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Existence of Solutions of the Functional Second Order Abstract Cauchy Problem with Nonlocal Conditions

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Existência de Soluções para um Problema de Cauchy Abstrato com Condições não Locais Associado a uma Equação Funcional de Segunda Ordem.

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Resumo

Neste artigo mostramos a existência de soluções fracas, fortes e clássicas para um problema de Cauchy não local de segunda ordem modelado na forma

$$\begin{aligned} x''(t) &= Ax(t) + f(t, x(t), x(a(t)), x'(t), x'(b(t))), \quad t \in [a, b], \\ x(0) &+ p(x, x') = x_0, \\ x'(0) &+ q(x, x') = x_1, \end{aligned}$$

onde A é o gerador infinitesimal de uma função coseno fortemente continua de operadores lineares num espaço Banach X; $q(\cdot), p(\cdot) : C(I; X) \to X$ são apropriadas e $a(\cdot), b(\cdot)$ são funções continuas não decrescentes.

Existence of Solutions of the Functional Second Order Abstract Cauchy Problem with Nonlocal Conditions

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Abstract

By using the theory of strongly continuous cosine functions, we study the existence of mild, strong and classical solutions for a functional second order abstract Cauchy problem with nonlocal conditions.

1 Introduction.

The purpose of this work is to study the existence of mild, strong and classical solutions of a class of functional second order differential equations modelled in the form

$$x''(t) = Ax(t) + f(t, x(t), x(a(t)), x'(t), x'(b(t))), \quad t \in I = [0, T], \tag{1.1}$$

$$x(0) + p(x, x') = x_0, \tag{1.2}$$

$$x'(0) + q(x, x') = x_1, \tag{1.3}$$

where A is the infinitesimal generator of a strongly cosine family of bounded linear operators $(C(t))_{t\in R}$ on a Banach space X; $a(\cdot), b(\cdot) : I \to I$; $p(\cdot), q(\cdot) : C(I; X) \times C(I; X) \to X$ and $f : I \times X^4 \to X$ are appropriate functions.

In general, the nonlocal conditions can be applied in Physics with better effect that the classical initial conditions $x(0) = x_0$. For the importance of nonlocal conditions in different fields, we refer to [2, 3, 4, 5, 6] and the references contained therein. The study of nonlocal initial value problems was initiated by Byszewski in [2, 6]. In [2], Byszewski prove the existence of mild, strong and classical solutions of the semilinear nonlocal Cauchy problem

$$\begin{aligned} x'(t) &= Ax(t) + f(t, x(t)), \qquad t \in I = [0, T] \\ x(0) &= x_0 + q(t_1, t_2, t_3, ..., t_n, u(\cdot)) \in X, \end{aligned} \tag{1.4}$$

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where A is the infinitesimal generator of a strongly continuous semigroup of linear operators on X; $0 < t_0 < t_1 < .... < t_n \leq T$; $f : [0,T] \times X \to X$, $q(\cdot,\cdot) : I^n \times C(I : X) \to X$ are appropriated functions and the symbol $q(t_1, t_2, t_3, ..., t_n, u(\cdot))$ is used in the sense that "$\cdot$" can be substitute only for the points t_i, for instance $q(t_1, t_2, t_3, ..., t_n, u(\cdot)) = \sum_{i=1}^{n} \alpha_i u(t_i)$.

On the other hand, some second order nonlocal initial value problems are studied by Ntouyas & Tsamatos in [15, 16, 17]. In these works the authors discuss the existence of solutions for a type of second order delay integrodifferential equations with nonlocal conditions described in the form

$$x''(t) = Ax(t) + f(t, x(\sigma_1(t)), \int_0^t k(t-s)h(s, x(\sigma_2(s)), x(\sigma_3'(s)))ds, x(\sigma_4(t))), \quad t \in (0,T),$$

$$x(0) = g(x) + x_0, \quad x'(0) = \eta,$$

where A is the generator of a strongly continuous cosine family, $(C(t))_{\in R}$, of bounded linear operators on X; $x_0, \eta \in X$ and $g : C(I : X) \to X$, $f, h : I \times X^3 \to X$ are appropriated functions. In general the result are obtained using the Leray-Schauder Alternative and the strong assumption that the cosine family $C(t)$ is compact for every $t > 0$, which in turn implies that $dim(X) < \infty$.

Throughout this paper X will be a Banach space endowed with a norm $\|\cdot\|$ and $C(t)$ denotes a strongly continuous operator cosine function defined on X with infinitesimal generator A. We refer the reader to [1, 19] for the necessary concepts about cosine functions. Next we only mention a few results and notations needed to establish our results. We denote by $S(t)$ the sine function associated with $C(t)$ which is defined by

$$S(t)x := \int_0^t C(s)x\, ds, \quad x \in X, \quad t \in I\!R. \tag{1.5}$$

For a closed operator $B : D(B) \to X$ we denote by $[D(B)]$ the space $D(B)$ endowed with the graph norm $\|\cdot\|_B$. In particular $[D(A)]$ is the space $D(A)$ endowed with the norm

$$\|x\|_A = \|x\| + \|Ax\|, \quad x \in D(A).$$

Moreover, in this work the notation E stands for the space formed by the vectors $x \in X$ for which the function $C(\cdot)x$ is of class C^1. It was proved by Kisińsky ([13]) that E endowed with the norm

$$\|x\|_1 = \|x\| + \sup_{0 \leq t \leq 1} \|AS(t)x\|, \quad x \in E,$$

is a Banach space. The operator valued function $G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is a strongly continuous group of linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. From this it follows that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0,\ t \to 0,$ for each $x \in E$. Furthermore, if $x : [0,\infty) \to X$ is a locally integrable function then

$$y(t) = \int_0^t S(t-s)x(s)\,ds$$

defines an E-valued continuous function. This is an immediate consequence of the fact that

$$\int_0^t G(t-s)\begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)\,ds \\ \int_0^t C(t-s)x(s)\,ds \end{bmatrix}$$

defines an $E \times X$-valued continuous function.

The existence of solutions of the second order abstract Cauchy problem

$$x''(t) = Ax(t) + h(t), \quad 0 \le t \le a, \tag{1.6}$$

$$x(0) = x_0, \quad x'(0) = x_1, \tag{1.7}$$

where $h : [0,a] \to X$ is an integrable function, has been discussed in [20]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem it has been treated in [21]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s)\,ds, \quad 0 \le t \le a, \tag{1.8}$$

is called a mild solution of (1.6)-(1.7). In the case in which $x_0 \in E$ then $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)x_0 + C(t)x_1 + \int_0^t C(t-s)h(s)\,ds. \tag{1.9}$$

Additional terminology and notations are those generally used in functional analysis. In particular, $\mathcal{L}(X)$ stands for the Banach space of bounded linear operators from X into X and $B_r(x)$ will denote the closed ball with center at x and radius r in an appropriated space. Moreover, for bounded functions $\xi : I \to I\!R$ and $\Lambda : Y \to I\!R$ we will employ the notations ξ_t and N_Λ for $\xi_t = \sup\{\xi(\theta) : \theta \le t\}$ and $N_\Lambda = \sup\{\| \Lambda(y) \| : y \in Y\}$.

The paper is organized as follows. In section 2 we establish existence of mild, strong and classical solutions of some second order nonlocal initial value Cauchy problems. The results are obtained using the ideas and techniques in Henriquez & Vasquez [12], Ntouyas & Tsamatos [15, 16, 17] and the Leray Schauder Alternative. Finally, in section 4 we present an application to the wave equation.

Lemma 1.1 *Let D a convex subset of a Banach space X and assume that $0 \in S$. Let $F : D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda F(x),\ 0 < \lambda < 1\}$ is unbounded or the map F has a fix point in D.*

2 Existence of solutions.

In this section we study existence of solutions of some nonlocal second order functional abstract Cauchy problem. We begin by introducing some notations. We assume that $M \geq 1$ is a constant such that $\|C(t)\| \leq M$ for all $t \in I$. Moreover, we consider the space of continuous functions $C(I;X)$ endowed with the norm of uniform convergence and we abbreviate to $C(I)$ whenever $I = X$.

Initially we consider the following second order initial value value problem

$$x''(t) = Ax(t) + f(t, x(t), x(a(t))), \quad 0 \leq t \leq t_1, \tag{2.10}$$

$$x(0) + p(x) = x_0, \tag{2.11}$$

$$x'(0) + q(x) = x_1, \tag{2.12}$$

where $x_0, x_1 \in X$.

To treat this initial value problem we always assume that the following general assumptions hold:

Assumption A

(i) The function $a : I \to I$ is continuous.

(ii) The function $f : I \times X \times X \to X$ satisfies the following Caratheodory conditions:

(a) $f(t, \cdot) : X \times X \to X$ is continuous a.e. $t \in I$;

(b) For each $x, y \in X$, the function $f(\cdot, x, y) : I \to X$ is strongly measurable.

(iii) The maps p, q are continuous.

Definition 2.1 *We say that a function $x(\cdot) : I \to X$ is a mild solution of the problem (2.10)-(2.11)-(2.12) if $x(\cdot)$ is continuous and satisfies the integral equation*

$$x(t) = C(t)(x_0 - p(x)) + S(t)(x_1 - q(x)) + \int_0^t S(t-s) f(s, x(s), x(a(s)))\, ds, \quad t \in I,$$

A cosine function defined on an infinite dimensional Banach space can be not compact. However the sine function, especially those that arise with practical situations, are frequently compact, see [20]. This facts leads naturally to include that property in the statements of our results. For the theory of condensing maps we refer to [10]. In what follows we denote by B_r the closed ball in $C(I;X)$ with center at 0 and radius r and, for a given $x \in C(I;X)$ we abbreviate the notations by writing $u(s) = (x(s), x(a(s)))$.

Theorem 2.1 *Assume that Assumption* **A** *and the following conditions hold:*

(H-1) *The functions* $p(\cdot)$ *and* $q(\cdot)$ *take bounded sets into bounded sets. For* $r > 0$, *let* $\alpha_r = \sup\{\| p(x) \| : x \in B_r\}$ *and* $\beta_r = \sup\{\| q(x) \| : x \in B_r\}$.

(H-2) *The map* $C(I;X) \to C(I;X)$, $x \to C(\cdot)p(x)$ *is condensing.*

(H-3) *The map* $C(I;X) \to C(I;X)$, $x \to S(\cdot)q(x)$ *is completely continuous.*

(H-4) *For each* $r \geq 0$ *there is a positive integrable function* $\gamma_r \in \mathcal{L}^1(I)$ *such that* $\sup\{\|f(t,x,y)\| : x, y \in B_r[0,X]\} \leq \gamma_r(t)$ *a.e.* $t \in [0,T]$.

(H-5) *For each* $t \in I$ *and* $r > 0$ *the set* $\{S(t)f(s,x,y) : 0 \leq s \leq T, \|x\|, \|y\| \leq r\}$ *is relatively compact.*

(H-6) $\liminf_{k\to\infty} \frac{M}{k}\left(\alpha_k + T\beta_k + \int_0^T (T-s)\,\gamma_k(s)\,ds\right) < 1.$

Then there is a mild solution of (2.10)-(2.11)-(2.12). In further, if the following conditions is fulfilled:

(H-6') $\limsup_{k\to\infty} \frac{M}{k}\left(\alpha_k + T\beta_k + \int_0^T (T-s)\,\gamma_k(s)\,ds\right) < 1,$

then the set $\mathcal{S}$ *formed by the mild solutions of (2.10)-(2.11)-(2.12) is compact in* $C(I;X)$

Proof. Let $\mathcal{T} : C(I;X) \to C(I;X)$ be the map defined by

$$\mathcal{T}(x)(t) = C(t)(x_0 - p(x)) \; + \; S(t)(x_1 - q(x)) \; + \; \int_0^t S(t-s)f(s,x(s),x(a(s)))\,ds. \quad (2.13)$$

Clearly Γ is well defined and the Lebesgue's dominated convergence Theorem implies that is continuous. We affirm that there exist $n \in I\!N$ such that $\mathcal{T} : B_n \to B_n$. In fact, if we assume that the assertion is false we can to select a sequence $(x_k)_{k\in N}$ in $C(I:X)$ such that $\| x_k \|_\infty \leq k$ and $\| \mathcal{T}(x_k) \|_\infty > k$ for every $k \in I\!N$. Consequently, for $k \in I\!N$ we see that

$$k \leq \|\mathcal{T}(x_k)\|_\infty \leq M(\|x_0\| + \alpha_k) + MT(\|x_1\| + \beta_k) + M\int_0^T (T-s)\gamma_k(s)ds$$

which yields

$$1 \leq \liminf_{k\to\infty} \frac{M}{k}\left(\alpha_k + T\beta_k + \int_0^T (T-s)\gamma_k(s)\,ds\right)$$

and this inequality contradicts **(H-6)**.

Now, we introduce the decomposition $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ where $\mathcal{T}_i : C(I;X) \to C(I;X)$ are defined by

$$\begin{aligned} \mathcal{T}_1(x) &= C(\cdot)(x_0 - p(x)), \\ \mathcal{T}_2(x)(t) &= S(t)(x_1 - q(x)) \; + \; \int_0^t S(t-s)f(s,x(s),x(a(s)))\,ds, \quad t \in I. \end{aligned}$$

It's clear that $\mathcal{T}_1$, $\mathcal{T}_2$ are continuous maps. Moreover, the hypothesis **H-2** implies that $\mathcal{T}_1$ is condensing. Next, we prove that $\mathcal{T}_2$ is completely continuous. Let us define

$$z(t) = \int_0^t S(t-s)f(s,x(s),x(a(s))ds.$$

for $x \in C(I;X)$. Applying **H-3**, it remains to prove that the set $\{z(\cdot) :\| x \|_\infty \leq r\}$ is relatively compact in $C(I;X)$, for every $r > 0$. We begin by establishing that this set is equicontinous. For this purpose, we fix $t \geq 0$ and $t \geq 0$ such that $t+h \leq T$. Since

$$z(t+h) - z(t) = \int_0^t [S(t+h-s) - S(t-s)]f(s,u(s))ds + \int_t^{t+h} S(t+h-s)f(s,u(s))ds$$

from **H-4**we obtain the elementary estimation

$$\| z(t+h) - z(t) \| = Mh \int_0^t \gamma_r(s)ds + MT \int_t^{t+h} \gamma_r(s)ds$$

which yields the assertion. Next we show that for every $t \in [0,T]$ the set $\{z(t) :\| x \|_\infty \leq r\}$ is relatively compact in X. Since $S(\cdot)$ is uniformly continuous on I, given $\epsilon > 0$, let $\delta > 0$ be such that $\| S(t_1) - S(t_2) \| \leq \epsilon$ when $| t_1 - t_2 | \leq \delta$. We select $0 = s_0 < s_1 < s_1 .... < s_k = t$ so that $| s_i - s_{i-1} | \leq \delta$. We can write

$$\begin{aligned} z(t) &= \sum_{i=1}^k \int_{s_{i-1}}^{s_i} S(t-s)f(s,u(s))ds \\ &= \sum_{i=1}^k \int_{s_{i-1}}^{s_i} [S(t-s) - S(t-s_i)]f(s,u(s))ds + \sum_{i=1}^k \int_{s_{i-1}}^{s_i} S(t-s_i)f(s,u(s))ds \end{aligned}$$

Applying **H-5** we obtain easily that the second term of the right side of the above expression is included in a compacts set. Since we can estimate the first term as

$$\begin{aligned} \| \sum_{i=1}^k \int_{s_{i-1}}^{s_i} [S(t-s) - S(t-s_i)]f(s,u(s))ds \| &\leq \epsilon \sum_{i=1}^k \int_{s_{i-1}}^{s_i} \gamma_r(s)ds \\ &\leq \epsilon \int_0^T \gamma_r(s)ds \end{aligned}$$

and ϵ was arbitrary chosen the assertion follows. Hence we obtain that $\mathcal{T}$ is a condensing map and combining this result with the first part of this proof and applying the fixed point Theorem of Sadovskii we infer that $\mathcal{T}$ has a fixed point $x \in B_n$. Clearly x is a mild solution of (2.10)-(2.11)-(2.12).

On the other hand, the continuity of $\mathcal{T}$ implies that $\mathcal{S}$ is closed. Moreover, proceeding as at the beginning of the proof, but using **H-6'** instead **H-6** we can see that $\mathcal{S}$ is bounded. In fact, if we assume that $\mathcal{S}$ is not bounded there is a sequence of functions $x_k \in \mathcal{S}$ such that $r_k = \| x_k \|_\infty \geq k$. Hence obtain

$$\begin{aligned} \| x_k(t) \| &= \| \mathcal{T} x_k(t) \| \\ &\leq M(\|x_0\| + \alpha_{r_k}) + MT(\|x_1\| + \beta_{r_k}) + M \int_0^T (T-s)\gamma_{r_k}(s)ds \end{aligned}$$

which yields

$$1 \leq \limsup_{k\to\infty} \frac{M}{r_k}\left(\alpha_{r_k} + T\beta_{r_k} + \int_0^T (T-s)\gamma_{r_k}(s)\,ds\right)$$

and this inequality contradicts **H-6'**. Finally, using that $\mathcal{T}$ is condensing we infer that $\mathcal{S}$ is compact. Thus the proof is complete.

For those operators f which are a perturbation of a linear operator we can to establish the following modifications of the precedent result.

Theorem 2.2 *Assume that conditions* **H-1**, **H-3** *and* **H-4** *are fulfilled, that $a(t) \leq t$, $t \in I$, and that p is completely continuous. Suppose furthermore that there are functions $g, l : I \times X \to X$ such that $f = g + l$ and that*

(a) *g, l satisfy the Caratheodory, conditions (ii) of Assumption* **A**;

(b) *$l(t,\cdot) : X^2 \to X$ is linear and the function $L(t) = \parallel l(t,\cdot) \parallel$ is integrable on I;*

(c) *For each $t \in I$ and $r > 0$ the set $\{S(t)g(s,x,y) : 0 \leq s \leq T, \|x\|, \|y\| \leq r\}$ is relatively compact.*

If **H-6** *hold then there is a mild solution of (2.10)-(2.11)-(2.12). In further, if* **H-6'** *hold then the set formed by the mild solutions of (2.10)-(2.11)-(2.12) is compact in C(I;X).*

Proof. We define the map $\tilde{\mathcal{T}} = \mathcal{T} + \mathcal{T}_0$ where $\mathcal{T}$ is the map defined by (2.13) with g instead f and $\mathcal{T}_0$ is given by

$$\mathcal{T}_0 x(t) = \int_0^t S(t-s)l(s, x(s), x(a(s)))ds.$$

It's clear from the proof of Theorem (2.1) that $\mathcal{T}$ is completely continuous. Moreover, $\mathcal{T}_0$ is bounded linear operator and

$$\begin{aligned}
\parallel \mathcal{T}_0 x(t) \parallel &\leq M\int_0^t (t-s)L(s)(\parallel x(s) \parallel + \parallel x(a(s)) \parallel\, ds \\
&\leq 2M\int_0^t (t-s)L(s) \max_{0\leq\xi\leq s} \parallel x(\xi) \parallel\, ds.
\end{aligned}$$

This estimation implies that $\mathcal{T}_0^m$ is a contraction for m large enough. Consequently, $\tilde{\mathcal{T}}^m$ is also a condensing map and the proof of the assertion is obtained arguing as in the proof of Theorem (2.1).

When p or q satisfies a Lipschitz condition we can establish a slightly different result. To state our following result we introduce the conditions:

(H-7) The function p satisfies $\|p(x) - p(y)\| \leq N_p\|x - y\|_\infty$, $x, y \in C(I;X)$.

(H-8) The function q satisfies $\|q(x) - q(y)\| \leq N_q\|x - y\|_\infty$, $x, y \in C(I;X)$.

Theorem 2.3 *Assume assumption* **A** *and conditions H-4, H-5 hold. Suppose also that at least one of the following conditions hold:*

(i) *Conditions H-1, H-3, H-7 are satisfied and*

$$MN_p + \liminf_{k\to\infty} \frac{M}{k}\left(T\beta_k + \int_0^T (T-s)\,\gamma_k(s)\,ds\right) < 1; \tag{2.14}$$

(ii) *Conditions H-7, H-8 are satisfied and*

$$MN_p + MTN_q + \liminf_{k\to\infty} \frac{M}{k}\int_0^T (T-s)\,\gamma_k(s)\,ds < 1, \tag{2.15}$$

then there exist a mild solution of (2.10)-(2.11)-(2.12).

Proof. We prove the statement in the case (i). Let $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ as in the proof of Theorem (2.1). Using condition (2.14) we can to select $n \in \mathbb{N}$ sufficiently large such that

$$MN_p n + M \parallel x_0 - p(0) \parallel + MT(\parallel x_1 \parallel + \beta_n) + M\int_0^T (T-s)\gamma_n(s)ds \leq n.$$

We affirm that the map $\mathcal{T} : B_n \to B_n$. In fact, for $x \in B_n$ and $t \in I$ we have that

$$\begin{aligned}
\|\mathcal{T}(x)(t)\| &\leq \|C(t)(x_0 - p(x)) + C(t)(x_0 - p(0))\| + \parallel S(t)(x_1 - q(x)) \parallel \\
&\quad + \int_0^t \parallel S(t-s)f(s, x(s), x(a(s))) \parallel ds \\
&\leq MN_p n + M \parallel x_0 - p(0) \parallel + MT(\|x_1\| + \beta_n) + M\int_0^T (T-s)\gamma_n(s)\,ds \\
&\leq n,
\end{aligned}$$

which yields the assertion. Moreover, it is immediate to see that $\mathcal{T}_1$ is a contraction, and proceeding as in the proof of Theorem (2.1), it follows that $\mathcal{T}_2$ is completely continuous. Thus, $\mathcal{T}$ is a condensing map and the Sadovskii Theorem implies that $\mathcal{T}$ has a fixed point in B_n.

The proof when condition (ii) holds is similar, except that in this case we modified the definition of $\mathcal{T}_1$ and $\mathcal{T}_2$ as

$$\begin{aligned}
\mathcal{T}_1(x)(t) &= C(t)(x_0 - p(x)) + S(t)(x_1 - q(x)), \\
\mathcal{T}_2(x)(t) &= \int_0^t S(t-s)f(s, x(s), x(a(s)))ds \quad t \in I.
\end{aligned}$$

We omit details. ■

When we can assure that the set $\mathcal{S}$ formed by the mild solutions is bounded we infer that it's compact. Thus, the following result can be proved with the arguments employed in the proof of Theorem 2.1. By the sake of brevity we omit this proof.

Corollary 2.1 *Assume that assumption* **A** *and conditions H-4, H-5 hold. Suppose further that at least one of the following conditions hold.*

(i) *Conditions H-1, H-3, H-7 are satisfied and*

$$MN_p + \limsup_{r\to\infty} \frac{M}{r}\left(T\beta_r + \int_0^T (T-s)\gamma_r(s)ds\right) < 1;$$

(ii) *Conditions H-7, H-8 and*

$$MN_p + MTN_q \limsup_{r\to\infty} \frac{M}{r}\int_0^T (T-s)\gamma_r(s)ds < 1;$$

then the set $\mathcal{S}$ is compact.

In most of situations of practical interest the sine function $S(t)$ is compact ([20]). For this reason we establish in separate form the following result.

Corollary 2.2 *Assume that assumption* **A** *holds and that the operator $S(t)$ is compact for every $t \in I\!R$. In further, if one of the following conditions is fulfilled*

(i) The map p is completely continuous and H-1, H-4 and H-6 hold;

(ii) Conditions H-1, H-4, H-7 and (2.14) hold;

(iii) Conditions H-4, H-7, H-8 and (2.15) hold,

then there is a mild solution of (2.10)-(2.11)-(2.12).

Proof. The compactness of $S(t)$ implies easily that condition H-3 is verified. Furthermore, the operator $\mathcal{T}_0 : C(I;X) \to C(I;X)$ defined by

$$\mathcal{T}_0 x(t) = \int_0^t S(t-s)f(s,x(s),x(a(s)))ds,$$

is completely continuous. The argument used in the proof of Theorem 2.1 to prove the same property for $\mathcal{T}_2$ permit us to affirm that it's sufficient to show that the set $\{\mathcal{T}_0 x(t) :\| x \|_\infty \leq r\}$, $r > 0$, is relatively compact in X for $t \in I$. In order to establish this fact we point out previously that if we take $\xi > 0$, then we can write $S(n\xi) = S(\xi)P_n$, $n \in I\!N$, where P_n is certain bounded linear operator on X.

Since $S(\cdot)$ is uniformly continuous on I, given $\epsilon > 0$ there is $\delta > 0$ such that $\| S(t_1) - S(t_2) \| \leq \epsilon$ when $| t_1 - t_2 | < \delta$. We select, $n \in I\!N$ such that $\xi = \frac{t}{n} \leq \delta$ and define $s_i = i\xi$, $i = 0, 1, ...n$. Hence we can write

$$\begin{aligned} z(t) &= \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} S(t-s)f(s,u(s))ds \\ &+ \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} [S(t-s) - S(t-s_i)]f(s,u(s))ds - S(\xi)\sum_{i=1}^{n-1} P_i \int_{s_{i-1}}^{s_i} f(s,u(s))ds. \end{aligned}$$

Applying (H-4) we can establish the estimation

$$\| \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} [S(t-s) - S(t-s_i)] f(s,u(s)) ds \| \leq \epsilon \int_0^T \gamma_r(s) ds,$$

$$\| \sum_{i=1}^{n-1} P_i \int_{s_i}^{s_{i+1}} f(s,u(s)) ds \| \leq \max_{i=1,\dots n-1} \| P_i \| \int_0^T \gamma_r(s) ds,$$

which are valid independent of $x(\cdot)$. Since $S(\xi)$ is a compact operator and ϵ was arbitrary chosen the assertion follows.

Finally we complete the proof of the Corollary by observing that if the condition (i) is verified, the statement is consequence of Theorem 2.1, while if conditions (ii) or (iii) hold the assertion is consequence of Theorem 2.3. ■

From the preliminaries we know that if $x_0 - p(x) \in E$ then the mild solution $x(\cdot)$ is continuously differentiable on I and

$$x'(t) = AS(t)(x_0 - p(x)) \ + \ C(t)(x_1 - q(x)) \ + \ \int_0^t C(t-s) f(s, x(s), x(a(s))\, ds. \quad (2.16)$$

Next we study the differentiability of the function $x'(\cdot)$. First we consider the following concept of strong solution.

Definition 2.2 *A function $x(\cdot) : [0,T] \to X$ is a strong solution of (2.10)-(2.11)-(2.12) if $x(\cdot) \in W^{2,1}([0,T];X)$, the equation (2.10) is satisfied a.e. $t \in I$ and the conditions (2.11) and (2.12) are verified.*

In our results we consider Banach spaces that have the Radom Nikodym property, (abbreviated RNP). We refer to [11] for this matter.

Theorem 2.4 *Assume that X has the RNP and that the hypotheses of Theorem 2.1 or 2.2 hold. If in further the following conditions are verified:*

(H-9) *$\mathcal{R}(p) \subseteq x_0 + D(A)$ and $\mathcal{R}(q) \subseteq x_1 + E$;*

(H-10) *The functions $\gamma_k \in \mathcal{L}^\infty(I)$;*

(H-11) *For each bounded set $D \subseteq X$, the functions $C(\cdot)f(t,x,y)$, $t \in I$, $x, y \in D$, are uniformly Lipschitz continuous on I,*

then the mild solution $x(\cdot)$ of (2.10)-(2.11)-(2.12) is a strong solution.

Proof. Let $x(\cdot)$ be a mild solution of (2.10)-(2.11)-(2.12). From (2.16) and **H-9** we find that

$$\begin{aligned} x'(t+s) - x'(t) &= A(S(t+s) - S(t))(x_0 - p(x)) \ + \ (C(t+s) - C(t))(x_1 - q(x)) \\ &+ \int_0^t (C(t+s-\xi) - C(t-\xi)) f(\xi, x(\xi), x(a(\xi)))\, d\xi \\ &+ \int_t^{t+s} C(t+s-\xi) f(\xi, x(\xi), x(a(\xi)))\, d\xi, \end{aligned}$$

which, applying **H-10** and **H-11** implies that $x'(\cdot)$ is Lipschitz continuous. From this and the fact that X has the RNP it follows that $x \in W^{2,1}(I; X)$. The assertion is now consequence of Proposition 3.3 in [12]. ■

Remark. From the results in [13], the condition **H-11** holds if $\mathcal{R}(f) \subseteq E$ and $f : I \times D \times D \to E$ is bounded.

Next we discuss the existence of classical solutions. We begin with the following definition.

Definition 2.3 *We say that a function $x(\cdot) : I \to X$ is a classical solution of the problem (2.10)-(2.11)- (2.12) if $x(\cdot)$ is a function of class C^2 that satisfies the equation (2.10) and the initial conditions (2.11) and (2.12).*

Theorem 2.5 *Assume that the hypotheses of Theorem 2.1 or (2.3) are verified. Suppose, furthermore, that X has the RNP and that the conditions* **H-9** *hold. If in further the following conditions are verified:*

(H-12) *The function $a(\cdot)$ is Lipschitz continuous.*

(H-13) *For each bounded set $D \subseteq X$, the function $f : I \times D \times D \to X$ is Lipschitz continuous,*

then each mild solution $x(\cdot)$ of (2.10)-(2.11)-(2.12) is a classical solution.

Proof. It follows from **H-9** that every mild solution $x(\cdot)$ is a function of C^1. Applying **H-12** and **H-13** it is easy to see that the function $t \to f(t, x(t), x(a(t)))$ is Lipschitz continuous. The assertion is consequence of Theorem 3.1 in [12]. ■

We can omit the condition X to have RNP if the functions $f(\cdot), a(\cdot)$ are differentiable. We establish the following result without proof (see [21]).

Proposition 2.1 *Let Assumption in Theorem 2.1 or 2.3 be satisfied. Suppose, furthermore, that* **H-9** *holds and that the functions $f(\cdot)$, $a(\cdot)$ are continuously differentiable. Then the mild solution $x(\cdot)$ of (2.10)-(2.11)-(2.12) is a classical solution.*

Next we consider briefly the linear inhomogeneous case. Henceforth we consider the space $X \times X$ endowed with the norm

$$\|(x, y)\| = \max\{\|x\|, \|y\|\}.$$

In the next result we assume that $f(t, x, y) = L(x, y) + h(t)$ where $L : X \times X \to X$ is a bounded linear map and $h \in \mathcal{L}^1(I; X)$. Furthermore, p, $q : C(I; X) \to X$ are bounded linear maps which can be represented as the Riemann-Stieltjes integral

$$\begin{aligned} p(x) &= \int_0^T d_s P(s) x(s), \\ q(x) &= \int_0^T d_s Q(s) x(s), \end{aligned}$$

where P, $Q : I \to \mathcal{L}(X)$ have bounded variation. We denote by $V(P)$ and $V(Q)$ the variation of P and Q respectively.

Theorem 2.6 *Assume that the following condition holds*

$$M\left(V(P) + TV(Q) + \frac{1}{2}T^2 \|L\|\right) < 1. \tag{2.17}$$

Then the problem (2.10)-(2.11)-(2.12) has a unique mild solution $x(\cdot)$. Furthermore, if X has the RNP, $h \in \mathcal{L}^\infty(I;X)$ and **H-9** *hold then $x(\cdot)$ is a strong solution. If in addition, $a(\cdot)$ and $h(\cdot)$ are Lipschitz continuous functions then $x(\cdot)$ is a classical solution.*

Proof. The inequality (2.17) implies that the operator $\mathcal{T} : C(I;X) \to C(I;X)$ defined in (2.13) is a contraction. Consequently the existence of a unique mild solution follows from the contraction mapping principle. The other assertions are immediate consequence of Theorem 2.4 and Theorem 2.5, respectively. The proof is fulfilled.

This case shows the importance to distinguish between strong and classical solutions.

In the sequel we consider the initial value problem defined by the equations (1.1)-(1.2)-(1.3). To treat this problem we assume that the following general conditions are satisfied.

Assumption $\mathbf{A_2}$

(a) The functions $a(\cdot)$, $b(\cdot) : I \to I$ are continuous, nondecreasing and $\max\{a(t), b(t)\} \leq t$ for every $t \in I$.

(b) The function $f : I \times X^4 \to X$ satisfies the following Caratheodory conditions:

(i) $f(t, \cdot) : X^4 \to X$ is continuous a.e. $t \in I$;

(ii) For each $x \in X^4$, the function $f(\cdot, x) : I \to X$ is strongly measurable.

(c) There exist a continuous functions $m : I \to [0, \infty)$ and a continuous nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that

$$\|f(t, x_1, x_2, x_3, x_4)\| \leq m(t)W(\sum_{i=1}^{4} \| x_i \|),$$

for every $t \in I$ and every $x_i \in X$.

(d) The functions p, $q : C(I;X)^2 \to X$ are continuous.

Definition 2.4 *A function $x(\cdot) : I \to X$ is a mild solution of the problem (1.1)-(1.2)-(1.3) if: $x(\cdot)$ and $x'(\cdot)$ are continuous, the initial conditions (1.2)-(1.3) are verified and $x(\cdot)$ satisfies the integral equation*

$$\begin{aligned} x(t) &= C(t)(x_0 - p(x,x')) + S(t)(x_1 - q(x,x')) \\ &\quad + \int_0^t S(t-s) f(s, x(s), x(a(s)), x'(s), x'(b(s)))\, ds, \qquad t \in I. \quad (2.18) \end{aligned}$$

It's clear that if x is a function of class C^1 that satisfies (2.18) then $x_0 - p(x,x') \in E$ and x' satisfies the equation

$$\begin{aligned} x'(t) &= AS(t)(x_0 - p(x,x')) + C(t)(x_1 - q(x,x')) \\ &\quad + \int_0^t C(t-s) f(s, x(s), x(a(s)), x'(s), x'(b(s)))\, ds. \end{aligned}$$

Since the operator map $AS(\cdot)$ is strongly continuous with values in $\mathcal{L}(E;X)$ there is a constant $N > 0$ such that $\| AS(t) \| \leq N$ on I. Next we establish our first existence result.

Theorem 2.7 *Assume that assumption* $\mathbf{A_2}$ *and the following conditions hold:*

(H-14) *The functions $p(\cdot)$ and $q(\cdot)$ are completely continuous. For $r > 0$ we denote $\alpha_r = \sup\{\| p(x,y) \| : x, y \in B_r\}$ and $\beta_r = \sup\{\| q(x,y) \| : x, y \in B_r\}$.*

(H-15) *Any of the following conditions hold.*

- (a) *The map $x_0 - p$ is completely continuous with values in E.*
- (b) *$S(t)$ is compact for $t > 0$ and the map $x_0 - p : X \times X \to [D(A)]$ takes closed bounded sets into the bounded sets.*

We employ the notation $e_r = \sup\{\| x_0 - p(x,y) \|_1 : x, y \in B_r\}$.

(H-16) *For each $r > 0$ the set $f(I \times B_r^4\}$ is relatively compact in X. We denote $\gamma_r = \sup\{\| z \| : z \in f(I \times B_r^4\}\}$.*

(H-17) $\liminf_{k \to \infty} \frac{M}{k} \left(\alpha_k + T\beta_k + \frac{1}{2} T^2 \gamma_k \right) < 1$ *and* $\liminf_{k \to \infty} \frac{1}{k} (N e_k + M\beta_k + MT\gamma_k) < 1$.

Then there is a mild solution of (1.1)-(1.2)-(1.3). In further, if the following conditions is fulfilled:

(H-18) $\limsup_{k \to \infty} \frac{M}{k} \left(\alpha_k + T\beta_k + \frac{1}{2} T^2 \gamma_k \right) < 1$ *and* $\limsup_{k \to \infty} \frac{1}{k} (N e_k + M\beta_k + MT\gamma_k) < 1$,

then the set $\mathcal{S}$ formed by the mild solutions of (1.1)-(1.2)-(1.3) is compact in $C^1(I;X)$

Proof. We define the map $\Gamma : C(I : X)^2 \to C(I : X)^2$ by

$$\mathcal{T}(x,y) = (\mathcal{T}^1(x,y), \mathcal{T}^2(x,y)) \tag{2.19}$$

where

$$\begin{aligned}
\mathcal{T}^1(x)(t) &= C(t)(x_0 - p(x,y)) + S(t)(x_1 - q(x,y)) \\
&\quad + \int_0^t S(t-s) f(s, u(s), v(s))\, ds, \qquad (2.20) \\
\mathcal{T}^2(x)(t) &= AS(t)(x_0 - p(x,y)) + C(t)(x_1 - q(x,y)) \\
&\quad + \int_0^t C(t-s) f(s, u(s), v(s)) ds, \qquad (2.21)
\end{aligned}$$

where in order to facilitate the reading of the text, we have abbreviate the notations by writing $u(s) = (x(s)), x(a(s))$ and $u(s) = (y(s), y(b(s)))$.

Now we can to repeat the arguments in the proof of Theorem 2.1 to conclude that $\mathcal{T}$ is a completely continuous map and that there exists $n \in I\!N$ such that $\mathcal{T} : B_n^2 \to B_n^2$. Hence applying the Schauder's Theorem we infer that $\mathcal{T}$ has a fixed point $(x,y) \in B_n^2$. It is clear from (2.20)-(2.21) that $y = x'$, which by (2.18) implies that x is a mild solution of (1.1)-(1.2)-(1.3).

Finally, in similar way we prove that the set $\tilde{\mathcal{S}} = \{(x,x') : x \in \mathcal{S}\}$ is compact in $C(I;X)^2$ which implies that $\mathcal{S}$ is compact in $C^1(I;X)$. The proof is finished.

Remark. Arguing as in the proof of Theorem 2.2 we can generalize this result to include some functions f which are a compact perturbation of a linear operator.

3 Applications

The one dimensional wave equation modeled as a abstract Cauchy problem it has been studied extensively, see for example [22]. In this section we illustrate some the results in this work with the wave equation with nonlocal conditions. Specifically we consider the boundary value problem

$$\begin{aligned}
\frac{\partial}{\partial t} u(t,\xi) &= \frac{\partial^2}{\partial \xi^2} u(t,\xi) + F(t,\xi, w(\xi,t), w(\xi, a(t))), \qquad \xi \in [0,\pi],\ t \in I, \qquad (3.22) \\
w(0,t) &= w(\pi,t) = 0, \qquad t \in [0,T], \qquad (3.23)
\end{aligned}$$

where F from $[0,\pi] \times [0,T] \times I\!R^2$ into $I\!R$ satisfies appropriate conditions. It's well know that this equation can be modelled as an abstract Cauchy problem on the space $X := L^2([0,\pi])$ defining $x(t) = \omega(\cdot, t)$. The operator A is given by $A\varphi := \varphi''$ on the domain

$$D(A) := \{x \in H^2(0,\pi) : x(0) = x(\pi) = 0\}.$$

This operator generated a cosine function C on X. Furthermore, A has discrete spectrum, the eigenvalues are $-n^2$, $n \in I\!N$, with corresponding normalized eigenvectors $z_n(\xi) := \left(\frac{2}{\pi}\right)^{1/2} \sin(n\xi)$ and the following conditions holds

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X.

(b) If $\varphi \in D(A)$ then $A\varphi = -\sum_{n=1}^{\infty} n^2 < \varphi, z_n > z_n$.

(c) For every $\varphi \in X$, $C(t)\varphi = \sum_{n=1}^{\infty} \cos(nt) < \varphi, z_n > z_n$, consequently $\| C(t) \|= 1$.

(d) The associated sine function is given by $S(t)\varphi = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < \varphi, z_n > z_n$. Follows from this expression that $S(t)$ is compact and that $\| S(t) \|= 1$.

(e) If G denotes the group of translations on X defined by $G(t)x(\xi) = \tilde{x}(\xi + t)$, where $\tilde{x}$ is the extension of x with period 2π, then $C(t) = \frac{1}{2}(G(t) + G(-t))$. Hence it follows, see [1], that $A = B^2$ where B is the infinitesimal generator of the group G and $E = \{x \in H^1(0,\pi) : x(0) = x(\pi) = 0\}$.

We assume that F satisfies the following Caratheodory conditions:

(a) $F(\xi, t, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continuous *a.e.* $\xi \in [0,\pi]$, $t \in \mathbb{R}$.

(b) For every $w_1, w_2 \in \mathbb{R}$ the function $F(\cdot, w_1, w_2) : [0,\pi] \times [0,T] \to \mathbb{R}$ is measurable.

(c) There exist a measurable positive function η on $[0,\pi] \times [0,T]$ and a positive constant N such that

$$| F(\xi, t, w_1, w_2) |\leq \eta(t,\xi) + N(| w_1 | + | w_2 |).$$

We assume that $\int_0^T (\int_0^\pi \eta^2(\xi,t)d\xi)^{\frac{1}{2}} dt < \infty$.

With this conditions the substitution operator $f : [0,T] \times X^2 \to X$ defined by

$$f(t,x,y) = F(\xi, t, x(\xi), y(\xi))$$

satisfies assumption **A**.

We consider the problem (3.22)-(3.23) submitted to the following initial conditions:

$$w(\xi, 0) + \int_0^T P(w(\cdot, s))(\xi) d\mu(s) = \varphi(\xi), \qquad \xi \in [0,\pi], \tag{3.24}$$

$$\frac{\partial w(\xi,0)}{\partial t} + \int_0^T Q(w(\cdot, s))(\xi) d\nu(s) = \psi(\xi), \qquad \xi \in [0,\pi], \tag{3.25}$$

where μ, ν are functions of bounded variation. We also assume that $P, Q : X \to X$ are continuous operators, P is completely continuous and Q takes closed bounded sets into bounded sets. We introduced the positive constants $\alpha_{1,r}$ and $\beta_{1,r}$ with the meaning that $\| P(x) \|\leq \alpha_{1,r}$ and $\| Q(x) \|\leq \beta_{1,r}$ when $\| x \|\leq r$. We refer to [14] for examples of operators with these properties. We define $p, q : C(I; X) \to X$ by

$$p(x) = \int_0^T P(x(s)) d\mu(s), \tag{3.26}$$

$$q(x) = \int_0^T Q(x(s)) d\nu(s). \tag{3.27}$$

The maps p, q are continuous and p is completely continuous. Furthermore, for $x \in C(I; X)$ with $\| x \|_\infty \leq r$, from (3.26) it follows that

$$\| p(x) \| \leq \max_{0 \leq s \leq T} \| P(x(s)) \| V(\mu) \leq \alpha_{1,r} V(\mu).$$

Similarly, from (3.27)

$$\| q(x) \| \leq \beta_{1,r} V(\nu).$$

On the other hand, if $x, y \in B_r$ then

$$\begin{aligned}
\| f(t, x, y) \| &\leq \left(\int_0^\pi | F(\xi, t, x(\xi), y(\xi)) |^2 \, d\xi \right)^{\frac{1}{2}} \\
&\leq \left(\int_0^\pi (\eta(\xi, t) + N(| x(\xi) | + | y(\xi) |)^2 \, d\xi \right)^{\frac{1}{2}} \\
&\leq \left(\int_0^\pi \eta^2(\xi, t) d\xi \right)^{\frac{1}{2}} + N(\| x \| + \| y \|) \\
&\leq \left(\int_0^\pi \eta^2(\xi, t) d\xi) \right)^{\frac{1}{2}} + 2Nr. \qquad (3.28)
\end{aligned}$$

We denote by $\gamma_r(t)$ the last right hand side. Applying our previous estimation it is clear that

$$\begin{aligned}
\alpha_k + T\beta_k + \int_0^T (T - s)\gamma_k(s) ds &\leq \alpha_{1,k} V(\mu) + T\beta_{1,k} V(\nu) + N_1 T^2 k \\
&\quad + \int_0^T (T - s) \left(\int_0^\pi \eta^2(\xi, t) d\xi \right)^{\frac{1}{2}} ds
\end{aligned}$$

which yields that

$$\liminf_{k \to \infty} \frac{M}{k} \left(\alpha_k + T\beta_k + \int_0^T (T - s)\gamma_k(s) ds \right) \leq NT^2 + \liminf_{k \to \infty} \frac{1}{k} (\alpha_{1,k} V(\mu) + T\beta_{1,k} V(\nu)).$$

Similarly we obtain that

$$\limsup_{k \to \infty} \frac{M}{k} \left(\alpha_k + T\beta_k + \int_0^T (T - s)\gamma_k(s) ds \right) \leq NT^2 + \limsup_{k \to \infty} \frac{1}{k} (\alpha_{1,k} V(\mu) + T\beta_{1,k} V(\nu)).$$

Employing these properties Corollary 2.2, Theorem 2.4 and Theorem 2.5, we can establish the following consequences:

(1) If

$$NT^2 + \liminf_{k \to \infty} \frac{1}{k} (\alpha_{1,k} V(\mu) + T\beta_{1,k} V(\nu)) < 1 \qquad (3.29)$$

then the problem (3.22)-(3.23)-(3.24)-(3.25) has a mild solution.

(2) If (3.29) and the following conditions hold:

(a) $\mathcal{R}(p) \subseteq \varphi + D(A)$ and $\mathcal{R}(q) \subseteq \psi + E$;

(b) $\sup_{0 \leq t \leq T} \int_0^\pi \eta^2(\xi, t) d\xi < \infty$;

(c) For each bounded set $D \subseteq X$ there is a sequence of positive numbers $(a_n)_{n \in N}$ with $\sum_{i=1}^\infty n a_n < \infty$ such that

$$| \int_0^\pi | F(\xi, t, x(\xi), y(\xi)) z_n(\xi) d\xi | \leq a_n,$$

for every $t \in I$ and every $x, y \in D$,

then the mild solutions of (3.22)-(3.23)-(3.24)-(3.25) are strong solutions.

(iii) If (3.29) and the following conditions hold:

(a) $\mathcal{R}(p) \subseteq \varphi + D(A)$ and $\mathcal{R}(q) \subseteq \psi + E$;

(b) The function a is Lipschitz continuous.

(c) There is a positive function $N_2 \in L^2(0, \pi)$ and a constant $N_3 > 0$ such that

$$\begin{aligned} | F(\xi, t_1, x, y) - F(\xi, t_2, w, z) | &\leq N_2(\xi) | t_1 - t_2 | \\ &+ N_3(| x - w | + | y - z |), \quad a.e. \xi \in [0, \pi], \end{aligned}$$

for all $t_i \in I$ and all $x_i, y_i \in \mathbb{R}$.

then each mild solutions of (3.22)-(3.23)-(3.24)-(3.25) is a classical solution.

(iv) If (3.29) and

$$NT^2 + \limsup_{k \to \infty} \frac{1}{k} (\alpha_{1,k} V(\mu) + T \beta_{1,k} V(\nu)) < 1$$

then the set of mild solutions of (3.22)-(3.23)-(3.24)-(3.25) is compact.

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