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W. L. MARAR

F. TARI

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On the geometry of simple germs of corank 1 maps from $\mathbb{R}^3$ to $\mathbb{R}^3$

W. L. Marar* F. Tari†

1 Introduction

In this paper we investigate the geometry of simple germs of corank 1 maps from $\mathbb{R}^3$ to $\mathbb{R}^3$. Those of codimension 1 have already been dealt with by several authors. In [2], V. I. Arnold considered the problem of evolution of galaxies (see Figure 2 (ii)). For a medium of non-interacting particles in $\mathbb{R}^3$ with an initial velocity distribution $v = v(x)$ (and a positive density distribution), the initial motion of particles defines a time-dependent map $g_t : \mathbb{R}^3 \to \mathbb{R}^3$ given by $g_t(x) = x + tv(x)$. At some time t singularities occur and the critical values of g_t correspond to points of condensation of particles. Arnold assumed the vector field v is a gradient, that is $v = \nabla S$, for some potential S. J. W. Bruce generalised these results in [3] by dropping the assumption on the velocity distribution and studied generic 1-parameter families of map germs $F : \mathbb{R}^3, 0 \to \mathbb{R}^3, 0$.

In [6], J. P. Duffour and P. Jean studied these map germs from a different perspective. They investigated 1-parameter families of smooth surfaces together with a projection, that is diagrams of the form $\mathbb{R} \xleftarrow{\sigma} \mathbb{R}^3 \xrightarrow{F} \mathbb{R}^3$, where σ is a submersion. This lead them to consider corank 1 germs of codimension ≤ 1.

As part of a program of classifying unimodal map germs $\mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, with $n + p \leq 6$, carried out in [10], a list of all simple map germs $F : \mathbb{R}^3, 0 \to \mathbb{R}^3, 0$ of corank 1 is obtained. The classification method and the computation of the normal forms in Table 1 are detailed in Section 2. The method of classification follows that introduced by A. A. duPlessis ([7], [8]) and consists of writing the germs as unfoldings of functions and reducing the $\mathcal{A}$-classification to that of a subgroup of the contact group $\mathcal{K}$. In Section 3, we describe the invariants in Table 1 and the way to compute them. The geometry of the normal forms is dealt with in Section 4. There we analyse the discriminants of the germs of Table 1 and compute its double point and cuspidaledge curves. We also look at the bifurcation sets of versal unfoldings of these germs and deduce the adjacency diagrams.

Theorem 1.1 *The $\mathcal{A}$-classes of simple singularities of map-germs $\mathbb{R}^3, 0 \to \mathbb{R}^3, 0$ of corank 1 are shown in Table 1.*

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Table 1:

Name	Normal Form	$\mathcal{A}_e$-cod	$\mu(\Sigma)$	$\mu(d)$	$\#A_3$	$\mu(c)$
A_1	(x,y,z^2)	0	-	-	-	-
*	$(x,y,z^3+P(x,y)z)$	$\mu(P)$	$\mu(P)$	-	-	$\mu(P)$
4_1^k	$(x,y,z^4+xz\pm y^kz^2), k\geq 1$	$k-1$	0	$k-1$	k	$k-1$
4_2^k	$(x,y,z^4+(y^2\pm x^k)z+xz^2)$ $k\geq 2$	k	1	$3k+1$	2	2
5_1	$(x,y,z^5+xz+yz^2)$	1	0	5	2	0
5_2	$(x,y,z^5+xz+y^2z^2+yz^3)$	2	0	10	3	1
5_3	$(x,y,z^5+xz+yz^3)$	3	0	10	3	1

* P is of type A_k, D_k, E_6, E_7, E_8. (μ denotes the Milnor number.)

2 The classification

Let $\mathcal{E}_n$ be the local ring of function-germs $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ and $\mathcal{M}_n$ the corresponding maximal ideal. We denote by $\mathcal{E}_n^p$ the p-tuples of elements in $\mathcal{E}_n$.

Given a map-germ $f \in \mathcal{E}_n^p$, θ_f denotes the $\mathcal{E}_n$-module of vector fields along f and $\theta_n = \theta_{I(\mathbb{R}^n,0)}$.

Let $f \in \mathcal{E}_n$ be an $\mathcal{R}$-finite germ and $\phi_1, \ldots, \phi_{\mu-1}$ a basis for $\mathcal{M}_n/J(f)$, where $J(f)$ denotes the Jacobian ideal of f. Set $\phi_0 = 1$ and let

$$F : \mathbb{R}^q \times \mathbb{R}^n, 0 \to \mathbb{R}^q \times \mathbb{R}, 0$$

be a $\mathbb{R}^q$-level preserving unfolding of f of the form $F(u,z) = (u, g(u,z))$, where $g(u,z) = f(z) + \sum_{i=1}^{\mu-1} P_i(u)\phi_i(z)$, $P_i \in \mathcal{M}_q$, $i = 1, \ldots, \mu-1$.

It follows from the Preparation Theorem that $(0,\phi_0), \ldots, (0,\phi_{\mu-1})$ project to a basis for $\theta_F/tF(\theta_{n+q})$ as a $\mathcal{E}_q$-module, and it is shown in [9] that they project to a free basis. This leads to an $\mathcal{E}_q$-module homomorphism

$$LF : \theta_F \to \mathcal{E}_q^\mu$$

so that any $\alpha \in \theta_F$ can be uniquely written as $\sum_{i=0}^{\mu-1} \alpha_i\phi_i$ modulo $tF(\theta_{n+q})$, with $\alpha_i \in \mathcal{E}_q$.

Let $\phi F : \mathcal{E}_q^\mu \to \theta_F$ be such that $\phi F(\alpha_0, \ldots, \alpha_{\mu-1}) = (0, \sum_{i=1}^{\mu-1} \alpha_i\phi_i)$, and denote by NF the composite $\phi F \circ LF$.

Proposition 2.1 *([10]) $TA_eF \cap \mathcal{E}_q\{(0,\phi_i),\ i = 0, \ldots, \mu-1\} =$
$\mathcal{E}_q\{NF(0,g^i), NF(0,g^jg_{u_k}),\ i = 0, \ldots, \mu-1,\ j = 0, \ldots, \mu-2,\ k = 1, \ldots, q\}$.*

A theorem of duPlessis asserts that two germs $(u, f(z) + \sum_{i=1}^{\mu-1} P_i(u)\phi_i(z))$ and $(u, f(z) + \sum_{i=1}^{\mu-1} Q_i(u)\phi(z))$ are $\mathcal{A}$-equivalent if and only if $(P_1, \ldots, P_{\mu-1})$ and $(Q_1, \ldots, Q_{\mu-1})$ are $\mathcal{G}$-equivalent, where $\mathcal{G}$ is a subgroup of the contact group $\mathcal{K}$, (see [8]). The tangent space $T\mathcal{G}_e(P_1, \ldots, P_{\mu-1})$ is given

$$TA_eF \cap \mathcal{E}_q\{(0,\phi_i),\ i = 1, \ldots, \mu-1\}.$$

The $\mathcal{A}$-classification is thus reduced to the $\mathcal{G}$-classification. This is a powerful method as it simplifies the difficulties arising from the mixed module structure of the $T\mathcal{A}$-tangent space. The $\mathcal{A}_e$-codimension of a finitely determined germ F is the $\mathcal{G}_e$-codimension of the associated germ $(P_1, \ldots, P_{\mu-1})$.

2.1 Map-germs $\mathbb{R}^3, 0 \to \mathbb{R}^3, 0$

Suppose $F : \mathbb{R}^3, 0 \to \mathbb{R}^3, 0$ is of corank 1 at the origin and of finite singularity type. We can set $F(x, y, z) = (x, y, g(x, y, z))$ with $f(z) = g(0, 0, z)$ having an A_k-singularity and write in suitable coordinates $g(x, y, z) = z^{k+1} + P_1(x, y)z + \ldots + P_{k-1}(x, y)z^{k-1}$. The map F, that we shall call of type k, can therefore be considered as an unfolding of $g(0, 0, z)$ with x, y playing the role of the unfolding parameters. We seek a classification of the $\mathcal{A}$-simple singularities of such an F.

The first task for this classification is to obtain the generators in Proposition 2.1, that is to express g^i, $g^j g_x$ and $g^j g_y$ in the form $\alpha_1 z + \alpha_2 z^2 + \ldots + \alpha_{k-1} z^{k-1}$ modulo $tF(\theta_3)$ (we denote this equivalence relation by $\simeq$). We shall write $\alpha_1 z + \alpha_2 z^2 + \ldots + \alpha_{k-1} z^{k-1}$ as a vector $(\alpha_1, \ldots, \alpha_{k-1})$ and denote the $\mathcal{G}$-tangent space by $T\mathcal{G}F$.

By Proposition 2.1, $T\mathcal{G}F$ is a finitely generated $\mathcal{E}_{x,y}$-module. So to check that the germ $(P_1, \ldots, P_{k-1})$ is finitely determined, that is, $\mathcal{M}_{x,y}^l \subset T\mathcal{G}F$ for some l, it is enough using Nakayama's lemma to show that $\mathcal{M}_{x,y}^l \subset T\mathcal{G}F + \mathcal{M}_{x,y}^{l+1}$.

The classification is carried out inductively on the jet-level of the map-germ $P = (P_1, \ldots, P_{k-1})$. Given a non-sufficient l-jet P, the orbits in the $(l+1)$-jet space whose l-jets is P are obtained by the *complete transversal* tool [4], stated below where $T\mathcal{G}_1$ denotes the subgroup of $\mathcal{G}$ whose elements have 1-jet the identity.

Proposition 2.2 *([4]) Let h be an l-jet in $J^l(n, p)$ and T a vector subspace of the space of germs of maps $\mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ of degree $l+1$, $H^{l+1}(n, p)$, with*

$$J^{l+1}(T\mathcal{G}_1 h) \cap H^{l+1} + T = H^{l+1}.$$

Then any $(l+1)$-jet g with l-jet h is $\mathcal{G}_1$-equivalent to $h + t$ for some $t \in T$.

The space T is called the complete (l+1)-transversal.

We shall now compute the normal forms for the singularities of F when $k = 2, 3$ and 4. There are no simple germs for $k > 4$.

(1). The A_1 type: $f(z) = z^2$.
$F(x, y, z) = (x, y, z^2)$ is stable.

(2). The A_2 type: $f(z) = z^3$.
Here $F(x, y, z) = (x, y, z^3 + P(x, y)z)$ with $P \in \mathcal{M}_2$. We have the following.

Proposition 2.3 *(Cf. [Proposition 4.9, [7]])*
F is finitely $\mathcal{A}$-determined if and only if P is finitely $\mathcal{K}$-determined.

Proof. The group $\mathcal{G}$ associated to germs of this type is in fact $\mathcal{K}$. Indeed, let $g(x, y, z) = z^3 + P(x, y)z$. Then $g_z(x, y, z) = 3z^2 + P(x, y)$ and $g(x, y, z) \simeq \frac{2}{3}P(x, y)z$. Thus,

$$T\mathcal{A}_e F \cap \mathcal{E}_{x,y}\{(0, 0, z)\} = \mathcal{E}_{x,y}\{P, P_x, P_y\} = T\mathcal{K}_e P.$$

In particular, the simple germs of the form $(x, y, z^3 + P(x, y)z)$ correspond to the singularity of P being one of the following:

$A_k : x^2 \pm y^{k+1}$; $D_k : x^2 y \pm y^{k-1}$; $E_6 : x^3 + y^4$; $E_7 : x^3 + xy^3$; $E_8 : x^3 + y^5$, and $\text{codim}_{\mathcal{A}_e} F = \mu(P)$.

(3). The A_3 type: $f(z) = z^4$.

In this case $F(x, y, z) = (x, y, z^4 + P(x, y)z + Q(x, y)z^2)$ with $P, Q \in \mathcal{M}_2$. Set $g = z^4 + Pz + Qz^2$ so that $g_z = 4z^3 + P + 2Qz$. Then $z^3 \simeq -\frac{1}{4}P - \frac{1}{2}Qz$, $z^4 \simeq -\frac{1}{4}Pz - \frac{1}{2}Qz^2$ and

$g \simeq \frac{3}{4}Pz + \frac{1}{2}Qz^2$,
$g.z \simeq -\frac{1}{8}PQ - \frac{1}{4}Q^2 z + \frac{3}{4}Pz^2$,
$g.z^2 \simeq -\frac{3}{16}PQ - \frac{1}{2}PQz - \frac{1}{4}Q^2 z^2$.

It is now easy to show that the generators of $T\mathcal{G}F$, displayed in a vector form, are $g \simeq (3P, 2Q)$, $g_x \simeq (P_x, Q_x)$, $g_y \simeq (P_y, Q_y)$, $g^2 \simeq (-4PQ^2, 9P^2) - Q^2(3P, 2Q)$, $gg_x \simeq (-2PQQ_x, 3PP_x)$ and $gg_y \simeq (-2PQQ_y, 3PP_y)$. The inductive classification starts as follows.

The 1-jet $(ax + by, cx + dy)$.

- If $ad - bc \neq 0$ then 1-jet $\simeq (x, y)$ which is stable.
- If $ad - bc = 0$, $a \neq 0$ or $b \neq 0$ then 1-jet $\simeq (x, 0)$.
- If $ad - bc = 0$, $a = b = 0$, $c \neq 0$ or $d \neq 0$ then 1-jet $\simeq (0, x)$.
- Else, 1-jet $= (0, 0)$.

For a given $(k-1)$-jet $g = (x, 0)$ we have $g \simeq (x, 0)$, $g_x \simeq (1, 0)$ and $gg_x \simeq (0, 3x)$. In the space $H^k(2, 2)$ (see Proposition 2.2) all germs of the form $(p, 0)$ are in $J^k\mathcal{G}_1 F \cap H^k(2, 2)$ as $p.g_x \simeq (p, 0)$, and so are those of the form $(0, xq)$ as $\frac{1}{3}q.gg_x \simeq (0, xq)$. The germ $(0, y^k)$ is not in $J^k\mathcal{G}_1 F \cap H^k(2, 2)$. Thus

$$J^k\mathcal{G}_1 F \cap H^k(2, 2) + \mathbb{R}\{(0, y^k)\} = H^k(2, 2).$$

By Proposition 2.2 any k-jet whose $(k-1)$-jet $(x, 0)$ is equivalent to (x, ay^k) for some $a \in \mathbb{R}$. It follows now by Mather's Lemma that there are three orbits of this form: $(x, \pm y^k)$ if $a \neq 0$ and $(x, 0)$ otherwise.

Proposition 2.4 *The germ $(x, \pm y^k)$ is $k - \mathcal{G}$-determined, $k \geq 2$.*

Proof. Indeed, if we consider $g \simeq (3x, \pm 2y^k)$, $g_x \simeq (1, 0)$, $g_y \simeq (0, \pm ky^{k-1})$ and $gg_x \simeq (0, 3x)$ we see that $\mathcal{M}_{x,y}^{k+1} \subset T\mathcal{G}F + \mathcal{M}_{x,y}^{k+2}$. The codimension of $(x, \pm y^k)$ is $k - 1$.

The 1-jet $(0, x)$.

We have $g \simeq (0, 2x), g_y \simeq (0, 1)$. All z-coefficients of the remaining generating elements of $T\mathcal{G}F$ are zero. A complete 2-transversal is $\{(x^2, 0), (xy, 0), (y^2, 0)\}$ and any 2-jet whose 1-jet is $(0, x)$ is equivalent to $(ax^2 + bxy + cy^2, x)$ for some a, b and c in $\mathbb{R}$. The orbits in $J^2(2, 2)$ of this form are:

$(y^2 \pm x^2, x)$ if $c \neq 0$, $b^2 - 4ac \neq 0$. This is 2-determined.
(y^2, x) if $c \neq 0$, $b^2 - 4ac = 0$.
(xy, x) if $c = 0$, $b \neq 0$.
(x^2, x) if $c = b = 0$, $a \neq 0$.
$(0, x)$ if $a = b = c = d = 0$.

The 2-jet (y^2, x).

Here $g \simeq (3y^2, x), g_x \simeq (0,1), g_y \simeq (2y, 0), gg_x \simeq (-xy^2, 0), gg_y \simeq (0, 6y^3)$ and $g^2 \simeq (-7x^2y^2, 9y^4)$. By the complete transversal argument, any k-jet whose 2-jet is (y^2, x) is equivalent to $(y^2 + ax^k, x)$ for some $a \in \mathbb{R}$. There are three orbits of this form: $(y^2 \pm x^k, x)$ if $a \neq 0$ and (y^2, x) otherwise.

Proposition 2.5 *The germ* $(y^2 \pm x^k, x)$ $k - \mathcal{G}$*-determined for* $k \geq 2$.

Proof. Considering $g \simeq (3(y^2 \pm x^k), 2x)$, $g_x \simeq (\pm k x^{k-1}, 1)$ and $g_y \simeq (2y, 0)$ we see that $\mathcal{M}_{x,y}^{k+1} \subset T\mathcal{G}F + \mathcal{M}_{x,y}^{k+2}$. Furthermore, the codimesion of g is k.

The remaining 2-jets with 1-jets $(0, x)$ lead to non-simple map-germs.

(4) The A_4 type: $f(z) = z^5$.

Germs in this class are written in the form $F = (x, y, g(x, y, z))$ with $g(x, y, z) = z^5 + P(x, y)z + Q(x, y)z^2 + L(x, y)z^3$ for some $P, Q, L \in \mathcal{M}_{x,y}$. The generators for the $T\mathcal{G}.F$ tangent space are the representatives of $g, g^2, g^3, g_x, gg_x, g^2g_x, g_y, gg_y, g^2g_y$ in E_2^3. The expression for these vectors are easy to compute using the fact that $g \simeq \frac{4}{5}Pz + \frac{3}{5}Qz^2 + \frac{2}{5}L$.

The 1-jet $(P, Q, L) = (ax + by, cx + dy, ex + fy)$

- If $ad - bc \neq 0$ then 1-jet$\simeq (x, y, 0)$
- If $af - be \neq 0$ then 1-jet$\simeq (x, 0, y)$
- Else 1-jet$\simeq (x, 0, \pm x), (x, 0, 0), (0, x, y), (0, x, x), (0, x, 0), (0, 0, x)$ or $(0, 0, 0)$

Proposition 2.6 *(i). The 1-jet $(x, y, 0)$ is 1-determined and of codimension 1.*

(ii). There are two 2-determined orbits whose 1-jets are $(x, 0, y)$ and (x, y^2, y) of codimension 2 and $(x, 0, y)$ of codimension 3.

(iii). The remaining 1-jets in this class lead to non-simple orbits.

Proof. We prove assertion (ii) here. For the 1-jet $(x, 0, y)$ we have

$g \simeq (4x, 0, 2y)$

$g_x \simeq (1, 0, 0)$

$g_y \simeq (0, 0, 1)$

$gg_x \simeq (0, 2x - \frac{3}{5}y^2, 0)$

$gg_y \simeq (0, 7xy - \frac{9}{5}y^3, 0)$

It is clear that the germ $(x, 0, y)$ is 2-determined. A complete 2-transversal is given by $\{(0, y^2, 0)\}$ and the orbits in the 2-jet space are (x, y^2, y) and $(x, 0, y)$. The first has codimension 2 and the second 3.

3 The Invariants

Let Σ denote the singular set of a map germ $F : \mathbb{R}^3, 0 \to \mathbb{R}^3, 0$, *i.e.* Σ is the locus of the points p in the source where the rank of the Jacobian matrix $JF(p)$ is less than 3. Let Δ be the discriminant of F, that is $\Delta = F(\Sigma)$.

When Σ is smooth, the restriction $F|\Sigma$ can be thought of as a map-gem $\phi : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ parametrizing Δ. The map germ ϕ is in general not finitely $\mathcal{A}$-determined due to the presence of cuspidaledges. Indeed, Arnold shows that the singularities ϕ are as in Figure 1.

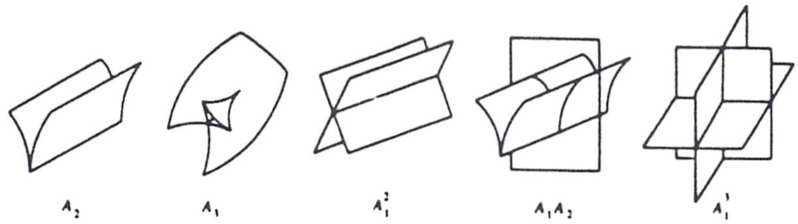

Figure 1: $A_2, A_3, A_1^2, A_1A_2, A_1^3$ singularities

A good deal of the geometry of the map germ ϕ from 2 to 3-space can be obtained by studying a plane curve singularity, namely the double point curve of ϕ. Such a curve is the locus of points p such that $\#\phi^{-1}(\phi(p)) = 2$ together with the points where ϕ fails to be immersive. The defining function $h : \mathbb{R}^2, 0 \to \mathbb{R}, 0$ of this double point curve can be obtained using the following.

Proposition 3.1 *([5]) Let $\phi : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ be an analytic map germ, with $\phi(x,y) = (a(x,y),\ b(x,y),\ c(x,y))$. Let $G : \mathbb{R}^3, 0 \to \mathbb{R}, 0$ be an analytic function such that $G(X,Y,Z) = 0$ is a defining equation of the image of ϕ. Then the defining function h of the double point curve of ϕ is given by*

$$h(x,y) = \frac{\frac{\partial G}{\partial Z}(a(x,y), b(x,y), c(x,y))}{\left|\frac{\partial(a,b)}{\partial(x,y)}\right|}.$$

(See also [18], where G and h are obtained using Fitting ideals.)

The formula above simplifies when ϕ is of corank 1 at the origin, which is the case for all but one of the map germs ϕ treated in here. Then we can choose a suitable system of coordinates where ϕ assumes the form $\phi = (x, \alpha(x,y), \beta(x,y))$ and the formula is transformed to

$$h(x,y) = \frac{\frac{\partial G}{\partial Z}(x, \alpha(x,y), \beta(x,y))}{\frac{\partial \alpha}{\partial y}(x,y)}.$$

The equation $G(X,Y,Z) = 0$ is obtained by eliminating the variable y from the equations $Y - \alpha(X,y) = 0$ and $Z - \beta(X,y) = 0$. In other words, $G(X,Y,Z)$ is the resultant of the polynomials $p(X,Y,Z,y) = Y - \alpha(X,y)$ and $q(X,Y,Z,y) = Z - \beta(X,y)$ with respect to the variable y. We use Maple to carry out the computation of G.

It turns out (see [5]) that the function h factors as

$$h(x,y) = d(x,y)(c(x,y))^2$$

where $d(x,y) = 0$ defines the closure of the curve of ordinary double points of ϕ, *i.e.* the curve whose image by F is the self intersection curve of the image of ϕ and $c(x,y) = 0$ defines the curve in the source that is mapped by F into the cuspidaledge of Δ.

Definition 3.2 *The curve given by $d(x,y) = 0$ will be called the double point curve and the curve given by $c(x,y) = 0$ the cuspidaledge curve of the discriminant of F.*

The Milnor numbers $\mu(d)$ and $\mu(c)$ of these plane curve singularities are analytic invariants of the map germ F.

The case 4_2^k, whose critical set Σ is itself singular, has to be analysed in a different way (see 4.2.*(ii)* below).

Another invariant attached to the map germ F in the number of swallowtails of the discriminant Δ. We shall denote this invariant by $\#A_3$ and use the following result to carry out the computations.

Proposition 3.3 *([14]) Let $F : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a corank 1 map germ with $F(x,y,z) = (x,y,g(x,y,z))$. Then*

$$\#A_3 = dim_{\mathbb{C}}\mathcal{E}_3/(g_z, g_{zz}, g_{zzz})\mathcal{E}_3.$$

In particular, if g is quasi-homogeneous of weights w_1, w_2 and w_3 and degree d, then,

$$\#A_3 = \frac{(d-w_3)(d-2w_3)(d-3w_3)}{w_1 w_2 w_3}.$$

4 The geometry

4.1 The A_2 type

We have in this case $F(x,y,z) = (x,y,z^3 + P(x,y)z)$. The critical set of F is

$$\Sigma : 3z^2 + P(x,y) = 0$$

and it is easy to see that the discriminant is defined by

$$\Delta : 27Z^2 + 4P(X,Y)^3 = 0.$$

It follows that Δ is singular if and only if $P(x,y) = 0$. Along the smooth part of $P(x,y) = 0$ the discriminant Δ is diffeomorphic to a cuspidaledge. (See Figure 2 for an illustration of the transitions on the discriminant corresponding to the A_1 singularities of P.)

The double point curve is empty, so there is no presence of swallowtail points in this situation. A versal deformation $\overline{F}$ of F is given by $\overline{F}(x,y,z,u) = (x,y,z^3 + \overline{P}(x,y,u)z)$, where $\overline{P}$ is a versal deformation of P. The bifurcation set of F is exactly that of P. In particular the adjacency diagram for F is just that of the simple singularities of functions [1] (Diagram 1).

$$\begin{array}{ccccccccccccc}
A_1 & \leftarrow & A_2 & \leftarrow & A_3 & \leftarrow & A_4 & \leftarrow & A_5 & \leftarrow & A_6 & \leftarrow & A_7 & \cdots \\
 & & & & & \nwarrow & & \nwarrow & & \nwarrow & & \nwarrow & & \\
 & & & & & & D_4 & \leftarrow & D_5 & \leftarrow & D_6 & \leftarrow & D_7 & \leftarrow & \cdots \\
 & & & & & & & & & \nwarrow & & \nwarrow & & \nwarrow & \\
 & & & & & & & & & & E_6 & \leftarrow & E_7 & \leftarrow & E_8
\end{array}$$

Diagram 1. Adjacency diagram for singularities of type A_2

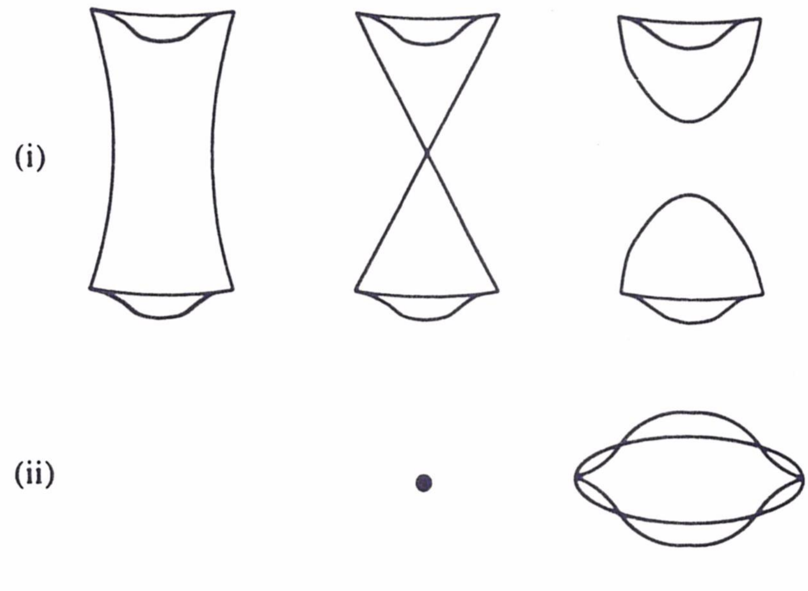

Figure 2: Transitions associated to an $A_1^{\pm}$-singularity of P

4.2 The A_3 type

(i). The series $F(x,y,z) = (x, y, z^4 + xz \pm y^k z^2), k \geq 1$
The critical set of F is a smooth surface given by

$$\Sigma : x \pm 2y^k z + 4z^3 = 0$$

and the discriminant Δ is parametrized by

$$\phi(y,z) = (-4z^3 \mp 2zy^k, y, -3z^4 \mp z^2 y^k).$$

Now, the defining function $G(X,Y,Z)$ of the discriminant Δ is obtained by means of the resultant of the two polynomials $X + 4z^3 \pm 2zY^k$ and $Z + 3z^4 \pm z^2 Y^k$ with respect to the variable z, which is

$$G(X,Y,Z) = 256Z^3 + 27X^4 + 144X^2Y^kZ + 128Y^{2k}Z^2 + 4X^2Y^{3k} + 16Y^{4k}Z.$$

Substituting $X, Y,$ and Z by their values as functions of x, y and z and using Proposition 3.1 yield the defining function $h(y,z)$ of the double point curve

$$h(y,z) = (2z^2 \pm y^k)(6z^2 \pm y^k)^2.$$

Thus, $c(y,z) = 6z^2 \pm y^k$ defines the cuspidaledge curve and $d(y,z) = 2z^2 \pm y^k$ defines the closure of the ordinary double points of ϕ. It follows that $\mu(c) = \mu(d) = k - 1$.

The number of swallowtails of Δ is, by Proposition 3.3, equal to k. Indeed, $g(x,y,z) = z^4 + xz \pm y^k z^2$ is quasi-homogeneous of degree $4k$ and weights $w_1 = 3k, w_2 = 2$ and $w_3 = k$.

A versal deformation $\overline{F}$ of F is given by $\overline{F}(x,y,z,u) = (x, y, z^4 + xz + \overline{Q}(y,u)z^2)$, where $\overline{Q}(y,u) = \pm y^k + u_1 y^{k-2} + \ldots + u_{k-2}y + u_{k-1}$, *i.e.* $\overline{Q}$ is a versal deformation of the A_{k-1} singularity y^k.

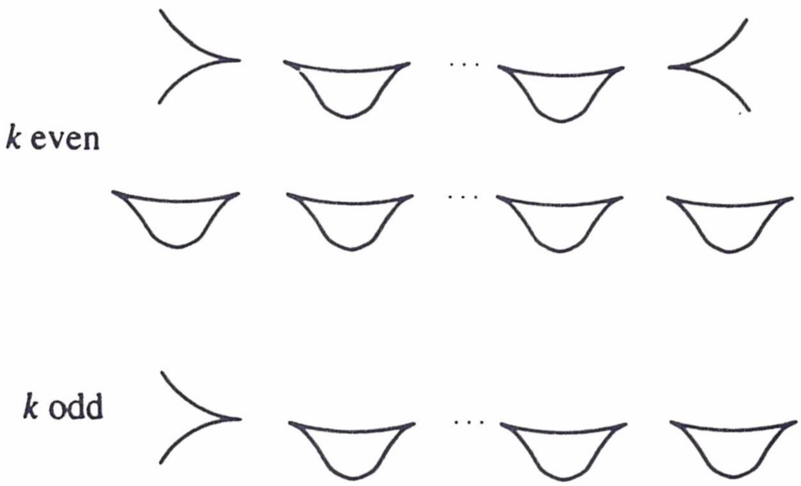

Figure 3: Bifurcation of the cuspidaledge curve of the 4_1^k singularities.

The bifurcation set and the adjacency diagram of F coincides with that of the A_{k-1} singularity: $A_1 \leftarrow A_2 \leftarrow \ldots \leftarrow A_{k-1}$.

One can visualise the way in which the cuspidaledge curve bifurcates in the target as follows. Let $g(x,y,z,u) = z^4 + xz + \overline{Q}(y,u)z^2$. Then the cuspidaledge curve is defined by $g_z = g_{zz} = 0$, that is $x + 2\overline{Q}z + 4z^3 = 0$ and $6z^2 + \overline{Q} = 0$. Representing the last equation in the (x, y, ξ) coordinates with $\xi = z^2$ and taking into account that the swallowtail points, which are the cusps of the cuspidaledge curve, correspond to points satisfying $g_{zzz} = z = 0$, we get the illustration in Figure 3.

Remark 4.1 One can think of the germ $F(x,y,z) = (x, y, z^4 + xz \pm y^k z^2)$ as a 1-parameter family of the swallowtail singularity $g(x,z) = (x, z^4 + xz)$, a map-germ $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$. A versal unfolding $G : \mathbb{R}^2 \times \mathbb{R}, 0 \to \mathbb{R}^2 \times \mathbb{R}, 0$ of g is given by $G(x,z,u) = (x, z^4 + xz + uz^2, u)$ and any 1-parameter unfolding of g can be derived from G in such a way that the diagram

$$\begin{array}{ccc} \mathbb{R}^2 \times \mathbb{R}, 0 & \xrightarrow{F} & \mathbb{R}^2 \times \mathbb{R}, 0 \\ \downarrow (h,l) & & \downarrow (k,l) \\ \mathbb{R}^2 \times \mathbb{R}, 0 & \xrightarrow{G} & \mathbb{R}^2 \times \mathbb{R}, 0 \end{array}$$

commutes; where $h, k : \mathbb{R}^2 \times \mathbb{R}, 0 \to \mathbb{R}^2, 0$ are such that $h(x,y,0) = k(x,y,0) = (x,y)$ and l is a smooth function $\mathbb{R}, 0 \to \mathbb{R}, 0$. Suppose that l has order k, then we can put $l(y) = \pm y^k$ and write $F(x,y,z) = (x, y, z^4 + xz \pm y^k z^2)$ in a suitable system of coordinates. Note that the order of l is the number of swallowtails of the discriminant of F.

Remark 4.2 After a change of coordinates, the parametrization ϕ of the discriminant of F can be written in the form

$$\phi(u,v) = (u, 2v^3 \pm vu^k, 3v^4 \pm v^2 u^k).$$

When $k = 1$, this is the map germ $P_3(3/2)$ of Mond's classification [15], [16]. When $k = 2$, it is the tangent developable of a regular space curve with a non-degenerate zero of curvature at the origin (see for example [17]).

(ii). The series $F(x, y, z) = (x, y, z^4 + (y^2 \pm x^k)z + xz^2), k \geq 2$

The critical set of F is given by

$$\Sigma : y^2 \pm x^k + 2xz + 4z^3 = 0.$$

In this case Σ is not smooth but has an A_1-singularity at the origin. The discriminant cannot be parametrized by a corank 1 map. An alternative approach is needed in this situation.

The double points curve $\mathcal{D} : d(x, y) = 0$ and the cuspidaledge curve $\mathcal{C}$: $c(x, y) = 0$ of the discriminant of F can be obtained as follows. Let $g(x, y, z) = z^4 + (y^2 \pm x^k)z + xz^2$. A point $p_1 = (x_1, y_1, z_1)$ is a double point of the discriminant if there is another point $p_2 = (x_2, y_2, z_2)$, with $p_1 \neq p_2$, such that $p_1, p_2 \in \Sigma$ and $F(p_1) = F(p_2)$. This is the case if and only if there is a function $\alpha(x, y)$ such that $g - \alpha$ has two real roots as a polynomial in z, that is, $g - \alpha = (z - z_1)^2(z - z_2)^2$. It is not difficult to see that this is equivalent to $y^2 \pm x^k = 0$.

The double point curve $\mathcal{D}$ is thus the intersection of the two surfaces $y^2 \pm x^k = 0$ and Σ. This is a complete intersection and Lê's formula [12] provides the tool for computing the Milnor number of $\mathcal{D}$. Set $r(x, y, z) = 4z^3 + y^2 \pm x^k + 2xz$ and $s(x, y, z) = y^2 \pm x^k$. Then $\mathcal{D} = (r, s)^{-1}(0)$. Now, Lê's formula asserts that

$$\mu(\mathcal{D}) = -\dim_{\mathbb{C}} \mathcal{E}_3 / J(r) + \dim_{\mathbb{C}} \mathcal{E}_3 / (J(r, s), r)\mathcal{E}_3$$

where $J(r)$ is the Jacobian ideal of r, *i.e.*, $J(r) = (r_x, r_y, r_z)\mathcal{E}_3$ and $J(r, s)$ is the ideal in $\mathcal{E}_3$ generated by the 2×2 minor of the Jacobian matrix of the mapping (r, s). Calculations, using a Maple program elaborated by N.P. Kirk [11], show that $\mu(\mathcal{D}) = 3k + 1$.

As for the cuspidaledge curve, a point $p = (x, y, z) \in \mathcal{C}$ if and only if $f_z(p) = f_{zz}(p) = 0$. So $\mathcal{C}$ is the intersection of the surfaces $12z^2 + x = 0$ and Σ. The first surface is smooth and we have $\mu(\mathcal{C}) = 2$.

The number $\#A_3$ of swallowtails is computed using the first formula in Proposition 3.3 as g is not quasi-homogeneous. We have $\#A_3 = 2$ for all k.

A versal unfolding has the form $\overline{F} = (x, y, z^4 + (y^2 \pm x^k + (u_1 x^{k-1} + \ldots + u_{k-1}x + u_k)z^2 + xz^3)$. Part of the bifurcation set of $\overline{F}$ is diffeomorphic to that of an A_k singularity of a function and consists of points u where $\overline{F}_u$ has a singularity of type 4_2^i for $i < k$. There is also a smooth hypersurface in the u-parameter space where $\overline{F}$ has a singularity of type 4_1^2. These correspond to a birth/annihilation of two swallowtail points on the discriminant. (See section 4.3 (ii) for the way to compute this stratum.) The adjacency diagram for this series is as in Diagram 2.

4.3 The A_4 type

(i). The case $F(x, y, z) = (x, y, z^5 + xz + yz^2)$.

(i) (ii)

Figure 4:

The critical set of F is a smooth surface given by

$$\Sigma : x + 2yz + 5z^4 = 0$$

and the discriminant Δ is parametrized by

$$\phi(y, z) = (-5z^4 - 2zy, y, -4z^5 - z^2y).$$

Setting $X = -5z^4 - 2yz$, $Y = y$ and $Z = -4z^5 - yz^2$ we obtain the defining equation $G(X, Y, Z) = 0$ by eliminating z from the equations above. We found

$$G(X, Y, Z) = 3125Z^4 - 24X^2Y^2 + 256X^5 + 2250XY^2Z^2 + 1600X^3YZ - 108Y^5Z$$

and the defining equation of the double point curve of ϕ is given by

$$h(y, z) = d(y, z)(c(y, z))^2 = (200z^6 + 140yz^3 + 27y^2)(10z^3 + y)^2.$$

It follows that $\mu(d) = 5$ (A_5 singularity) and $\mu(c) = 0$.

The function $g(x, y, z) = z^5 + xz + yz^2$ is quasi-homogeneous of degree 5 with weights $w_1 = 4, w_2 = 3$ and $w_3 = 1$. By Proposition 3.3 we have $\#A_3 = 2$.

A versal unfolding has the form $(x, y, z^5 + xz + yz^2 + uz^3)$. Figure 4 (i) illustrates the transition in the cuspidaledge curve of the discriminant.

(ii). The case $F(x, y, z) = (x, y, z^5 + xz + y^2z^2 + yz^3)$.

The critical set of F is also in this case a smooth surface with equation

$$\Sigma : x + 2y^2z + 3yz^2 + 5z^4 = 0$$

and the discriminant Δ is parametrized by

$$\phi(y, z) = (-5z^4 - 2zy^2 - 3yz^2, y, -4z^5 - z^2y^2 - 2yz^3).$$

Let $X = -5z^4 - 2zy^2 - 3yz^2$, $Y = y$ and $Z = -4z^5 - y^2z^2 - 2yz^3$. The defining equation $G(X, Y, Z) = 0$ is obtained by eliminating z from the equations above and the defining equation of the double point curve of ϕ can be obtained by Proposition 3.1. We have

$$h(y, z) = (200z^6 + 140y^2z^3 + 220yz^4 + 68z^2y^2 + 4y^3 + 72y^3z + 27y^4)(10z^3 + 3yz + y^2)^2.$$

Thus $\mu(d) = 10$ (J_{10} singularity) and $\mu(c) = 1$ (A_1 singularity).

The germ g is not quasi-homogeneous. To obtain the number of swallowtail points we use Proposition 3.3 and get

$$\# A_3 = \dim_{\mathbb{C}} \mathcal{E}_2/(5z^4 + x + 2y^2z + 3yz^2, 20z^3 + 2y^2 + 6yz, 60z^2 + 6y)\mathcal{E}_2 = 3.$$

A versal unfolding of F has the form $(x, y, z^5 + xz + (y^2 + uy + v)z^2 + yz^3)$. On the u-axis we have a singularity of type 5_1. Singularities of type 4_1^2 occur at a birth/annihilation of two swaillowtail points, that is when the curve giving the cuspidaledge in the parameter space is singular, *i.e.* the defining equation of this curve has $A_{\geq 1}$ singularity.

The cuspidalcurve is given by $g_{zz} = 2(y^2 + uy + v)z^2 + 3yz^2 = 0$ on the surface $g_z = 0$. It is singular when $g_{zzy} = g_{zzz} = 0$. It follows that $u = -3z - 20z^2$ and $v = -10z^3 - 300z^4$. The (u, v) curve is smooth with an inflexion along the u-axis (Figure 4,*(ii)*).

(iii). The case $F(x, y, z) = (x, y, z^5 + xz + yz^3)$

The critical set of F is given by $\Sigma : x + 3yz^2 + 5z^4 = 0$ and the discriminant Δ is parametrized by

$$\phi(y, z) = (-5z^4 - 3z^2y, y, -4z^5 - 2z^3y).$$

Setting $X = -5z^4 - 3yz^2$, $Y = y$ and $Z = -4z^5 - 2yz^3$ we obtain the defining equation $G(X, Y, Z) = 0$ by eliminating z from the equations above. We found

$G(X, Y, Z) = 3125Z^4 + 256X^5 + 2000X^2YZ^2 - 900XY^3Z^2 - 128X^4Y^2 + 16X^3Y^4 + 108Y^5Z^2$.

It follows by Proposition 3.1 that the defining equation of the double point curve of ϕ is given by

$$h(y, z) = d(y, z)(c(y, z))^2 = (y + 2z^2)(25z^4 + 15z^2y + y^2)(10z^3 + 3zy)^2.$$

So $\mu(d) = 10$ (J_{10} singularity) and $\mu(c) = 1$ (A_1 singularity).

The function $f(x, y, z) = z^5 + xz + yz^3$ is quasi-homogeneous of degree 5 with weights $w_1 = 4, w_2 = 2$ and $w_3 = 1$. By Proposition 3.3, we have $\# A_3 = 3$.

A versal unfolding of F has the form $(x, y, z^5 + xz + (uy^2 + vy + w)z^2 + yz^3)$. Calculations similar those in the previous case show that singularities of type 5_1 occur on the plane $w = 0$ and those of type 4_1^2 occur on the surface $(u, z) \to (u, -3z - 20uz^2, -10z^3 - 300uz^4)$.

We summarise this section by drawing the adjacency diagram for the A_3 and A_4 types in Diagram 2.

$$\begin{tikzcd}
4_1^1 & 4_1^2 \arrow[l] & 4_1^3 \arrow[l] & 4_1^4 \arrow[l] & \cdots \arrow[l] & 4_1^k \arrow[l] & \arrow[l] \\
5_1 \arrow[u] & 5_2 \arrow[l] \arrow[u] & 4_2^2 \arrow[ul] & 4_2^3 \arrow[l] & \cdots \arrow[l] & 4_2^{k-1} \arrow[l] & \arrow[l] \\
 & 5_3 \arrow[u] & & & & &
\end{tikzcd}$$

Diagram 2

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Address
W. L. Marar: Instituto de Ciências, Matemáticas de São Carlos, Universidade de São Paulo, Caixa Postal 668, 13560 São Carlos (SP), Brazil.
Email: walmarar@ICMSC.USP.BR

F. Tari: University of Liverpool, Department of Pure Mathematics, P.O. Box 147, Liverpool L69 3BX, Britain.
Email: tari@liverpool.ac.uk

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