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Modified C^{ℓ} - Equivalence of Real Functions

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MODIFIED C^{ℓ}-EQUIVALENCE OF REAL FUNCTIONS

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Resumo

Kuo [K] introduziu a noção de trivialização analítica modificada de família de funções analíticas reais. Introduzimos aqui o conceito de equivalência C^{ℓ}-modificada de funções reais e trivialização C^{ℓ}-modificada de famílias de funções reais. Este conceito é mais interessante porque permite-nos estudar os germes que não são analíticos, mas que são de classe C^{ℓ}.

Abstract

Kuo [K] introduced the notion of modified analytic trivialization of family of real analytic functions. Here we introduce the concepts of modified C^{ℓ}-equivalence of real functions and modified C^{ℓ}-trivialization for families of real functions. This concept is more interesting because allows us to study the germs that are not analytic, but are of class C^{ℓ}.

1. Introduction

Kuo [K], motivated by the Whitney's example

$$W(x,y,t) = y(y-x)(y-2x)(y-tx),\ 2 < t < \infty,$$

introduced the notion of modified analytic trivialization of family of real analytic functions. This concept is stronger than topological triviality and weaker than bianalytic triviality. However this notion is not related to C^{ℓ}-triviality. Fukui [Fu], Kuo [K], Paunescu [Pa] and Yoshinaga [Yo] investigated problems of classification on real analytic function-germs induced by the modified analytic trivialization. On the other hand there are few works wkere the classification of $C^{\ell}-$functions are studied. For instance, see [B-M], [R-S].

Here we introduce the concepts of *modified C^{ℓ}-equivalence* of real functions and *modified C^{ℓ}-trivialization* for families of real functions. These concepts are intermediate between the C^{ℓ} and topological equivalence (triviality). The ideas presented here are inspirated by the work [K], where the concept of modified analytic trivialization

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is introduced and the work [B-K] where the topological triviality of families defined by weighted homogeneous functions of class C^k is shown under some conditions.

We show that for analytic functions which are modified C^ℓ- equivalent, the multiplicity is invariant. We also show that a $C^{\ell+1}$ deformation of a weighted homogeneous real function, under certain conditions, admits a modified C^ℓ-trivialization. We give an example showing that the modified $C^\ell-$equivalence is stronger than topological equivalence.

2. Modified C^ℓ-equivalence

We fix a systen of coordinates $x = (x_1, ..., x_n)$ of $\mathbb{R}^n$ and denote by $\mathbb{R}[x_1, ..., x_n]$ the associated ring of polynomials.

Definition Let X be an analytic $n-$dimensional manifold. We say that a mapping $N : X \to X$ is $C^\ell-$*liftable* if there exists an analytic mapping $\pi : \mathbb{R}^n \to X$ and a local $C^\ell-$diffeomorphism, also denoted by N, $N : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{N} & \mathbb{R}^n \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow[N]{} & X \end{array}$$

commuts.

For instance, when X is an $n-$dimensional analytic manifold with $\mathbb{R}^n$ as its universal covering and $N : X \to X$ is a $C^\ell-$diffeomorphism.

Definition Consider functions $f, g : U \subseteq \mathbb{R}^n \to \mathbb{R}$. We say that f and g are *modified* $C^\ell-$*equivalent* if there exist an $n-$dimensional analytic manifold X, a proper analytic mapping $\varphi : X \to \mathbb{R}^n$, such that $\varphi(X) = \mathbb{R}^n$ and a $C^\ell-$liftable mapping $N : U_1 \to U_2$ where U_1, U_2 are neighbourhoods of $\varphi^{-1}(0)$ in $\varphi^{-1}(U)$, with $N(\varphi^{-1}(0)) = \varphi^{-1}(0)$, such that

$$f \circ \varphi \circ N = g \circ \varphi.$$

Moreover, N induces a homeomorphism $\widetilde{N} : \varphi(U_1) \to \varphi(U_2)$ with $\widetilde{N}(0) = 0$ such that the diagram

$$\begin{tikzcd} U_1 \arrow[rr, "N"] \arrow[d, "\varphi"'] & & U_2 \arrow[d, "\varphi"] \\ \varphi(U_1) \arrow[rr, "\widetilde{N}"'] \arrow[dr, "g"'] & & \varphi(U_2) \arrow[dl, "f"] \\ & \mathbb{R} & \end{tikzcd}$$

commuts.

Remark: We can lift the mapping $\varphi : X \to \mathbb{R}^n$ to a mapping, also denoted by φ, $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ wich satisfies the same properties, and is such that the following diagram commuts, i. e., $\varphi \circ \pi = \varphi$.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "N"] \arrow[d, "\pi"'] & \mathbb{R}^n \arrow[d, "\pi"] \arrow[dr, "\varphi"] & \\ X \arrow[r, "N"'] & X \arrow[r, "\varphi"'] & \mathbb{R}^n \end{tikzcd}$$

Lemma 2.1. *Let $P \in \mathbb{R}[x_1, ..., x_n]\backslash\{0\}$ be a polynomial of degree k and $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, $\varphi = (\varphi_1, ..., \varphi_n)$ an analytic mapping, with $\varphi(0) = 0$ and $\varphi \neq 0$. Then,*

$$\lim_{x \to 0} \frac{P \circ \varphi(x)}{\|\varphi(x)\|^l} \neq 0, \forall\, l \text{ with } m_P \leq l \leq k,$$

where m_P is the multiplicity of P.

Proof: We decompose the polynomial P into its homogeneous parts, i.e., $P = P_{m_P} + ... + P_k$ and call $\varphi_m = (\varphi_{1,m_1}, ..., \varphi_{n,m_n})$ where each φ_{i,m_i} is the homogeneous part of lower degree of φ_i and m_i denotes its degree. Let $m_\varphi = \min\{m_i,\ i = 1, ..., n\}$, and we can write $\varphi = \varphi_m + \xi$. Hence we conclude that $\varphi(tv) = t^{m_\varphi}\widetilde{\varphi}_t(v)$ with $\widetilde{\varphi}_0(v) \neq 0$ and $t \in \mathbb{R}$.

Since there exists generic v such that $\widetilde{\varphi}_0(v) \neq 0$, we consider such v with $v \notin \widetilde{\varphi}_0^{-1}(0)$. Thus,

$$\begin{aligned} &\frac{P \circ \varphi(tv)}{\|\varphi(tv)\|^\ell} = \\ = \;& \frac{P_{m_P}(t^{m_\varphi}\widetilde{\varphi}_t(v)) + ... + P_{l-1}(t^{m_\varphi}\widetilde{\varphi}_t(v))}{\|t^{m_\varphi}\widetilde{\varphi}_t(v)\|^l} + \frac{P_l(t^{m_\varphi}\widetilde{\varphi}_t(v))}{\|t^{m_\varphi}\widetilde{\varphi}_t(v)\|^l} + \\ + \;& \frac{P_{l+1}(t^{m_\varphi}\widetilde{\varphi}_t(v)) + ... + P_k(t^{m_\varphi}\widetilde{\varphi}_t(v))}{\|t^{m_\varphi}\widetilde{\varphi}_t(v)\|^l} = \\ = \;& \frac{t^{m_P m_\varphi}[P_{m_P}(\widetilde{\varphi}_t(v)) + t^{m_\varphi}P_{m_P+1}(\widetilde{\varphi}_t(v)) + ... + t^{(l-1-m_P)m_\varphi}P_{l-1}(\widetilde{\varphi}_t(v))]}{t^{lm_\varphi}\|\widetilde{\varphi}_t(v)\|^l} + \\ + \;& \frac{t^{lm_\varphi}P_l(\widetilde{\varphi}_t(v))}{t^{lm_\varphi}\|\widetilde{\varphi}_t(v)\|^l} + \\ + \;& \frac{t^{(l+1)m_\varphi}[P_{l+1}(\widetilde{\varphi}_t(v)) + t^{m_\varphi}P_{l+2}(\widetilde{\varphi}_t(v)) + ... + t^{k-l-1}P_k(\widetilde{\varphi}_t(v))]}{t^{lm_\varphi}\|\widetilde{\varphi}_t(v)\|^l}. \end{aligned}$$

Therefore,

$$\lim_{t\to 0}\frac{P\circ\varphi(tv)}{\|\varphi(tv)\|^l}=\widetilde{P_l}+\frac{P_l(\widetilde{\varphi}_0(v))}{\|\widetilde{\varphi}_0(v)\|^l}+0\neq 0.$$

where

$$\widetilde{P_l}=\begin{cases}0, & \text{if } l=m_P\\ sgn(P_{m_P}(\widetilde{\varphi}_0(v)))\infty, & \text{otherwise}\end{cases}$$

■

Lemma 2.2. *Let $f,g:U\subseteq\mathbb{R}^n\to\mathbb{R}$ be functions of class C^k and modified C^1-equivalent. Then, for all $l\leq k$*

$$j^lf\equiv 0\Leftrightarrow j^lg\equiv 0$$

Proof: Let X be a $n-$dimensional analytic manifold, $\varphi:X\to\mathbb{R}^n$ an analytic mapping with $\varphi(X)=\mathbb{R}^n$ and $N:U_1\to U_2$ a C^1-liftable mapping, with U_1,U_2 neighbourhoods of $\varphi^{-1}(0)$ in $\varphi^{-1}(U)$, such that

$$f\circ\varphi\circ N=g\circ\varphi$$

Now, as f,g are of class C^l, $\forall\, l\leq k$, we have by the Taylor's infinitesimal formula that

$$f(x)=j^lf(x)+R_l(x)\ and\ g(x)=j^lg(x)+\xi_l(x)$$

where $\lim\limits_{u\to 0}\frac{R_l(u)}{\|u\|^l}=0$ and $\lim\limits_{u\to 0}\frac{\xi_l(u)}{\|u\|^l}=0$

Thus,

$$j^lf\circ\varphi\circ N+R_l\circ\varphi\circ N=j^lg\circ\varphi+\xi_l\circ\varphi$$

Then , if $j^lf\equiv 0$, we have that

$$j^lg\circ\varphi=R_l\circ\varphi\circ N-\xi_l\circ\varphi$$

Claim 1: $j^lg\circ\varphi\equiv 0$.

In fact, $j^lg\circ\varphi(v)=R_l(\varphi(N(v)))-\xi_l(\varphi(v))$ and therefore

$$\frac{j^lg(\varphi(tv))}{\|\varphi(tv)\|^l}=\frac{R_l(\varphi(N(tv)))}{\|\varphi(tv)\|^l}-\frac{\xi_l(\varphi(tv))}{\|\varphi(tv)\|^l}$$

Remark: $\lim\limits_{t\to 0}\frac{\xi_l(\varphi(tv))}{\|\varphi(tv)\|^l}=0$, since $\varphi(0)=0$.

Now,

$$\frac{R_l(\varphi(N(tv)))}{\|\varphi(tv)\|^l}=\frac{R_l(\varphi(N(tv)))}{\|\varphi(N(tv))\|^l}\frac{\|\varphi(N(tv))\|^l}{\|\varphi(tv)\|^l}.$$

Claim 2: $\frac{\|\varphi(N(tv))\|}{\|\varphi(tv)\|}$ is limited.

In fact, write $\varphi = \varphi_m + \xi$, where φ_m is the initial part of φ. Since φ_m is a nonzero polynomial, for generic v, $\varphi_m(tv) \neq 0, \forall\, t \ll 1$.

Therefore,

$$\|\varphi(tv)\| \approx \|tv\|^{m_\varphi}$$

where m_φ is the minimum of the multiplicities of the coordinate functions of φ_m.

Since $\|\varphi(y)\| \preceq \|y\|^{m_\varphi}$ generically, we have that

$$\|\varphi(N(x))\| \quad \preceq \quad \|N(x)\|^{m_\varphi} \underbrace{\preceq}_{N\ lips} \|x\|^{m_\varphi} \Rightarrow$$

$$\Rightarrow \quad \|\varphi(tv)\| \approx \|tv\|^{m_\varphi} \succeq \|\varphi(N(tv))\|$$

Therefore,

$$\lim_{t\to} \frac{\|\varphi(N(tv))\|}{\|\varphi(tv)\|} < \infty$$

hence $\dfrac{\|\varphi(N(tv))\|}{\|\varphi(tv)\|}$ is limited, and we conclude that

$$\lim_{t\to} \frac{\|R_l(\varphi(N(tv)))\|}{\|\varphi(tv)\|^l} = 0$$

Thus, $\lim\limits_{t\to 0} \dfrac{j^l g \circ \varphi(tv)}{\|\varphi(tv)\|^l} = 0$ and by the Lemma (2.1) we have that $j^l g \circ \varphi \equiv 0$. Therefore $j^l g \equiv 0$ (since φ is surjective). ■

Theorem 2.3. (I) *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be analytic functions which are modified C^1-equivalent. Then $mult(f) = mult(g)$;*

(II) *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be functions of class C^k which are modified C^1-equivalent. Then $mult(j^k f) = mult(j^k g)$.*

Proof: (I) Follows by the Lemma (2.2).

(II) Suppose that $mult(j^k f) = l_1 < l_2 = mult(j^k g)$. Then $j^{l_1} g \equiv 0$ and by the lemma (2.2), $j^{l_1} f \equiv 0$, but this contradicts the fact that $l_1 = mult(j^k f)$.

Therefore, $mult(j^k f) = mult(j^k g)$. ■

Theorem 2.4. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be functions of class C^k and modified C^1-equivalent. Then $j^l f$ and $j^l g$ are modified C^1-equivalent, where $l = mult(j^k f)$ $(= mult(j^k g))$.*

To prove this Theorem we need the following

Lemma 2.5. *Let $P, Q \in \mathbb{R}[x_1, ..., x_n] \backslash \{0\}$ be homogeneous polynomials of degree l, $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ a nonzero analytic mapping and $N : \mathbb{R}^n \to \mathbb{R}^n$ a nonzero diffeomorphism* *of class C^ℓ. Then,*

$$\lim_{x\to 0}\frac{(P\circ\varphi\circ N - Q\circ\varphi)(x)}{\|\varphi(x)\|^l}\neq 0$$

Remark 1: We can write $N = T + R$ where T is the linear part of n and T is invertible since n is of class C^ℓ.

Remark 2: $N(tv) = T(tv) + R(tv) = t[T(v) + \frac{R(tv)}{t}]$, i.e., $N(tv) = tu(tv)$, where $u(0) = T(v) \neq 0$.

Proof of the Lemma 2.5: Since $\varphi(tv) = t^{m_\varphi}\widetilde{\varphi}_t(v)$, $\widetilde{\varphi}_0(v)\neq 0$, we have

$$\begin{aligned}\frac{P(\varphi(N(tv))) - Q(\varphi(tv))}{\|\varphi(tv)\|^l} &= \frac{P(t^{m_\varphi}\widetilde{\varphi}_t(u(tv))) - Q(t^{m_\varphi}\widetilde{\varphi}_t(v))}{t^{lm_\varphi}\|\widetilde{\varphi}_t(v)\|^l} = \\ &= \frac{P\circ\widetilde{\varphi}_t\circ u(tv) - Q\circ\widetilde{\varphi}_t(v)}{\|\widetilde{\varphi}_t(v)\|^l}\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{t\to 0}\frac{(P\circ\varphi\circ N - Q\circ\varphi)(tv)}{\|\varphi(tv)\|^l} &= \frac{P\circ\widetilde{\varphi}_0\circ u(0) - Q\circ\widetilde{\varphi}_0(v)}{\|\widetilde{\varphi}_0(v)\|^l} = \\ &= \frac{(P\circ\widetilde{\varphi}_0)(T(v)) - Q\circ\widetilde{\varphi}_0(v)}{\|\widetilde{\varphi}_0(v)\|^l}\neq 0\end{aligned}$$

for generic $v\notin (P\circ\widetilde{\varphi}_0\circ T - Q\circ\widetilde{\varphi}_0)^{-1}(0)$. ■

Proof of the Theorem 2.4: By the Taylor's infinitesimal formula we can write

$$f = j^l f + R_l \text{ and } g = j^l g + \xi_l$$

where $\lim\limits_{u\to 0}\frac{R_l(u)}{\|u\|^l} = \lim\limits_{u\to 0}\frac{\xi_l(u)}{\|u\|^l} = 0$.

Therefore,

$$j^l f\circ\varphi\circ N - j^l g\circ\varphi = \xi_l\circ\varphi - R_l\circ\varphi\circ N$$

Hence,

$$\frac{j^l f\circ\varphi\circ N - j^l g\circ\varphi}{\|\varphi\|^l} = \frac{\xi_l\circ\varphi - R_l\circ\varphi\circ N}{\|\varphi\|^l}$$

Thus, we have that

$$\lim_{x\to 0}\frac{(j^l f\circ\varphi\circ N - j^l g\circ\varphi)(x)}{\|\varphi(x)\|^l} = 0.$$

Therefore, since $j^l f$ and $j^l g$ are homogeneous polynomials of degree l, we can apply the Lemma (2.5), and obtain that $j^l f \circ \varphi \circ N = j^l g \circ \varphi$ ■

Corollary 2.6. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be functions of class C^k and modified C^1-equivalent. Then $j^r f$ and $j^r g$ are modified C^1- equivalent, $\forall\ l \leq r \leq k$, where $l = mult(j^k f)$ $(= mult(j^k g))$.*

Proof: We can write $f = j^l f + H_{l+1} + R_{l+1}$ and $g = j^l g + L_{l+1} + \xi_{l+1}$ where $H_{l+1} = j^{l+1} f - j^l f$ and $L_{l+1} = j^{l+1} g - j^l g$ are homogeneous polynomials of degree $l+1$.

Then, $\exists\ \varphi : \mathbb{R}^n \to \mathbb{R}^n$ analytic and $N : \mathbb{R}^n \to \mathbb{R}^n$ a diffeomorphism of class C^ℓ such that

$$j^l f \circ \varphi \circ N + H_{l+1} \circ \varphi \circ N + R_{l+1} \circ \varphi \circ N = j^l g \circ \varphi + L_{l+1} \circ \varphi + \xi_{l+1} \circ \varphi,$$

thus, by the theorem (2.4), we have that

$$H_{l+1} \circ \varphi \circ N - L_{l+1} \circ \varphi = \xi_{l+1} \circ \varphi - R_{l+1} \circ \varphi \circ N$$

Hence

$$\lim_{x \to 0} \frac{(H_{l+1} \circ \varphi \circ N - L_{l+1} \circ \varphi)(x)}{\|\varphi(x)\|^{l+1}} = 0$$

Therefore, by the Lemma (2.5), $H_{l+1} \circ \varphi \circ N = L_{l+1} \circ \varphi$

Thus, $j^{l+1} f \circ \varphi \circ N = j^{l+1} g \circ \varphi$, i.e., $j^{l+1} f$ and $j^{l+1} g$ are modified C^1-equivalent and the result follows by induction. ■

3. Modified $C^\ell-$triviality

Fix X a $n-$dimensional analytic manifold and I a $p-$dimensional compact cube.

Definition Let $M : X \times I \to X \times I$ be a $t-$level preserving mapping. We say that M is *$C^\ell-$liftable* when there exist an analytical mapping $\pi : \mathbb{R}^n \to X$ and a $t-$level preserving local $C^\ell-$diffeomorphism, also denoted by M, $M : \mathbb{R}^n \times I \to \mathbb{R}^n \times I$ such that $(\pi \times Id_I) \circ M = M \circ (\pi \times Id_I)$, *i.e.*,

$$\begin{array}{ccc} \mathbb{R}^n \times I & \xrightarrow{M} & \mathbb{R}^n \times I \\ {\scriptstyle \pi \times Id_I}\downarrow & & \downarrow{\scriptstyle \pi \times Id_I} \\ X \times I & \xrightarrow[M]{} & X \times I \end{array}$$

commuts.

Let $\varphi : X \to \mathbb{R}^n$ be a proper analytic mapping such that $\varphi(X) = \mathbb{R}^n$.

Consider a family $F_t : \mathbb{R}^n \to \mathbb{R}$ of real functions parametrized by $t = (t_1, ..., t_p) \in I$ such that $F(0, t) = 0$ for any $t \in I$.

Definition F admits a *modified $C^\ell-$trivialization along I* if there exist a neighbourhood U of the origin in $\mathbb{R}^n$ and a $C^\ell-$liftable $t-$level preserving mapping

$$M : U_1 \times I \to U_2 \times I$$

where U_1 and U_2 are neighbourhoods of $\varphi^{-1}(0)$ in $\varphi^{-1}(U)$ such that $F\circ(\varphi\times Id_I)\circ M$ is independent of t, where Id_I denotes the identity of I. Moreover, M induces a $t-$level preserving homeomorphism $m : \varphi(U_1) \times I \to \varphi(U_2) \times I$, such that the diagram

$$\begin{tikzcd} U_1 \times I \arrow[r, "M"] \arrow[d, "\varphi\times Id_I"'] & U_2 \times I \arrow[d, "\varphi\times Id_I"] & \\ \varphi(U_1) \times I \arrow[r, "m"'] & \varphi(U_2) \times I \arrow[r, "F"'] & \mathbb{R} \end{tikzcd}$$

commuts.

Now, we show that a $C^{\ell+1}$ deformation of a weighted homogeneous real function, under certain conditions, admits a modified C^ℓ-trivialization.

Let $\omega = (\omega_1, ..., \omega_n)$ be a $n-$tuple of positive integers and d a positive integer. A function $P : \mathbb{R}^n \to \mathbb{R}$ is called weighted homogeneous of type (ω, d) if $P(r^{\omega_1}x_1, ..., r^{\omega_n}x_n) = r^d P(x)$, $\forall r \in \mathbb{R}, x = (x_1, ..., x_n)$.

Given an analytic function $f : \mathbb{R}^n \to \mathbb{R}$, we define $fil(f)$ as the minimum of the numbers $\omega_1 i_1 + ... + \omega_n i_n$ where the monomial $x_1^{i_1}\cdots x_n^{i_n}$ appears with coefficient different of zero in the Taylor's expansion of f.

Theorem 3.1. *Let $F : U \times \mathbb{R}^p \to \mathbb{R}$ be a family of functions, where U is a neighbourhood of origin at $\mathbb{R}^n$. Suppose that we can write F on the form $F = f + g$ with $f, g : U \times \mathbb{R}^p \to \mathbb{R}$ satisfying the following conditions:*

(i) *f and g are of class $C^{\ell+1}$ and g, $\frac{\partial g}{\partial t_s}$ are of class C^k in the x-variable where $k > \max\{\frac{d}{\omega_1}, \dots, \frac{d}{\omega_n}\} + \ell$;*

(ii) *for each $t \in \mathbb{R}^p$, $f_t = f(.,t)$ is the restriction to U of a weighted homogeneous function of type (ω, d) with isolated singularity at origin;*

(iii) *$fil(j^{k-1}g_t) > d$, $\forall t \in \mathbb{R}^p$, where $j^k(.)$ denotes the k-jet at origin (the Taylor's polynomial of degree k).*

Then given a point t_0 in $\mathbb{R}^p$ and a $p-$dimensional cube I with $t_0 \in I$, F admits a modified C^ℓ-trivialization along I.

Proof: We remark that $fil(j^k g_t) > d$ since from the item (1) we conclude that $fil(x_1^{i_1}\cdots x_n^{i_n}) > d$ if $i_1 + ... + i_n \geq k$.

Without loss of generality, we may assume that $t_0 = 0$ and I is the cube $[-L, L]^p$ for some positive real number L.

Consider $\varphi : S^{n-1} \times (-\epsilon, \epsilon) \to \mathbb{R}^n$ defined by

$$\varphi(x, r) = (r^{\omega_1}x_1, ..., r^{\omega_n}x_n)$$

with $\epsilon > 0$ is choosen such that φ has its values in U.

We define $H : S^{n-1} \times (-\epsilon, \epsilon) \times \mathbb{R}^p \to \mathbb{R}$ by

$$H(x,r,t) = r^{-d} g(\varphi(x,r),t)$$

for $r \neq 0$ and $H(x,0,t) = 0$.

We call $\pi : S^{n-1} \times (-\epsilon, \epsilon) \times \mathbb{R}^p \to S^{n-1} \times \mathbb{R}^p$ the standard projection $\pi(x,r,t) = (x,t)$.

Lemma 3.2. *The conditions (1) and (3) imply that $H, \frac{\partial H}{\partial x_i}, \frac{\partial H}{\partial t_s}$ are at least of class C^ℓ, and equal to 0 if $r = 0$.*

Proof of the Lemma 3.2: It is easy to see that

$$\begin{aligned} j^{k-q}\left(\frac{\partial^q g_t}{\partial x_{i_1}\cdots\partial x_{i_q}}\right) &= \frac{\partial^q g_t}{\partial x_{i_1}\cdots\partial x_{i_q}} j^k(g_t) \\ fil\left(j^{k-q}\left(\frac{\partial^q g_t}{\partial x_{i_1}\cdots\partial x_{i_q}}\right)\right) &> d - \sum_{j=1}^{q} \omega_{i_j} \end{aligned} \tag{3.1}$$

Now, by the Taylor's infinitesimal formula we have that

$$\frac{\partial^q}{\partial x_{i_1}\cdots\partial x_{i_q}}(g_t \circ \varphi(x,r)) = j^{k-q}\left[\frac{\partial^q}{\partial x_{i_1}\cdots\partial x_{i_q}}(g_t \circ \varphi(x,r))\right] + R_{k-q}(\varphi(x,r))$$

with $\lim\limits_{r\to 0} \frac{R_{k-q}(\varphi(x,r))}{\|\varphi(x,r)\|^{k-q}} = 0$.

Therefore,

$$\frac{\partial^q H}{\partial x_{i_1}\cdots\partial x_{i_q}} = \frac{1}{r^{(d-\sum_{j=1}^{q}\omega_{i_j})}}\left(j^{k-q}\left(\frac{\partial^q g_t}{\partial x_{i_1}\cdots\partial x_{i_q}}\right)(\varphi(x,r))\right) + \frac{R_{k-q}(\varphi(x,r))}{r^d}$$

where $q = 0, 1, ..., \ell + 1$.

From (3.1) $\lim\limits_{r\to 0} \frac{1}{r^{(d-\sum_{j=1}^{q}\omega_{i_j})}}\left(j^{k-q}\left(\frac{\partial^q g_t}{\partial x_{i_1}\cdots\partial x_{i_q}}\right)(\varphi(x,r))\right) = 0.$

Thus it is sufficient to prove that $\lim\limits_{r\to 0} \frac{R_{k-q}(\varphi(x,r))}{r^d} = 0.$

In fact,

$$\frac{R_{k-q}(\varphi(x,r))}{r^d} = \frac{R_{k-q}(\varphi(x,r))}{\|\varphi(x,r)\|^{k-q}} \frac{\|\varphi(x,r)\|^{k-q}}{r^d}$$

As $\lim\limits_{r\to 0} \frac{R_{k-q}(\varphi(x,r))}{\|\varphi(x,r)\|^{k-q}} = 0$ and

$$\left(\frac{\|\varphi(x,r)\|^{k-q}}{r^d}\right)^2 = \left[\frac{r^{2\omega_1}x_1^2 + \ldots + r^{2\omega_n}x_n^2}{r^{\frac{2d}{k-q}}}\right]^{k-q} \leq r^{2(\omega'(k-q)-d)}$$

where $\omega' = \min\{\omega_1, ..., \omega_n\}$ and $k - q \geq k - \ell - 1 > \frac{d}{\omega'} - 1 \Rightarrow k - q \geq \frac{d}{\omega'} \Rightarrow \omega'(k-q) - d \geq 0$, hence the result follows. $\square$

Now, let us consider the mapping $\widetilde{F} : S^{n-1} \times (-\epsilon, \epsilon) \times \mathbb{R}^p \to \mathbb{R}^n$ given by $\widetilde{F}(x, r, t) = f \circ \pi(x, r, t) + H(x, r, t)$.

As f_t has isolated singularity at origin, $u(x, r, t) := \frac{\frac{\partial \widetilde{F}}{\partial t_p}}{\|\nabla_x \widetilde{F}\|^2} \nabla_x \widetilde{F}$ is well defined in $S^{n-1} \times (-\epsilon, \epsilon) \times \mathbb{R}^p$ and is of class C^ℓ.

Thus, we can write

$$\frac{\partial \widetilde{F}}{\partial t_p} = \langle \nabla_x \widetilde{F}, u \rangle \tag{3.2}$$

Now multiplying the equation (3.2) by r^d we obtain

$$\frac{\partial F}{\partial t_p}(\varphi(x, r), t) = \langle \nabla_{\varphi(x,r)} F(\varphi(x, r), t), w(x, r, t) \rangle \tag{3.3}$$

with $(x, r, t) \in S^{n-1} \times ((-\epsilon, 0) \cup (0, \epsilon)) \times (-L-\delta, L+\delta)^p$ and $w(x, r, t) = \varphi(u(x, r, t)) = (r^{\omega_1} u_1(x, r, t), ..., r^{\omega_n} u_n(x, r, t))$ is of class C^ℓ in $S^{n-1} \times (-\epsilon, \epsilon) \times (-L - \delta, L + \delta)^p$.

Now we can rewrite (3.3) as

$$\frac{\partial F}{\partial t_p}(y, t) = \langle \nabla_y F(y, t), w(\varphi^{-1}(y), t) \rangle \tag{3.4}$$

where $y \in \varphi(S^{n-1} \times (-\epsilon, \epsilon)) \setminus \{0\}$.

Let $W(y, t) = \frac{\partial}{\partial t_p} - w(\varphi^{-1}(y), t)$ be the vector field defined in $[\varphi(S^{n-1} \times (-\epsilon, \epsilon)) \setminus \{0\}] \times \mathbb{R}^p$ which is of class C^ℓ.

Note that $W \perp \nabla F$, since

$$\langle W, \nabla F \rangle = \frac{\partial F}{\partial t_p} - \langle \nabla_y F, w(\varphi^{-1}(y), t) \rangle = 0.$$

Remark that $\lim\limits_{y \to 0} W(y, t) = \frac{\partial}{\partial t_p} - \lim\limits_{r \to 0} w(x, r, t) = \frac{\partial}{\partial t_p}$, hence W has a continuous extension to $(0, t)$ with $W(0, t) = \frac{\partial}{\partial t_p}$.

Consider the problem of initial value

$$\left\{ \begin{array}{rcl} \frac{\partial \tau^p}{\partial t_p}(y, t) & = & W(\tau^p(y, t)) \\ \tau^p(y, t_1, ..., t_{p-1}, 0) & = & (y, t_1, ..., t_{p-1}, 0) \end{array} \right. \tag{3.5}$$

Suppose, that exist ϵ_p, δ_p and τ^p such that $0 < \epsilon_p \leq \epsilon, 0 < \delta_p \leq \delta$ and τ^p is a mapping from

$$[\varphi(S^{n-1} \times (-\epsilon_p, \epsilon_p))\backslash\{0\}] \times (-L - \delta_p, L + \delta_p)^p \to \varphi(S^{n-1} \times (-\epsilon, \epsilon))\backslash\{0\}$$

satisfying (3.5).

Remark: Since W is constant on the t_p-parameter, we obtain that $\tau^p(y,t) = (\overline{\tau}^p(y,t), t)$.

As W is ortoghonal to ∇F, we have that $F(\overline{\tau}^p(y,t), t) = F(y, t_1, ..., t_{p-1}, 0)$.

We know by the existence and unicity theorem of ordinary differential equations that there exists an unique solution

$$\overline{\tau}_t^p : \varphi(S^{n-1} \times (-\epsilon_p, \epsilon_p))\backslash\{0\} \to \varphi(S^{n-1} \times (-\epsilon, \epsilon))\backslash\{0\}$$

of class C^ℓ.

Remark: In the course of the construction of $\overline{\tau}^p(y,t)$, we show that if we define $\overline{\tau}_t^p(0) = 0$, i.e., $\tau_t^p(0,t) = (0,t)$ then

$$\overline{\tau}_t^p : \varphi(S^{n-1} \times (-\epsilon_p, \epsilon_p)) \to \varphi(S^{n-1} \times (-\epsilon, \epsilon))$$

is a homeomorphism. In fact, this homeomorphism is induced by a C^ℓ-diffeomorphism from $S^{n-1} \times (-\epsilon_p, \epsilon_p) \to S^{n-1} \times (-\epsilon, \epsilon)$

Thus, let us consider the lifting of $w_t \circ \varphi^{-1}$ to $S^{n-1} \times [(-\epsilon, 0) \cup (0, \epsilon)]$, i.e., let us consider the vector field

$$V_t : S^{n-1} \times [(-\epsilon, 0) \cup (0, \epsilon)] \to T(S^{n-1} \times [(-\epsilon, 0) \cup (0, \epsilon)])$$

such that $d_{(x,r)}\varphi.V_t = w_t(x,r)$

Lemma 3.3. *V_t has a C^ℓ extension to $S^{n-1} \times (-\epsilon, \epsilon)$ which is tangent to $S^{n-1} \times \{0\}$ at all points of $S^{n-1} \times \{0\}$.*

Proof of Lemma 3.3 Let us identify the tangent bundle of S^{n-1} with a subbundle of the trivial bundle $S^{n-1} \times \mathbb{R}^n$. Thus the equation to V_t becomes:

$$\begin{bmatrix} r^{\omega_1} & 0 & \cdots & 0 & \omega_1 x_1 r^{\omega_1 - 1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & r^{\omega_n} & \omega_n x_n r^{\omega_n - 1} \\ x_1 & x_2 & \cdots & x_n & 0 \end{bmatrix} \begin{bmatrix} V_{1t} \\ \vdots \\ V_{nt} \\ V_{(n+1)t} \end{bmatrix} = \begin{bmatrix} r^{\omega_1} u_1(x,r,t) \\ \vdots \\ r^{\omega_n} u_n(x,r,t) \\ 0 \end{bmatrix}. \tag{3.6}$$

Where the above equation of V_t is deduced as follows:

Let $\gamma(s) = (x(s), r(s))$ with $\gamma'(0) = (x'(0), r'(0)) = v = (v_1, ..., v_n, v_{n+1})$ and $\gamma(0) = (x, r)$.

Therefore

$$\begin{aligned}\frac{\partial}{\partial s}(\varphi\circ\gamma)_{|_{s=0}} &= \frac{\partial}{\partial s}(r(s)^{\omega_1}x_1(s),...,r(s)^{\omega_n}x_n(s))_{|_{s=0}} = \\ &= (\omega_1 x_1 r^{\omega_1-1}v_{n+1}+r^{\omega_1}v_1,...,\omega_n x_n r^{\omega_n-1}v_{n+1}+r^{\omega_n}v_n) = \\ &= \begin{bmatrix} r^{\omega_1} & 0 & \cdots & 0 & \omega_1 x_1 r^{\omega_1-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & r^{\omega_n} & \omega_n x_n r^{\omega_n-1} \\ x_1 & x_2 & \cdots & x_n & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ v_n \\ v_{(n+1)} \end{bmatrix}.\end{aligned}$$

Solving the equation (3.6) by the Cramer's rule, we have that $V_{it}=\frac{\Delta_i}{\Delta}$, $i=1,...,n$, and $V_{(n+1)t}=\frac{\Delta_{n+1}}{\Delta}$. Then we see that Δ is the product between $r^{(\sum_i \omega_i)-1}$ and a nonvanishing term and each Δ_i has a factor of $r^{(\sum_i \omega_i)-1}$ with $i=1,...,n$. Since Δ_i are functions that depend on $u_1,...,u_n$, this proves that V_i has a C^ℓ extension. Finally Δ_{n+1} has a factor of $r^{\sum_i \omega_i}$, justifying the claim that the extension is tangent to $S^{n-1}\times\{0\}$. □

Then, the Theorem follows by induction with respect to the number of parameters. ■

4. Examples

Example 1: Let $F(x,y,t)=f+g=x^2-y^5+(t^7\sin(\frac{1}{t}))(y^{17}\sin(\frac{1}{y}))$ be a deformation of class C^3 of the germ $f=x^2-y^5$.

Note that f is a real weighted homogeneous function of type $(5,2;10)$ and $g=(t^7\sin(\frac{1}{t}))(y^{17}\sin(\frac{1}{y}))$ is of class C^3 with $fil(j^8(g_t))=\infty$. Moreover, g_t, $\frac{\partial g}{\partial t}$ are of class C^8 on the $(x,y)-$variable, and obviously $8>\max\{\frac{10}{5},\frac{10}{2}\}+2$. Therefore, by the Theorem (2.3) we have that F admits a modified C^2-trivialization.

We can observe that the estimates obtained in [B-M] do not even guarantee the C^1-triviality of F.

The following examples shown that the modified $C^\ell-$equivalence is stronger than topological equivalence.

Example 2: Let $f,g:\mathbb{R},0\to\mathbb{R},0$ be given by $f(x)=x$ and $g(x)=x^3$. Naturally f and g are topologically equivalent, but are not modified C^1-equivalent since $mult(f)=1$ and $mult(g)=3$.

References

[B-K] Buchner - Kucharz, *Topological triviality of family of zero-sets.*, Proc. Amer. Math. Soc., vol. 102, No. 3, 1988.

[B-M] Bromberg - Medrano, *C^r-sufficiency of quasihomogeneous functions.*, Real and Complex Singularities, Pitman Research Notes in Math. Series 333, 1995, 179-189.

[Fu] Fukui, *Modified analytic trivialization of a family of real analytic mappings.*, Contemporary Math., Vol. 90, 1999.

[K] Tzee-Char Kuo, *The modified analytic trivialization of singularities*, J. Math Soc. Japan, Vol. 32, No. 4, 1980.

[R-S] Ruas - Saia, *C^ℓ-determinacy of weighted homogeneous germs.*, Hokkaido Math. Journal, Vol. 26, 1997, pp. 89-99.

[Pa] Paunescu, *An example of blow analytic homeomorphism.*, Real analytic and algebraic singularities, Pitman Research Notes in Mathematics Series, Addison Wesley Longman Inc, 138 (1998), 62-63.

[Yo] Yoshinaga, *The modified analytic trivialization of real analytic family via blowing-ups.*, J. Math. Soc. Japan, 40, 1988, pp. 161-179.

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