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Global Solutions for a Functional Second Order Abstract Cauchy Problem with Nonlocal Conditions

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Soluções Globais para um Problema de Cauchy Abstrato com Condições não Locais Associado a uma Equação Funcional de Segunda Ordem.

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Resumo

Neste artigo mostramos a existência de soluções asimtoticamente quase periódicas para um problema de Cauchy não local de segunda ordem modelado na forma

$$\begin{aligned} x''(t) &= Ax(t) + f(t, x(t), x(a(t)), x'(t), x'(b(t))), \quad t \in [0, \infty), \\ x(0) &+ p(x, x') = x_0, \\ x'(0) &+ q(x, x') = x_1, \end{aligned}$$

onde A é o gerador infinitesimal de uma função coseno fortemente continua de operadores lineares num espaço Banach X; $q(\cdot), p(\cdot) : C(I; X) \to X$ são apropriadas e $a(\cdot), b(\cdot)$ são funções continuas não decrescentes.

Global Solutions for a Functional Second Order Abstract Cauchy Problem with Nonlocal Conditions

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Abstract

By using the theory of strongly continuous cosine family and the Leray Schauder Alternative, we study the existence of mild, strong and classical solutions for a functional second order abstract Cauchy problem with nonlocal conditions.

1 Introduction.

The purpose of this work is to study the existence of mild, strong and classical solutions of a class of functional second order differential equations modelled in the form

$$
\begin{aligned}
x''(t) &= Ax(t) + f(t, x(t), x(a(t)), x'(t), x'(b(t))), \quad t \in I, && (1.1)\\
x(0) &+ p(x, x') = x_0, && (1.2)\\
x'(0) &+ q(x, x') = x_1, && (1.3)
\end{aligned}
$$

where A is the infinitesimal generator of a strongly cosine family of bounded linear operators $(C(t))_{t\in R}$ on a Banach space X; I is a finite interval $[0, T]$ or the interval $[0, \infty)$ and $a(\cdot), b(\cdot) : I \to I$; $p(\cdot), q(\cdot) : C(I; X) \times C(I; X) \to X$ and $f : I \times X^4 \to X$ are appropriate functions.

In general, the nonlocal conditions can be applied in Physics with better effect that the classical initial conditions $x(0) = x_0$. For the importance of nonlocal conditions in different fields, we refer to [1, 2, 3, 4, 5] and the references contained therein. The study of nonlocal initial value problems was initiated by Byszewski in [3, 5]. In [3], Byszewski prove the existence of mild, strong and classical solutions of the semilinear nonlocal Cauchy problem

$$
\begin{aligned}
x'(t) &= Ax(t) + f(t, x(t)), \qquad t \in I = [0, T]\\
x(0) &= x_0 + q(t_1, t_2, t_3, ..., t_n, u(\cdot)) \in X,
\end{aligned} \tag{1.4}
$$

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where A is the infinitesimal generator of a strongly continuous semigroup of linear operators on X; $0 < t_0 < t_1 < .... < t_n \leq T$; $f : [0,T] \times X \to X$, $q(\cdot,\cdot) : I^n \times C(I : X) \to X$ are appropriated functions and the symbol $q(t_1, t_2, t_3, ..., t_n, u(\cdot))$ is used in the sense that "$\cdot$" can be substitute only for the points t_i, for instance $q(t_1, t_2, t_3, ..., t_n, u(\cdot)) = \sum_{i=1}^{n} \alpha_i u(t_i)$.

On the other hand, some second order nonlocal initial value problems are studied by Ntouyas & Tsamatos in [15, 16, 17]. In these works the authors discuss the existence of solutions for a type of second order delay integrodifferential equations with nonlocal conditions described in the form

$$x''(t) = Ax(t) + f(t, x(\sigma_1(t)), \int_0^t k(t-s)h(s, x(\sigma_2(s)), x(\sigma_3'(s)))ds, x(\sigma_4(t))), \quad t \in (0,T),$$

$$x(0) = g(x) + x_0, \quad x'(0) = \eta,$$

where A is the generator of a strongly continuous cosine family of bounded linear operators on X; $x_0, \eta \in X$ and $g : C(I : X) \to X$, $f, h : I \times X^3 \to X$ are appropriated functions. In general the result are obtained using the Leray-Schauder Alternative and the strong assumption that $C(t)$ is compact for $t > 0$, which in turn implies that $dim(X) < \infty$.

Throughout this paper X will be a Banach space endowed with a norm $\|\cdot\|$ and $C(t)$ denotes a strongly continuous operator cosine function defined on X with infinitesimal generator A. We refer the reader to [10, 19] for the necessary concepts about cosine functions. Next we only mention a few results and notations needed to establish our results. We denote by $S(t)$ the sine function associated with C which is defined by

$$S(t)x := \int_0^t C(s)x\, ds, \quad x \in X, \quad t \in I\!R. \tag{1.5}$$

For a closed operator $B : D(B) \to X$ we denote by $[D(B)]$ the space $D(B)$ endowed with the graph norm $\|\cdot\|_B$. In particular $[D(A)]$ is the space $D(A)$ endowed with the norm

$$\|x\|_A = \|x\| + \|Ax\|, \quad x \in D(A).$$

Moreover, in this work the notation E stands for the space formed by the vectors $x \in X$ for which the function $C(\cdot)x$ is of class C^1. It was proved by Kisińsky ([13]) that E endowed with the norm

$$\|x\|_1 = \|x\| + \sup_{0 \leq t \leq 1} \|AS(t)x\|, \quad x \in E,$$

is a Banach space. The operator valued function $G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is a strongly continuous group of linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. From this it follows that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0$, $t \to 0$, for each $x \in E$. Furthermore, if $x : [0, \infty) \to X$ is a locally integrable function then

$$y(t) = \int_0^t S(t-s)x(s)\, ds$$

defines an E-valued continuous function. This is an immediate consequence of the fact that

$$\int_0^t G(t-s)\begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)\,ds \\ \int_0^t C(t-s)x(s)\,ds \end{bmatrix}$$

defines an $E \times X$-valued continuous function.

The existence of solutions of the second order abstract Cauchy problem

$$\begin{aligned} x''(t) &= Ax(t) + h(t), \;\; 0 \le t \le a, \\ x(0) &= x_0, \;\; x'(0) = x_1, \end{aligned}$$

where $h : [0, a] \to X$ is an integrable function, has been discussed in [20]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem it has been treated in [21]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s)\,ds, \;\; 0 \le t \le a, \tag{1.6}$$

is called a mild solution of (2.8)-(2.9). In the case in which $x_0 \in E$ then $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)x_0 + C(t)x_1 + \int_0^t C(t-s)h(s)\,ds. \tag{1.7}$$

If $C(t)$ is uniformly bounded then $-A$ is a non negative operator so that the fractional power $(-A)^\alpha$, $\alpha > 0$, is well defined as a closed operator ([14]). The following properties are well know.

Lemma 1.1 *If the cosine function $(C(t))_{t\in R}$ is uniformly bounded then the following conditions hold:*

(i) If $x \in E$, then $S(t)x \in D(A)$ and $\frac{d}{dt}C(t)x = AS(t)x$.

(ii) Let $0 < \alpha < 1$. $(-A)^{-\alpha}$ is compact if and only if $(-A)^{-1}$ is compact.

(iii) $(-A)^\alpha C(t)x = C(t)(-A)^\alpha x$ and $(-A)^\alpha S(t)x = S(t)(-A)^\alpha x$ for every $t \in I\!R$ and every $x \in D((-A)^\alpha)$

(iv) For $\alpha, \beta > 0$, $(-A)^\alpha(-A)^\beta x = (-A)^{\alpha+\beta}x$, for every $x \in D((-A)^{\alpha+\beta})$.

Additional terminology and notations are those generally used in functional analysis. In particular, $\mathcal{L}(X)$ stands for the Banach space of bounded linear operators from X into X and $B_r(x)$ will denote the closed ball with center at x and radius r in an appropriated space. Moreover, for bounded functions $\xi : I \to I\!R$ and $\Lambda : Y \to I\!R$ we will employ the notations ξ_t and N_Λ for $\xi_t = \sup\{\xi(\theta) : \theta \le t\}$ and $N_\Lambda = \sup\{\| \Lambda(y) \| : y \in Y\}$.

The paper is organized as follows. In section 2 we establish existence of mild, strong and classical solutions of some second order nonlocal initial value Cauchy problems in a finite interval and in section 3 we consider existence of g-bounded solutions and asymptotically almost periodic solutions on the interval $[0, \infty)$. Finally, in section 4 we present an application to the wave equation. The results are obtained using the techniques introduced in Henríquez [12], Ntouyas & Tsamatos [15, 16, 17] and the Leray Schauder Alternative. For completeness we include here this result.

Lemma 1.2 *Let D a convex subset of a Banach space X and assume that $0 \in S$. Let $F : D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda F(x),\ 0 < \lambda < 1\}$ is unbounded or the map F has a fix point in D.*

2 Existence of solutions.

In this section we study existence of solutions of some nonlocal second order functional abstract Cauchy problem in the interval $I = [0, T]$. We begin by introducing some notations. In this section we assume that $M \geq 1$ and $N \geq 1$ are constants such that $\|C(t)\| \leq M$, $\|S(t)\| \leq N$ for every $t \in I$. Moreover, we consider the space of continuous functions $C(I; X)$ endowed with the norm of uniform convergence $\|\cdot\|_\infty$, and we abbreviate the notation to $C(I)$ instead of $C(I; I)$.

In this section we always assume that the following general assumptions hold:

Assumption $\mathbf{A_1}$

(a) The function $a(\cdot) : I \to I$ is continuous and $a(t) \leq t$ for every $t \in I$.

(b) The function $f : I \times X \times X \to X$ satisfies the following Caratheodory conditions:

(i) $f(t, \cdot) : X \times X \to X$ is continuous a.e. $t \in I$;

(ii) For each $x, y \in X$, the function $f(\cdot, x, y) : I \to X$ is strongly measurable.

(c) There exist a continuous functions $m : I \to [0, \infty)$ and a continuous nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that

$$\|f(t, x, y)\| \leq m(t)W(\| x \| + \| y \|),$$

for every $t \in I$ and $x, y \in X$.

(d) The functions p, $q : C(I; X) \to X$ are continuous.

Initially we consider the following second order initial value problem

$$x''(t) = Ax(t) + f(t, x(t), x(a(t))), \quad t \in I, \tag{2.8}$$

$$x(0) + p(x) = x_0, \tag{2.9}$$

$$x'(0) + q(x) = x_1, \tag{2.10}$$

where $x_0, x_1 \in X$.

Expression (1.6) motivates the following definition.

Definition 2.1 *We say that a function* $x(\cdot) : I \to X$ *is a mild solution of the problem (2.8)-(2.9)-(2.10) if* $x(\cdot)$ *is continuous and satisfies the integral equation*

$$x(t) = C(t)(x_0 - p(x)) + S(t)(x_1 - q(x)) + \int_0^t S(t-s)f(s, x(s), x(a(s))\, ds, \quad t \in I. \quad (2.11)$$

Theorem 2.1 *Assume that the following conditions hold:*

(H-1) *The functions* $p(\cdot)$ *and* $q(\cdot)$ *are bounded and both* p *as the map* $C(I; X) \to C(I; X)$, $x \to S(\cdot)q(x)$ *are completely continuous.*

(H-2) *For each* $t \in I$, $t' \leq t$ *and every constant* $L \geq 0$ *the set* $\{S(t')f(s, x, y) : 0 \leq s \leq t, \|x\|, \|y\| \leq L\}$ *is relatively compact.*

Let $c = M(\| x_0 \| + N_p) + N(\| x_1 \| + N_q)$. *If* $2N \int_0^T m(s)\, ds < \int_{2c}^{\infty} \frac{1}{W(s)}\, ds$, *then there exist a mild solution of (2.8)-(2.9)-(2.10).*

Proof. In order to use the Leray Schauder Alternative, we obtain an *a priori* bound for the solution of the integral equation

$$x_\lambda(t) = \lambda C(t)(x_0 - p(x_\lambda)) + \lambda S(t)(x_1 - q(x_\lambda)) + \lambda \int_0^t S(t-s)f(s, x_\lambda(s), x_\lambda(a(s)))\, ds,$$

for $t \in I$. Using the notations introduced in section 1, for $t \in I$ and $\lambda \in (0, 1)$ we get

$$\begin{aligned} \| x_\lambda(t) \| \leq & \ M(\| x_0 \| + N_p) + N(\| x_1 \| + N_q) \\ & + N \int_0^t m(s) W(\| x_\lambda(s) \| + \| x_\lambda(a(s)) \|)\, ds. \end{aligned}$$

Designating by $\alpha_\lambda(t)$ the right hand side of the above expression and using the condition $a(t) \leq t$ we obtain that

$$\alpha'_\lambda(t) \leq N m(t) W(2\alpha_\lambda(t)).$$

¿From this it follows that

$$\int_{2\alpha_\lambda(0)}^{2\alpha_\lambda(t)} \frac{ds}{W(s)} \leq 2N \int_0^t m(s)\, ds \int_{2\alpha_\lambda(0)}^{\infty} \frac{ds}{W(s)}\, ds,$$

which implies that $\alpha_\lambda(\cdot)$ is uniformly bounded respect to λ and consequently that the set $\{x_\lambda : \lambda \in (0, 1)\}$ is bounded in $C(I : X)$.

Next, we prove that the operator $\mathcal{T} : C(I : X) \to C(I : X)$ defined by

$$\mathcal{T}(x)(t) = C(t)(x_0 - p(x)) + S(t)(x_1 - q(x)) + \int_0^t S(t-s)f(s, x(s), x(a(s)))\, ds, \quad t \in I, \quad (2.12)$$

is completely continuous. This expression leads us to define the maps $\mathcal{T}_1, \mathcal{T}_2 : C(I : X) \to C(I : X)$ by means of the expressions

$$\begin{aligned} \mathcal{T}_1(x) &= C(\cdot)(x_0 - p(x)) + S(\cdot)(x_1 - q(x)), \\ \mathcal{T}_2(x)(t) &= \int_0^t S(t-s)f(s, x(s), x(a(s)))\, ds, \quad t \in I. \end{aligned}$$

¿From the hypothesis **(H-1)** it is clear that $\mathcal{T}_1$ is completely continuous. Thus, it only remains to prove that $\mathcal{T}_2$ is completely continuous. The continuity of $\mathcal{T}_2$ is consequence of the Lebesgue dominated convergence theorem. To prove the compactness condition we denote by B_r the closed ball in $C(I; X)$ with center at 0 and radius r. Initially we establish that the set $\mathcal{T}_2(B_r)$ is equicontinuous. We fix $t \in I$ and let h be such that $t + h \in I$. For $x \in B_r$ from the definition of $\mathcal{T}_2$ it follows that

$$\begin{aligned} \| \mathcal{T}_2(x)(t+h) - \mathcal{T}_2(x)(t) \| &= \int_0^t \| (S(t+h-s) - S(t-s))f(s, x(s), x(a(s))) \| \, ds \\ &\quad + \int_t^{t+h} \| S(t+h-s)f(s, x(s), x(a(s))) \| \, ds \\ &\leq Mh \int_0^t m(s)W(\| x(s) \| + \| x(a(s)) \|)\, ds \\ &\quad + N \int_t^{t+h} m(s)W(\| x(s) \| + \| x(a(s)) \|) ds \\ &\leq MhW(2r) \int_0^t m(s) ds + NW(2r) \int_t^{t+h} m(s) ds, \end{aligned}$$

which shows that

$$\| \mathcal{T}_2(x)(t+h) - \mathcal{T}_2(x)(t) \| \to 0, \quad h \to 0,$$

uniformly with respect to $x \in B_r$.

Next we establish that $\mathcal{T}_2(B_r)(t) = \{\mathcal{T}_2 x(t) : x \in B_r\}$ is relatively compact in X, for each $0 \leq t \leq T$. We put $U = \{f(t-s, x(t-s), x(a(t-s))) : 0 \leq s \leq t, \ x \in B_r\}$. Since

$$\|y\| \leq m(t-s)W(2r) \leq m_T W(2r)$$

for all $y \in U$ and the operator map $S(\cdot) : [0, T] \to \mathcal{L}(X)$ is uniformly continuous we infer that for each $\varepsilon > 0$ there are $t_1, t_2, \cdots, t_n \in I$ so that for each $0 \leq s \leq t$ and $y \in U$ we can select t_i for which

$$\|S(s)y - S(t_i)y\| \leq \varepsilon.$$

In view of that by hypothesis (H-2) the set $\bigcup_{i=1}^n S(t_i)U$ is relatively compact we obtain that the set $\{S(s)y : 0 \leq s \leq t, \ y \in U\}$ is also relatively compact. From the definition of $\mathcal{T}_2$ and the main value theorem it follows that

$$\mathcal{T}_2(B_r)(t) \subseteq t\, \overline{co\{S(s)y : 0 \leq s \leq t, \ y \in U\}},$$

where *co* is used to denote the convex hull, which shows the assertion.

Finally, an application of the Ascoli-Arzela theorem completes our proof that $\mathcal{T}_2(B_r)$ is relatively compact and that $\mathcal{T}_2$ is completely continuous.

The existence of a mild solution of the abstract second order Cauchy problem (2.8)-(2.9)-(2.10) is now consequence of Lemma 1.2. The proof is complete. ■

Corollary 2.1 *Assume that condition* **(H-1)** *holds. Suppose, in addition that* $(-A)^{-1}$ *is compact and that*

(i) *The function* p *is bounded and there exists* $0 < \alpha_1 < 1$ *such that* $x_0 + p(\cdot)$ *is continuous and bounded with values in* $D((-A)^{\alpha_1})$.

(ii) *The function* q *is bounded and there exist* $0 < \alpha_2 < 1$ *such that* $x_1+q(\cdot)$ *is continuous and bounded with values in* $D((-A)^{\alpha_2})$.

Let $x_0 \in X$ *and* $c = M(\| x_0 \| + N_p) + N(\| x_1 \| + N_q)$. *If* $2N \int_0^T h(s)ds < \int_{2c}^{\infty} \frac{1}{W(s)}ds$, *then there exist a mild solution of (2.8)-(2.9)-(2.10).*

From the properties of the abstract Cauchy problem mentioned in the preliminaries we know that if $x_0 - p(x) \in E$ then the mild solution x is continuously differentiable on I and

$$x'(t) = AS(t)(x_0 - p(x)) + C(t)(x_1 - q(x)) + \int_0^t C(t-s)f(s, x(s), x(a(s)))ds. \quad (2.13)$$

Next we study the differentiability of the function x'. Firstly we consider the following concept of strong solution.

Definition 2.2 *A function* $x(\cdot) : I \to X$ *is a strong solution of (2.8)-(2.9)-(2.10) if* $x(\cdot) \in W^{2,1}(I; X)$, *the equation (2.8) is satisfied a.e.* $t \in I$ *and the conditions (2.9) and (2.10) are verified.*

Theorem 2.2 *Assume that* X *has the RNP and that the hypotheses of Theorem 2.1 hold. If in further the following conditions are verified:*

(H-3) $\mathcal{R}(p) \subseteq x_0 + D(A)$ *and* $\mathcal{R}(q) \subseteq x_1 + E$;

(H-4) *For each bounded set* $D \subseteq X$, *the functions* $C(\cdot)f(t, x, y)$, $t \in I$, $x, y \in D$, *are uniformly Lipschitz continuous on* I,

then a mild solution of (2.8)-(2.9)-(2.10) is a strong solution.

Proof. Let $x(\cdot)$ be a mild solution of (2.8)-(2.9)-(2.10). From **H-3** and the preliminaries we get

$$\begin{aligned} x'(t+s) - x'(t) &= (S(t+s) - S(t))A(x_0 - p(x)) + (C(t+s) - C(t))(x_1 - q(x)) \\ &+ \int_0^t (C(t+s-\xi) - C(t-\xi))f(\xi, x(\xi), x(a(\xi)))\, d\xi \\ &+ \int_t^{t+s} C(t+s-\xi)f(\xi, x(\xi), x(a(\xi)))\, d\xi, \end{aligned}$$

Which implies that $x'(\cdot)$ is Lipschitz continuous. Since X has the RNP, it follows that $x \in W^{2,1}([0,b];X)$. Now, the assertion is consequence of Proposition 3.3 in [12]. ■

Remark. Related with this result, we know from the results in [13] that condition **H-4** holds if $\mathcal{R}(f) \subseteq E$ and $f : I \times X \times X \to E$ takes bounded sets into the bounded sets.

In order to discuss the existence of classical solutions we introduce the following concept.

Definition 2.3 *A function $x(\cdot) : I \to X$, is a classical solution of the problem (2.8)-(2.9)- 2.10) if $x(\cdot)$ is a function of class C^2 that satisfies the equation (2.8) and the initial conditions (2.9) and (2.10).*

Theorem 2.3 *Assume that X has the RNP and that the hypothesis of Theorem 2.1 and condition* **H-3** *hold. If in addition*

(H-5) *The function $a(\cdot)$ is Lipschitz continuous,*

(H-6) *For each bounded set $D \subseteq X$, the function $f : I \times D \times D \to X$ is Lipschitz continuous,*

then a mild solution of (2.8)-(2.9)-(2.10) is a classical solution.

Proof. Let $x(\cdot)$ be a mild solution of (2.8)-(2.9)-(2.10). From **H-3**, **H-5** and the relation

$$\begin{aligned} x(t+s) - x(t) &= (C(t+s) - C(t))(x_0 - p(x)) + (S(t+s) - S(t))(x_1 - q(x)) \\ &\quad + \int_0^t (S(t+s-\xi) - S(t-\xi)) f(\xi, x(\xi), x(a(\xi)))\, d\xi \\ &\quad + \int_t^{t+s} S(t+s-\xi) f(\xi, x(\xi), x(a(\xi)))\, d\xi, \end{aligned}$$

it follows that $h(t) = f(t, x(t), x(a(t)))$ is Lipschitz continuous and consequently that $x \in W^{1,1}([0,b];X)$, since X has the RNP. The assertion is now consequence of Theorem 3.1 in [12]. ■

In what follow we consider the following second order nonlocal initial value problem

$$x''(t) = Ax(t) + f(t, x(t), x(a(t)), x'(t), x'(b(t))), \qquad t \in I, \tag{2.14}$$

$$x(0) + p(x, x') = x_0, \tag{2.15}$$

$$x'(0) + q(x, x') = x_1, \tag{2.16}$$

where $a(\cdot), b(\cdot) : I \to I$; $p, q : C(I;X)^2 \to X$ and $f : I \times X^4 \to X$ are appropriated functions.

Definition 2.4 *We say that a function* $x(\cdot): I \to X$ *is a mild solution of the problem (2.14)-(2.15)-(2.16) if:* $x(\cdot)$ *and* $x'(\cdot)$ *are continuous, the initial conditions (2.15)-(2.16) are verified and* $x(\cdot)$ *satisfies the integral equation*

$$\begin{aligned} x(t) &= C(t)(x_0 - p(x,x')) + S(t)(x_1 - q(x,x')) \\ &\quad + \int_0^t S(t-s) f(s, x(s), x(a(s)), x'(s), x'(b(s)))\, ds, \qquad t \in I. \end{aligned}$$

In order to establish the existence of a mild solution for the nonlocal initial value problem (2.14)-(2.15)-(2.16), in the rest of this section we always assume that the following assumptions hold.

Assumption $\mathbf{A_2}$

(a) The functions $a(\cdot)$, $b(\cdot): I \to I$ are continuous, nondecreasing and $\max\{a(t), b(t)\} \leq t$ for every $t \in I$.

(b) The function $f: I \times X^4 \to X$ satisfies the following Caratheodory conditions:

(i) $f(t, \cdot): X^4 \to X$ is continuous a.e. $t \in I$;

(ii) For each $x \in X^4$, the function $f(\cdot, x): I \to X$ is strongly measurable.

(c) There exist a continuous functions $m: I \to [0, \infty)$ and a continuous nondecreasing function $W: [0,\infty) \to (0,\infty)$ such that

$$\|f(t, x_1, x_2, x_3, x_4)\| \leq m(t) W(\sum_{i=1}^{4} \| x_i \|),$$

for every $t \in I$ and every $x_i \in X$.

(d) The functions p, $q: C(I;X)^2 \to X$ are continuous.

Theorem 2.4 *Let assumption* $\mathbf{A_2}$ *be satisfied and assume that the following conditions are verified.*

(H-7) *The functions* $p(\cdot)$, $q(\cdot)$ *are completely continuous and bounded.*

(H-8) *Any of the following conditions hold.*

(a) *The map* $x_0 - p$ *is completely continuous with values in* E.

(b) $S(t)$ *is compact and the map* $x_0 - p$ *is bounded with values in* $D(A)$.

Let $N_p^1 = \sup\{\| x_0 - p(x,y) \|_1 \, x, y \in C(I;X)\}$.

(H-9) *For every* $r > 0$, *the set* $f(I \times B_r^4)$ *is relatively compact.*

If $2(M+N)\int_0^T h(s)ds < \int_{2c}^{\infty}\frac{ds}{W(s)}$, *where* $c = M(\| x_0 \| + N_p) + \| AS(\cdot)x_0 \|_T + \| B(\cdot) \|_T N_g + (M+N)(\| x_1 \| + N_q)$, *then there exist a mild solution of (2.14)-(2.15)-(2.16).*

Proof. We define the map $\Gamma : C(I:X)^2 \to C(I:X)^2$ by

$$\mathcal{T}(x,y)(t) = (\mathcal{T}^1(x,y)(t), \mathcal{T}^2(x,y)(t)) \tag{2.17}$$

where

$$\begin{aligned}
\mathcal{T}(x)(t) &= C(t)(x_0 - p(x)) + S(t)(x_1 - q(x)) + \int_0^t S(t-s)f(s,x(s),x(a(s)))\,ds,\\
\mathcal{T}(x)(t) &= AS(t)(x_0 - p(x)) + C(t)(x_1 - q(x)) + \int_0^t C(t-s)f(s,x(s),x(a(s)))ds,
\end{aligned}$$

At first, we obtain a *priori* bound for the solutions of $(x_\lambda, y_\lambda) = \lambda\mathcal{T}(x_\lambda, y_\lambda), \lambda \in (0,1)$. If $\lambda \in (0,1)$ and (x_λ, y_λ) is a referred solution we get

$$\begin{aligned}
\| x_\lambda(t) \| &\le M(\| x_0 \| + N_p) + N(\| x_1 \| + N_q)\\
&\quad + N\int_0^t m(s)W(\| x_\lambda(s) \| + \| x_\lambda(a(s)) \| + \| x'_\lambda(s) \| + \| x'_\lambda(b(s)) \|)\,ds\\
&= u_\lambda(t).
\end{aligned}$$

Since $a(\cdot)$ is nondecreasing, it's follows that

$$u'_\lambda(t) \le Nm(t)W(2u_\lambda(t) + \| y_\lambda(t) \| + \| y'_\lambda(b(t)) \|). \tag{2.18}$$

Similarly, from **(H-8)**, we find that

$$\begin{aligned}
\| y_\lambda(t) \| &\le N_p^1 + M(\| x_1 \| + N_q) + M\int_0^t m(s)W(2u_\lambda(s) + \| y'_\lambda(s) \| + \| y_\lambda(b(s)) \|)ds\\
&= v_\lambda(t),
\end{aligned}$$

and hence

$$v'_\lambda(t) \le Mm(t)W(2u_\lambda(t) + 2v_\lambda(t)), \qquad t \in [0,T]. \tag{2.19}$$

Integrating between 0 and t the sum of (2.18), (2.19) we get

$$\int_{2(u_\lambda+v_\lambda)(0)}^{2(u_\lambda(t)+v_\lambda)(t)} \frac{ds}{W(s)} \le 2(M+N)\int_0^t m(s)ds < \int_{2(u_\lambda+v_\lambda)(0)}^{\infty}\frac{ds}{W(s)},$$

which implies that the set $\{x_\lambda : \lambda \in (0,1)\}$ is bounded.

It's easy to prove that $\mathcal{T}$ is continuous. Moreover, repeating the arguments in the proof of Theorem 2.1 we can to conclude that $\mathcal{T}^1$ is completely continuous. In both, if **a** or **a** hold, the function $AS()(x_0 + p)$ is completely, which jointly with **(H-9)** and **(H-7)** permit to conclude that $\mathcal{T}^2$ is completely continuous.

The existence of a mild solution for the initial value problem (2.14)-(2.15)-(2.16) is now consequence of Lemma 1.2. The proof is complete. ∎

We conclude this section with a regularity result for the mild solutions of the initial value problem (2.14)-(2.15)-(2.16). Previously we introduce some definitions.

Definition 2.5 *A function $x(\cdot) : I \to X$ is a strong solution of (2.14)-(2.15)-(2.16) if $x(\cdot) \in W^{2,1}(I; X)$, the equation (2.14) is satisfied a.e. $t \in I$ and the conditions (2.15) and (2.16) are verified.*

Definition 2.6 *We say that a function $x(\cdot) : I \to X$, is a classical solution of the problem (2.14)-(2.15)- (2.16) if $x(\cdot)$ is a function of class C^2 that satisfies the equation (2.14) and the initial conditions (2.15) and (2.16).*

The proof of the next result, follows using the ideas in the proof of the Theorems 2.2 and 2.3.

Theorem 2.5 *Assume that X has the RNP and that the hypotheses of Theorem 2.4 are verified. If the conditions* **(H-3)** *and*

(H-4)' *For each bounded set $D \subseteq X$, the functions $C(\cdot)f(t, x_1, x_2, x_3, x_4)$, $t \in I$, $x_i \in D$, are uniformly Lipschitz continuous on I,*

hold, then a mild solution $x(\cdot)$ of (1.1)-(1.2)-(1.3) is a strong solution.

Theorem 2.6 *Assume that X has the RNP and that the hypotheses of Theorem 2.4 are verified. If the conditions* **(H-3)** *and*

(H-10) *The functions $a(\cdot), b(\cdot)$ are Lipschitz continuous,*

(H-11) *For each bounded set $D \subseteq X$, the function $f : I \times D^4 \to X$ is Lipschitz continuous,*

hold, then a mild solution $x(\cdot)$ of (1.1)-(1.2)-(1.3) is a classical solution.

3 Existence of global solutions.

In this section we study the problem (2.8)-(2.9)-(2.10) on the interval $I = [0, \infty)$. We suppose that the general assumptions considered in section 2 are satisfied on I and that M and N are positive constants such that $||C(t)|| \leq M$ and $||S(t)|| \leq N$, for all $t \geq 0$. Moreover, to generalize the arguments employed in the previous section to include continuous functions defined on $[0, \infty)$ we need an appropriate version of the Ascoli-Arzela theorem. For this reason we consider two cases. First, we study existence of solutions in the space of continuous functions with weight and afterwards we establish existence of asymptotically almost periodic solutions.

Let $g : [0, \infty) \to (0, \infty)$ be a continuous function such that $g(t) \to \infty$, as $t \to \infty$. To simplify our developments we also assume that g is increasing and $g(0) = 1$. Next the notation $C_g^0(X)$ stands for the Banach space formed by the continuous functions $x : [0, \infty) \to X$ such that $\dfrac{x(t)}{g(t)} \to 0,\ t \to \infty,$ endowed with the norm

$$\|x\|_g = \sup_{t \geq 0} \frac{\|x(t)\|}{g(t)}.$$

We recall here the following result of compactness in the space $C_0(X)$ formed by the continuous functions which vanish at infinity.

Lemma 3.3 *A set $B \subseteq C_0(X)$ is relatively compact if, and only if, the following conditions are fulfilled:*

(a) *B is equicontinuous;*

(b) *$x(t) \to 0,\ t \to \infty,$ uniformly for $x \in B$;*

(c) *The orbits $B(t)$ are relatively compact in X, for all $t \geq 0$.*

Next, without any danger of confusion, we abbreviate our notations by writing and $u(t) = (x(t), x(a(t)))$, for a well determined function x.

Theorem 3.7 *Assume that the following conditions are verified.*

(a) *The functions $p,\ q : C_g^0(X) \to X$ are continuous, bounded and p is completely continuous;*

(b) *The function $S(t)q : C_g^0(X) \to X$ is completely continuous for each $t \geq 0$;*

(c) *For each $t \in I,\ t' \leq t$ and a constant $L \geq 0$ the set $\{S(t')f(s, x, y) : 0 \leq s \leq t,\ \|x\|, \|y\| \leq L\}$ is relatively compact.*

(d) *For every constant $L \geq 0$, $\dfrac{1}{g(t)} \displaystyle\int_0^t m(s) W(Lg(s))\, ds \to 0,\ t \to \infty$;*

(e) *$2N \int_0^\infty m(s)\, ds < \int_{2c}^\infty \frac{1}{W(s)}\, ds$, where $c = M(\|x_0\| + N_p) + N(\|x_1\| + N_q)$.*

Then there exist a mild solution $x \in C_g^0(X)$ of (2.8)-(2.9)-(2.10).

Proof. For each $x \in C_g^0(X)$ we define $\mathcal{T}(x)(t)$ by means of (2.12). Hence,

$$\|\mathcal{T}(x)(t)\| \leq M(\|x_0\| + N_p) + N(\|x_1\| + N_q) + N \int_0^t m(s) W(\|x(s)\| + \|x(a(s))\|)\, ds.$$

In view of that $\|x(s)\| \leq \|x\|_g g(s)$, applying condition (d) in the above expression it follows that

$$\frac{1}{g(t)} \|\mathcal{T}(x)(t)\| \leq \frac{M}{g(t)}(\|x_0\| + N_p) + \frac{N}{g(t)}(\|x_1\| + N_q) + \frac{N}{g(t)} \int_0^t m(s) W(2\|x\|_g g(s))\, ds$$

converges to zero as $t \to \infty$. This shows that $\mathcal{T}$ is a map well defined from $C_g^0(X)$ into $C_g^0(X)$. To prove the continuity of $\mathcal{T}$, we select a sequence $(x_n)_n$ in $C_g^0(X)$ which converges to x. It is clear that $f(s, u_n(s)) \to f(s, u(s))$, $n \to \infty$, a.e. and, in view of that,

$$\|f(s, u_n(s))\| \leq m(s)W(Lg(s)),$$

for some constant $L \geq 0$, and the right hand side is an integrable function on $[0, t]$ we conclude that $\mathcal{T}(x_n)(t) \to \mathcal{T}(x)(t)$, $n \to \infty$. It is also clear that the convergence is uniform for t in bounded intervals. Furthermore, given $\varepsilon > 0$, applying (a) and (d) we can chose t_0 so that

$$\begin{aligned}
\frac{1}{g(t)}\|\mathcal{T}(x_n)(t) - \mathcal{T}(x)(t)\| &\leq \frac{M}{g(t)}\|p(x_n) - p(x)\|) + \frac{N}{g(t)}\|q(x_n) - q(x)\| \\
&\quad + \frac{2N}{g(t)} \int_0^t m(s)W(Lg(s))\, ds \\
&\leq \varepsilon,
\end{aligned}$$

for $t \geq t_0$. This implies that $\mathcal{T}(x_n) \to \mathcal{T}(x)$, $n \to \infty$ in the space $C_g^0(X)$. Consequently, $\mathcal{T}$ is a continuous map.

On the other hand, for $0 < \lambda < 1$ let $x_\lambda \in C_g^0(X)$ be such that $\lambda\mathcal{T}(x_\lambda) = x_\lambda$. Proceeding as above we can estimate

$$\|x_\lambda(t)\| \leq M(\|x_0\| + N_p) + N(\|x_1\| + N_q) + N \int_0^t m(s)W(\|x_\lambda(s)\| + \|x_\lambda(a(s))\|)\, ds.$$

If $\alpha_\lambda(t)$ denotes the right hand side , repeating the argument used in the proof of Theorem 2.1 we conclude that the set $\{\alpha_\lambda(t) : 0 < \lambda < 1,\ t \geq 0\}$ is bounded. Therefore, the set $\{\|x_\lambda(t)\| : 0 < \lambda < 1,\ t \geq 0\}$ is bounded and, in turn, $\{\|x_\lambda\|_g : 0 < \lambda < 1\}$ is also bounded.

Finally we show that $\mathcal{T}$ is completely continuous. Since we already establish that $\mathcal{T}$ is continuous, it only remains to prove that $\mathcal{T}(B)$ is relatively compact in $C_g^0(X)$ for every bounded set B. Thus, we need to verify that the functions $\dfrac{1}{g}\mathcal{T}(x)$, $x \in B$, satisfy the conditions of Lemma 3.1. To prove that $\dfrac{1}{g}\mathcal{T}(x)$, $x \in B$, are equicontinuous, we observe first that $\frac{1}{g}[C(\cdot)(x_0 - p(x)) + S(\cdot)(x_1 - q(x))]$ are equicontinuous in view of that p is completely continuous and q is bounded. Moreover, for $x \in B$ and $h \geq 0$, there is a constant $L \geq 0$ for which the estimation

$$\begin{aligned}
&\|\int_0^{t+h} S(t+h-s)f(s, u(s))\, ds - \int_0^t S(t-s)f(s, u(s))\, ds\| \\
&\leq \|\int_0^t [S(t+h-s) - S(t-s)]f(s, u(s))\, ds\| + \|\int_t^{t+h} S(t+h-s)f(s, u(s))\, ds\| \\
&\leq Mh \int_0^t m(s)W(Lg(s))\, ds + N \int_t^{t+h} m(s)W(Lg(s))\, ds
\end{aligned}$$

is satisfied. This shows that $\frac{1}{g}\mathcal{T}(x)(t+h) - \frac{1}{g}\mathcal{T}(x)(t) \to 0$, when $h \to 0$, independent of $x \in B$. The same argument and condition (d) also permit us to affirm that $\frac{1}{g(t)}\mathcal{T}(x)(t) \to 0$, $t \to \infty$, uniformly with respect to $x \in B$. Finally, applying our hypotheses (a), (b), (c) and proceeding as in the proof of Theorem 2.1 we derive that the set $\{\mathcal{T}(x)(t) : x \in B\}$ is relatively compact in X, for each $t \geq 0$. ■

We study now the existence of asymptotically almost periodic solutions of (2.8)-(2.9)-(2.10). For the basic concepts about almost periodic and asymptotically almost periodic functions we refer to [22]. In particular, we denote by $AP(X)$ (resp. $AAP(X)$) the space formed by the functions $x : [0, \infty) \to X$ which are almost periodic (resp. asymptotically almost periodic). We will also utilize the properties of almost periodic cosine functions and almost periodic sine functions. We refer to [9] for the characterization of almost periodic cosine functions and to [11] for similar properties for almost periodic sine functions. The following lemma ([22]) establish a result of Ascoli-Arzela type for the characterization of compactness in $AAP(X)$.

Lemma 3.4 *Let $B \subseteq AAP(X)$ be a set with the following properties:*

(a) *B is equicontinuous uniformly;*

(b) *B is equi-asymptotically almost periodic. This means that for every $\varepsilon > 0$ there are $T_\varepsilon \geq 0$ and relatively dense set P_ε in $[0, \infty)$ such that*

$$\|x(t+\tau) - x(t)\| \leq \varepsilon,$$

for all $x \in B$, and all $t \geq T_\varepsilon$, $\tau \in P_\varepsilon$.

(c) *for each $t \geq 0$, $B(t)$ is relatively compact in X.*

Then B is relatively compact in $AAP(X)$.

To establish our result we need the following property of compactness of integrable functions.

Lemma 3.5 *Let $B \in \mathcal{L}^1([0, \infty); X)$ be a set for which the following properties are fulfilled.*

(a) *B is equiintegrable at ∞. This means that $\int_L^\infty \|x(s)\|\, ds \to 0$, $L \to \infty$, uniformly for $x \in B$;*

(b) *For each $t \geq 0$, the set $\{x(s) : x \in B,\ 0 \leq s \leq t\}$ is relatively compact in X;*

(c) *The function $F : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous and uniformly bounded.*

Then there is a compact set $U \subseteq X$ such that $\int_0^t F(s)x(s)\, ds \in U$, for all $x \in B$ and $0 \leq t \leq \infty$.

Proof. For each $\varepsilon > 0$ we can select $L \geq 0$ such that $\| \int_L^\infty F(s)x(s)\,ds\| \leq \varepsilon$, for all $x \in B$. Using (b) we can affirm that there is a compact set $U_0 \subseteq X$ such that $x(s) \in U_0$, for all $0 \leq s \leq L$ and $x \in B$. In view of that the map $[0, L] \times U_0 \to X$, $(s, x) \to F(s)x$ is continuous we infer that $\{F(s)x(s) : 0 \leq s \leq L,\ x \in B\}$ is relatively compact and the main value theorem for integrals allows us to assure that the same property holds for the set $\{\int_0^L F(s)x(s)\,ds :\ x \in B\}$. The decomposition

$$\int_0^t F(s)x(s)\,ds = \int_0^L F(s)x(s)\,ds + \int_L^t F(s)x(s)\,ds$$

completes the proof. ∎

Now we combine these results to establish the following property for the of mild solutions of the abstract Cauchy problem. In this statement we denote by y_x the function defined by

$$y_x(t) = \int_0^t S(t-s)x(s)\,ds.$$

Proposition 3.1 *Assume that the sine function $S(\cdot)$ is almost periodic. Let $B \subseteq \mathcal{L}^1([0,\infty); X)$ be a set that satisfies conditions (a) and (b) of Lemma 3.3 as well as the following condition of equiintegrability: for each $t \geq 0$, $\int_t^{t+h} \|x(s)\|\,ds \to 0$, $h \to 0$, uniformly for $x \in B$. Then the set $\{y_x : x \in B\}$ is relatively compact in $AAP(X)$.*

Proof. we can rewrite

$$\begin{aligned} y_x(t) &= S(t)\int_0^t C(s)x(s)\,ds - C(t)\int_0^t S(s)x(s)\,ds \\ &= S(t)\int_0^\infty C(s)x(s)\,ds - S(t)\int_t^\infty C(s)x(s)\,ds \\ &\quad - C(t)\int_0^\infty S(s)x(s)\,ds + C(t)\int_t^\infty S(s)x(s)\,ds. \end{aligned}$$

¿From the properties of almost periodic sine functions ([11]) we deduce that the first and third term on the right hand side are almost periodic functions while the second and fourth term are functions that vanish at ∞. We infer from this that $y_x \in AAP(X)$. From Lemma 3.3 we know that the integrals $\int_0^\infty C(s)x(s)\,ds$ and $\int_0^\infty S(s)x(s)\,ds$, $x \in B$, are included in a compact set. This implies that the set of functions $S(\cdot)\int_0^\infty C(s)x(s)\,ds - C(\cdot)\int_0^\infty S(s)x(s)\,ds$, $x \in B$, is relatively compact in $AP(X)$. On the other hand, employing the properties of B it is not difficult to see that the functions $S(\cdot)\int_t^\infty C(s)x(s)\,ds - C(\cdot)\int_t^\infty S(s)x(s)\,ds$, $x \in B$, satisfy the conditions of Lemma 3.2 from which we can assure that this set is relatively compact in the space $C_0(X)$. Collecting these results it follows easily the assertion. ∎

We are in conditions to establish the main result.

Theorem 3.8 *Assume that $S(\cdot)$ is almost periodic and that the following conditions hold.*

(a) *The functions* $p, q : AAP(X) \to X$ *are completely continuous and bounded;*

(b) *For each* $t \in I$ *and a constant* $L \geq 0$ *the set* $\{f(s,x,y) : 0 \leq s \leq t, \|x\|, \|y\| \leq L\}$ *is relatively compact;*

(c) $2N \int_0^\infty m(s)\, ds < \int_{2c}^\infty \frac{1}{W(s)}\, ds$, *where* $c = M(\|x_0\| + N_p) + N(\|x_1\| + N_q)$.

Then there exist a mild solution $x \in AAP(X)$ *of (2.8)-(2.9)-(2.10).*

Proof. For each $x \in AAP(X)$ we define $\mathcal{T}(x)(t)$ by means of (2.12). From our hypotheses and proceeding as in the proof of Proposition 3.1 we deduce easily that $\mathcal{T}(x) \in AAP(X)$, for each $x \in AAP(X)$. To prove the continuity of $\mathcal{T} : AAP(X) \to AAP(X)$ we fix a sequence $(x_n)_n$ in $AAP(X)$ that converges to x. Clearly $S(t-s)f(s,u_n(s)) \to S(t-s)f(s,u(s))$, $n \to \infty$, a.e. for $s \in [0,t]$. Since

$$\|S(t-s)f(s,u_n(s)) - S(t-s)f(s,u(s))\| \leq 2Nm(s)W(L),$$

for some constant $L \geq 0$, and the right hand side is an integrable function on $[0,\infty)$ we conclude that $\mathcal{T}(x_n)(t) \to \mathcal{T}(x)(t)$, $n \to \infty$, and that the convergence is uniform on $[0,\infty)$. Hence we conclude that $\mathcal{T}$ is a continuous map.

On the other hand, for $0 < \lambda < 1$, let $x_\lambda \in AAP(X)$ be such that $\lambda\mathcal{T}(x_\lambda) = x_\lambda$. Proceeding as in the proof of Theorem 2.1 we conclude that the set $\{x_\lambda : 0 < \lambda < 1\}$ is bounded.

Finally, we show that $\mathcal{T}$ is completely continuous. We take a bounded set $B \subseteq AAP(X)$. Since p and q are completely continuous, it is clear that the set of functions $C(\cdot)(x_0 - p(x)) + S(\cdot)(x_1 - q(x)), x \in B$, is relatively compact in $AAP(X)$. In addition, since the functions $f(s,u(s))$, $x \in B$, satisfy the hypotheses of Proposition 3.1 we infer that the set $\{y_x : x \in B\}$ is relatively compact in $AAP(X)$. The proof is complete. ■

Next we consider a situation where the derivative of the mild solution x defined previously also is asymptotically almost periodic. To establish our result we need the following property of almost periodic sine functions.

Lemma 3.6 *Assume that* $S(\cdot)$ *is almost periodic. Then, for each* $x \in X$ *the function* $S(\cdot)x$ *is almost periodic for the norm in* E. *Furthermore, if* $x : [0,\infty) \to X$ *is integrable then the function* $y : [0,\infty) \to E$ *given by*

$$y(t) = \int_0^t S(s)x(s)\, ds$$

is continuous with range relatively compact in E.

Proof. Since $C(\cdot)x$ and $S(\cdot)x$ are almost periodic functions it follows that $(C(\cdot)x, S(\cdot)x)$ is almost periodic in $X \times X$. Thus, given $\varepsilon > 0$, there is a relatively dense set P_ε such that

$$\|C(t+\tau)x - C(t)x\| + \|S(t+\tau)x - S(t)x\| \leq \varepsilon, \quad t \geq 0, \quad \tau \in P_\varepsilon.$$

Using that $C(\cdot)$ is uniformly bounded and

$$C(t+s)x = C(t)C(s)x \; + \; AS(t)S(s)x \tag{3.20}$$

we obtain

$$\begin{aligned}
\|S(t+\tau)x \; &- \; S(t)x\|_1 = \|S(t+\tau)x - S(t)x\| \; + \; \sup_{0\leq h\leq 1} \|AS(h)[S(t+\tau)x - S(t)x]\| \\
&\leq \; \varepsilon + \sup_{0\leq h\leq 1} \|C(t+h+\tau)x - C(h)C(t+\tau)x + C(t)C(h)x - C(t+h)x\| \\
&\leq \; \varepsilon + \sup_{0\leq h\leq 1} \|C(h)\| \, \|C(t+\tau)x - C(t)x\| \\
&\quad + \sup_{0\leq h\leq 1} \|C(t+h+\tau)x - C(t+h)x\| \\
&\leq \; (M+2)\varepsilon,
\end{aligned}$$

for all $t \geq 0$ and $\tau \in P_\varepsilon$, which proves the first assertion.

On the other hand, in the introduction we have already mentioned that $y(\cdot)$ is a continuous E-valued function. Moreover, turning to use (3.20) we infer that $S(\cdot)x(\cdot) \in \mathcal{L}^1([0,\infty); E)$ which allows us to complete the demonstration with the same argument used in the proof of Lemma 3.3. ■

The following result is consequence of the previous property.

Corollary 3.2 *Assume that the conditions of Theorem 3.2 hold, $x_0 - p(x) \in E$, and that for each $y \in E$ the function $AS(\cdot)y$ is uniformly bounded on $[0,\infty)$. Let x be the mild solution of (2.8)-(2.9)-(2.10). Then x' also is asymptotically almost periodic.*

Proof. From the uniform boundedness principle follows that the operators $AS(t) : E \to X$ are uniformly bounded for $t \geq 0$. This implies that for each $y \in E$ the function $AS(\cdot)y$ is uniformly continuous on $[0,\infty)$. In fact,

$$\|AS(t+h)y - AS(t)y\| \leq \|AS(t)\| \, \|(C(h) - I)y\| \; + \; \|C(t)\| \|AS(h)y\|$$

converges to zero as $h \to 0$, uniformly with respect to $t \geq 0$.

On the other hand, from(1.7) we can write

$$x'(t) = AS(t)(x_0 - p(x)) + C(t)(x_1 - q(x)) + \int_0^t C(t-s)f(s,u(s))\,ds$$

Since $AS(t)(x_0 - p(x))$ is the derivative of $C(t)(x_0 - p(x))$, the previous remark and the Theorem 5.2 in [22] imply that it is almost periodic. Consequently, the function $AS(t)(x_0 - p(x)) + C(t)(x_1 - q(x))$ is almost periodic. It remains to prove that the function given by

$$y(t) = \int_0^t C(t-s)f(s,u(s))\,ds$$

is asymptotically almost periodic. Turning to use Theorem 5.2 in [22], it is sufficient to show that y is uniformly continuous. To this purpose we decompose

$$\begin{aligned} y(t+h)-y(t) &= \int_0^t [C(t+h-s)-C(t-s)]f(s,u(s))\,ds \\ &\quad + \int_t^{t+h} C(t+h-s)f(s,u(s))\,ds \\ &= (C(h)-I)y(t) \;+\; AS(h)\int_0^t S(t-s)f(s,u(s))\,ds \\ &\quad + \int_t^{t+h} C(t+h-s)f(s,u(s))\,ds. \end{aligned}$$

Since $f(s,u(s))$ is integrable on $[0,\infty)$, the range of $y(\cdot)$ is relatively compact, which implies that $(C(h)-I)y(t)\to 0,\ h\to 0,$ uniformly for $t\geq 0$. The same argument serves to show that the third term of the above expression converges to zero when $h\to 0$, uniformly with respect to $t\geq 0$. Similarly, employing the preceding lemma we have that the values $\int_0^t S(t-s)f(s,u(s))\,ds,\ t\geq 0,$ are included in a relatively compact subset of E and, from the introduction, we can assert that $AS(h)\int_0^t S(t-s)f(s,u(s))\,ds\to 0,\ h\to 0,$ uniformly with respect to $t\geq 0$. This completes the proof. ∎

4 Applications

The one dimensional wave equation modeled as a abstract Cauchy problem it has been studied extensively, see for example [23]. In this section we illustrate some the results in this work with this equation. At first we introduce some relative technical remarks.

On the space $X:=L^2([0,\pi])$ we consider the operator $Af(\xi):=f''(\xi)$ with domain

$$D(A):=\{f(\cdot)\in H^2(0,\pi): f(0)=f(\pi)=0\}.$$

It is well known that A is the generator of strongly continuous cosine function $(C(t))_{t\in R}$. Furthermore, A has discrete spectrum, the eigenvalues are $-n^2,\ n\in \mathbb{N},$ with corresponding normalized eigenvectors $z_n(\xi):=\left(\frac{2}{\pi}\right)^{1/2}\sin(n\xi)$ and the following conditions holds

(a) $\{z_n : n\in \mathbb{N}\}$ is an orthonormal basis of X.

(b) If $\varphi\in D(A)$ then $A\varphi=-\sum\limits_{n=1}^{\infty} n^2 <\varphi,z_n> z_n$.

(c) For every $\varphi\in X,\ C(t)\varphi=\sum\limits_{n=1}^{\infty}\cos(nt)<\varphi,z_n>z_n$, consequently $\| C(t)\|=1$.

(d) The associated sine function is given by $S(t)\varphi=\sum_{n=1}^{\infty}\frac{\sin(nt)}{n}<\varphi,z_n>z_n$. Follows from this expression that $S(t)$ is compact, 2π-periodic and that $\| S(t)\|=1$.

(e) If G denotes the group of translations on X defined by $G(t)x(\xi) = \tilde{x}(\xi + t)$, where $\tilde{x}$ is the extension of x with period 2π, then $C(t) = \frac{1}{2}(G(t)+G(-t))$. Hence it follows, see [10], that $A = B^2$ where B is the infinitesimal generator of the group G and $E = \{x \in H^1(0,\pi) : x(0) = x(\pi) = 0\}$.

Now we consider the boundary value problem

$$\begin{aligned} \frac{\partial}{\partial t}u(t,\xi) &= \frac{\partial^2}{\partial \xi^2}u(t,\xi) + F(t,\xi,w(\xi,t),w(\xi,a(t))), && (4.21)\\ w(0,t) &= w(\pi,t) = 0, \qquad t \in [0,T], && (4.22)\end{aligned}$$

where the function $F : I\!R^4 \to I\!R$ satisfies the following Caratheodory conditions.

(a) $F(\xi,t,\cdot) : I\!R^2 \to I\!R$ is continuous $a.e.$ $\xi \in [0,\pi]$, $t \in I\!R$.

(b) For every $w_1, w_2 \in I\!R$ the function $F(\cdot,w_1,w_2) : [0,\pi] \times I\!R \to I\!R$ is measurable.

(c) There exist a measurable and bounded positive function η on $[0,\pi] \times I\!R$ such that

$$\mid F(\xi,t,w_1,w_2) \mid \leq \eta(t,\xi)(\mid w_1 \mid + \mid w_2 \mid).$$

Associated to problem (4.21)-(4.21), we consider the following nonlocal initial conditions

$$\begin{aligned} w(\xi,0) + \int_0^{T_1} P(w(\cdot,s))(\xi)d\mu(s) &= x_0(\xi), \qquad \xi \in [0,\pi], && (4.23)\\ \frac{\partial w(\xi,0)}{\partial t} + \int_0^{T_2} Q(w(\cdot,s))(\xi)d\nu(s) &= x_1(\xi), \qquad \xi \in [0,\pi], && (4.24)\end{aligned}$$

where $T_i > 0$ and μ, ν are functions of bounded variation. We also assume that $P, Q : X \to X$ are bounded continuous operators and that P is completely continuous. Let $p, q : C(I;X) \to X$ the maps defined by

$$\begin{aligned} p(x) &= \int_0^{T_1} P(x(s))d\mu(s), && (4.25)\\ q(x) &= \int_0^{T_2} Q(x(s))d\nu(s). && (4.26)\end{aligned}$$

The maps p, q are continuous and p is completely continuous. Under these conditions, the problem (4.21)-(4.22)-(4.23)-(4.24) can be to modeled as (2.8)-(2.9)-(2.10), where $f(t,x,y)(\xi) = F(t,\xi,x(\xi),y(\xi))$ satisfies the assumption $\mathbf{A_1}$. Moreover, if $x, y \in C(I;X)$ then

$$\begin{aligned} \| f(t,x,y) \| &\leq \left(\int_0^{\pi} \mid F(\xi,t,x(\xi),y(\xi)) \mid^2 d\xi\right)^{\frac{1}{2}}\\ &\leq \left(\int_0^{\pi} \eta^2(\xi,t)\left(\mid x(\xi) \mid + \mid y(\xi)\right)^2 d\xi\right)^{\frac{1}{2}}\\ &\leq \eta(\cdot,t)_{0,\pi}\left(\| x \| + \| y \|\right).\end{aligned}$$

(e) If G denotes the group of translations on X defined by $G(t)x(\xi) = \tilde{x}(\xi + t)$, where $\tilde{x}$ is the extension of x with period 2π, then $C(t) = \frac{1}{2}(G(t) + G(-t))$. Hence it follows, see [10], that $A = B^2$ where B is the infinitesimal generator of the group G and $E = \{x \in H^1(0,\pi) : x(0) = x(\pi) = 0\}$.

Now we consider the boundary value problem

$$\begin{aligned} \frac{\partial}{\partial t}u(t,\xi) &= \frac{\partial^2}{\partial \xi^2}u(t,\xi) + F(t,\xi,w(\xi,t),w(\xi,a(t))), && (4.21)\\ w(0,t) &= w(\pi,t) = 0, \qquad t \in [0,T], && (4.22) \end{aligned}$$

where the function $F : \mathbb{R}^4 \to \mathbb{R}$ satisfies the following Caratheodory conditions.

(a) $F(\xi,t,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continuous *a.e.* $\xi \in [0,\pi]$, $t \in \mathbb{R}$.

(b) For every $w_1, w_2 \in \mathbb{R}$ the function $F(\cdot, w_1, w_2) : [0,\pi] \times \mathbb{R} \to \mathbb{R}$ is measurable.

(c) There exist a measurable and bounded positive function η on $[0,\pi] \times \mathbb{R}$ such that

$$\mid F(\xi,t,w_1,w_2) \mid \leq \eta(t,\xi)(\mid w_1 \mid + \mid w_2 \mid).$$

Associated to problem (4.21)-(4.21), we consider the following nonlocal initial conditions

$$\begin{aligned} w(\xi,0) + \int_0^{T_1} P(w(\cdot,s))(\xi)d\mu(s) &= x_0(\xi), \qquad \xi \in [0,\pi], && (4.23)\\ \frac{\partial w(\xi,0)}{\partial t} + \int_0^{T_2} Q(w(\cdot,s))(\xi)d\nu(s) &= x_1(\xi), \qquad \xi \in [0,\pi], && (4.24) \end{aligned}$$

where $T_i > 0$ and μ, ν are functions of bounded variation. We also assume that $P, Q : X \to X$ are bounded continuous operators and that P is completely continuous. Let $p, q : C(I;X) \to X$ the maps defined by

$$\begin{aligned} p(x) &= \int_0^{T_1} P(x(s))d\mu(s), && (4.25)\\ q(x) &= \int_0^{T_2} Q(x(s))d\nu(s). && (4.26) \end{aligned}$$

The maps p, q are continuous and p is completely continuous. Under these conditions, the problem (4.21)-(4.22)-(4.23)-(4.24) can be to modeled as (2.8)-(2.9)-(2.10), where $f(t,x,y)(\xi) = F(t,\xi,x(\xi),y(\xi))$ satisfies the assumption $\mathbf{A_1}$. Moreover, if $x, y \in C(I;X)$ then

$$\begin{aligned} \| f(t,x,y) \| &\leq \left(\int_0^{\pi} \mid F(\xi,t,x(\xi),y(\xi)) \mid^2 d\xi\right)^{\frac{1}{2}}\\ &\leq \left(\int_0^{\pi} \eta^2(\xi,t)\left(\mid x(\xi) \mid + \mid y(\xi)\right)^2 d\xi\right)^{\frac{1}{2}}\\ &\leq \eta(\cdot,t)_{0,\pi}\left(\| x \| + \| y \|\right). \end{aligned}$$

Employing the results of the previous sections we can to prove the followings properties.

Theorem 4.9 *Under the previous conditions the following properties hold.*

(a) *If* $\int_0^T \eta(\cdot)_{0,\pi} < \infty$ *then there exist a mild solution of (2.8)-(2.9)-(2.10) for every* $x, y \in X$.

(b) *If* $g : [0,\infty) \to (0,\infty)$ *is nondecreasing and continuous,* $g(t) \to \infty$ *as* $t \to \infty$, $g(0) = 1$ *and* $g(t)^{-1} \int_0^t \eta(\cdot, s)_{0,\pi} g(s) ds \to 0$ *as* $t \to \infty$, *then there exits a mild solution* $x \in C_g^0(X)$ *of (2.8)-(2.9)-(2.10).*

(c) *If for each* $t > 0$ *and* $L \geq 0$ *the set* $\{f(s, x, y) : 0 \leq s \leq t, \|x\|, \|y\| \leq L\}$ *is relatively compact; then there exist a almost periodic solution of (2.8)-(2.9)-(2.10).*

Proof. The proof of (a), (b)(c) follows from Theorems 2.1, 3.7 and 3.8 respectively.

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