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DEFORMATIONS WITH CONSTANT MILNOR NUMBER AND MULTIPLICITY OF NON-DEGENERATE COMPLEX HYPERSURFACES

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Resumo

Neste trabalho é estudada a questão de se determinar condições para que uma deformação a um parâmetro de um germe de função $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ com singularidade isolada na origem tenha número de Milnor constante. O principal resultado mostra que se uma deformação $f_t = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ é tal que os ideais Jacobianos $J(f_t) = \langle \partial f_t/\partial x_1, \ldots, \partial f_t/\partial x_n \rangle$ são não degenerados em relação a algum poliedro de Newton Γ_+, então a família $f_t(x)$ é μ-constante para pequenos valores de t, se e somente se todos os germes g_s têm Γ-ordem não decrescente com respeito à Γ-ordem de f. Para famílias de germes que satisfazem esta condição, obtemos também uma resposta positiva para a seguinte questão de Zariski: *Quando a multiplicidade é um invariante para o tipo topológico de germes de hipersuperfícies?*

Abstract

We investigate the μ-constant deformations at one parameter of holomorphic germs of functions $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated singularity, in terms of some Newton polyhedra associated to such germs. We consider familes $f_t = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ and if the Jacobian ideals $J(f_t) = \langle \partial f_t/\partial x_1, \ldots, \partial f_t/\partial x_n \rangle$ are non-degenerate on some fixed Newton polyhedron Γ_+, we show that a deformation $f_t(x)$ is μ-constant for small values of t, if and only if all germs g_s have non-decreasing Γ-order with respect to f. For this kind of germs we have a positive answer for the Zariski's question: *Whether for a hypersurface singularity the multiplicity is an invariant of the topological type?*

1. μ-CONSTANT DEFORMATIONS AND MULTIPLICITY

The topological triviality of families of germs of functions $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is one of the most interesting questions in singularity theory, and it is well known that in the case of germs with isolated singularity, the constancy of the Milnor number of the family is the condition for the topological triviality of the family f_t. The determination of necessary and sufficient conditions for a family of germs of functions $f_t(x)$, to be μ-constant (or topologically trivial) has been investigated by several authors. Varchenko

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gives in [10] a full answer for this question for the case of weighted homogeneous germs with isolated singularity.

Theorem 1.1. [10] *Let* $F(x,t) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ *be a deformation of a weighted homogeneous polynomial germ* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *with isolated singularity at* 0, *where* $\delta_s : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ *and* $g_s : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *are holomorphic germs of functions and* $\delta_s \neq 0$. *Then the family* $f_t(x) = F(x,t)$ *is* μ*-constant for small values of* t, *if and only if all monomials of each germ* g_s *have weighted degree higher or equal than the weighted degree of* f.

Another number that is associated to germs of functions is the multiplicity, and Zariski asked in [12] the following question:

Whether for a hypersurface singularity the multiplicity is an invariant of the topological type?

A positive answer for Zariski's question was previously known for the case of plane curves, for homogeneous surfaces and for quasi homogeneous complex hypersurfaces with isolated singularity, as shown by Greuel in [4].

In the case of semi quasi homogeneous germs, a positive answer for Zariski's question was done by Greuel in [4] and independently by D. O'Shea in [7]. Both authors applied the theorem 1.1 of Varchenko, but used different methods.

Theorem 1.2. [4], [7] *Let* $F(x,t) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ *be a* μ*-constant deformation of a weighted homogeneous polynomial germ* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *with isolated singularity at* 0. *Then the family is also* m*-constant (*m *denotes the multiplicity).*

Trotman gives in [9] a positive answer for Zariski's question for the case of deformations of type $F(x,t) = f(x) + tg(x)$ (a first order deformation) of a complex germ f with isolated singularity.

Theorem 1.3. [9] *Let* $F(x,t) = f(x) + tg(x)$ *be a* μ*-constant deformation of a holomorphic germ* f *with isolated singularity at* 0. *Then the family* $f_t(x) = F(x,t)$ *is* m*-constant.*

In this article we investigate the relationship between these questions and some Newton polyhedra associated to the germ f. The use of Newton polyhedra to give sufficient conditions for the μ-constancy was done by Kouchnirenko in [5]. Yoshinaga in [11] and Damon-Gaffney in [3] also dealt with the Newton polyhedron to obtain sufficient conditions for the topological triviality of families in terms of the Newton polyhedron of f.

In the first part we study the μ-constant families and give necessary and sufficient conditions, in terms of the Newton polyhedron defined by the Jacobian ideal of f, for the μ-constancy for a deformation at one parameter of f with isolated singularity. Then we consider the case of families of germs which are non-degenerate on some fixed Newton polyhedron Γ_+ (see 4.6). We give necessary and sufficient conditions for the μ-constancy of deformations of such germs, in terms of the Newton polyhedron Γ_+.

In the final section we apply the results on non-degeneracy to show that μ-constant deformations of non-degenerate germs on some Newton polyhedron also have constant multiplicity, giving a positive answer for Zariski's question for this kind of germs.

2. μ-CONSTANT DEFORMATIONS AND INTEGRAL CLOSURE

We fix a system of local coordinates x of $\mathbb{C}^n$, consider the ring $\mathcal{O}_n$ of holomorphic germs $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ and denote by m_n its maximal ideal. Due to the identification between $\mathcal{O}_n$ and the ring of convergent power series $\mathbb{C}\{x_1, \dots, x_n\}$ we identify a germ $f \in \mathcal{O}_n$ with its power series $f(x) = \sum a_\alpha x^\alpha$, with $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

The Milnor number, denoted by $\mu(f)$, of a germ $f \in \mathcal{O}_n$ is algebraically defined as the $\dim_{\mathbb{C}} \mathcal{O}_n/J(f)$, where $J(f)$ denotes the ideal generated by the partial derivatives $\{\partial f/\partial x_i, i = 1, \dots, n\}$ of f. A deformation $F\colon (\mathbb{C}^n \times \mathbb{C}^p, 0) \to (\mathbb{C}, 0)$ of f is said to be μ-constant if $\mu(f_t) = \mu(f)$ for small values of t.

Greuel gives in [4] a very interesting result showing the relationship between the μ-constant deformations of f and the integral closure of the Jacobian ideal $J(F) = \langle \partial F/\partial x_1, \dots, \partial F/\partial x_n \rangle$ of F with respect to the x-coordinates.

Definition 2.1. *Given an ideal I in a ring R, the* **integral closure** *of I is the ideal $\overline{I}$, of the elements $h \in R$ which satisfies a relation $h^k + a_1 h^{k-1} + \dots + a_{k-1}h + a_k = 0$, with $a_i \in I^i$.*

Teissier gave in [8, p.288] the following characterization of the integral closure of ideals in $R = \mathcal{O}_n$.

Proposition 2.2. *If I is an ideal in $\mathcal{O}_n$, the following statements are equivalent:*

(i) $h \in \overline{I}$,
(ii) *For each system of generators $h_1, \dots, h_r$ of I there exists a neighbourhood U of 0 and a constant $C > 0$ such that $|h(x)| \leq C \sup\{|h_1(x)|, \dots, |h_r(x)|\}$, for all $x \in U$.*
(iii) *For each analytic curve $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$, $h \circ \varphi$ lies in $(\varphi^*(I))\,\mathcal{O}_1$.*

The item (iii) of this proposition is called *valuative criterion* since it is equivalent to the condition $\nu(h \circ \varphi) \geq \inf\{\nu(h_1 \circ \varphi), \dots, \nu(h_r \circ \varphi)\}$, where ν denotes the usual valuation of a complex curve. In this case, the valuation is the multiplicity of the curve, see 5.1 for the definition of multiplicity.

Definition 2.3. *An element $h \in \mathcal{O}_n$ is in the* **strict integral closure,** *denoted by $\overline{I}^+$, of an ideal in $\mathcal{O}_n$, if for each system of generators $h_1, \dots, h_r$ of I there exists a neighbourhood U of 0 and a constant $C > 0$ such that $|h(x)| < C \sup_j\{|h_j(x)|\}$, for all $x \in U$, or equivalently, $\nu(h \circ \varphi) > \inf\{\nu(g_1 \circ \varphi), \dots, \nu(g_s \circ \varphi)\}$, for each analytic curve $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$.*

Theorem 2.4. [4, p.161] *Let $F\colon (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a one parameter deformation of a holomorphic germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated singularity. The following statements are equivalent*

(i) F is a μ-constant deformation of f

(ii) $\frac{\partial F}{\partial t} \in \overline{J(F)}^{+}$
(iii) $\frac{\partial F}{\partial t} \in \overline{J(F)}$
(iv) $\frac{\partial F}{\partial t} \in \sqrt{J(F)}$
(v) The polar curve of F with respect to $\{t=0\}$ does not split, i.e.,

$$\left\{(x,t) \in \mathbb{C}^n \times \mathbb{C} \mid \frac{\partial F}{\partial x_i}(x,t) = 0, \forall\, i = 1, \dots, n\right\} = \{0\} \times \mathbb{C} \text{ near } (0,0).$$

Next we give necessary and sufficient conditions for a deformation $F(x,t)$ to be μ-constant. These conditions will be given in terms of some suitable Newton polyhedra associated to the germ f.

3. μ-CONSTANT DEFORMATIONS AND NEWTON POLYHEDRA

For a germ $f(x) = \sum a_k x^k$, we call *supp* $f = \{k \in \mathbb{Z}^n \; : \; a_k \neq 0\}$.

Definition 3.1. *Let I be an ideal in $\mathcal{O}_n$, we call supp $I = \cup\{supp\ g \; : \; g \in I\}$.*

Definition 3.2. The Newton polyhedron of I, denoted by $\Gamma_+(I)$, is the convex hull in $\mathbb{R}^n_+$ of the set $\cup\{k + v \; : \; k \in supp\ I, \; v \in \mathbb{R}^n_+\}$. $\Gamma(I)$ denotes the union of all compact faces of $\Gamma_+(I)$.

We call $J(f)$ the system of generators $\{\partial f/\partial x_1, \dots, \partial f/\partial x_n\}$, of the Jacobian ideal of f. The integral closure of this ideal is useful to give conditons for the μ-constancy.

Definition 3.3. *We call $T(f)$ the convex hull in $\mathbb{R}^n_+$ of $\cup\left\{m + \mathbb{R}^n_+ : x^m \in \overline{J(f)}\right\}$.*

In the next lemma we show a necessary condition for the μ-constancy of the families given by first order deformations.

Lemma 3.4. *Let $F(x,t) = f(x) + tg(x)$ be a first order deformation of a complex germ f with isolated singularity. A necessary condition for the μ-constancy of the family f_t is $\Gamma_+(g) \subseteq T(f)$.*

Proof. If $\Gamma_+(g) \not\subseteq T(f)$, it follows from the valuative criterion that there exists an holomorphic curve $\gamma \colon (\mathbb{C},0) \to (\mathbb{C}^n,0)$, such that

$$\nu(g \circ \gamma) < \inf\left\{\nu\left(\frac{\partial f}{\partial x_i} \circ \gamma\right), \; \forall i = 1, \dots, n\right\}.$$

Consider the curve $\psi \colon (\mathbb{C},0) \to (\mathbb{C}^n \times \mathbb{C}, 0)$, defined as $\psi = (\gamma, 0)$. Since $\frac{\partial F}{\partial x_i} = \frac{\partial f}{\partial x_i} + t\frac{\partial g}{\partial x_i}$, we obtain $\frac{\partial F}{\partial x_i} \circ \psi = \frac{\partial f}{\partial x_i} \circ \gamma$ and the result follows from Theorem 2.4. □

We describe now sufficient conditions for the μ constancy.

Yoshinaga in [11] gave conditions for the topological triviality of families of type $F(x,t) = f(x) + tg(x)$ in terms of the gradient polyhedron $\Lambda_+(f)$, defined as the convex hull of the set $\bigcup\left\{m + \mathbb{R}^n_+ \; : \; \left|x_1 \frac{\partial f}{\partial x_1}\right| + \dots + \left|x_n \frac{\partial f}{\partial x_n}\right| \geq \mathcal{E}\,|x^m|\right\}$, for a positive

$\mathcal{E}(m)$ in a neighbourhood of the origin in $\mathbb{C}^n$. In the Theorem 1.6 of [11] it is shonw that if $\Gamma_+(g) \subset \Lambda_+(f)$, then $F(x,t) = f(x) + tg(x)$ is topologically trivial for sufficiently small values of t. Damon-Gaffney in [3] also gave similar results for the topological triviality. We remember that if a germ g satisfies the condition $\Gamma_+(g) \subset \Lambda_+(f)$, it is equivalent to say that g is in integral closure of the ideal generated by the system $\left\{x_1\frac{\partial f}{\partial x_1}, \dots, x_n\frac{\partial f}{\partial x_n}\right\}$.

As an immediate consequence of the results of Yoshinaga and Damon-Gaffney, we have the following

Proposition 3.5. *Let $F(x,t) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$, be a deformation of f with isolated singularity at 0. If $\Gamma_+(g_s) \subset \Lambda_+(f)$ for all $s = 1, \dots, \ell$, then $F(x,t)$ is μ-constant for sufficiently small values of t.*

Next we give a sufficient condition for the μ-constance in terms of the polyhedron $T(f)$.

Proposition 3.6. *Let $F(x,t) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s$ be a deformation of a complex germ f with isolated singularity. If $\Gamma_+(J(g_s)) \subseteq T(f)$ for all $s = 1, \dots, \ell$, then $F(x,t)$ is μ-constant for small values of t.*

Proof. The first part of the proof follows the proof of the Lemma 2.1 in [11]. Suppose that $\Gamma_+(J(g_s)) \subseteq T(f)$, from item (ii) of proposition 2.2, we obtain that for each $i = 1, \dots, n$ and $s = 1, \dots, \ell$ there exist a neighbourhood $U_{i,s}$ of 0 and a constant $C_{i,s} > 0$, such that $\left|\frac{\partial g_s}{\partial x_i}\right| \leq C_{i,s} \sup\left\{\left|\frac{\partial f}{\partial x_1}\right|, \dots, \left|\frac{\partial f}{\partial x_n}\right|\right\}$, and

$$\sum_{s=1}^{\ell} |\delta_s(t)| \left|\frac{\partial g}{\partial x_i}\right| \leq \sum_{s=1}^{\ell} |\delta_s(t)|\; C_{i,s}\; \sup\left\{\left|\frac{\partial f}{\partial x_1}\right|, \dots, \left|\frac{\partial f}{\partial x_n}\right|\right\}.$$

Therefore, for all x in a neighbourhood $U \subset U_{i,s}$ for all $s = 1, \dots, \ell$ and $i = 1, \dots, n$,

$$\begin{aligned}
\sup_i \left|\frac{\partial F}{\partial x_i}\right| = \sup_i \left|\frac{\partial f}{\partial x_i} + \sum_{s=1}^{\ell} \delta_s(t) \cdot \frac{\partial g_s}{\partial x_i}\right| &\geq \sup_i \left|\frac{\partial f}{\partial x_i}\right| - \sup_i \sum_{s=1}^{\ell} |\delta_s(t)| \left|\frac{\partial g}{\partial x_i}\right| \\
&\geq \sup_i \left|\frac{\partial f}{\partial x_i}\right| - \sum_{s=1}^{\ell} |\delta_s(t)| C_{i,s} \sup_i \left\{\left|\frac{\partial f}{\partial x_i}\right|\right\} \\
&\geq (1 - \sum_{s=1}^{\ell} |\delta_s(t)| C_{i,s}) \sup_i \left|\frac{\partial f}{\partial x_i}\right| \\
&\geq (1 - \alpha) \sup_i \left|\frac{\partial f}{\partial x_i}\right|,
\end{aligned}$$

for some $0 < \alpha < 1$ with $|\delta_s(t) C_{i,s}| \leq \alpha$ for all $i = 1, \dots, n$ and $s = \dots, \ell$.

This inequality implies that $\overline{\left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle} \subset \overline{\left\langle \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right\rangle}$.

Now we show that for each analytic curve $\psi : (\mathbb{C},0) \to (\mathbb{C}^{n+1},0)$,

$$\nu\left(\frac{\partial F}{\partial t}\circ\psi\right) \geq \min_i\left\{\nu\left(\frac{\partial F}{\partial x_i}\circ\psi\right)\right\}.$$

We write $\psi = (\varphi, \lambda)$, hence

$$\nu\left(\frac{\partial F}{\partial t}\circ\psi\right) \geq \min_s\left\{\nu\left(\delta_s'\circ\lambda\right) + \nu\left(g_s\circ\varphi\right)\right\} \geq \min_s\left\{1 + \nu\left(g_s\circ\varphi\right)\right\}.$$

From the hypothesis, for each $s = 1, \dots, \ell$, we have $g_s \in \overline{\left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right\rangle}$, hence

$$\nu\left(g_s\circ\varphi\right) \geq \min_i\left\{\nu\left(\frac{\partial f}{\partial x_i}\circ\varphi\right)\right\} \geq \min_i\left\{\nu\left(\frac{\partial F}{\partial x_i}\circ\psi\right)\right\},$$

$$\text{therefore } \nu\left(\frac{\partial F}{\partial t}\circ\psi\right) \geq \min_s\left\{1 + \nu\left(g_s\circ\varphi\right)\right\} \geq \min_i\left\{\nu\left(\frac{\partial F}{\partial x_i}\circ\psi\right)\right\}$$

and the result follows from 2.4. □

We see in the example below that the conditions shown in the propositions 3.5 and 3.6 gives rise to different classes of μ-constant deformations $F(x,t) = f(x) + tg(x)$.

Example 3.7. Let $f(x,y) = y^7 + x^4y + x^9$.

From the Theorem 1.3 of Yoshinaga (see corollary 4.2), we know that $\Lambda_+(f) = \Gamma_+(f)$ is the polygon with vertices $(0,7)$, $(4,1)$ and $(9,0)$. From the corollary 4.2 we see that the polygon $T(f)$ has vertices $(0,6)$, $(3,1)$ and $(4,0)$.

If we consider $g(x,y) = x^2y^4$, we have $\Gamma_+(g) \subset \Lambda_+(f)$, hence the deformation $F(x,y,t) = y^7 + x^4y + x^9 + tx^2y^4$ is μ-constant from 3.5, but $\Gamma_+(J(g)) \not\subset T(f)$.

We also see in this example that the condition $\Gamma_+(J(g)) \subset T(f)$ does not imply that $\Gamma_+(g) \subset \Lambda_+(f)$, for this, consider $g(x,y) = x^6$. Here we conclude that the deformation $F(x,y,t) = y^7 + x^4y + x^9 + tx^6$ is μ-constant from 3.6, and $\Gamma_+(g) \not\subset \Lambda_+(f)$.

4. Non-degenerate ideals and μ-constancy

In this section we describe how to obtain μ-constant deformations in terms of non-degeneracy conditions given by some Newton polyhedra associated to the germ f.

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Let I be an ideal of finite codimension in $\mathcal{O}_n$, for each face $\Delta \subseteq \Gamma(I)$ we denote by $C(\Delta)$ the union of half-rays emanating from the origin and passing through Δ. We call $\mathcal{A}_\Delta$ the subring with unity of $\mathcal{O}_n$ given by $\mathcal{A}_\Delta = \{g \in \mathcal{O}_n : \operatorname{supp} g \subseteq C(\Delta) \cap \mathbb{Z}^k\}$. If D is a fixed subset of $\Gamma_+(I)$ and $g = \sum_k a_k x^k$, we set $g_D = \sum_{k\in D} a_k x^k$.

Definition 4.1. Let I be an ideal of finite codimension in $\mathcal{O}_n$, we say that I is *Newton non-degenerate* if there exists a system of generators $\{g_1, \dots, g_s\}$ of I such that, for each compact face $\Delta \subseteq \Gamma(I)$, the ideal I_Δ generated by $\{g_{1_\Delta}, \dots, g_{s_\Delta}\}$ has finite codimension in $\mathcal{A}_\Delta$.

A germ $g = \sum_k a_k x^k$ is *Newton non-degenerate*, if the ideal generated by the system $\{x_1 \partial g/\partial x_1, \ldots, x_n \partial g/\partial x_n\}$ is of finite codimension in $\mathcal{O}_n$ and is Newton non-degenerate.

A germ g is said to have non-decreasing Newton order with respect to $\Gamma_+(I)$ if $\Gamma_+(g) \subseteq \Gamma_+(I)$.

When the germ f is Newton non-degenerate, Damon-Gaffney and Yoshinaga also gave a similar result as an immediate consequence of 3.5;

Corollary 4.2. *Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a Newton non-degenerate germ with an isolated singularity. If each g_s has non-decreasing Newton order with respect to $\Gamma_+(f)$, the deformation $f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ is μ-constant for sufficiently small values of t.*

From 3.6, we also have the following:

Corollary 4.3. *Suppose that the system of generators $\{\partial f/\partial x_1, \ldots, \partial f/\partial x_n\}$ of the Jacobian ideal $J(f)$ is Newton non-degenerate and for each germ g_s, $\Gamma_+(J(g_s)) \subseteq \Gamma_+(J(f))$. Then $F(x,t)$ is μ-constant for small values of t.*

The next example shows that these two conditions are not enough to give all μ-constant deformations of any germ with isolated singularity.

Example 4.4. Briançon and Speder showed in [1] that the family of hypersurfaces $X_t \in \mathbb{C}^3$ defined by the equations $f_t(x,y,z) = z^5 + y^7 x + x^{15} + t y^6 z = 0$ has constant topological type, but the monomial $y^6 z$ does not satisfies the condition of corollary 4.3. We can not apply the corollary 4.2 either, since f is not Newton non-degenerate because the ideal generated by the system $\{x\partial f/\partial x, y \partial f/\partial y, z\partial f/\partial z\}$ does not have finite codimension in $\mathcal{O}_n$.

The germ $f(x,y,z) = z^5 + y^7 x + x^{15}$ has isolated singularity at 0 and is weighted homogeneous of weights $(1,2,3)$ and degree $\mathrm{d}(f) = 15$, hence the μ-constancy of this family follows from the Theorem 1.1 of Varchenko.

In order to generalize these results for a bigger class of germs that contains the Newton non-degenerate germs and also the class of semi weighted homogeneous germs, we apply the results of Bivia-Fukui-Saia, given in [2] to show a necessary and sufficient condition for the μ-constancy of families defined by germs which are non-degenerate on some Newton polyhedron. We recover here the basic results for this definition.

A subset $\Gamma_+ \subseteq \mathbb{R}^n_+$ is a *Newton polyhedron* if there exist some $k_1, \ldots, k_r \in \mathbb{Q}^n_+$ such that Γ_+ is the convex hull in $\mathbb{R}^n_+$ of the set $\{k_i + v : v \in \mathbb{R}^n_+,\ i = 1, \ldots, r\}$ and Γ_+ intersects all the coordinate axis. We denote by Γ the union of the compact faces of Γ_+ and by Γ_- the set $\mathbb{R}^n \smallsetminus \Gamma_+$ and by $\mathrm{V}_n(\Gamma_-)$ the n-dimensional volume of Γ_-.

From the boundary of a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n$ we construct a piecewise-linear function $\phi_\Gamma : \mathbb{R}^n \to \mathbb{R}$ with the following properties:

(i) ϕ_Γ is linear on each cone $C(\Delta)$, where Δ is a compact face of Γ;
(ii) ϕ_Γ takes positive integer values on the lattice points of $\mathbb{R}^n_+ \smallsetminus \{0\}$;
(iii) there exists a positive integer M such that $\phi_\Gamma(k) = M$, for all $k \in \Gamma$.

This map ϕ_Γ induces a filtration of $\mathcal{O}_n = \mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \dots$, by the ideals $\mathcal{A}_q = \{g \in \mathcal{O}_n : \operatorname{supp} g \subseteq \phi_\Gamma^{-1}(q+\mathbb{N})\}$, for all $q \in \mathbb{N}$, called the *Newton filtration of* $\mathcal{O}_n$ *associated to* Γ_+ (see [5](2.1)).

For any compact face Δ of Γ, this filtration induces a filtration on $\mathcal{A}_\Delta$ in a natural way.

Definition 4.5. The Γ-*order* of a germ $g = \sum_k a_k x^k$, with respect to the filtration induced by Γ is defined as

$$\mathrm{d}(g) = \min\{\phi_\Gamma(k) : k \in \operatorname{supp} g\} = \max\{q : g \in \mathcal{A}_q\}.$$

The *principal part of* g, denoted by $\mathrm{in}(g)$, is the sum of terms $a_k x^k$ such that $\phi_\Gamma(k) = \mathrm{d}(g)$. Given a face Δ of Γ, *the principal part of* g *over* Δ, denoted by $\mathrm{in}_\Delta(g)$, is the polynomial

$$\mathrm{in}_\Delta(g) = \sum\{a_k x^k : k \in \operatorname{supp} g \cap C(\Delta) \text{ and } \phi_\Gamma(k) = \mathrm{d}(g)\}.$$

Definition 4.6. A system of generators $g_1, \dots, g_s$ of a finite codimension ideal I is *non-degenerate on* Γ_+ if, for each compact face $\Delta \subseteq \Gamma$, the ideal of $\mathcal{A}_\Delta$ generated by $\mathrm{in}_\Delta(g_1), \dots, \mathrm{in}_\Delta(g_s)$ has finite codimension in $\mathcal{A}_\Delta$. When the system $g_1, \dots, g_s$ does not satisfy the above definition, we say that this system is *degenerate on* Γ_+.

We remark that this definition depends of the system of generators, while the definition of Newton non-degeneracy does not depend of the system on generators of the ideal.

The next theorem is essential to prove the main results of this section.

Theorem 4.7. *([2], Theorem 3.3.) Let $g_1, \dots, g_n$ be a system of generators of an ideal I with finite codimension in $\mathcal{O}_n$ and let $\Gamma_+ \subseteq \mathbb{R}^n$ be a Newton polyhedron. If M is the value on Γ of the filtration induced by Γ_+ and $d_1 = \mathrm{d}(g_1), \dots, d_n = \mathrm{d}(g_n)$ are the levels of the given set of generators of I with respect to this filtration, then*

(i) *we have the inequality* $\dim_\mathbb{C} \mathcal{O}_n/I \geq \dfrac{d_1 \dots d_n}{M^n} n!\, \mathrm{V}_n(\Gamma_-)$,

(ii) *the equality holds if and only if the system $g_1, \dots, g_n$ is non-degenerate on Γ_+.*

From now on we consider a deformation $F(x,t) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$, of a germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated singularity at 0 and fix a Newton polyhedron Γ_+.

We denote by D_i, D_i^s, D_{ti}, the Γ-order of the partial derivatives $\frac{\partial f}{\partial x_i}$, $\frac{\partial g_s}{\partial x_i}$, $\frac{\partial f_t}{\partial x_i}$ respectively.

Theorem 4.8. *Suppose that the system of generators $\{\partial f/\partial x_1, \dots, \partial f/\partial x_n\}$ of the Jacobian ideal $J(f)$ is non-degenerate on Γ_+. If $D_i^s \geq D_i$ for all $i = 1, \dots, n$ and all $s = 1, \dots, \ell$. Then, for small values of t, the system $\{\partial f_t/\partial x_1, \dots, \partial f_t/\partial x_n\}$ of generators of the Jacobian ideal $J(f_t)$ is non-degenerate on Γ_+ and $\mu(f_t)$ is constant.*

Proof. Suppose first that $D_i^s > D_i$ for all $i = 1, \dots, n$ and all $s = 1, \dots, \ell$, hence, $D_{ti} = D_i$ for all $i = 1, \dots, n$ and, for all t the principal part of each $\partial f_t/\partial x_i$ is equal to the principal part of $\partial f/\partial x_i$. From the non-degeneracy of $J(f)$ on Γ_+, we obtain the non-degeneracy of each Jacobian ideal $J(f_t)$ and the equality $\mu(f_t) = \mu(f)$ follows from the part (ii) of the theorem 4.7.

If there exist $D_i^s = D_i$, the non-degeneracy of each Jacobian ideal $J(f_t)$ for small values of t follows from the fact that the set of non-degenerate ideals on some Newton polyhedron is open, for the Zariski topology. Hence the equality $\mu(f_t) = \mu(f)$ follows from the part (ii) of the theorem 4.7. $\square$

The following example ilustrates this result.

Example 4.9. Let $f(x, y) = x^{12} + y^8$. Fix the Newton polyhedron $\Gamma_+ = \Gamma_+(f)$, since f is weighted homogeneous with respect to the weight $w = (2, 3)$, we obtain that Γ_+ has only one compact face with vertices $(12, 0)$ and $(0, 8)$. The jacobian ideal $J(f) = \langle f_x, f_y\rangle$ is non-degenerate on Γ_+, with $\mathrm{d}(f_x) = 22$ and $\mathrm{d}(f_y) = 21$.

Consider the family $f_t(x, y) = x^{12} + y^8 - 2tx^6y^4$. Here $\mathrm{d}(f_{tx}) = 22 = \mathrm{d}(f_x)$ and $\mathrm{d}(f_{ty}) = 21 = \mathrm{d}(f_y)$. The principal part of the generators $\{f_{tx}, f_{ty}\}$ of the Jacobian ideal $J(f_t)$ is different from the principal part of the generators $\{f_x, f_y\}$ of the Jacobian ideal $J(f)$, but for each $0 \le t < 1$, the system of generators $\{f_{tx}, f_{ty}\}$ is also non-degenerate, hence $\mu(f) = \mu(f_t)$ for all $0 \le t < 1$. For $t = 1$, we see that the germ $f_1 = (x^6 - y^4)^2$ has not isolated singularity, hence $\mu(f_1) = \infty$.

On the other side, we can also prove the folowing

Theorem 4.10. *Suppose that the system of generators $\{\partial f/\partial x_1, \dots, \partial f/\partial x_n\}$ of the Jacobian ideal $J(f)$ is non-degenerate on Γ_+. If for small values of t, the system of generators $\{\partial f_t/\partial x_1, \dots, \partial f_t/\partial x_n\}$ of the Jacobian ideal $J(f_t)$ is non-degenerate on Γ_+ and $\mu(f_t)$ is constant, then $D_i^s \ge D_i$ for all $i = 1, \dots, n$ and all $s = 1, \dots, \ell$.*

Proof. If $\mu(f_t) = \mu(f)$, since $J(f_t)$ is non-degenerate on Γ_+ , it follows from part (ii) of the theorem 4.7 that $\mu(f_t) = \dfrac{D_{t1} \dots D_{tn}}{M^n} n!\, \mathrm{V}_n(\Gamma_-)$. From the hypothesis we have $\mu(f_t) = \mu(f) = \dfrac{D_1 \dots D_n}{M^n} n!\, \mathrm{V}_n(\Gamma_-)$, where the last equality holds from part (ii) of the theorem 4.7 . Hence $D_{t1} \dots D_{tn} = D_1 \dots D_n$.

But for small values of t, we have

$$D_{ti} = min.\,\{D_i, D_i^s,\}\,, \text{ for all i} = 1 \dots, \text{n};\ \text{s} = 1, \dots, \ell.$$

Hence we obtain $D_{ti} = D_i$ for all $i = 1 \dots, n$. Therefore $D_i^s \ge D_i$ for all $i = 1 \dots, n$ and $s = 1, \dots, \ell$. $\square$

Example 4.11. Generalized Briançon-Speder Example

We shall consider here families of type $f_t(x, y, z) = z^5 + y^7x + x^{15} + tx^ay^bz^c$ for a fixed monomial $x^ay^bz^c$.

In order to study the μ-constancy of this family, we first check when $x^ay^bz^c$ satisfies the necessary condition $\Gamma_+(x^ay^bz^c) \subseteq T(f)$ given in the lemma 3.4.

The Jacobian ideal $J(f)$, generated by the system $\{15x^{14} + y^7, 7y^6x, 5z^4\}$, is Newton non-degenerate, hence $T(f) = \Gamma_+(J(f))$, whose vertices are $(14, 0, 0)$, $(0, 7, 0)$, $(1, 6, 0)$ and $(0, 0, 4)$ in $\mathbb{R}^3_+$. Therefore, we consider the monomials $x^ay^bz^c$ such that $(a, b, c) \in \Gamma_+(J(f))$.

For instance, if we fix the monomial $x^a y^b z^c = y^6 z$, the Newton polyhedron Γ_+ that we consider is associated to the set of weights $(1,2,3)$ and has only one compact face with vertices $(3,0,0)$, $(0,2,0)$ and $(0,0,1)$.

An easy calculation shows that the Γ-order of the partial derivatives of f are: $\mathrm{d}(15x^{14}+y^7)=14$, $\mathrm{d}(7y^6x)=13$, $\mathrm{d}(5z^4)=12$ and $J(f)$ is non-degenerate on Γ_+.

The Γ-order of the partial derivatives of the monomial y^6z are: $\mathrm{d}(\partial(y^6z)/\partial x)=\infty$, $\mathrm{d}(6y^5z)=13$ and $\mathrm{d}(y^6)=12$, hence we apply the theorem 4.8 to conclude that the system of generators $\{15x^{14}+y^7, 7y^6x+t6y^5z, 5z^4+ty^6\}$ is non-degenerate on Γ_+ and the family f_t is μ-constant for small values of t.

An analogous argument can be applied to any family $f_t(x)=f+\sum_{s=1}^{\ell}\delta_s(t)g_s$ satisfying the conditions of the Theorem 4.8 for this Newton polyhedron.

But there are some monomials $x^ay^bz^c$ satisfying the condition $(a,b,c)\in\Gamma_+(J(f))$ that do not satisfy the conditions of the Theorem 4.8 for this Newton polyhedron. For example, if we consider $x^ay^bz^c=yz^4$, the Γ-order of $\mathrm{d}(\partial(yz^4)/\partial y)=11$ and we can not apply the theorem 4.8.

On the other side we can apply the theorem 4.10 to show that the family $f_t(x,y,z)=z^5+y^7x+x^{15}+tyz^4$ is not μ-constant. For this we fix the Newton polyhedron $\Gamma_+=\Gamma_+(J(f_t))$. An easy calculation shows that $J(f_t)$ is Newton non-degenerate and $J(f_t)$ is degenerate on this polyhedron, hence $\mu(f_t)<\mu(f)$ for all $t\neq 0$.

Analogous argument can be applied to any family $f_t(x)=f+\sum_{s=1}^{\ell}\delta_s(t)g_s$ such that $J(f_t)$ is Newton non-degenerate.

5. μ-CONSTANT DEFORMATIONS AND THE MULTIPLICITY

Zariski propposed in [12] the following question:

Whether for a hypersurface singularity the multiplicity is an invariant of the topological type?

It is well known that this question has a positive answer in the case of plane curves and for quasi homogeneous complex hypersurfaces, as is shown by Greuel in [4].

Definition 5.1. The multiplicity of a germ $f(x)=\sum a_kx^k$, is defined as the lowest degree in the power series of $f(x)$.

From the results given in the section 4 we obtain a positive answer for Zariski's question for families of germs which are non-degenerate on some Newton polyhedron Γ_+.

Corollary 5.2. *Let $F(x,t)=f(x)+\sum_{s=1}^{\ell}\delta_s(t)g_s(t)$ be a deformation of a germ f with isolated singularity such that for small values of t, the system of generators $\{\partial f_t/\partial x_1,\dots,\partial f_t/\partial x_n\}$ of each Jacobian ideal $J(f_t)$ is non-degenerate on Γ_+. If $\mu(f_t)$ is constant, then $F(x,t)$ is m-constant.*

Proof. To prove this corollary we apply the theorem 4.10 to conclude that $D_i^s\geq D_i$ for all $i=1,\dots,n$ and all $s=1,\dots,\ell$, therefore $\Gamma_+(g_s)\subset\Gamma_+(f)$ for all $s=1,\dots,\ell$ and the equality $m(f_t)=m(f)$ follows. $\square$

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