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A Massera Type Criterion for Partial Neutral Functional Differential Equations

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Um Critério do Tipo Massera para uma Equação Funcional do Tipo Neutro.

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Resumo

Neste artigo usamos um critério de tipo Massera, para mostrar a existência de soluções periódicas de uma equação funcional do tipo neutro modelada na forma

$$\frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F(t, x_t), \qquad t > 0, \tag{1}$$

$$x_0 = \varphi \in \mathcal{D}, \tag{2}$$

onde A é o gerador infinitesimal de um semigrupo analítico e compacto de operadores lineares num espaço de Banach X; $\mathcal{D}$ é um apropriado espaço de fase e $G, F : \mathbb{R} \times \mathcal{D} \to X$ são funções continuas e periódicas.

A Massera Type Criterion for Partial Neutral Functional Differential Equations.

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Abstract

We prove the existence of a periodic solution for a partial neutral functional differential equation with delay, using a Massera type criterion.

1 Introduction

By using a Massera type criterion we prove the existence of a periodic solution for a class of partial neutral functional differential equation modeled in the form

$$\frac{d}{dt}(x(t)+G(t,x_t)) = Ax(t)+F(t,x_t), \qquad t>0, \tag{1}$$

$$x_0 = \varphi\in\mathcal{D} \tag{2}$$

where A is the infinitesimal generator of an compact analytic semigroup of linear operators, $(T(t))_{t\geq 0}$, on a Banach space X; the history x_t, $x_t(\theta)=x(t+\theta)$, belongs to a appropriate phase space $\mathcal{D}$ and $G, F:\mathbb{R}\times\mathcal{D}\to X$ are continuous functions.

The Massera Theorem [9] is a pioneer work in the study of the relations between the boundedness of solutions and the existence of periodic solutions. In Massera [9], is explain this relation for a two dimensional periodic ordinary differential equations. Subsequently, several authors considered this problem, see for example Yoshizawa for a n-dimensional differential equation; Lopes and Hale, for n-dimensional ordinary and functional equations with delay and Yong for a functional differential equations.

Recently Ezzinbi [1], using a Massera type criterion showed the existence of a periodic solution for a partial functional differential equation modeled in the form

$$\begin{aligned} \dot{x}(t) &= Ax(t)+F(t,x_t),\\ x_0 &= \varphi\in\mathcal{C}=C([-r,0]:X), \end{aligned}$$

where A is the infinitesimal generator of a compact semigroup of bounded linear operators on a Banach space.

Neutral differential equations arises in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature of neutral functional differential equations with finite delay is the Hale book [4] and the references therein. The work in partial neutral functional differential equations with infinite delay, was initiated by Hernández & Henríquez in [6, 5]. In these papers Hernández & Henríquez proved the existence of mild, strong and periodic solutions for a neutral equation modeled in the form

$$\begin{cases} \frac{d}{dt}(x(t)+G(t,x_t)) &= Ax(t)+F(t,x_t), \\ x_0 &= \varphi \in \mathcal{B}. \end{cases} \tag{3}$$

where A is the infinitesimal generator of an analytic semigroup of linear operators on a Banach space and $\mathcal{B}$ is a phase space defined axiomatically. In general, the results was obtained using the semigroup theory and the Sadovskii fixed point Theorem.

Our purpose in this paper is establish similar existence results at those in Ezzinby [1], now for the case of partial neutral differential equation with delay modeled as (1)-(2).

Throughout this paper, X will be a Banach space provided with norm $\| \cdot \|$ and $A : D(A) \to X$ will be the infinitesimal generator of an compact analytic semigroup, $(T(t))_{t\geq 0}$, of linear operators on X. For the theory of strongly continuous semigroup, refer to Pazy [10]. We will point out here some notations and properties that will be used in this work. It is well known that there exist $\tilde{M} \geq 1$ and $\rho \in \mathbb{R}$ such that $\| T(t) \| \leq \tilde{M}e^{\rho t}$, for every $t \geq 0$. If $(T(t))_{t\geq 0}$ is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)^{\alpha}$, for $\alpha \in (0,1]$, as a closed linear operator on its domain $D(-A)^{\alpha}$. Furthermore, the subspace $D(-A)^{\alpha}$ is dense in X and the expression

$$\| x \|_{\alpha} = \| (-A)^{\alpha} x \|, \qquad x \in D(-A)^{\alpha},$$

defines a norm in $D(-A)^{\alpha}$. If X_{α} represent the space $D(-A)^{\alpha}$ endowed with the norm $\| \cdot \|_{\alpha}$, then the following properties are well known ([10], pp. 74):

Lemma 1 *If the previous conditions hold:*

1. *Let $0 < \alpha \leq 1$. Then X_{α} is a Banach space.*
2. *If $0 < \beta < \alpha \leq 1$ then $X_{\alpha} \hookrightarrow X_{\beta}$ and the imbedding is compact whenever the resolvent operator of A is compact.*
3. *For every $0 < \alpha \leq 1$ there exists $C_{\alpha} > 0$ such that*

$$\| (-A)^{\alpha} T(t) \| \leq \frac{C_{\alpha}}{t^{\alpha}}, \qquad t > 0.$$

In what follows, to avoid unnecessary notations, we suppose that $0 \in \rho(A)$ and that for $0 < \vartheta \leq 1$

$$\| T(t) \| \leq \tilde{M}, \quad t \geq 0, \quad \text{and} \quad \| (-A)^{\vartheta} T(t) \| \leq \frac{C_{\vartheta}}{t^{\vartheta}}, \quad t > 0, \tag{4}$$

for some positive constant C_{θ}.

Along of this paper $0 < \beta \leq 1$ and $\omega > 0$ are fixed numbers, $G, F : \mathbb{R} \times \mathcal{D} \to X$ are continuous and we use the notation $\mathbf{H_i}$ for the following conditions

H_1 The function G is X_β-valued and $(-A)^\beta G$ is continuous.

H_2 $G(t,\psi)=V(t,\psi)+h(t)$ where V, h are X_β-valued; $(-A)^\beta V$, $(-A)^\beta h$ are continuous; $(-A)^\beta V(\cdot,\psi)$, $(-A)^\beta h$ are ω-periodic and $(-A)^\beta V(t,\cdot)$ is linear.

H_3 $G(t,\psi)=V(t,\psi)+G_1(t,\psi)$ where V, G_1 are X_β-valued; $(-A)^\beta V$, $(-A)^\beta G_1$ are continuous; $(-A)^\beta V(\cdot,\psi)$, $(-A)^\beta G_1(\cdot,\psi)$ are ω-periodic and $(-A)^\beta V(t,\cdot)$ is linear.

H_4 $F(t,\psi)=L(t,\psi)+f(t)$ where L, f are continuous; $L(\cdot,\psi)$, f are ω-periodic and $L(t,\cdot)$ is linear.

H_5 $F(t,\psi)=L(t,\psi)+F_1(t,\psi)$ where L, F_1 are continuous; $L(\cdot,\psi)$, $F_1(\cdot,\psi)$ are ω-periodic and $L(t,\cdot)$ is linear.

H_6 For every $R>0$ and all $T>0$ the set of functions

$$\{s\to G(s,x_s): x\in C([-r,T]:X),\ \sup_{\theta\in[-r,T]}\parallel x(\theta)\parallel\leq R\}$$

is equicontinuous on $[0,T]$.

H_7 For every $R>0$ and all $T>0$ the set of functions

$$\{s\to G(s,x_s): x\in C_b((-\infty,T]:X),\ \sup_{\theta\in[-\infty,T]}\parallel x(\theta)\parallel\leq R\}$$

is equicontinuous on $[0,T]$.

We remark that for the proofs of our results we will use the following results.

Theorem 1 *Let Y be a Banach space and $\Gamma:=\Gamma_1+y$ where $\Gamma_1: Y\to Y$ is a bounded linear operator and $y\in Y$. If there exist $x_0\in Y$ such that the set $\{\Gamma^n(x_0): n\in\mathbb{N}\}$ is relatively compact in Y, then Γ has a fixed point in Y.*

Theorem 2 *Let X, Y be Banach spaces and M be a nonempty convex subset of X. If $\Gamma: M\to 2^Y$ is a multivalued map such that*

(i) *For every $x\in M$, the set $\Gamma(x)$ is nonempty, convex and closed,*

(ii) *The set $\Gamma(M)=\bigcup_{x\in M}\Gamma x$ is relatively compact,*

(iii) *Γ is upper semi-continuous,*

then Γ has a fixed point in M.

The paper has four sections. In sections 2 and 3 we discuss the existence of a periodic solution for partial neutral differential equations with bounded and unbounded delay respectively. In section 4 some examples are discussed. Our results are based on the properties of analytic semigroups and the ideas and techniques in Hernández & Henríquez [6, 5] and Ezzinbi [1].

Throughout this paper $B_r(x:Z)$, ($B_r[x:Z]$) will be the open (the closed) ball in a metric space Z with center at x and radius r. For a bounded function $\xi:[a,b]\to[0,\infty)$ and $a\leq t\leq b$ we will employ the notation $\xi_{a,t}$ for

$$\xi_{a,t}=\sup\{\xi(s): s\in[a,t]\}. \tag{5}$$

Finally, if $\mathcal{D}$ is a Banach phase space, the norm in $\mathcal{D}$ will be denote by $\parallel\cdot\parallel_{\mathcal{D}}$.

2 A periodic solution for a partial neutral differential equation with bounded delay

In this section we prove the existence of an periodic solution of the initial value problem

$$\frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F(t, x_t), \tag{6}$$

$$x_0 = \varphi \in \mathcal{C} = C([-r, 0] : X). \tag{7}$$

.

Definition 1 *A function $x : [-r, T] \to X$ is a mild solution of the abstract Cauchy problem (6)-(7) if: $x_0 = \varphi$; the restriction of $x(\cdot)$ to the interval $[0, T]$ is continuous; for each $0 \leq t < T$ the function $AT(t-s)G(s, x_s)$, $s \in [0, t)$, is integrable and*

$$\begin{aligned} x(t) &= T(t)(\varphi(0) + G(0, \varphi)) - G(t, x_t) - \int_0^t AT(t-s)G(s, x_s)ds \\ &\quad + \int_0^t T(t-s)F(s, x_s)ds, \qquad t \in [0, T]. \end{aligned} \tag{8}$$

The existence of mild solutions for the abstract Cauchy problem (6), (7) follows from Theorems 2.1 and 2.2 in [5]; for this reason we choose to omit the proof of the next two results.

Theorem 3 *Let $\varphi \in \mathcal{C}$, $T > 0$ and assume that the following conditions hold:*

(a) *There exist constants $\beta \in (0, 1)$ and $L \geq 0$ such that the function G is X_β-valued, $L \parallel (-A)^{-\beta} \parallel < 1$ and*

$$\parallel (-A)^\beta G(t, \psi_1) - (-A)^\beta G(s, \psi_2) \parallel \leq L(|t - s| + \parallel \psi_1 - \psi_2 \parallel_{\mathcal{C}}), \tag{9}$$

for every $0 \leq s, t \leq T$ and $\psi_1, \psi_2 \in \mathcal{C}$.

(b) *The function F is continuous and takes bounded sets into the bounded sets.*

Then there exists a mild solution $x(\cdot, \varphi)$ of the abstract Cauchy problem (6)-(7) defined on $[-r, b]$, for some $0 < b \leq T$.

Theorem 4 *Let $\varphi \in \mathcal{C}$ and $T > 0$. Assume that condition* (a) *of the previous Theorem holds and that there exists $N > 0$ such that*

$$\parallel F(t, \varphi) - F(t, \psi) \parallel \leq N \parallel \varphi - \psi \parallel_{\mathcal{C}}, \tag{10}$$

for all $0 \leq t \leq T$ and every $\varphi, \psi \in \mathcal{C}$. Then there exists a unique mild solution $x(\cdot, \varphi)$ of (6)-(7) defined on $[-r, b]$ for some $0 < b \leq T$. Moreover, b can be to chosen as $\min\{T, b_0\}$, where b_0 is a positive constant independent of φ.

To prove the main result of this section, is fundamental the next result.

Theorem 5 *Let $T > r$ and assume that assumption $\mathbf{H_1}, \mathbf{H_6}$ hold. Suppose, furthermore, that the following conditions hold.*

(a) *For every $\varphi \in \mathcal{C}$ the set*

$$\mathrm{X}(\varphi) = \{\mathrm{x} \in \mathrm{C}([-\mathrm{r},\mathrm{T}]:\mathrm{X}) : \mathrm{x} \text{ is solution of (6)-(7)}\}$$

is nonempty.

(b) *For every $R > 0$, the set*

$$\{(-A)^{\beta} G(s, x_s),\ F(s, x_s) : s \in [0, T],\ x \in \mathrm{X}(\mathrm{x_0}) \text{ and } \| \mathrm{x} \|_{-\mathrm{r},\mathrm{T}} \leq \mathrm{R}\},$$

is bounded.

Then the multivalued map $\Upsilon : \mathcal{C} \to 2^{\mathcal{C}};\ \varphi \to \mathrm{X_T}(\varphi) = \{\mathrm{x_T} : \mathrm{x} \in \mathrm{X}(\varphi)\}$ is compact, that is, for every $R > 0$ the set $\mathcal{U}_{\mathcal{R},\mathcal{T}} = \bigcup_{\|\varphi\|_{\mathcal{C}} \leq \mathcal{R}} \mathrm{X_T}(\varphi)$ is relatively compact in $\mathcal{C}$.

Proof: Let $R > 0$ and $\mathcal{U}_{\mathcal{R}} = \bigcup_{\|\varphi\|_{\mathcal{C}} \leq \mathcal{R}} \mathrm{X}(\varphi)$. From **(b)**, we fix $N > 0$ such that $\| (-A)^{\beta} G(s, x_s) \| \leq N$ and $\| F(s, x_s) \| \leq N$ for every $x \in \mathcal{U}_{\mathcal{R}}$ and every $s \in [0, T]$. In order to use the Ascoli Theorem, we divide the proof in two steps.

Step 1 The set $\mathcal{U}_R(t) = \{x(t) : x \in \mathcal{U}_R\}$ is relatively compact for $t \in (0, T]$.

Let $0 < \epsilon < t \leq T$. Since $(T(t))_{t \geq 0}$ is analytic, the operator function $s \to AT(s)$ is continuous in the uniform operator topology on $(0, T]$, which by the estimate

$$\| (-A)^{1-\beta} T(t-s) (-A)^{\beta} G(s, x_s) \| \leq \frac{C_{1-\beta} N}{(t-s)^{1-\beta}}, \qquad s \in [0, t),\ x \in \mathcal{U}_{\mathcal{R}} \tag{11}$$

implies that the function $s \to \| (-A)^{1-\beta} T(t-s) G(s, x_s) \|$ is integrable on $[0, t)$ for every $x \in \mathcal{U}_{\mathcal{R}}$. Under the previous conditions, for $x \in \mathcal{U}_{\mathcal{R}}$ we get

$$\begin{aligned} x(t) &= T(\epsilon) T(t-\epsilon)(\varphi(0) + G(0, \varphi)) + (-A^{-\beta})(-A^{\beta}) G(t, x_t) \\ &\quad + T(\epsilon) \int_0^{t-\epsilon} (-A)^{1-\beta} T(t-s-\epsilon)(-A)^{\beta} G(s, x_s) ds \\ &\quad + \int_{t-\epsilon}^{t} (-A)^{1-\beta} T(t-s)(-A)^{\beta} G(s, x_s) ds \\ &\quad + T(\epsilon) \int_0^{t-\epsilon} T(t-s-\epsilon) F(s, x_s) ds + \int_{t-\epsilon}^{t} T(t-s) F(s, x_s) ds, \end{aligned}$$

and hence

$$x(t) \in T(\epsilon) \tilde{M} B_{R+N}[0, X] + (-A^{-\beta}) B_N[0, X] + W_t^1 + C_\epsilon^1 + W_t^2 + C_\epsilon^2,$$

where each W_t^i is compact, $diam(C_\epsilon^1) \leq 2 C_{1-\beta} N \frac{\epsilon^{\beta}}{\beta}$ and $diam(C_\epsilon^2) \leq 2 \tilde{M} N \epsilon$. Since $(-A)^{-\beta}$ is compact, these remarks implies that $\mathcal{U}_{\mathcal{R}}(\sqcup)$ is totally bounded and consequently relatively compact in X.

Step 2 $\mathcal{U}_{\mathcal{R}}$ is equicontinuous on $(0,T]$.

Let $0<\epsilon<t_0<t\le T$. The strong continuity of $(T(t))_{t\ge 0}$ implies that the set of functions $\{s\to T(s)x : x\in T(\epsilon)B_{R+N}(0,X)\}$ is equicontinuous on $[0,T]$. Let $0<\delta<\epsilon$ be such that

$$\begin{aligned}
\| T(s)x-T(s')x \| &< \epsilon, \quad x\in T(\epsilon)B_{R+N}(0,X),\\
\| G(t,u_t)-G(t_0,u_{t_0}) \| &< \epsilon, \quad u\in C([-r,T]:X),\ \| u\|_{-r,T}\le R+N,
\end{aligned}$$

when $|s-s'|<\delta,\ \ 0\le s,s'\le T$ and $0\le t-t_0<\delta$.

Under the previous conditions, for $x\in\mathcal{U}_{\mathcal{R}}$ and $0\le t-t_0<\delta$ we get

$$\begin{aligned}
&\| x(t_0)-x(t) \|\\
&\le \| (T(t_0-\epsilon)-T(t-\epsilon))T(\epsilon)(\varphi(0)+G(0,\varphi)) \|\\
&\quad + \| G(t_0,x_{t_0})-G(t,x_t) \|\\
&\quad + \int_0^{t_0-\epsilon} \| (-A)^{1-\beta}T(t_0-s-\epsilon)(I-T(t-t_0))T(\epsilon)(-A)^{\beta}G(s,x_s) \| \,ds\\
&\quad + \int_{t_0-\epsilon}^{t_0} \| I-T(t-t_0) \| \| (-A)^{1-\beta}T(t_0-s)(-A)^{\beta}G(s,x_s) \| \,ds\\
&\quad + \int_{t_0}^{t} \| (-A)^{1-\beta}T(t-s)(-A)^{\beta}G(s,x_s) \| \,ds\\
&\quad + \int_0^{t_0-\epsilon} \| T(t_0-s-\epsilon)(I-T(t-t_0))T(\epsilon)F(s,x_s) \| \,ds\\
&\quad + \int_{t_0-\epsilon}^{t_0} \| (T(t_0-s)-T(t-s))F(s,x_s) \| \,ds + \int_{t_0}^{t} \| T(t-s)F(s,x_s) \| \,ds\\
&\le 2\epsilon + \epsilon C_{1-\beta}\frac{(t_0-\epsilon)^{\beta}}{\beta} + 2\tilde{M}C_{1-\beta}N\frac{\epsilon^{\beta}}{\beta} + NC_{1-\beta}\frac{(t-t_0)^{\beta}}{\beta} + \epsilon\tilde{M}(t_0-\epsilon)\\
&\quad +2\tilde{M}N\epsilon + \tilde{M}N(t-t_0),
\end{aligned}$$

thus

$$\| x(t_0)-x(t) \| \le c_1\epsilon + c_2\epsilon^{\beta} + c_3\delta^{\beta} + c_4\delta,$$

where the constants c_i are independent of $x(\cdot)$. This proves that $\mathcal{U}_{\mathcal{R}}$ is equicontinuous from the right side at $t_0>0$. The equicontinuity in $t_0>0$ is proved in similar form, we omit details. Thus, $\mathcal{U}_{\mathcal{R}}$ is equicontinuous on $(0,T]$.

From the steps 1 and 2 follow that the set $\{x|_{[\mu,T]} : x\in\mathcal{U}_{\mathcal{R}}\}$ is relatively compact in $C([\mu,T]:X)$ for every $\mu>0$, which in turn implies that $\mathcal{U}_{\mathcal{R},T}$ is relatively compact in $C([-r,0]:X)$. This completes the proof.

Corollary 1 *Assume that the hypothesis in Theorems 4 and 5 are fulfilled. Then the map $\varphi\to x_T(\cdot,\varphi)$ is a completely continuous function.*

Proof: The assertion follow from the Lebesgue dominated convergence Theorem, the estimate (11) and Theorem 5. We omit details.

Definition 2 *A function $x:\mathbb{R}\to X$ is an ω-periodic solution of equation (6) if: $x(\cdot)$ is a mild solution of (6) and $x(t+\omega)=x(t)$ for every $t\in\mathbb{R}$.*

Using the ideas and techniques in [6] is possible to establish sufficient conditions for the existence of global solutions of (6). In what follows, we always assume that the mild solutions are defined on $[0,\infty)$.

Theorem 6 *Let conditions* $\mathbf{H_2}$, $\mathbf{H_4}$ *and* $\mathbf{H_6}$ *be satisfied. If the equation (6) has a bounded mild solution, then there exist an* ω*-periodic solution of (6).*

Proof: For a mild solution $x(\cdot) = x(\cdot,\varphi)$, we introduce the decomposition $x(\cdot) = v(\cdot) + z(\cdot)$, where $v(\cdot)$ is the mild solution of

$$\begin{aligned} \frac{d}{dt}(u(t) + V(t,u_t)) &= Au(t) + L(t,u_t), \\ u_0 &= \varphi, \end{aligned}$$

and $z(\cdot)$ is the mild solution of

$$\begin{aligned} \frac{d}{dt}(u(t) + V(t,u_t) + h(t)) &= Au(t) + L(t,u_t) + f(t), \\ u_0 &= 0. \end{aligned}$$

Let $y : [-r,\infty) \to X$ be a bounded mild solution of (6) and $\Gamma : \mathcal{C} \to \mathcal{C}$ be the map $\Gamma(\varphi) := \Gamma_1(\varphi) + z_\omega := v_\omega + z_\omega$. Since Γ_1 is a bounded linear operator and $\bigcup_{n\geq 1}\Gamma^n(y_0) = \{y_{n\omega} : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{C}$, see Theorem 5, follow from Theorem 1 that Γ has a fixed point in $\mathcal{C}$. This fixed point give a periodic solution. The proof is complete.

Observation 1 *In the rest of this paper* CP *is the space*

$$CP = \{u : \mathbb{R} \to X : u \textit{ is } \omega\textit{-periodic}\},$$

endowed with the uniform converge topology.

Now we prove the main result of this work.

Theorem 7 *Let conditions* $\mathbf{H_3}, \mathbf{H_5}$ *and* $\mathbf{H_6}$ *be satisfied and assume that the following conditions are fulfilled.*

(a) *The functions* $(-A)^\beta V$, $(-A)^\beta G_1$, F *and* F_1 *takes bounded sets into the bounded sets.*

(b) *There is* $\rho > 0$ *such that for every* $u \in B_\rho[0, CP]$ *the neutral equation*

$$\frac{d}{dt}(y(t) + V(t,y_t) + G_1(t,u_t)) = Ay(t) + L(t,y_t) + F_1(t,u_t),$$

has an ω*-periodic solution* $x(\cdot,u) \in B_\rho[0, CP]$.

Then the equation (6) has an ω*-periodic solution.*

Proof: On $B_\rho = B_\rho[0, CP]$, we define the multivalued map $\Gamma : B_\rho \to 2^{B_\rho}$ by: $x \in \Gamma(u)$ if, and only if,

$$\begin{aligned} x(t) &= T(t-s)(x(s) + V(s, x_s) + G_1(s, x_s)) - V(t, x_t) - G_1(t, u_t) \\ &\quad - \int_s^t AT(t-\tau)(V(\tau, x_\tau) + G_1(\tau, u_\tau))d\tau \\ &\quad + \int_s^t T(t-\tau)(L(\tau, x_\tau) + F_1(\tau, u_\tau))d\tau, \qquad t > s. \end{aligned}$$

Next we prove that Γ verifies the hypothesis of Theorem 2. Clearly assumption **(i)** holds. The condition **(ii)** follow from the proof of Theorem 5. In relation with **(iii)** we observe that from **(ii)**, is sufficient to show that Γ is closed. If $(u^n)_{n\in\mathbb{N}}$, $(x^n)_{n\in\mathbb{N}}$ are sequences in B_ρ such that $x^n \in \Gamma u^n$; $u^n \to u$ and $x_n \to x$ then

$$\begin{aligned} (-A)^\beta(V(\tau, x_\tau^n) + G_1(\tau, u_\tau^n)) &\to (-A)^\beta(V(\tau, x_\tau) + G_1(\tau, u_\tau)), \\ T(t)(L(\tau, x_\tau^n) + F_1(\tau, u_\tau^n)) &\to T(t)(L(\tau, x_\tau) + F_1(\tau, u_\tau)), \end{aligned}$$

for $\tau \in \mathbb{R}$ and $t \geq s$. From the Lebesgue dominated convergence Theorem, **(a)** and the estimate

$$\| AT(t-s)(V(\tau, x_\tau^n) + G_1(\tau, u_\tau^n)) \| \leq C_{1-\beta} \frac{\| (-A)^\beta(V(\tau, x_\tau^n) + G_1(\tau, u_\tau^n)) \|}{(t-s)^{1-\beta}},$$

we conclude that

$$\begin{aligned} x(t) &= T(t-s)(x(s) + V(s, x_s) + G_1(s, x_s)) - V(t, x_t) - G_1(t, u_t) \\ &\quad - \int_s^t AT(t-\tau)(V(\tau, x_\tau) + G_1(\tau, u_\tau))d\tau \\ &\quad + \int_s^t T(t-\tau)(L(\tau, x_\tau) + F_1(\tau, u_\tau))d\tau, \qquad t > s, \end{aligned}$$

which proves that $x \in \Gamma u$. Thus Γ is closed and consequently upper semi-continuous.

Finally, from Theorem 1 the operator Γ has a fixed point. This fixed point is an ω-periodic solution of (6). The proof is finished.

3 A periodic solution for a partial neutral differential equation with unbounded delay

In this section we discuss the existence of an ω-periodic solution for a partial functional neutral differential equation with unbounded delay modeled in the form

$$\frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F(t, x_t), \tag{12}$$

$$x_\sigma = \varphi \in \mathcal{B}, \tag{13}$$

where the history $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically and $F, G : \mathbb{R} \times \mathcal{B} \to X$ are appropriate continuous functions.

In the rest of this paper $\mathcal{B}$ will be a abstract phase space defined axiomatically as in Hale and Kato [2]. To establish the axioms of the space $\mathcal{B}$ we follows the terminology used in Hino-Murakami-Naito [7], and thus, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X, endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$. We will assume that $\mathcal{B}$ satisfies the following axioms:

(A) If $x : (-\infty, \sigma + a) \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$, the following conditions hold:

i) x_t is in $\mathcal{B}$.

ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$.

iii) $\| x_t \|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\| x(s) \| : \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \to [0, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in **(A)**, x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + a)$.

(B) The space $\mathcal{B}$ is complete.

(C2) If a uniformly bounded sequence $(\varphi^n)_n$ in C_{00} converges to a function φ in the compact-open topology, then $\varphi \in \mathcal{B}$ and $\|\varphi^n - \varphi\|_{\mathcal{B}} \to 0$ as $n \to \infty$.

Example 1

We consider the phase space $\mathcal{B} := C_r \times L^p(g; X)$, $r \geq 0$, $1 \leq p < \infty$ in [7], which consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable and $g \mid \varphi(\cdot) \mid^p$ is Lebesgue integrable on $(-\infty, -r)$, where $g : (-\infty, -r) \to \mathbb{R}$ is a positive Lebesgue integrable function. The seminorm in $\|\cdot\|_{\mathcal{B}}$ is defined by

$$\| \varphi \|_{\mathcal{B}} := \sup\{\| \varphi(\theta) \| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} g(\theta) \| \varphi(\theta) \|^p \, d\theta \right)^{1/p}.$$

We will assume that g satisfies conditions (g-6) and (g-7) in the terminology of [7]. This means that g is integrable on $(-\infty, -r)$ and that there exists a non-negative and locally bounded function γ on $(-\infty, 0]$ such that

$$g(\xi + \theta) \leq \gamma(\xi) \, g(\theta),$$

for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. In this case, $\mathcal{B}$ is a phase space which verifies axioms **(A)**, **(A1)**, **(B)** and **(C2)** (see [7], Theorem 1.3.8).

Definition 3 *A function $x : (-\infty, T] \to X$ is a mild solution of the abstract Cauchy problem (12)-(13) if: $x_0 = \varphi$; the restriction of $x(\cdot)$ to the interval $[0, T]$ is continuous; for each $0 \leq t < T$ the function $AT(t - s)G(s, x_s)$, $s \in [0, t)$, is integrable and*

$$\begin{aligned} x(t) &= T(t)(\varphi(0) + G(0, \varphi)) - G(t, x_t) - \int_0^t AT(t - s)G(s, x_s) ds \\ &+ \int_0^t T(t - s)F(s, x_s) ds, \qquad t \in [0, T]. \end{aligned} \tag{14}$$

Lemma 2 *Let Assumption* $\mathbf{H_1}, \mathbf{H_7}$ *be satisfied. If* $x : (-\infty, T] \to X$ *is a bounded mild solution of (12), then the set* $U = \{x_{n\omega} : n \in \mathbb{N}\}$ *is relatively compact in* $\mathcal{B}$.

Proof: Let $(x_{n_k\omega})_{k\in\mathbb{N}}$ a sequence in U. For $n > 0$ we define the set $U(n) = \{x_{n_k\omega} : n_k \geq n\}$. Using the same arguments in the proof of Theorem 5, follows that $U(n)$ is relatively compact in $C([-r,,0]; X)$ when $n\omega > r$. Using a diagonalized process we can to obtain a subsequence of $(x_{n_k\omega})_{k\in\mathbb{N}}$; which be indicated by the same index, that converges uniformly on compact subsets of $(-\infty, 0]$ to some function $x \in C_b((-\infty, 0] : X)$. Since $\mathcal{B}$ verifies axiom **C 2**, follow that $x \in \mathcal{B}$ and that $x_{n_k\omega} \to x$ in $\mathcal{B}$. These remarks implies that U is relatively compact in $\mathcal{B}$.

Definition 4 *A function* $x : \mathbb{R} \to X$ *is an* ω*-periodic solution of equation (12) if:* $x(\cdot)$ *is a mild solution of (12) and* $x(t+\omega) = x(t)$ *for every* $t \in \mathbb{R}$.

The proofs of the following results are similar to the proofs of Theorems 6 and 7. We only remark that the continuity of $\varphi \to x_\omega(\cdot, \varphi)$ is discussed in [6].

Theorem 8 *Let conditions* $\mathbf{H_2}$, $\mathbf{H_4}$ *and* $\mathbf{H_7}$ *be satisfied. If the equation (12) has a bounded mild solution then there exist an* ω*-periodic solution of (12).*

Theorem 9 *Let conditions* $\mathbf{H_3}, \mathbf{H_5}$ *and* $\mathbf{H_7}$ *be satisfied and assume that the following conditions are fulfilled.*

(a) *The functions* $(-A)^\beta V$, $(-A)^\beta G_1$, F *and* F_1 *takes bounded sets into the bounded sets.*

(b) *There is* $\rho > 0$ *such that for every* $u \in B_\rho[0, CP]$ *the neutral equation*

$$\frac{d}{dt}(y(t) + V(t, y_t) + G_1(t, u_t)) = Ay(t) + L(t, y_t) + F_1(t, u_t),$$

has an ω*-periodic solution* $x(\cdot, u) \in B_\rho[0, CP]$.

Then the neutral equation (12) has an ω*-periodic solution.*

4 Applications

In this section we illustrate some of the results in this work. Along of this section, $X = L^2([0, \pi])$ and A is the operator $Ax = x''$ with domain

$$D(A) := \{f(\cdot) \in L^2([0, \pi]) : f''(\cdot) \in L^2([0, \pi]),\ f(0) = f(\pi) = 0\}.$$

It's well known that A is the infinitesimal generator of a strongly continuous semigroup, $(T(t))_{t\geq 0}$, on X, which is compact, analytic and self-adjoint. Furthermore, A has discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with corresponding normalized eigenvectors $z_n(\xi) := \left(\frac{2}{\pi}\right)^{1/2} \sin(n\xi)$ and the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X.

(b) If $f \in D(A)$ then $A(f) = -\sum_{n=1}^{\infty} n^2 < f, z_n > z_n$.

(c) For every $f \in X$, $(-A)^{-\alpha} f = \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} < f, z_n > z_n$. In particular, $\|(-A)^{-1/2}\| = 1$.

(d) The operator $(-A)^{\alpha}$ is given by $(-A)^{\alpha} f = \sum_{n=1}^{\infty} n^{2\alpha} < f, z_n > z_n$ on the space $D((-A)^{\alpha}) = \{f \in X : \sum_{n=1}^{\infty} n^{2\alpha} < f, z_n > z_n \in X\}$.

(e) For every $f \in X$, $T(t)f = \sum_{n=1}^{\infty} e^{-n^2 t} < f, z_n > z_n$. Moreover, there exist constants $\tilde{M} \geq 1$, $\mu > 0$ such that

$$\| T(t) \| \leq \tilde{M} e^{-\mu t}, \ t \geq 0, \ \text{and} \ \| (-A)^{\vartheta} T(t) \| \leq \frac{C_{\vartheta} e^{-\mu t}}{t^{\vartheta}}, \ t > 0,$$

for some $C_{\vartheta} > 0$.

4.1 A neutral equation with bounded Delay

Considering the example in Ezzimby [1], in this section we study the following neutral equation

$$\begin{aligned} \frac{d}{dt}[u(t,\xi) + \int_{-r}^{0} \int_{0}^{\pi} b(s,\eta,\xi) u(t+s,\eta) d\eta ds] &= \frac{\partial^2}{\partial \xi^2} u(t,\xi) \\ &+ a_1(t) x(t-r,\xi) + a_2(t) p(t, x(t-r,\xi)) + q(t,\xi), \end{aligned} \tag{15}$$

$$u(t,0) = u(t,\pi) = 0, \qquad t \geq 0, \tag{16}$$

$$u(\tau,\xi) = \varphi(\tau,\xi), \qquad \tau \in [-r,0], \ 0 \leq \xi \leq \pi. \tag{17}$$

where

(i) The function $b(\cdot)$ is measurable and

$$\sup_{t \in [-r,\infty)} \int_{0}^{\pi} \int_{0}^{\pi} b^2(t,\eta,\xi) d\eta d\xi < \infty.$$

(ii) The function $\frac{\partial}{\partial \zeta} b(\tau,\eta,\zeta)$ is measurable; $b(\tau,\eta,\pi) = 0$; $b(\tau,\eta,0) = 0$ and

$$N_1 := \int_{0}^{\pi} \int_{-r}^{0} \int_{0}^{\pi} \left(\frac{\partial}{\partial \zeta} b(\tau,\eta,\zeta) \right)^2 d\eta d\tau d\zeta < 1.$$

(iii) The functions $p, q : \mathbb{R}^2 \to \mathbb{R}$ are continuous and ω-periodic in the first variable.

(iv) The substitution operators $g, f : \mathbb{R} \times X \to X$ defined by $g(t,x)(\xi) = p(t,x,\xi)$ and $f(t,x)(\xi) = q(t,x,\xi)$ are ω-periodic, continuous and there exist $k > 0$ such that

$$\| g(t,x) \| \leq k \| x \|, \qquad (t,x) \in \mathbb{R} \times X.$$

(v) The functions $a_i : \mathbb{R} \to \mathbb{R}$ are ω-periodic, continuous and there exist constants l, μ such that

$$-\mu + \| a_1(t) \| + \| a_2(t) \| k \leq -l, \quad t \geq 0.$$

On the space $\mathbb{R} \times \mathcal{C}$ we define the maps G, F, F_1 by

$$\begin{aligned} G(t,\psi)(\xi) &:= \int_{-r}^{0}\int_0^{\pi} b(s,\eta,\xi)\psi(s,\eta)d\eta ds \\ L(t,\psi)(\xi) &:= a_1(t)\psi(-r)(\xi), \\ F_1(t,\psi)(\xi) &:= a_2(t)g(t,\psi(-r))(\xi) + f(t)(\xi) \end{aligned}$$

With the previous notations, the initial boundary value problem (15)-(17) can be modeled as the abstract Cauchy problem

$$\frac{d}{dt}(x(t) + G(t,x_t)) = Ax(t) + L(t,x_t) + F_1(t,x_t), \qquad t \geq 0. \tag{18}$$

$$x_0 = \varphi \in \mathcal{B}. \tag{19}$$

A straithforward estimation using **(i)**-**(iv)** shows that G, L and F_1 are continuous functions. Moreover, from **(d)** in the first part of this section and **(ii)** follow that G is a $X_{\frac{1}{2}}$-valued bounded linear operator and that $\| (-A)^{1/2}G(t,\cdot) \| \leq N_1$ for every $t \in \mathbb{R}$.

Next we prove that G satisfies $\mathbf{H_6}$. Let $R > 0$ and $x \in C([-r,T];X)$ such that $\| x \|_{-r,T} \leq R$. For $t > 0$ we get

$$\begin{aligned} &G(t+h, x_{t+h})(\xi) - G(t,x_t)(\xi) \\ &\quad = \int_{-r+t+h}^{-r+t}\int_0^{\pi} b(\theta - t - h,\eta,\xi)x(\theta,\eta)d\eta d\theta \\ &\quad\quad + \int_{-r+t}^{t}\int_0^{\pi}(b(\theta - t - h,\eta,\xi) - b(\theta - t,\eta,\xi))x(\theta,\eta)d\eta d\theta \\ &\quad\quad + \int_t^{t+h}\int_0^{\pi} b(\theta - t - h,\eta,\xi)x(\theta,\eta)d\eta d\theta \end{aligned}$$

and hence

$$\begin{aligned} \left| \frac{d}{dt}G(t,x_t) \right|_X^2 &\leq 4 \| x(-r+t) \|^2 \int_0^{\pi}\int_0^{\pi} b^2(-r,\eta,\xi)^2 d\eta d\xi \\ &\quad + 4 \| x \|_\infty^2 \, r \int_0^{\pi}\int_{-r+t}^{t}\int_0^{\pi}\left(\frac{\partial b}{\partial \theta}(\theta - t,\eta,\xi)\right)^2 d\eta d\theta d\xi \\ &\quad + 4 \| x \|_\infty^2 \int_0^{\pi}\int_0^{\pi} b^2(0,\eta,\xi)d\eta d\xi. \end{aligned}$$

Thus,

$$\left| \frac{d}{dt}G(t,x_t) \right|_X^2 \leq C(\rho, \frac{\partial b}{\partial \theta}, b), \tag{20}$$

where $C(\rho, \frac{\partial b}{\partial \theta}, b) > 0$ is independent of $t > 0$ and x. This implies that the set

$$\mathcal{U} = \{ s \to G(s,x_s) : x \in C([-r,T];X), \ \sup_{\theta \in [-r,T]} \| x(\theta) \| \leq R \}$$

is equicontinuous from the right side at $t > 0$. The equicontinuity of $\mathcal{U}$ on $\mathbb{R}$ is proved in similar form. Thus condition $\mathbf{H_6}$ hold.

Proposition 1 *Assume that the previous condition hold. If* $\rho = 1 + \frac{\|f\|_\infty}{l}$ *and*

$$\frac{l}{\mu}(1 - e^{-\mu t_0}) > N_1\rho(2 + C_{\frac{1}{2}}(\frac{1}{\mu} + 2)), \tag{21}$$

then there exist an ω*-periodic solution of (15)-(17).*

Proof: Let $v \in B_\rho[0, CP]$ and $\varphi \in B_\rho[0, \mathcal{C}]$. We affirm that the mild solution, $x(\cdot, \varphi)$, of

$$\begin{aligned} \frac{d}{dt}(x(t) + G(t, x_t)) &= Ax(t) + L(t, x_t) + F_1(t, v_t), \qquad t \geq 0, \\ u_0 &= \varphi, \end{aligned}$$

is bounded by ρ on $[0, a_\varphi)$, where $[0, a_\varphi)$ is the maximal interval of definition of $x(\cdot, \varphi)$. Assume that the affirmation is false and let $t_0 = \inf\{t > 0 : \| u(t) \| > \rho\}$. Clearly, $u(t_0) = \rho$. If $t_0 > 1$, by employing the estimates in Ezzinbi [1], pp. 227, we have that

$$\begin{aligned} \| x(t_0) \| &\leq \rho - \frac{l}{\mu}(1 - e^{-\mu t_0}) + e^{-\mu t_0} \| G(0, \varphi) \| + N_1 \| x_{t_0} \|_{\mathcal{C}} \\ &\quad + C_{\frac{1}{2}} N_1 \int_0^{t_0 - 1} e^{-\mu(t_0 - s)} \| x_s \|_{\mathcal{C}}\, ds \\ &\quad + C_{\frac{1}{2}} N_1 \int_{t_0 - 1}^{t_0} \frac{e^{-\mu(t_0 - s)}}{(t_0 - s)^{\frac{1}{2}}} \| x_s \|_{\mathcal{C}}\, ds \\ &\leq \rho - \frac{l}{\mu}(1 - e^{-\mu t_0}) + e^{-\mu t_0} N_1 \rho + N_1 \rho + C_{\frac{1}{2}} N_1 \rho(\frac{1}{\mu} + 2), \end{aligned}$$

thus,

$$\| x(t_0) \| \leq \rho - \frac{l}{\mu}(1 - e^{-\mu t_0}) + N_1\rho(2 + C_{\frac{1}{2}}(\frac{1}{\mu} + 2)). \tag{22}$$

Similarly, if $t_0 \in (0, 1]$

$$\| x(t_0) \| \leq \rho - \frac{l}{\mu}(1 - e^{-\mu t_0}) + N_1\rho(2 + 2C_{\frac{1}{2}}). \tag{23}$$

From (21), (22) and (23) follows $\| u(t_0) \| < \rho$, which is a absurd.

Now we prove that $a_\varphi = \infty$. Assume that $a_\varphi < \infty$ and let N be the number $N = \rho(| a_1 |_\infty + | a_2 |_\infty k) + \| f \|_\infty$. For $\epsilon > 0$ we fix $0 < \delta < \frac{a_\varphi}{2}$ such that

$$\| T(s)x - T(s')x \| < \epsilon, \qquad x \in T(\epsilon) B_{\rho + N}(0, X),$$

when $| s - s' | < \delta$ and $s, s' \in [0, a_\varphi]$. Let $\frac{a_\varphi}{2} < t < t_0 < a_\varphi$. Using the estimate in step 2 in the proof of Theorem 5, we have

$$\begin{aligned} \| x(t_0) - x(t) \| &\leq \epsilon + \| G(t_0, x_{t_0}) - G(t, x_t) \| + 2\epsilon C_{\frac{1}{2}} a_\varphi^{\frac{1}{2}} + 2\tilde{M} C_{\frac{1}{2}} N_1 \rho \epsilon^{\frac{1}{2}} \\ &\quad + 2N_1 \rho C_{\frac{1}{2}} \delta^{\frac{1}{2}} + \epsilon \tilde{M} a_\varphi + 2\tilde{M} N \epsilon + \tilde{M} N \delta \end{aligned}$$

which from (20) implies that x is uniformly continuous on $[\frac{a_\varphi}{2}, a_\varphi]$, since the constants c_i are independent of $t, t_0 \in [\frac{a_\varphi}{2}, a_\varphi]$. Let $\tilde{x} : [-r, a_\varphi] \to X$ the unique continuous extension of x. From Theorem 4, there exist a mild solution $y(\cdot)$ of (18) with initial condition $y_{a_\varphi} = \tilde{x}_{a_\varphi}$. This solution give a extension of $x(\cdot, \varphi)$, which is a contradiction. Thus $x(\cdot, \varphi)$ is defined on $\mathbb{R}$.

These remarks and Theorem 6, implies that for every $v \in B_\rho[0, CP]$ there exist a ω-periodic, $u(\cdot, v)$, of (18) and that $u(\cdot, v) \in B_\rho[0, CP]$. Finally, the existence of an ω-periodic solution of (18) follows from Theorem 7. The proof is complete.

4.2 A neutral equation with unbounded delay

Next we consider the following boundary value problem

$$\begin{aligned}\frac{\partial}{\partial t}[u(t,\xi)+\int_{-\infty}^{t}\int_{0}^{\pi}b(s-t,\eta,\xi)u(s,\eta)d\eta ds] &= \frac{\partial^2}{\partial \xi^2}u(t,\xi)+a_0(\xi)u(t,\xi)\\ &+\int_{-\infty}^{t}a(s-t)u(s,\xi)\,ds+a_1(t,\xi) \qquad (24)\end{aligned}$$

$$\begin{aligned}u(t,0) &= u(t,\pi)=0, && t\geq 0, && (25)\\ u(\tau,\xi) &= \varphi(\tau,\xi), && \tau\leq 0,\ \ 0\leq\xi\leq\pi, && (26)\end{aligned}$$

which is studied by Hernández and Henriquez in [6].

Let $\mathcal{B}:=C_r\times L^p(g;X)$, $r\geq 0$, $1\leq p<\infty$ be the phase space studied in Example 1. In this case $H=1$; $M(t)=\gamma(-t)^{\frac{1}{2}}$ and $K(t)=1+\left(\int_{-t}^{0}g(\tau)d\tau\right)^{\frac{1}{2}}$ for all $t\geq 0$. Assuming the conditions (i) – (iii) of Example 3.1 in [6], this problem can be modeled in the form

$$\begin{aligned}\frac{d}{dt}(x(t)+G(t,x_t)) &= Ax(t)+F(t,x_t)+f(t),\\ x_0 &= \varphi\in\mathcal{B}.\end{aligned}$$

where

$$\begin{aligned}G(t,\psi)(\xi) &:= \int_{-\infty}^{0}\int_{0}^{\pi}b(s,\,\eta,\,\xi)\psi(s,\,\eta)d\eta ds,\\ F(t,\psi)(\xi) &:= a_0(\xi)\psi(0,\xi)+\int_{-\infty}^{0}a(s)\psi(s,\xi)ds,\\ f(t) &:= a_1(t,\cdot).\end{aligned}$$

Moreover, the range of G is contained in $X_{\frac{1}{2}}$, $\|(-A)^{1/2}G(t,\cdot)\|\leq N_1$ and $\|F(t,\cdot)\|\leq N_3$ where

$$\begin{aligned}N_1 &:= \int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\frac{1}{g(s)}\left(\frac{\partial}{\partial\zeta}b(s,\eta,\zeta)\right)^2 d\eta ds d\zeta,\\ N_3 &:= \max\{\|a_0\|_\infty,\left(\int_{-\infty}^{0}\frac{a^2(\theta)}{g(\theta)}d\theta\right)^2\}.\end{aligned}$$

Theorem 10 *Assume that the function $g(\cdot)$ satisfies the following conditions*

$\mathrm{g_i}$) $\ln(g)$ *is uniformly continuous.*

$\mathrm{g_{ii}}$) $k_1=\int_{-\infty}^{0}g(\theta)e^{-\rho\theta}d\theta<\infty$,

$\mathrm{g_{iii}}$) $\gamma(t)\to 0$ *as* $t\to\infty$.

and that

$$K\left[\|(-A)^{-\beta}\|N_1+\frac{1}{\rho}\tilde{M}C_\beta N_1e^{-\rho}+C_\beta N_1\chi(\rho)+\frac{1}{\rho}\tilde{M}N_3\right]<1$$

where $\chi(\rho):=\frac{1}{\rho^\beta}\int_0^\rho s^{\beta-1}e^{-s}ds$. *Then there exist an ω-periodic solution of (24).*

Proof: Using similar estimates that those in the subsection 4.1, follows that condition $\mathbf{H_7}$ hold. From Lemma 3.1 and Proposition 3.4 in [6], each mild solution of (24) is bounded on $[0, \infty)$. Now the existence of an ω-periodic solution for (24)-(25)-(26) is consequence of Theorem 8. The proof is finished.

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