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Divergent Diagrams of Folds Associated with a Set of Involutions

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Abstract

In this work we show that the smooth classification of divergent diagrams of folds $(f_1, \ldots, f_s) : \mathbb{R}^n, 0 \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n, 0$ can be reduced to the classification of the s-tuples $(\varphi_1, \ldots, \varphi_s)$ of associated involutions. We apply the result to obtain normal forms when $s \leq n$ and $\{\varphi_1, \ldots, \varphi_s\}$ is a transversal set of linear involutions. A complete description is given when $n = s = 2$. We also present a brief discussion on applications of our results to the study of discrete reversible dynamical systems.

1 Introduction

In this work we are interested in the smooth classification of divergent diagrams

$$(f_1, \ldots, f_s) : \mathbb{R}^n, 0 \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n, 0,$$

where each germ $f_i : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ is a fold, $i = 1, \ldots, s$. More precisely, we discuss the relation between the classification of divergent diagrams of folds $(f_1, \ldots, f_s)$ and the classification of the corresponding s-tuples $(\varphi_1, \dots, \varphi_s)$ of involutions associated with these diagrams, that is, s-tuples of germs of diffeomorphisms $\varphi_i : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ such that $\varphi_i \circ \varphi_i = \mathrm{I}$, $\varphi_i \neq \mathrm{I}$ and $f_i \circ \varphi_i = f_i$, for $i = 1, \dots, s$. The definitions of the equivalence relations of s-tuples of involutions and of divergent diagrams in these classifications are given in Definition 2.2 and Definition 3.2. This work is contained in an area of study that reveals an important connection between singularities of smooth mappings and discrete dynamical systems, with particular applications to reversible systems. For a survey on this topic we refer to [2].

For any two divergent diagrams of pairs of folds (topologically) equivalent by (h, k_1, k_2), where h is a germ of diffeomorphism (homeomorphism) acting on the source and k_1, k_2 are germs of diffeomorphisms (homeomorphisms) acting on each target, h realizes the equivalence of the pairs of involutions associated with the corresponding diagrams. Also, the same h realizes a conjugacy of the compositions of the involutions. Problems related to topological stability of pairs of involutions and applications to the study of divergent diagrams of folds, reversible systems and discontinuous vector fields have been treated by Teixeira in [4], [5], [6] and [7]. In [6], he studies pairs of involutions on the plane with the help of their compositions. In [8] Voronin presents a list of problems including the analytic classification of divergent diagrams of pairs of folds and pairs of associated involutions on $\mathbb{C}, 0$. This problem is concerned with the following question: Let $(f_1, \dots, f_s)$ and $(g_1, \dots, g_s)$ be divergent diagrams of folds on $\mathbb{R}^n, 0$ and let $(\varphi_1, \dots, \varphi_s)$ and $(\psi_1, \dots, \psi_s)$ be the s-tuples of involutions associated with each of the diagrams respectively. Then, on what conditions the equivalence of $(\varphi_1, \dots, \varphi_s)$ and $(\psi_1, \dots, \psi_s)$ implies the equivalence of $(f_1, \dots, f_s)$ and $(g_1, \dots, g_s)$? Under C^0-equivalence, this question is answered by Teixeira in [5] for the case $n = s = 2$ when the fixed-point space of the involutions coincide.

The main result of this work answers the question above for the general smooth case. In fact, in section 3 we prove that two divergent diagrams of folds on $\mathbb{R}^n, 0$ are equivalent if, and only if, the associated s-tuples of involutions are equivalent. Following this result, we have an invariant for the equivalence class of a divergent diagram of folds $(f_1, \dots, f_s)$, namely the trace of the linearization at the origin of the composition $\varphi_1 \circ \cdots \circ \varphi_s$, denoted by $(d(\varphi_1 \circ \cdots \circ \varphi_s))_0$, where the s-tuple $(\varphi_1, \dots, \varphi_s)$ is associated with $(f_1, \dots, f_s)$. In section 4 we apply the main result to obtain normal forms when $s \leq n$ and $\{\varphi_1, \dots, \varphi_s\}$ is a transversal set of linear involutions. We also obtain the normal form for the situation where the set of involutions is transversal and generates an Abelian group, the involutions not necessarily linear. In addition, we present in section 5 a characterization of the orbits based on the parameters that appear in the normal forms of section 4. In section 6 we treat the particular case $n = s = 2$ to describe the stratification of the plane of parameters whose strata correspond to these orbits, and show that for almost all diagrams (f_1, f_2), the knowledge of the invariant $\mathrm{tr}(\varphi_1 \circ \varphi_2)$ determines the equivalence class of (f_1, f_2). Finally, in section 7 we apply the results to a special case of reversible planar diffeomorphisms.

2 Definitions and preliminary results

Definition 2.1 *An* involution *is a germ of diffeomorphism* $\varphi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ *satisfying* $\varphi \circ \varphi = I$.

Let $\mathrm{Fix}(\varphi)$ denote the fixed-point set of φ,

$$\mathrm{Fix}(\varphi) = \{x \in \mathbb{R}^n, 0 : \varphi(x) = x\}.$$

Definition 2.2 *Two s-tuples* $(\varphi_1, \ldots, \varphi_s)$ *and* $(\psi_1, \ldots, \psi_s)$ *of involutions on* $\mathbb{R}^n, 0$ *are said to be equivalent if there exists a germ of diffeomorphism* h *of* $\mathbb{R}^n, 0$ *such that* $\psi_i = h \circ \varphi_i \circ h^{-1}$, *for all* $i = 1, \ldots, s$.

Note that in the situation of Definition 2.2 the germ of diffeomorphism h satisfies

$$h(\mathrm{Fix}(\varphi_i)) = \mathrm{Fix}(\psi_i), \quad i = 1, \ldots, s.$$

Consider a set $G_s = \{\varphi_1, \ldots, \varphi_s\}$ of involutions on $\mathbb{R}^n, 0$ and let $\Lambda_s = [\varphi_1, \ldots, \varphi_s]$ denote the group generated by the φ_i's. The following theorem appears in [3], (Theorem 1, p.206) and it is restated here in our particular context:

Theorem 2.3 *If* Λ_s *is a compact group, then the s-tuple* $(\varphi_1, \ldots, \varphi_s)$ *is equivalent to an s-tuple* $(\psi_1, \ldots, \psi_s)$ *of linear involutions.*

It is an immediate consequence of Theorem 2.3 that any involution $\varphi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ is conjugate to a linear involution. This implies that $\mathrm{Fix}(\varphi)$ is locally diffeomorphic to a linear subspace of $\mathbb{R}^n$; therefore, $\mathrm{Fix}(\varphi)$ is a submanifold in $\mathbb{R}^n, 0$. This also implies that one of the following holds: (a) φ is the identity I; (b) φ is conjugate to -I; (c) if $\dim \mathrm{Fix}(\varphi) = n - k$ ($k \neq 0, n$), then φ is conjugate to the canonical form

$$(x_1, \ldots, x_n) \mapsto (-x_1, \ldots, -x_k, \ldots, x_n). \tag{2.1}$$

Definition 2.4 *Given an involution* φ *on* $\mathbb{R}^n, 0$ *and a fold* $f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$, *we say that* f *is associated with* φ, *or* φ *is associated with* f, *if* $\varphi \neq I$ *and* $f \circ \varphi = f$.

We now give some general results concerned with an involution and a fold associated with it.

Lemma 2.5 *Given a fold* $f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$, *there exists a unique involution associated with* f.

Proof. Since f is a fold, then it is $\mathcal{A}$-equivalent to the fold f^0 given by

$$f^0 : (x_1, \ldots, x_n) \mapsto (x_1^2, \ldots, x_n), \tag{2.2}$$

that is, there exist germs of diffeomorphisms h and k of $\mathbb{R}^n, 0$ such that $f = k \circ f^0 \circ h^{-1}$. Now, it is not difficult to show that there exists a unique involution

φ^0 associated with f^0, namely $\varphi^0(x_1,\dots,x_n) = (-x_1,\dots,x_n)$. Therefore, $\varphi = h\circ\varphi^0\circ h^{-1}$ is the unique involution associated with f.

□

From the proof of Lemma 2.5 above, for the involution φ associated with the fold f we have that dim Fix$(\varphi) = n-1$. In addition, it is easy to see that

$$\mathrm{Fix}(\varphi) = \Sigma(f),$$

where $\Sigma(f)$ denotes the set of singular points of f.

Lemma 2.6 *Given an involution φ on $\mathbb{R}^n,0$ with* dim Fix$(\varphi) = n-1$, *there exists a fold $f : \mathbb{R}^n,0 \to \mathbb{R}^n,0$ associated with φ.*

Proof. Being the dimension of Fix(φ) equal to $n-1$, φ is conjugate to the involution φ^0, $\varphi^0(x_1,\dots,x_n) = (-x_1,x_2,\dots,x_n)$. Hence, there exists a germ of diffeomorphism h of $\mathbb{R}^n,0$ such that $\varphi = h\circ\varphi^0\circ h^{-1}$. The germ f^0 in (2.2) is a fold associated with φ^0, so $f = f^0\circ h^{-1}$ is a fold associated with φ. □

Remark 2.7 *A fold associated with an involution φ is not uniquely determined. In fact, if f is a fold associated with φ, then any germ $g \in \Big(\mathcal{S}(\varphi)\ \times\ \mathcal{L}\Big)\cdot f$ is also a fold associated with φ, where $\mathcal{L}$ is the group of left equivalences and $\mathcal{S}(\varphi)$ is the subgroup of the group $\mathcal{R}$ of right equivalences given by*

$$\mathcal{S}(\varphi) = \{h\in\mathcal{R} : h\circ\varphi\circ h^{-1} = \varphi\}.$$

Corollary 2.9 below states that the set of all folds associated with φ is precisely the orbit $\Big(\mathcal{S}(\varphi)\ \times\ \mathcal{L}\Big)\cdot f$.

For our purposes we have only to consider involutions φ on $\mathbb{R}^n,0$ for which dim Fix$(\varphi) = n-1$, so this condition is assumed from now on.

Proposition 2.8 *Let φ_1, φ_2 be involutions on $\mathbb{R}^n,0$ and let $f_1 : \mathbb{R}^n,0 \to \mathbb{R}^n,0$ be a fold associated with φ_1. Then $f_2 : \mathbb{R}^n,0 \to \mathbb{R}^n,0$ is a fold associated with φ_2 if, and only if, there exist germs of diffeomorphisms h and k of $\mathbb{R}^n,0$ such that*

$$\varphi_2 = h\circ\varphi_1\circ h^{-1} \quad \text{and} \quad f_2 = k\circ f_1\circ h^{-1}.$$

Proof. Sufficiency is immediate. To prove the necessity, suppose that f_2 is a fold associated with φ_2. Since f_1 and f_2 are folds, they are $\mathcal{A}$-equivalent, so there exists germs of diffeomorphisms h and k of $\mathbb{R}^n,0$ such that $f_2 = k\circ f_1\circ h^{-1}$. Also, $f_2\circ\varphi_2 = f_2$, then the involution $h^{-1}\circ\varphi_2\circ h$ is associated with f_1. By the uniqueness of of the involution associated with a fold, it follows that $h^{-1}\circ\varphi_2\circ h = \varphi_1$, or $\varphi_2 = h\circ\varphi_1\circ h^{-1}$.

□

If we take $\varphi_1 = \varphi_2 = \varphi$ in Proposition 2.8, we have the following:

Corollary 2.9 *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ be a fold associated with φ. Then $g : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ is also a fold associated with φ if, and only if, $g \in \Big(\mathcal{S}(\varphi) \times \mathcal{L}\Big) \cdot f$.*

3 The main result

We start with two definitions.

Definition 3.1 *Let $\varphi_1, \ldots, \varphi_s$ be involutions on $\mathbb{R}^n, 0$ and $(f_1, \ldots, f_s) : \mathbb{R}^n, 0 \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n, 0$ a divergent diagram of folds. We say that $(f_1, \ldots, f_s)$* is associated with the s-tuple $(\varphi_1, \ldots, \varphi_s)$, *or* $(\varphi_1, \ldots, \varphi_s)$ is associated with $(f_1, \ldots, f_s)$, *if f_i is a fold associated with φ_i for all $i = 1, \ldots, s$.*

In view of Lemma 2.5, for a given divergent diagram of folds $(f_1, \ldots, f_s)$, there exists a unique s-tuple of involutions $(\varphi_1, \ldots, \varphi_s)$ associated with it.

Consider the space of the divergent diagrams

$$(f_1, \ldots, f_s) : \mathbb{R}^n, 0 \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n, 0. \tag{3.3}$$

The concept of equivalence in this space is given by the following definition:

Definition 3.2 *Two divergent diagrams $(f_1, \ldots, f_s), (g_1, \ldots, g_s) : \mathbb{R}^n, 0 \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n, 0$ are equivalent if there exist germs of diffeomorphisms $h, k_1, \ldots, k_s$ of $\mathbb{R}^n, 0$ such that $g_i = k_i \circ f_i \circ h^{-1}$, for all $i = 1, \ldots, s$.*

In this section we show that the classification of divergent diagrams of folds can be reduced to the classification of the s-tuples of involutions associated with the diagrams. This is given in Theorem 3.9. We now present some general results for s-tuples of involutions and their associated divergent diagrams of folds.

The next two results are the corresponding for diagrams of Proposition 2.8 and Corollary 2.9 respectively.

Lemma 3.3 *Let $(f_1, \ldots, f_s)$ be a divergent diagram of folds associated with $(\varphi_1, \ldots, \varphi_s)$. Then a divergent diagram $(g_1, \ldots, g_s)$ is a diagram of folds associated with $(\psi_1, \ldots, \psi_s)$ and equivalent to $(f_1, \ldots, f_s)$ if, and only if, there exist germs of diffeomorphisms $h, k_1, \ldots, k_s$ such that*

$$\psi_i = h \circ \varphi_i \circ h^{-1}$$

and

$$g_i = k_i \circ f_i \circ h^{-1}$$

for all $i = 1, \ldots, s$.

Given the involutions $\varphi_1, \ldots, \varphi_s$ on $\mathbb{R}^n, 0$, let $\mathcal{S}(\varphi_1, \ldots, \varphi_s)$ denote the subgroup of $\mathcal{R}$ given by the intersection $\mathcal{S}(\varphi_1) \cap \cdots \cap \mathcal{S}(\varphi_s)$. From Lemma 3.3, by taking $\psi_1 = \varphi_1, \ldots, \psi_s = \varphi_s$, we have the following result:

Corollary 3.4 *Let $(f_1, \dots, f_s)$ be a divergent diagram of folds associated with $(\varphi_1, \dots, \varphi_s)$. Then a divergent diagram $(g_1, \dots, g_s)$ is a diagram of folds associated with $(\varphi_1, \dots, \varphi_s)$ and equivalent to $(f_1, \dots, f_s)$ if, and only if,*

$$(g_1, \dots, g_s) \in \Big(\mathcal{S}(\varphi_1, \dots, \varphi_s) \times (\mathcal{L} \times \cdots \times \mathcal{L})\Big) \cdot (f_1, \dots, f_s).$$

Let i be a fixed integer, $1 \leq i \leq n$. Consider the involution φ_i^0 given by $\varphi_i^0(x_1, \dots, x_n) = (x_1, \dots, -x_i, \dots, x_n)$ and the fold f_i^0, $f_i^0(x_1, \dots, x_n) = (x_1, \dots, x_i^2, \dots, x_n)$, associated with φ_i^0. Then,

Lemma 3.5 *A germ $g : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ is a fold associated with φ_i^0 if, and only if, g is $\mathcal{L}$-equivalent to f_i^0.*

Proof. Suppose that g is a fold associated with φ_i^0. The equality $g \circ \varphi_i^0 = g$ implies that the components of g are even in x_i. So we can write

$$\begin{aligned} g(x_1, \dots, x_n) &= \Big(k_1(x_1, \dots, x_i^2, \dots, x_n), \dots, k_n(x_1, \dots, x_i^2, \dots, x_n)\Big) = \\ &= (k \circ f_i^0)(x_1, \dots, x_n), \end{aligned}$$

where $k = (k_1, \dots, k_n)$. Now, as g is a fold, k is a germ of diffeomorphism. Therefore, g is $\mathcal{L}$-equivalent to f_i^0. The converse follows immediately from Corollary 2.9. □

Comparing Corollary 2.9 and Lemma 3.5, we have

$$(\mathcal{S}(\varphi_i^0) \times \mathcal{L}) \cdot f_i^0 \;=\; \mathcal{L} \cdot f_i^0. \tag{3.4}$$

Let us also note that, alternatively, we could prove Lemma 3.5 by first showing (3.4), and then applying Corollary 2.9. However, the proof presented above is more direct.

The next proposition generalizes Lemma 3.5 above.

Proposition 3.6 *Let φ be an involution on $\mathbb{R}^n, 0$, and let h be a germ of diffeomorphism of $\mathbb{R}^n, 0$ such that $\varphi = h \circ \varphi_i^0 \circ h^{-1}$. Consider the fold $f_i^0 \circ h^{-1}$ associated with φ. Then a germ $g : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ is also a fold associated with φ if, and only if, g is $\mathcal{L}$-equivalent to $f_i^0 \circ h^{-1}$.*

Proof. If g is a fold associated with φ, then $g \circ h$ is a fold associated with φ_i^0. From Lemma 3.5 it follows that $g \circ h$ is $\mathcal{L}$-equivalent to f_i^0, that is, g is $\mathcal{L}$-equivalent to $f_i^0 \circ h^{-1}$. The converse of each statement above also holds. □

A generalization for divergent diagrams of Proposition 3.6 is given by

Proposition 3.7 *Let $i_1, \dots, i_s$ be fixed integers, $1 \leq i_1, \dots, i_s \leq n$. Let $\psi_1, \dots, \psi_s$ be involutions on $\mathbb{R}^n, 0$ and let $h_1, \dots, h_s$ be germs of diffeomorphism of $\mathbb{R}^n, 0$ such that $\psi_j = h_j \circ \varphi_{i_j}^0 \circ h_j^{-1}$, for all $j = 1, \dots, s$. Then, a divergent diagram $(g_1, \dots, g_s) : \mathbb{R}^n, 0 \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n, 0$ is a diagram of folds associated with $(\psi_1, \dots, \psi_s)$ if, and only if,*

$$(g_1, \dots, g_s) \in (\mathcal{L} \times \cdots \times \mathcal{L}) \cdot (f_{i_1}^0 \circ h_1^{-1}, \dots, f_{i_s}^0 \circ h_s^{-1}).$$

The following proposition is useful in calculations. It allows us to replace a given s-tuple of involutions for another with a simpler treatment.

Proposition 3.8 *Let $(\varphi_1, \ldots, \varphi_s)$ be an s-tuple of involutions equivalent to $(\psi_1, \ldots, \psi_s)$ given in Proposition 3.7. Then every divergent diagram of folds $(g_1, \ldots, g_s)$ associated with $(\varphi_1, \ldots, \varphi_s)$ is equivalent to $(f^0_{i_1} \circ h_1^{-1}, \ldots, f^0_{i_s} \circ h_s^{-1})$.*

Proof. Let h be a germ of diffeomorphism of $\mathbb{R}^n, 0$ such that $\varphi_j = h \circ \psi_j \circ h^{-1}$ for all $j = 1, \ldots, s$. The divergent diagram of folds $(g_1 \circ h, \ldots, g_s \circ h)$ is associated with $(\psi_1, \ldots, \psi_s)$. Then, by Proposition 3.7, we have

$$(g_1 \circ h, \ldots, g_s \circ h) \in (\mathcal{L} \times \cdots \times \mathcal{L}) \cdot (f^0_{i_1} \circ h_1^{-1}, \ldots, f^0_{i_s} \circ h_s^{-1}),$$

which concludes the proof. □

We are now in position to state the key result to perform the classification of divergent diagrams of folds via the associated involutions.

Theorem 3.9 *Let $(f_1, \ldots, f_s)$ be a divergent diagram of folds associated with $(\varphi_1, \ldots, \varphi_s)$ and let $(g_1, \ldots, g_s)$ be a divergent diagram of folds associated with $(\psi_1, \ldots, \psi_s)$. Then, $(f_1, \ldots, f_s)$ and $(g_1, \ldots, g_s)$ are equivalent if, and only if, $(\varphi_1, \ldots, \varphi_s)$ and $(\psi_1, \ldots, \psi_s)$ are equivalent.*

Proof. Suppose that there exists a germ of diffeomorphism h of $\mathbb{R}^n, 0$ such that $\varphi_j = h \circ \psi_j \circ h^{-1}$, $j = 1, \ldots, s$. Then, $(f_1 \circ h, \ldots, f_s \circ h)$ is a divergent diagram of folds associated with $(\psi_1, \ldots, \psi_s)$. By Proposition 3.7,

$$(f_1 \circ h, \ldots, f_s \circ h), (g_1, \ldots, g_s) \in (\mathcal{L} \times \ldots \times \mathcal{L}) \cdot (f^0_{i_1} \circ h_1^{-1}, \ldots, f^0_{i_s} \circ h_s^{-1}),$$

for certain germs of diffeomorphism $h_1, \ldots, h_s$ of $\mathbb{R}^n, 0$. Therefore, $(f_1, \ldots, f_s)$ and $(g_1, \ldots, g_s)$ are equivalent.

The converse follows from Lemma 3.3. □

As a consequence of the result above, for the case of divergent diagrams of folds $(f_1, \ldots, f_s)$, we have that $\mathrm{tr}(d(\varphi_1 \circ \cdots \circ \varphi_s))_0$ is an invariant up to equivalence, where $(\varphi_1, \ldots, \varphi_s)$ is the s-tuple associated with $(f_1, \ldots, f_s)$.

4 Transversal sets of involutions

In this section we obtain normal forms for special classes of divergent diagrams of folds. These are given in Theorem 4.3 and Theorem 4.5. We start with a definition.

Definition 4.1 *A set $G_s = \{\varphi_1, \ldots, \varphi_s\}$ of involutions on $\mathbb{R}^n, 0$, $s \le n$, is said to be* transversal *if $\mathrm{Fix}(\varphi_i)$ is transversal to $\mathrm{Fix}(\varphi_j)$ at 0 for $i \neq j$ and $\mathrm{codim} \cap_{i=1}^s \mathrm{Fix}(\varphi_i) = \sum_{i=1}^s \mathrm{codim}\ \mathrm{Fix}(\varphi_i)$.*

The next result is essentially a result from Linear Algebra.

Lemma 4.2 *Let $G_s = \{\varphi_1, \dots, \varphi_s\}$ be a transversal set of linear involutions. Then $(\varphi_1, \dots, \varphi_s)$ is linearly equivalent to $(\psi_1, \dots, \psi_s)$ such that, for each ψ_i, $\mathrm{Fix}(\psi_i)$ is given by the equation $x_i = 0$ and, therefore, ψ_i has the form*

$$\psi_i(x_1, \dots, x_n) = (x_1 + a_{i1}x_i, \dots, -x_i, \dots, x_n + a_{in}x_i), \tag{4.5}$$

for some constants a_{ij}, $j \neq i$, $1 \leq j \leq s$.

Theorem 4.3 *Let $G_s = \{\varphi_1, \dots, \varphi_s\}$ be as in Lemma 4.2. Then any divergent diagram of folds $(g_1, \dots, g_s)$ associated with $(\varphi_1, \dots, \varphi_s)$ is equivalent to the diagram of folds $(f_1, \dots, f_s)$ associated with $(\psi_1, \dots, \psi_s)$, where ψ_i is given by (4.5) and*

$$f_i(x_1, \dots, x_n) = (x_1 + \frac{a_{i1}}{2}x_i, \dots, x_i^2, \dots, x_n + \frac{a_{in}}{2}x_i). \tag{4.6}$$

Proof. By Lemma 4.2, $(\varphi_1, \dots, \varphi_s)$ is equivalent to $(\psi_1, \dots, \psi_s)$. Consider now the germ at zero of the isomorphism h_i of $\mathbb{R}^n$ given by

$$h_i(x_1, \dots, x_n) = (x_1 - \frac{a_{i1}}{2}x_i, \dots, x_i, \dots, x_n - \frac{a_{in}}{2}x_i).$$

We have $\psi_i = h_i \circ \varphi_i^0 \circ h_i^{-1}$. Moreover, the formula (4.6) defines the fold $f_i = f_i^0 \circ h_i^{-1}$, which is associated with ψ_i. Hence, by Proposition 3.8, $(g_1, \dots, g_s)$ is equivalent to $(f_1, \dots, f_s)$. □

The following two results are concerned with the case when a transversal set of involutions generates an Abelian group, the involutions not being necessarily linear.

Lemma 4.4 *If G_s is transversal and Λ_s is Abelian, then $(\varphi_1, \dots, \varphi_s)$ is equivalent to $(\varphi_1^0, \dots, \varphi_s^0)$, where*

$$\varphi_i^0(x_1, \dots, x_n) = (x_1, \dots, -x_i, \dots, x_n), \quad i = 1, \dots, s. \tag{4.7}$$

Proof. Since Λ_s is Abelian and its generators are involutions, then it is a finite group. By Theorem 2.3, $(\varphi_1, \dots, \varphi_s)$ is equivalent to $(\bar{\varphi}_1, \dots, \bar{\varphi}_s)$, with each $\bar{\varphi}_i$ linear. Since G_s is transversal, then $\bar{G}_s = \{\bar{\varphi}_1, \dots, \bar{\varphi}_s\}$ is transversal, and using again the property of Λ_s being Abelian we also have $\bar{\Lambda}_s = [\bar{\varphi}_1, \dots, \bar{\varphi}_s]$ Abelian. Now we can argue in two ways: One would be to use the fact that $\bar{\Lambda}_s$ is Abelian and deal with the normal forms (4.5). The other, more direct, is to use the simultaneous diagonalization of commuting operators. This result implies that we can assume that the linear involutions $\bar{\varphi}_i$'s have each diagonal matricial form. Finally, by using a permutation matrix if necessary, $(\bar{\varphi}_1, \dots, \bar{\varphi}_s)$ is equivalent to $(\varphi_1^0, \dots, \varphi_s^0)$, where φ_i^0 is given by (4.7). □

From the two previous results we get the following:

Theorem 4.5 *If G_s is transversal and Λ_s is Abelian, then any divergent diagram of folds $(g_1, \dots, g_s)$ associated with $(\varphi_1, \dots, \varphi_s)$ is equivalent to the divergent diagram of folds $(f_1^0, \dots, f_s^0)$*

$$f_i^0(x_1, \dots, x_n) = (x_1, \dots, x_i^2, \dots, x_n).$$

5 Characterization of orbits

In this section we describe the orbits of divergent diagrams of folds $(f_1, \dots, f_s) : \mathbb{R}^n, 0 \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n, 0$ associated with s-tuples $(\varphi_1, \dots, \varphi_s)$ of linear involutions on $\mathbb{R}^n, 0$ when the set $G_s = \{\varphi_1, \dots, \varphi_s\}$ is transversal. According to Theorem 3.9 and Theorem 4.3, it suffices to describe the orbits of the s-tuples $(\psi_1, \dots, \psi_s)$, where ψ_i is as (4.5). This is given in the following result:

Proposition 5.1 *Consider the s-tuples of linear involutions $(\psi_{1_a}, \dots, \psi_{s_a})$ and $(\psi_{1_b}, \dots, \psi_{s_b})$, where*

$$\psi_{i_a}(x_1, \dots, x_n) = (x_1 + a_{i1}x_i, \dots, -x_i, \dots, x_n + a_{in}x_i),$$

$$\psi_{i_b}(x_1, \dots, x_n) = (x_1 + b_{i1}x_i, \dots, -x_i, \dots, x_n + b_{in}x_i),$$

for all $i = 1, \dots, s$. Then $(\psi_{1_a}, \dots, \psi_{s_a})$ and $(\psi_{1_b}, \dots, \psi_{s_b})$ are equivalent if, and only if, there exists a nonsingular matrix

$$H = \left(\begin{array}{cccc|ccc} \alpha_1 & & & & & & \\ & \alpha_2 & 0 & & & 0 & \\ & 0 & \ddots & & & & \\ & & & \alpha_s & & & \\ \hline \delta_{s+1} & \gamma_{s+1,2} & \cdots & \gamma_{s+1,s} & \beta_{s+1,s+1} & \cdots & \beta_{s+1,n} \\ \vdots & & \ddots & & & \ddots & \\ \delta_n & \gamma_{n2} & & \gamma_{ns} & \beta_{n,s+1} & & \beta_{nn} \end{array}\right) \tag{5.8}$$

such that $\alpha_1 = 1$ and if $1 \le i, j \le s$, $i \ne j$, then

$$b_{ij} = \frac{\alpha_j}{\alpha_i} a_{ij}. \tag{5.9}$$

If $s + 1 \le j \le n$, then

$$b_{1j} = \sum_{k=2}^{s} \gamma_{jk} a_{1k} + \sum_{k=s+1}^{n} \beta_{jk} a_{1k} - 2\delta_j, \tag{5.10}$$

and for $s + 1 \le j \le n$ and $2 \le i \le s$,

$$b_{ij} = \frac{1}{\alpha_i}(\delta_j a_{i1} + \sum_{k=2, k\ne i}^{s} \gamma_{jk} a_{ik} + \sum_{k=s+1}^{n} \beta_{jk} a_{ik} - 2\gamma_{ji}). \tag{5.11}$$

Proof. The s-tuples $(\psi_{1_a}, \dots, \psi_{s_a})$ and $(\psi_{1_b}, \dots, \psi_{s_b})$ are equivalent if, and only if, there exists a germ of diffeomorphism h of $\mathbb{R}^n, 0$ such that

$$\psi_{i_b} = h \circ \psi_{i_a} \circ h^{-1}. \tag{5.12}$$

By taking the derivatives at zero we can assume that h is linear. It is now straightforward to verify that the linear diffeomorphism has matrix of the form (5.8). The fact that α_1 can be taken to be 1 follows from the property that if h satisfy (5.12), then for any nonzero constant α, αh also satisfy (5.12). □

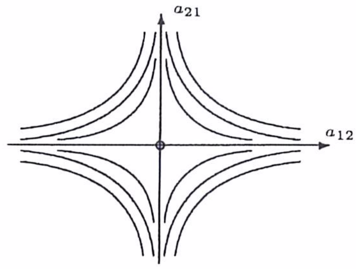

Figure 1: Stratification of the plane (a_{12}, a_{21}) determined by the orbits of the pairs of involutions (ψ_{1_a}, ψ_{2_a}).

6 The case $n = s$=2

In this section we apply the results of sections 4 and 5 to the particular case of pairs of transversal linear involutions on the plane. We start by describing the orbits on the plane of parameters. In subsection 6.1 we present the normal forms for the pairs of involutions and in subsection 6.2 the normal forms for the associated divergent diagrams. As we shall see, the normal forms for almost all pairs depend on one parameter, namely the trace of the composition $\varphi_1 \circ \varphi_2$. This fact is applied in section 7 in the discussion of reversible diffeomorphisms.

Following the notation of section 5, for the particular case of $s = n = 2$ we can easily explicit the stratification of the plane (a_{12}, a_{21}) whose strata correspond to the orbits of the pairs (ψ_{1_a}, ψ_{2_a}).

More precisely, consider the pairs (ψ_{1_a}, ψ_{2_a}) and (ψ_{1_b}, ψ_{2_b}), where

$$\psi_{1_a}(x_1, x_2) = (-x_1, x_2 + a_{12}x_1),$$

$$\psi_{2_a}(x_1, x_2) = (x_1 + a_{21}x_2, -x_2)$$

and

$$\psi_{1_b}(x_1, x_2) = (-x_1, x_2 + b_{12}x_1),$$

$$\psi_{2_b}(x_1, x_2) = (x_1 + b_{21}x_2, -x_2).$$

From (5.9) we have that (ψ_{1_a}, ψ_{2_a}) and (ψ_{1_b}, ψ_{2_b}) are equivalent if, and only if, there exists a nonzero constant α such that

$$b_{12} = \alpha a_{12}$$

$$b_{21} = \frac{1}{\alpha} a_{21}.$$

Therefore, the required stratification of the plane (a_{12}, a_{21}) is given in Fig.1. Each orbit is given by a hyperbole, by an axis minus the origin or by the origin itself.

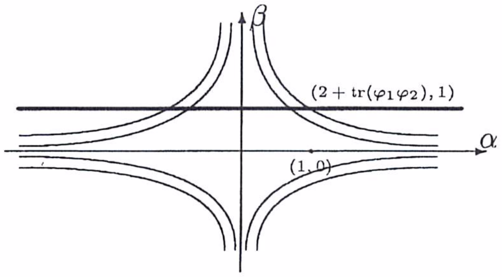

Figure 2: The points on the horizontal thick line together with the point (1,0) marked on the α-axis represent all the orbits of pairs of transversal linear involutions that generate a non-Abelian group, the point (α,β) representing the pair (ψ_1,ψ_2), where $\psi_1(x,y)=(-x,y+\alpha x)$ and $\psi_2(x,y)=(x+\beta y,-y)$.

6.1 Normal forms of pairs of involutions

In this subsection we present the classification of pairs (φ_1,φ_2) of linear involutions on $\mathbb{R}^2$ when $G_2=\{\varphi_1,\varphi_2\}$ is transversal. If the group $\Lambda_2=[\varphi_1,\varphi_2]$ generated by φ_1,φ_2 is Abelian, then according to Lemma 4.4 for $n=s=2$, (φ_1,φ_2) is equivalent to the canonical pair $(\varphi_1^0,\varphi_2^0)$, where

$$\varphi_1^0(x,y)=(-x,y),\quad \varphi_2^0(x,y)=(x,-y). \tag{6.13}$$

So in what follows we treat the case Λ_2 non-Abelian.

First we notice that the general matricial form of a linear involution φ on the plane is

$$\varphi=\begin{pmatrix} a & b \\ c & -a\end{pmatrix},\quad a^2+bc=1.$$

From Fig. 1, we clearly have that each pair (φ_1,φ_2) is equivalent to (ψ_1,ψ_2), where $\psi_1(x,y)=(-x,y+\alpha x)$ and $\psi_2(x,y)=(x+\beta y,-y)$, with a unique representative point (α,β) on a straight line or on a point as in Fig. 2. It is an interesting fact that in this classification the line can be parametrized by the trace $\mathrm{tr}(\varphi_1\circ\varphi_2)$, an invariant under conjugacy. This is given in the following result.

Theorem 6.1 *Consider any pair of transversal linear involutions on* $\mathbb{R}^2$

$$\varphi_1=\begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1\end{pmatrix},\quad \varphi_2=\begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2\end{pmatrix},\quad a_i^2+b_ic_i=1,\quad i=1,2,$$

with $\Lambda_2=[\varphi_1,\varphi_2]$ *non-Abelian. If* $a_1=1$, *take the number* $b=-c_1a_2+c_2-\frac{1}{4}c_1^2b_2$; *if* $a_1\neq 1$, *take* $b=\frac{1}{2(1-a_1)}(-2a_1b_1a_2+2b_1a_2-b_1^2c_2+a_1^2b_2-2a_1b_2+b_2)$. *Then, if* $b\neq 0$, *then* (φ_1,φ_2) *is equivalent to* (ψ_1,ψ_2), *where*

$$\psi_1=\begin{pmatrix} -1 & 0 \\ 2+\mathrm{tr}(\varphi_1\circ\varphi_2) & 1\end{pmatrix},\quad \psi_2=\begin{pmatrix} 1 & 1 \\ 0 & -1\end{pmatrix}, \tag{6.14}$$

and distinct values of the trace corresponding to distinct orbits. If $b = 0$, then (φ_1, φ_2) is equivalent to (ψ_1, ψ_2), with

$$\psi_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.15}$$

Proof. We can reduce the first involution to a canonical form such that (φ_1, φ_2) is equivalent to $(\tilde{\varphi}_1, \tilde{\varphi}_2)$, with

$$\tilde{\varphi}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{\varphi}_2 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

with $a^2 + bc = 1$; note that $a = -\frac{1}{2}\ \mathrm{tr}(\varphi_1 \circ \varphi_2)$. Also, the equivalence can be taken such that

$$\begin{array}{ll} a_1 = 1 \Rightarrow & b = -c_1 a_2 + c_2 - \frac{1}{4} c_1^2 b_2 \\ a_1 \neq 1 \Rightarrow & b = \frac{1}{2(1-a_1)}(-2a_1 b_1 a_2 + 2b_1 a_2 - b_1^2 c_2 + a_1^2 b_2 - 2a_1 b_2 + b_2) \end{array}.$$

Λ_2 non-Abelian implies that $b^2 + c^2 \neq 0$. Our aim now is to reduce the pair $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ to (ψ_1, ψ_2) as in (4.5). If $b \neq 0$ (which already implies in transversality for any a and c), we take the linear isomorphism

$$h = \frac{1}{b} \begin{pmatrix} 1 & 0 \\ a-1 & b \end{pmatrix}$$

and obtain

$$\psi_1 = h \circ \tilde{\varphi}_1 \circ h^{-1} = \begin{pmatrix} -1 & 0 \\ 2(1-a) & 1 \end{pmatrix}, \qquad \psi_2 = h \circ \tilde{\varphi}_2 \circ h^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Let us note that, by the stratification given in Fig. 1, distinct values of a give pairs in distinct orbits.

If $b = 0$, then $c \neq 0$ and by transversality $a = 1$. We then take the linear isomorphism

$$k = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ -c & 2 \end{pmatrix}$$

and obtain

$$k \circ \tilde{\varphi}_1 \circ k^{-1} = \begin{pmatrix} -1 & 0 \\ 2c & 1 \end{pmatrix}, \qquad k \circ \tilde{\varphi}_2 \circ k^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By the same stratification, we can take $c = 1/2$, so we have

$$\psi_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

□

The following result is a consequence of Theorem 6.1 above and has important aplications in the classification of divergent diagrams of two folds, as discussed in subsection 6.2, and in reversible systems, as discussed in section 7.

Corollary 6.2 *Two pairs of transversal linear involutions* (φ_1, φ_2) *and* $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ *with orbit representatives out of the* a_{12}*-axis on the plane of Fig. 1 are equivalent if, and only if, the compositions* $\varphi_1 \circ \varphi_2$ *and* $\tilde{\varphi}_1 \circ \tilde{\varphi}_2$ *are conjugate.*

6.2 Normal forms of divergent diagrams of folds

In this subsection we present the classification of divergent diagrams of folds

$$(f_1, f_2) : \mathbb{R}^2, 0 \to \mathbb{R}^2 \times \mathbb{R}^2, 0$$

associated with pairs (φ_1, φ_2) of transversal linear involutions (φ_1, φ_2) on $\mathbb{R}^2, 0$. This classification is obtained via the classification of pairs of involutions presented in the previous subsection and is given by the following theorem:

Theorem 6.3 *Let* $(f_1, f_2) : \mathbb{R}^2, 0 \to \mathbb{R}^2 \times \mathbb{R}^2, 0$ *be a divergent diagram of folds associated with a pair* (φ_1, φ_2) *of transversal linear involutions on* $\mathbb{R}^2, 0$*. Consider the group* $\Lambda_2 = [\varphi_1, \varphi_2]$.
(a) If Λ_2 *is Abelian, then* (f_1, f_2) *is equivalent to the canonical diagram* (f_1^0, f_2^0)*, where*

$$f_1^0(x, y) = (x^2, y), \quad f_2^0(x, y) = (x, y^2). \tag{6.16}$$

(b) Suppose Λ_2 *non-Abelian. If* (φ_1, φ_2) *is equivalent to* (ψ_1, ψ_2) *given in (6.14), then* (f_1, f_2) *is equivalent to* (g_1, g_2) *with*

$$g_1(x, y) = (x^2, y + (1 + \frac{1}{2}\mathrm{tr}(\varphi_1\varphi_2)x), \quad g_2(x, y) = (x + \frac{1}{2}y, y^2). \tag{6.17}$$

If (φ_1, φ_2) *is equivalent to* (ψ_1, ψ_2) *given in (6.15), then* (f_1, f_2) *is equivalent to* (g_1, g_2) *with*

$$g_1(x, y) = (x^2, y + \frac{1}{2}x), \quad g_2(x, y) = (x, y^2). \tag{6.18}$$

Proof. Part (a) is given by Theorem 4.5. Part (b) is a consequence of Theorem 3.9, Theorem 4.3 and Theorem 6.1. □

As we have already remarked, the number $\mathrm{tr}(\varphi_1 \circ \varphi_2)$ is an invariant for the equivalence class of (f_1, f_2). It is now a consequence of Theorem 6.1 that given the pairs (f_1, f_2) and $(\tilde{f}_1, \tilde{f}_2)$ associated with (φ_1, φ_2) and $(\tilde{\varphi}_1, \tilde{\varphi}_2)$, respectively, with orbit representatives out of the horizontal axis of Fig. 2, then they are equivalent if, and only if, $\mathrm{tr}(\varphi_1 \circ \varphi_2) = \mathrm{tr}(\tilde{\varphi}_1 \circ \tilde{\varphi}_2)$.

7 Reversible planar diffeomorphisms

In this section we discuss how Theorem 6.1 can be applied to the study of reversible diffeomorphisms on the plane. Given an involution φ on $\mathbb{R}^n, 0$, we say that a germ of diffeomorphism $F : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ is φ-reversible if

$$\varphi \circ F = F^{-1} \circ \varphi.$$

The presence and importance of reversing symmetry has been recognized since the early days of dynamical systems by Birkhoff (for more detais, see [2]). Note that for any two involutions φ_1 and φ_2 on $\mathbb{R}^n, 0$, the composition $F = \varphi_1 \circ \varphi_2$ is φ_1-reversible. Conversely, notice that a germ of diffeomorphism F with an involutory reversing symmetry φ_1 can always be written as the composition of two involutions:

$$F = \varphi_1 \circ \varphi_2.$$

As a consequence, some authors have addressed the study of $F = \varphi_1 \circ \varphi_2$ to the study of the pair of involutions (φ_1, φ_2). This is the approach used by Jacquemard and Teixeira in [1] in the study of germs of reversible planar diffeomorphisms of a degenerate cusp. But there is one point we want to remark. For any two pairs of involutions (φ_1, φ_2) and (ψ_1, ψ_2) that are equivalent, the compositions $\varphi_1 \circ \varphi_2$ and $\psi_1 \circ \psi_2$ generate conjugate reversible systems. The converse of this property does not hold in general. For example, if we take

$$\varphi_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 1 & a \\ 0 & -1 \end{pmatrix}, \quad a \neq 0$$

and

$$\psi_1 = \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} -1 & -a-b \\ 0 & 1 \end{pmatrix},$$

for any b, then the compositions are equal, but (φ_1, φ_2) and (ψ_1, ψ_2) are not equivalent (by Theorem 6.1).

So we may ask what kinds of restrictions the study of the pair (φ_1, φ_2) can impose to the study of the dynamics associated to the composition $\varphi_1 \circ \varphi_2$. It is again a consequence of Corollary 6.2 that there is no restriction for almost all pairs (φ_1, φ_2) of transversal linear involutions on the plane.

We end this section with the description of reversible linear diffeomorphisms given by the composition of involutions that occur in Corollary 6.2. First we present Lemma 7.1 below, which is a simple exercise of Linear Algebra but presented here for completeness.

Lemma 7.1 *Any rotation*

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi),$$

can be written as the composition of two reflections. For $\theta \neq 0, \pi$, if R_θ is the composition of two linear involutions, then these are necessarily reflections.

Proof. First we notice that given any reflection ρ, $\rho \circ R_\theta$ is also a reflection, so the first statement holds. We now suppose that $R_\theta = \varphi_1 \circ \varphi_2$, where φ_1 and φ_2 are linear involutions. So φ_1 is of the form

$$\varphi_1 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad a^2 + bc = 1.$$

Since $\varphi_2 = \varphi_1 \circ R_\theta$ is an involution and $\theta \neq 0, \pi$, it follows that $b = c$, that is, φ_1 is of the form

$$\varphi_1 = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix},$$

where $\alpha = (\arccos a)/2$, namely the reflection in the line at angle α to the positive x-axis. But then φ_2 is also a reflection. □

Suppose that φ is a linear diffeomorphism given as the compositions of two involutions, $\varphi = \varphi_1 \circ \varphi_2$, with (φ_1, φ_2) in the conditions of Corollary 6.2. Since $\det\varphi = 1$, then $-2 < \mathrm{tr}(\varphi) < 2$ if, and only if, φ is conjugate to a rotation. Now, by Theorem 6.1, the pair (φ_1, φ_2) is represented on the horizontal line $(\alpha, \beta) = (2 + \mathrm{tr}(\varphi), 1)$ of Fig. 2, that is, we can assume that the involutions are given by

$$\varphi_1 = \begin{pmatrix} -1 & 0 \\ 2+\mathrm{tr}(\varphi) & 1 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

By Lemma 7.1 it follows that $-2 < \mathrm{tr}(\varphi) < 2$ gives the segment on this line of the pairs equivalent to pairs of reflections. If $\mathrm{tr}(\varphi) > 2$ or $\mathrm{tr}(\varphi) < -2$, then φ correspond to a linear hyperbolic φ_1-reversible diffeomorphism.

One last remark is concerned with a geometrical interpretation of the equivalence between two pairs of planar reflections with respect to the fixed-point lines of the reflections. Notice that if φ_1 and φ_2 are reflections, then the angle between the lines $\mathrm{Fix}(\varphi_1)$ and $\mathrm{Fix}(\varphi_2)$ is half the angle θ of rotation $R_\theta = \varphi_1 \circ \varphi_2$. It is now a consequence of Corollary 6.2 that two pairs of transversal planar reflections are equivalent if, and only if, the angles between the lines of fixed point of each pair are equal.

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Resumo

Neste trabalho mostramos que a classificação C^∞ de diagramas divergentes de dobras $(f_1, \ldots, f_s) : \mathbb{R}^n, 0 \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n, 0$ pode ser reduzida à classificação das s-uplas $(\varphi_1, \ldots, \varphi_s)$ de involuções associadas. Aplicamos o resultado para obter formas normais quando $s \leq n$ e $\{\varphi_1, \ldots, \varphi_s\}$ é um conjunto transversal de involuções lineares. Uma descrição completa é dada quando $n = s = 2$. Apresentamos também uma breve discussão sobre aplicações de nossos resultados ao estudo de sistemas dinâmicos reversíveis discretos.

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