$ is completely bounded and therefore relatively compact in X. This proves that Γ_2 is completely continuous.

Γ_3 The assertion can be proved arguing as in the previous case.

Γ_4 In this case we include only the principal ideas and estimates. Let $0 < t_0 < t \le T$ and $0 < \epsilon < t_0$. For $x \in B_r$ we find

$$\begin{aligned}
&\| \Gamma_4 x(t) - \Gamma_4 x(t_0) \| \\
&\le \int_0^{t_0-\epsilon} \| (T(t-s) - T(t_0-s))F(s, x_s + y_s, \int_0^s h(s,\tau,x_\tau + y_\tau)d\tau) \| ds \\
&\quad + \int_{t_0-\epsilon}^{t_0} \| [T(t-s) - T(t_0-s))]F(s, x_s + y_s, \int_0^s h(s,\tau,x_\tau + y_\tau)d\tau) \| ds \\
&\quad + \int_{t_0}^{t} \| T(t-s)F(s, x_s + y_s, \int_0^s h(s,\tau,x_\tau + y_\tau)d\tau) \| ds = I_1 + I_2 + I_3.
\end{aligned} \tag{14}$$

We estimate each I_i separately. Let $r^{**} := \max\{r^*, \Omega_h(r^*) \cdot \int_0^T m_h(\tau)d\tau\}$ and $W^2_{\epsilon,r^{**}}$ be the compact set in (**a**). Since $T(\cdot)$ is strongly continuous, there is $0 < \delta < \epsilon$ such that

$$\| T(s)x - T(s')x \| < \epsilon$$

for every $x \in W^2_{\epsilon,r^{**}}$ and every $0 \le s, s' \le T$ with $| s - s' | < \delta$. Then

$$\begin{aligned}
I_1 &\le \int_0^{t_0-\epsilon} \| (T(t-s-\epsilon) - T(t_0-s-\epsilon))T(\epsilon)F(s, x_s + y_s, \int_0^s h(s,\tau,x_\tau + y_\tau)d\tau) \| ds \\
&\le (t_0-\epsilon)\epsilon
\end{aligned}$$

since $\| x_s + y_s \| \le r^{**}$ and $\int_0^s \| h(s,\tau,x_\tau + y_\tau) \| d\tau \le r^{**}$ on $[0,T]$, thus

$$I_1 \le (t_0-\epsilon)\epsilon, \tag{15}$$

for $x \in B_r$ and $| t - t_0 | < \delta$.

On the other hand,

$$I_2 \le 2\tilde{M} \int_{t_0-\epsilon}^{t_0} m_f(s)\Omega_f\left(\| x_s + y_s \|_{\mathcal{B}} + \int_0^s m_h(\tau)\Omega_h(\| x_\tau + y_\tau \|_{\mathcal{B}})d\tau\right) ds$$

thus

$$I_2 \le 2\tilde{M}\Omega_f\left(r^* + \Omega_h(r^*) \cdot \int_0^T m_h(\tau)d\tau\right) \cdot \int_{t_0-\epsilon}^{t_0} m_f(s)ds. \tag{16}$$

Similarly

$$I_3 \le \tilde{M}\Omega_f\left(r^* + \Omega_h(r^*) \cdot \int_0^T m_h(\tau)d\tau\right) \cdot \int_{t_0}^{t} m_f(s)ds. \tag{17}$$

From the inequalities (15), (16), (17) and (14) follows that $\Gamma_4(B_r)$ is equicontinuous from the right side at t_0. In the same manner we can to prove that $\Gamma_4(B_r)$ is equicontinuous at any $t_0 \geq 0$.

Now, we discuss the compactness of $\Gamma_4 B_r(t)$. The case $t = 0$ is trivial. Let $0 < t \leq T$ and $0 < \epsilon < t$. Let $r^{**} := \max\{r^*, \Omega_h(r^*) \cdot \int_0^T m_h(\tau)d\tau\}$ and $W^2_{\epsilon,r^{**}}$ be the compact set in condition (**a**). From the mean value theorem we get

$$\begin{aligned}
\Gamma_4 x(t) &= \int_0^{t-\epsilon} T(t-s-\epsilon)T(\epsilon)F(s, x_s + y_s, \int_0^s h(s,\tau,x_\tau + y_\tau)d\tau)ds \\
&\quad + \int_{t-\epsilon}^t T(t-s)F(s, x_s + y_s, \int_0^s h(s,\tau,x_\tau + y_\tau)d\tau)ds \\
&\in (t-\epsilon)\overline{Conv(B_\epsilon)} + C_\epsilon
\end{aligned}$$

where $Conv(B_\epsilon)$ is the convex hull of B_ϵ,

$$B_\epsilon = \{T(s)x : s \in [0,T],\, x \in W^2_{\epsilon,r^{**}}\}$$

and

$$Diam(C_\epsilon) \leq 2\tilde{M}\Omega_f\left(r^* + \Omega_h(r^*) \cdot \int_0^T m_h(\tau)d\tau\right) \cdot \int_{t-\epsilon}^t m_f(s)ds.$$

This means that $\Gamma_4 B_r(t)$ is totally bounded and consequently relatively compact in X. Thus Γ_4 is completely continuous.

These remarks enable us to assert that Γ is a completely continuous map on $S(T)$.

Finally, the Leray Schauder Alternative asserts that Γ has a fixed point which is a mild solution of the integrodifferential problem (1). The proof is complete.

The next result is a immediate consequence of theorem (1), we omit the proof.

Proposition 1 *Let $\varphi \in \mathcal{B}$ and assume that $\mathbf{H_1} - \mathbf{H_4}$ and condition $\mathbf{G}_\varphi$ hold. If any of the following conditions hold*

(a) The functions F, $(-A)^\beta G$ are completely continuous.

(b) The function F takes bounded set into bounded sets and the semigroup $T(\cdot)$ is compact.

Then there exist a mild solution of the integrodifferential equation (1).

3 Example 3.1

We consider briefly the Example 2.1 in [6]. Let $\mathcal{B}$ be the space $C_r \times L^2(g; X)$, with $r = 0$, see [6, 11], and $X = L^2([0,\pi])$. In this case $H = 1$; $K(t) = 1 + \left(\int_{-t}^0 g(\tau)d\tau\right)^{\frac{1}{2}}$ and $M(t) = \gamma(-t)^{\frac{1}{2}}$ for all $t \geq 0$. Now, we consider the boundary value problem

$$\begin{aligned}
\frac{\partial}{\partial t}[u(t,\xi) + \int_{-\infty}^t \int_0^\pi b(s-t,\eta,\xi)u(s,\eta)d\eta ds] &= \frac{\partial^2}{\partial \xi^2}u(t,\xi) + a_0(\xi)u(t,\xi) \\
&\quad + \int_{-\infty}^t a(s-t)u(s,\xi)\,ds, \quad t \geq 0,\ 0 \leq \xi \leq \pi,
\end{aligned} \tag{18}$$

$$
\begin{aligned}
u(t,0) &= u(t,\pi)=0, \quad t\geq 0, && (19)\\
u(\tau,\xi) &= \varphi(\tau,\xi), \;\; \tau\leq 0, \;\; 0\leq \xi\leq \pi. && (20)
\end{aligned}
$$

In addition to the condition $(i)-(vi)$ in [6] we assume that

(vii) The function $a(\cdot)\in L^{\infty}(\mathbb{R})$; $\frac{\partial}{\partial s}b(s,\eta,\xi)$ is measurable and

$$\sup_{t\in[0,T]}\int_0^{\pi}\int_0^{t}\int_0^{\pi}\left(\frac{\partial b}{\partial s}(s-t,\eta,\xi)\right)^2 d\eta ds d\xi<\infty,$$

$$\sup_{t\in[0,T]}\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\left(\frac{\partial b}{\partial s}(s-t,\eta,\xi)\right)^2 g(s)^{-1}d\eta ds d\xi<\infty.$$

Under these conditions we define $G:[0,\infty)\times\mathcal{B}\to X$ and $F:[0,\infty)\times\mathcal{B}\times X\to X$ by

$$
\begin{aligned}
G(t,\psi)(\xi) &:= \int_{-\infty}^{0}\int_0^{\pi} b(s,\,\eta,\,\xi)\psi(s,\,\eta)d\eta ds,\\
F(t,\psi,x)(\xi) &:= a_0(\xi)\psi(0,\xi)+\int_{-\infty}^{0}a(s-t)\varphi(s,\xi)ds+x(\xi).
\end{aligned}
$$

With the previous notation, we can to represent (18)−(20) as the abstract Cauchy problem

$$\frac{d}{dt}(x(t)+G(t,x_t))=Ax(t)+F(t,x_t,\int_0^t h(t,s,x_s)ds), \quad t\geq 0, \tag{21}$$

where $h:I\times I\times\mathcal{B}\to X$ is defined by $h(t,s,\psi)(\xi)=a(s-t)\psi(0,\xi)$ and A is the operator $Ax=x''$ with domain

$$D(A):=\{f(\cdot)\in L^2([0,\pi]):f''(\cdot)\in L^2([0,\pi]),\;\; f(0)=f(\pi)=0\}.$$

It's well know that A is the infinitesimal generator of a analytic and compact semigroup on X. Moreover, see [6] $G([0,T]\times\mathcal{B})\subseteq D((-A)^{1/2})$ and

$$\|\,(-A)^{1/2}G(t,\psi)\,\|\leq N_1\,\|\,\psi\,\|_{\mathcal{B}}$$

for every $(t,\psi)\in[0,T]\times\mathcal{B}$, where

$$N_1:=\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\frac{1}{g(s)}\left(\frac{\partial}{\partial\zeta}b(s,\eta,\zeta)\right)^2 d\eta ds d\zeta.$$

We affirm that condition $\mathbf{G}_{\varphi}$ hold for every $\varphi\in\mathcal{B}$. To prove this affirmation we take $\varphi\in\mathcal{B}$ and $x:(-\infty,T]\to X$ such that $x_0=\varphi$, $x\in C([0,T]:X)$ and $\|\,x(t)\,\|_{L^2[0,\pi]}\leq r$ for every $t\in[0,\pi]$. In this condition, for $h>0$ we have that

$$
\begin{aligned}
G(t+h,x_{t+h})(\xi)-G(t,x_t)(\xi) &= \int_{-\infty}^{t}\int_0^{\pi}(b(s-t-h,\eta,\xi)-b(s-t,\eta,\xi))x(s,\eta)d\eta ds\\
&\quad+\int_t^{t+h}\int_0^{\pi}b(s-t-h,\eta,\xi)x(s,\eta)d\eta ds
\end{aligned}
$$

then

$$
\begin{aligned}
\mid \frac{d}{dt}G(t,x_t)(\xi) \mid &\leq \int_{-\infty}^{t}\int_0^{\pi} \mid \frac{\partial b}{\partial s}(s-t,\eta,\xi)x(s,\eta) \mid d\eta ds + \int_0^{\pi} \mid b(0,\eta,\xi)x(t,\eta) \mid d\eta \\
&\leq \| \varphi \|_{\mathcal{B}} \left(\int_{-\infty}^{0}\int_0^{\pi} \left(\frac{\partial b}{\partial s}(s-t,\eta,\xi)\right)^2 g^{-1}(s) d\eta ds \right)^{\frac{1}{2}} \\
&\quad + \int_0^t \left(\int_0^{\pi} \left(\frac{\partial b}{\partial s}(s-t,\eta,\xi)\right)^2 d\eta \right)^{\frac{1}{2}} \| x(s) \|_{L^2} ds \\
&\quad + \left(\int_0^{\pi} b^2(0,\eta,\xi) d\eta \right)^{\frac{1}{2}} \| x(t) \|_{L^2}
\end{aligned}
$$

thus

$$
\begin{aligned}
\| \frac{d}{dt}G(t,x_t) \|_{L^2}^2 &\leq 4 \| \varphi \|_{\mathcal{B}}^2 \int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi} \left(\frac{\partial b}{\partial s}(s-t,\eta,\xi)\right)^2 g(s)^{-1} d\eta ds d\xi \\
&\quad +4r^2T \int_0^{\pi}\int_0^{t}\int_0^{\pi} \left(\frac{\partial b}{\partial s}(s-t,\eta,\xi)\right)^2 d\eta ds d\xi \\
&\quad +4r^2 \int_0^{\pi}\int_0^{\pi} b^2(0,\eta,\xi) d\eta d\xi
\end{aligned}
$$

which implies that the set of functions

$$
\{t \to G(t,x_t) : \ x : (-\infty, T] \to X, \ x_0 = \varphi, \ x \in C([0,T] : X), \ \| x(t) \| \leq r, \ t \in [0,T]\}
$$

is equicontinuous. Thus condition $\mathbf{G}_{\varphi}$ hold for every $\varphi \in \mathcal{B}$.

On the other, for $\psi \in \mathcal{B}$ and $t \geq 0$ we have that

$$
\begin{aligned}
\| F(t,\psi,x) \|_{L^2} &\leq N_{\varphi} + (\| a_0(\cdot) \|_{\infty} + 1)(\| \psi \|_{\mathcal{B}} + \| x \|_{L^2}), \\
\| h(t,s,\psi) \|_{L^2} &\leq \| a(\cdot) \|_{\infty} \| \psi \|_{\mathcal{B}} \leq \| a(\cdot) \|_{\infty} + \| a(\cdot) \|_{\infty} \| \psi \|_{\mathcal{B}}^2,
\end{aligned}
$$

thus $m_f(s) = 1$, $\Omega_f(s) = N_{\varphi} + (\| a_0(\cdot) \|_{\infty} + 1)s$, $m_h(s) = 1$ and $\Omega_h(s) = \| a(\cdot) \|_{\infty} + \| a(\cdot) \|_{\infty} s^2$. A inspection of the constants a, b, c_1, c_2 in the prof of Theorem 2, permit to infer that for $c_1 = N_1$ sufficiently small condition (**b**) is verified. The existence of mild solution is consequence of Theorem 2.

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