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A Impulsive Functional Second Order Differential Equation

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Uma Equação Diferencial Funcional de Segunda Ordem com Impulsos.

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Resumo

Neste artigo estudamos existência de soluções fracas, fortes e clássicas para uma equação de segunda ordem com impulsos modelada na forma

$$\begin{aligned} x''(t) &= Ax(t) + f(t, x(t), x(a(t)), x'(t), x'(b(t))), \quad t \in I = [0, T], \ t \neq t_i, \\ x(0) &= x_0, \quad x'(0) = y_0, \\ \triangle x(t_i) &= I_i^1(x(t_i), x'(t_i)), \\ \triangle x'(t_i) &= I_i^2(x(t_i), x'(t_i)), \end{aligned}$$

onde A é o gerador infinitesimal de uma função cosseno fortemente contínua de operadores lineares num espaço de Banach X; $0 < t_1 < t_2 < .... < t_n < T$ e $a(\cdot)$, $b(\cdot), f(\cdot), I_i^j, i = 1, 2...n, j = 1, 2$, são funções apropriadas.

A Impulsive Functional Second Order Differential Equation.

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Abstract

By using the theory of strongly continuous cosine functions, in this paper we study existence of mild, strong and classical solutions for a class of second order functional differential equations with impulses.

1 Introduction

In this paper we establish some existence results of mild, strong and classical solutions for a class of impulsive second order functional differential equations modelled in the form

$$\begin{aligned} x''(t) &= Ax(t) + f(t, x(t), x(a(t)), x'(t), x'(b(t))), \quad t \in I = [0,T], \; t \neq t_i, & (1)\\ x(0) &= x_0, \quad x'(0) = y_0, & (2)\\ \triangle x(t_i) &= I_i^1(x(t_i), x'(t_i)), & (3)\\ \triangle x'(t_i) &= I_i^2(x(t_i), x'(t_i)), & (4) \end{aligned}$$

where A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators, $(C(t))_{t\in\mathbb{R}}$, on a Banach space X; $0 < t_1 < t_2 < .... < t_n < T$; $a(\cdot)$, $b(\cdot) : I \to I$, $I_i^j(\cdot) : X \to X$, $i = 1, 2, ...n, j = 1, 2$ and $f(\cdot) : I \times X^4 \to X$ are appropriated functions and $\triangle\xi(t_i)$ denote the jump of a function $\xi(\cdot)$ in the point t_i, that is, $\triangle\xi(t_i) = \xi(t_i^+) - \xi(t_i^-)$.

The theory of impulsive differential equations has become an important area of investigation in recent years. Relative to this theory we only refer to the reader for the works: Cabada [1], Hernndez [7], Liu [9], Rogovchenko [12],[13] and Sun [15].

Motivated for numerous applications, recently Liu [9], Hernndez [7] and Benchohra [2] studied some impulsive differential problems modelled using the semigroup and the cosine functions theories of bounded linear operators on abstract Banach spaces. In [9], Liu proved existence of mild and classical solutions for the first order abstract impulsive Cauchy problem

$$\left\{ \begin{array}{ll} u'(t) = Au(t) + f(t, u(t)), & t \in I, \; t \neq t_i, \\ u(0) = x_0, & \\ \triangle u(t_i) = I_i(u(t_i)), & i = 1, 2, ...n, \end{array} \right. \qquad (5)$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on a Banach space X; $0 < t_1 < ..... < t_n < T$ and $I_i(\cdot)$, $f(\cdot)$ are appropriated functions. To prove your results, Liu used the contraction mapping principle and the ideas in [11]. Using similar arguments, Hernndez studied in [7] the impulsive Cauchy problem

$$\left\{\begin{array}{lll} u''(t) & = Au(t) + f(t, u(t), u'(t)), & t \in (-T_0, T_1),\ t \neq t_i, \\ u(0) & = x_0, \quad u'(0) = y_0, & \\ \Delta u(t_i) & = I_i^1(u(t_i)), \quad \Delta u'(t_i) = I_i^2(u'(t_i^+)), & \end{array}\right. \tag{6}$$

where A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators, $-T_0 < 0 < t_1 < ..... < t_n < T_1$ and $I_i^j(\cdot)$, $f(\cdot)$ are continuous functions. On the other hand, Benchohra discuss in [2] the existence of mild solutions for the neutral problem

$$\begin{array}{rcll} \frac{d}{dt}[y'(t) + g(t, y_t)] & \in & Ay(t) + f(t, y_t), & t \in [0, T],\ t \neq t_i, \\ y(0) & = & y_0, \quad y'(0) = y_0, & \\ \Delta y(t_i) & = & I_i^1(y(t_i^-))), \quad \Delta y'(t_i) = I_i^2(y(t_i^-)), & \\ y_0 & = & \phi, \quad \phi \in C([-r, 0] : X), & \end{array}$$

where A is the generator of a strongly continuous cosine function. To obtain his results Benchohra used Leray Shauders Alternative, see Theorem 1. To this end, Benchohra employed some standard growth restrictions on g, f and a severe condition on the cosine function, $(C(t))_{t\in \mathbb{R}}$, generated by A. Specifically, the author assume that $C(t)$ is compact for every $t \in (0, T]$; which is only possible if X is finite dimensional, see Travis [16] pp. 557. This fact is relevant, since under these conditions, the model studied by Benchohra, for instance, not include partial differential equations and in this sense, our results aren't consequence of the ideas and estimates in [2].

The goal of this paper is to present some results of existence and regularity of solutions for the abstract Cauchy problem (1)-(4). To prove existence of mild solutions, we used some point fixed criterions for condensing operator, specifically, Leray Shauders Alternative and the Sadovskii Point fixed Theorem. To study regularity of mild solutions, we consider some resents results reported by Henrquez in [6].

In this paper, henceforth $C(\cdot) = (C(t))_{t\in \mathbb{R}}$ is a strongly continuous cosine function of bounded linear operators on a Banach space X with infinitesimal generator A. We refer the reader to [4] for the necessary concepts about cosine functions. Next we only mention a few results and notations needed to establish our results. We denote by $S(t)$ the sine function associated with $C(t)$, that is

$$S(t)x := \int_0^t C(s)x ds, \quad x \in X, \quad t \in \mathbb{R}.$$

Along of this paper, $[D(A)]$ is the space

$$D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable}\},$$

endowed with the norm $\| x \|_A = \| x \| + \| Ax \|$, $x \in D(A)$. Moreover, in this work the notation E stands for the space formed by the vectors $x \in X$ for which the function $C(\cdot)x$ is of class C^1. It was proved by Kisińsky [8] that E endowed with the norm

$$\|x\|_1 = \|x\| + \sup_{0\le t\le 1} \|AS(t)x\|, \quad x \in E,$$

is a Banach space. The operator valued function $G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is a strongly continuous group of linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. From this it follows that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0,\ t \to 0,$ for each $x \in E$. Furthermore, if $x : [0, \infty) \to X$ is a locally integrable function then

$$y(t) = \int_0^t S(t-s)x(s)ds$$

defines an E-valued continuous function. This is an consequence of the fact that

$$\int_0^t G(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)\, ds \\ \int_0^t C(t-s)x(s)\, ds \end{bmatrix}$$

defines an $E \times X$-valued continuous function.

The existence of solutions of the second order abstract Cauchy problem

$$\begin{cases} x''(t) &= Ax(t) + h(t), \quad 0 \le t \le T, \\ x(0) &= x_0, \quad x'(0) = x_1, \end{cases} \tag{7}$$

where $h : [0, T] \to X$ is an integrable function, has been discussed in [16]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem it has been treated in [17]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s)\, ds, \quad 0 \le t \le T, \tag{8}$$

is called, mild solution of (7) and that when $x_0 \in E$, $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)x_0 + C(t)x_1 + \int_0^t C(t-s)h(s)\, ds.$$

This work contain three sections. In section 2 we establish existence of mild, strong and classical solutions for some second order impulsive Cauchy problems. The results are proved using point fixed Theorems for condensing map and the techniques introduced by Henríquez in [6]. In section 3 an exemple is considered.

The terminologies and notations are those generally used in functional analysis. In particular, for Banach spaces Z, W; $\mathrm{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W and we abbreviate to $\mathrm{L}(Z)$ when $Z = W$. Moreover, $B_r(x : Z)$ denotes the closed ball with center at x and radius r in Z and we will write simply B_r for $B_r(0 : Z)$ when no confusion arises. Additionally, we will employ the prefix $\mathcal{R}$ to indicate the image of a map and for a bounded function $\xi : [0, b] \to [0, \infty)$ and $t \in [0, b]$, we will use the notation ξ_t for

$$\xi_t = \sup\{\xi(s) : s \in [0, t]\}.$$

To finish this section and for completeness, we include the following well known result.

Theorem 1 *(Leray Schauder Alternative) Let D be a convex subset of a Banach space X and assume that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda F(x),\ 0 < \lambda < 1\}$ is unbounded or the map F has a fix point in D.*

2 Existence Results

In this section we study existence of solutions for some impulsive second order functional differential equations. We begin by introducing some notations. In this section, we assume that $M \geq 1$ and $N \geq 0$ are constants such that $\|C(t)\| \leq M$ and $\|S(t)\| \leq N$ for every $t \in I$. In what follow, $\mathcal{PC}$ will denote the space

$$\mathcal{PC} = \left\{ u : [0,T] \to X; \quad \begin{array}{l} u \text{ is continuous in } t \neq t_i,\ u(t_i) = u(t_i^-) \text{ and } u(t_i^+) \\ \text{exists for every } i = 1, ..., n \end{array} \right\},$$

endowed with the uniform convergence norm, $\| \cdot \|_T$. Similarly, $\mathcal{PC}^1$ will be the space $\mathcal{PC}^1 = \{u \in \mathcal{PC} : \sqcap' \in \mathcal{PC}\}$ provided with the norm $||| \, u \, ||| = \| \, u \, \|_T + \| \, u' \, \|_T$.

For a function $u \in \mathcal{PC}$; $\tilde{u}_0 \in C([0,t_1] : X)$, $\tilde{u}_i \in C([t_i, t_{i+1}] : X)$, $i = 1, 2, ...n-1$, and $\tilde{u}_n \in C([t_n, T] : X)$ are the functions defined by

$$\tilde{u}_0 = u \text{ on } [0, t_1], \tag{1}$$

$$\tilde{u}_i(t) = \begin{cases} u(t), & \text{for } t \in (t_i, t_{i+1}], \\ u(t_i^+), & \text{for } t = t_i, \end{cases} \tag{2}$$

$$\tilde{u}_n(t) = \begin{cases} u(t), & \text{for } t \in (t_n, T], \\ u(t_n^+), & \text{for } t = t_n. \end{cases} \tag{3}$$

Moreover, for $B \subset \mathcal{PC}$ we employ the notation $\widetilde{B}_i$, $i = 0, 1, ...n$, for the sets

$$\widetilde{B}_i = \{\tilde{u}_i : u \in B\}. \tag{4}$$

The proof of Proposition 1 below, follows essentially from the classical Ascoli-Arzela Theorem and will be omit.

Proposition 1 *A set $B \subset \mathcal{PC}$ is relatively compact if and only if each set $\widetilde{B}_i$ is relatively compact.*

At first we consider the following second order initial value problem

$$u''(t) = Au(t) + f(t, u(t), u(a(t))), \qquad t \in I,\ t \neq t_i, \tag{5}$$

$$u(0) = x_0, \quad u'(0) = y_0, \tag{6}$$

$$\Delta u(t_i) = I_i^1(u(t_i)), \tag{7}$$

$$\Delta u'(t_i) = I_i^2(u(t_i)). \tag{8}$$

To study the impulsive Cauchy problem (5)-(8) we always assume that the following assumptions hold:

Assumption A_1

(a) The function $a(\cdot) : I \to I$ is continuous and $a(t) \leq t$ for every $t \in I$.

(b) The function $f : I \times X^2 \to X$ satisfies the following Caratheodory conditions:

(i) $f(t,\cdot) : X \times X \to X$ is continuous a.e. $t \in I$;

(ii) For each $x, y \in X$, the function $f(\cdot, x, y) : I \to X$ is strongly measurable.

(d) There exist a continuous function $m : I \to [0, \infty)$ and a continuous nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that

$$\|f(t,x,y)\| \leq m(t)W(\| x \| + \| y \|), \qquad \forall\ (t,x,y) \in I \times X^2.$$

(c) The functions I_j^i are continuous.

By comparison with the second order abstract Cauchy problem (7), we introduce the following definition.

Definition 1 *A function $u \in \mathcal{PC}(I : X)$ is a mild solution of the impulsive problem (5)-(8) if condition (7) is satisfied and*

$$\begin{aligned} u(t) &= C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s,u(s),u(a(s)))ds \\ &\quad + \sum_{t_i<t} C(t-t_i)I_i^1(u(t_i)) + \sum_{t_i<t} S(t-t_i)I_i^2(u(t_i)), \qquad t \in I. \end{aligned}$$

Now, we establish our first existence result.

Theorem 1 *Assume that the followings conditions are verified.*

(H-1) *The functions I_i^1 are completely continuous and there exist positive constants c_i^1, $i = 1, 2, ...n$, such that $\| I_i^1(x) \| \leq c_i^1 \| x \| + c_i^2$ for every $x \in X$.*

(H-2) *For every $s \in I$ and every $r > 0$, the set $\{S(s)I_i^2(x) : \| x \| \leq r,\ i = 1, 2...n\}$ is relatively compact in X and there exist positive constants d_i^2, $i = 1, 2, ...n$, such that $\| I_j^2(x) \| \leq d_j^1 \| x \| + d_j^2$ for every $x \in X$.*

(H-3) *For each $t \in I$, $t' \leq t$ and each $r \geq 0$ the set $U(t,t',r) = \{S(t')f(s,x,y) : 0 \leq s \leq t,\ \|x\|, \|y\| \leq r\}$ is relatively compact.*

If

$$\mu = \sum_{i=1}^{n}(Mc_i^1 + Nd_i^1) < 1 \quad \text{and} \quad \frac{2N}{1-\mu}\int_0^T m(s)ds < \int_{2c}^{\infty} \frac{ds}{W(s)}\, ds \tag{9}$$

where $c = \dfrac{1}{1-\mu}[M \| x_0 \| + N \| y_0 \| + \sum_{i=1}^{n}(Mc_i^2 + Nd_i^2)]$, then there exist a mild solution of (5)-(8).

Proof: Let $\Gamma : \mathcal{PC} \to \mathcal{PC}$ the map defined by

$$\begin{aligned} \Gamma u(t) &= C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s,u(s),u(a(s)))ds \\ &\quad + \sum_{t_i<t} C(t-t_i)I_i^1(u(t_i)) + \sum_{t_i<t} S(t-t_i)I_i^2(u(t_i)). \end{aligned} \tag{10}$$

It's clear that Γ is well defined and continuous. In order to use Theorem 1, we make an *a priori* estimate for the solutions of the integral equation $\lambda\Gamma u_\lambda = u_\lambda$, $\lambda \in (0,1)$. If $t \in I$, from the definition of Γ we find that

$$\begin{aligned} \| u_\lambda(t) \| &\leq M \| x_0 \| + N \| y_0 \| + N \int_0^t m(s) W(\| u_\lambda(s) \| + \| u_\lambda(a(s)) \|) ds \\ &+ \sum_{t_i \leq t} (M c_i^1 + N d_i^1) \| u_\lambda(t_i) \| + \sum_{i=1}^n (M c_i^2 + N d_i^2). \end{aligned}$$

Using the notation $\alpha_\lambda(t)$ for the right hand side of the last inequequality, follow that

$$\begin{aligned} \alpha_\lambda(t) &\leq M \| x_0 \| + N \| y_0 \| + N \int_0^t m(s) W(2\alpha_\lambda(s)) ds \\ &+ \sum_{i=1}^n (M c_i^1 + N d_i^1) \alpha_\lambda(t) + \sum_{i=1}^n (M c_i^2 + N d_i^2) \end{aligned}$$

which, from the assumption on μ imply that

$$\alpha_\lambda(t) \leq \frac{1}{1-\mu} \left[M \| x_0 \| + N \| y_0 \| + \sum_{i=1}^n (M c_i^2 + N d_i^2) \right] + \frac{N}{1-\mu} \int_0^t m(s) W(2\alpha_\lambda(s)) ds.$$

Denoting by $\beta_\lambda(t)$ the right hand side of the last inequality, we get

$$\beta_\lambda'(t) \leq \frac{N}{1-\mu} m(t) W(2\beta_\lambda(t)),$$

and consequently

$$\int_{2\beta_\lambda(0)}^{2\beta_\lambda(t)} \frac{ds}{W(s)} \leq \frac{2N}{1-\mu} \int_0^t m(s)\, ds < \int_{2c}^\infty \frac{ds}{W(s)} ds.$$

From the last inequality and the hypothesis, follows that $\beta_\lambda(\cdot)$ is uniformly bounded respect to λ on I and consequently that the set $\{u_\lambda : \lambda \in (0,1)\}$ is bounded in $\mathcal{PC}(I : X)$.

To prove that Γ is completely continuous, we introduce the decomposition $\Gamma = \sum_{i=1}^3 \Gamma_i$ where

$$\Gamma_1 u(t) = C(t)x_0 + S(t)y_0, \tag{11}$$

$$\Gamma_2 u(t) = \int_0^t S(t-s) f(s, u(s), u(a(s))) ds, \tag{12}$$

$$\Gamma_3 u(t) = \sum_{t_i < t} C(t-t_i) I_i^1(u(t_i)) + \sum_{t_i < t} S(t-t_i) I_i^2(u(t_i)). \tag{13}$$

Next, we prove that Γ_2 and Γ_3 are completely continuous.

Step 1 The map Γ_2 is completely continuous.

Let $r > 0$ and $B_r = B_r(0, \mathcal{PC})$. By using the classical Ascoli-Arzela Theorem, we will prove that $\Gamma_2 B_r$ is relatively compact in $C([0,T] : X)$. Initially, we establish that the set

$\Gamma_2(B_r)$ is equicontinuous. We fix $t \in I$ and let h be such that $t+h \in I$. If $u \in B_r$, from the definition of Γ_2 we have that

$$\begin{aligned}
\| \Gamma_2 u(t+h) - \Gamma_2 u(t) \| &= \int_0^t \| (S(t+h-s) - S(t-s))f(s,u(s),u(a(s))) \| ds \\
&+ \int_t^{t+h} \| S(t+h-s)f(s,u(s),u(a(s))) \| ds \\
&\leq MhW(2r)\int_0^t m(s)ds + NW(2r)\int_t^{t+h} m(s)ds,
\end{aligned}$$

which shows that

$$\| \Gamma_2 u(t+h) - \Gamma_2 u(t) \| \to 0, \quad h \to 0,$$

uniform respect to $u \in B_r$.

Next we show that $\Gamma_2(B_r)(t) = \{\Gamma_2 u(t) : u \in B_r\}$ is relatively compact in X, for every $t \in [0,T]$. For $t \in I$, we put $U = \{f(t-s, u(t-s), u(a(t-s))) : 0 \leq s \leq t, \ u \in B_r\}$. Since

$$\|y\| \leq m(t-s)W(2r) \leq m_T W(2r) = r^*, \qquad y \in U,$$

and the operator map $S(\cdot) : [0,T] \to \mathrm{L(X)}$ is uniformly continuous, we infer that for each $\epsilon > 0$ there are points $0 = s_0 < s_1 < s_2 ... < s_k = t$ so that for each $i = 1,2...n$, and $s, s' \in [t_i, t_{i+1}]$

$$\|S(s)y - S(s')y\| \leq \epsilon, \qquad \| y \| \leq r^*.$$

Using **(H-3)** and the mean value theorem for the Bochner integral, see [10] Lemma 2.1.3, we get

$$\begin{aligned}
\Gamma_2 u(t) &= \sum_{i=0}^{k-1} \int_{s_i}^{s_{i+1}} (S(s) - S(s_i))f(t-s, u(t-s), u(a(t-s))ds \\
&+ \sum_{i=0}^{k-1} \int_{s_i}^{s_{i+1}} S(s_i)f(t-s, u(t-s), u(a(t-s))ds \\
&\in TB_\epsilon(0,X) + \sum_{i=0}^{k-1}(s_{i+1} - s_i)\overline{co(U(t,s_i,r^*))},
\end{aligned}$$

where co denote the convex hull. These remarks show that $\Gamma_2(B_r)(t)$ is totally bounded and consequently relatively compact in X. Thus Γ_2 is completely continuous.

Step 2 The map Γ_3 is completely continuous.

Let $r > 0$ and $B_r = B_r(0, \mathcal{PC})$. To prove the assertion we use Proposition 1 and the notations introduced in (1)-(4). At first we discuss the equicontinuity of $\Gamma_3 B_r$ in $t \neq t_i$. Let $t \in I \setminus \{t_1, ...t_n\}$ and $\epsilon > 0$. Since $C(\cdot)$ is strongly continuous, from condition **(H-1)** follows that the functions $C(\cdot)x$, $x \in \bigcup_{i=1}^n I_i^1(B_r(0,X))$, are uniformly equicontinuous on I. Let $\delta > 0$ such that $[t-\delta, t+\delta] \bigcap \{t_1, ...t_n\} \neq \emptyset$ and

$$\| C(s)x - C(s+h)x \| \leq \epsilon, \quad x \in \bigcup_{i=1}^n I_i^1(B_r(0,X)), \quad s \in [0,T], \tag{14}$$

when $| h |< \delta$. Under this conditions, for $u \in B_r$ and $| h |< \delta$ with $t+h \in I$, we get

$$\begin{aligned}\| \Gamma_3 u(t+h) - \Gamma_3 u(t) \| &= \sum_{t_i<t} \| (C(t+h-t_i) - C(t-t_i))I_i^1(u(t_i)) \| \\ &+ \sum_{t_i<t} \| (S(t+h-t_i) - S(t-t_i))I_i^2(u(t_i)) \| \\ &\leq n\epsilon + Mh\sum_{i=1}^{n}(d_i^1 r + d_i^2),\end{aligned}$$

which implies the equicontinuity in $t \neq t_i$.

Now, we study the equicontinuity of $[\widetilde{\Gamma_3 B_r}]_i$, $i = 1, 2, ...n$, at the right hand side at t_i. Since the cosine function is strongly continuous, from the definition of Γ_3 follows that

$$(\widetilde{\Gamma_3 u})_i(t_i) = \sum_{j=1}^{i-1} C(t_i - t_j)I_j^1(u(t_j)) + I_i^1(u(t_i)) + \sum_{j=1}^{i-1} S(t_i - t_j)I_j^2(u(t_j)), \qquad u \in B_r.$$

Let $\delta > 0$ such that (14) hold. For $u \in B_r$ and $0 < h < \delta$ with $t_i + h \in I$ we get

$$\begin{aligned}&\| (\widetilde{\Gamma_3 u})_i(t_i+h) - (\widetilde{\Gamma_3 u})_i(t_i) \| = \\ &\quad \sum_{j=1}^{i-1} \| (C(t_i+h-t_j) - C(t_i-t_j))I_j^1(u(t_j)) \| + \| C(h)I_i^1(u(t_i)) - I_i^1(u(t_i)) \| \\ &\quad + \sum_{j=1}^{i-1} \| (S(t_i+h-t_j) - S(t_i-t_j))I_j^2(u(t_j)) \| + \| S(h)I_i^2(u(t_i)) \| \\ &\leq n\epsilon + Mh\sum_{j=1}^{n}(d_j^1 r + d_j^2),\end{aligned}$$

which proves that $[\widetilde{\Gamma_3 B_r}]_i$ is equicontinuous at the right side at t_i. Similarly, $[\widetilde{\Gamma_3 B_r}]_i$, $i = 1, 2, ...n-1$, is equicontinuous at the left hand side at t_{i+1}.

These remarks prove that the sets $[\widetilde{\Gamma_3 B_r}]_i$, $i = 0, 1, 2, ...n$, are equicontinuous .

Using **(H-1)** and **(H-2)**, we infer that the set $\Gamma_3 B_r(t)$ is relatively compact in X for $t \neq t_i$. Moreover, since for $t = t_i$ and $u \in B_r$

$$\begin{aligned}\Gamma_3 u(t_i) &= \sum_{j=1}^{i-1} C(t_i - t_j)I_j^1(u(t_j)) + \sum_{j=1}^{i-1} S(t_i - t_j)I_j^2(u(t_j)), \\ \widetilde{\Gamma_3 u}(t_i) &= \sum_{j=1}^{i-1} C(t_i - t_j)I_j^1(u(t_j)) + I_i^1(u(t_i)) + \sum_{j=1}^{i-1} S(t_i - t_j)I_j^2(u(t_j)),\end{aligned}$$

follow from **(H-1)** and **(H-2)** that $\Gamma_3 B_r(t_i)$, $i = 1, 2, ..n$, and $[\widetilde{\Gamma_3 B_r}]_i(t_i) = \{(\widetilde{\Gamma_3 u})_i(t_i) : u \in B_r\}$ are relatively compact in X for every $i = 1, 2, .., n$. Now, Proposition 1 imply that Γ_3 is completely continuous.

Finally, by application of Theorem 1 we conclude that Γ has a fixed point in $\mathcal{PC}$. This point is a mild solution of (5)-(8). The proof is complete. ■

In the most of the situation of practical interest the sine function is compact, it's the motivation of the next result.

Corollary 1 *Assume that* **(H-1)** *holds, that $S(t)$ is a compact for every $t \in \mathbb{R}$ and that the followings assumptions are verified:*

(H-2)' *There exist positive constants d_i^2, $i = 1, 2, ...n$, such that $\| I_i^2(x) \| \leq d_i^1 \| x \| + d_i^2$ for every $x \in X$,*

(H-3)' *The function f takes bounded sets into bounded sets.*

If the conditions in (9) are verified, then there exists a mild solution of (5)-(8).

When the functions I_j^i are Lipschitz continuous we have the following result.

Theorem 2 *Assume that assumption* **(H-3)** *hold. Suppose, furthermore, that*

(H-4) *For every function I_j^i there exists a positive constant L_j^i such that*

$$\| I_i^j(x) - I_i^j(y) \| \leq L_i^j \| x - y \|, \qquad x, y \in X.$$

If $\liminf_{r \to \infty} \frac{MW(2r)}{r} \int_0^T m(s)ds + \sum_{i=1}^{n} (ML_i^1 + NL_i^2) < 1$, *then there exists a mild solution of (5)-(8).*

Proof: Let $\Gamma : \mathcal{PC} \to \mathcal{PC}$ the map defined by (10) and consider the decomposition $\Gamma = \overline{\Gamma}_1 + \overline{\Gamma}_2$ where

$$\overline{\Gamma}_2 u(t) = \int_0^t S(t-s) f(s, u(s), u(a(s))) ds.$$

We affirm that there exists $r > 0$ such that $\Gamma(B_r) \subset B_r$. In fact, if the affirmation is false, for every $r > 0$ there exists x_r in B_r such that $\| \Gamma x_r \|_T > r$ and then

$$\begin{aligned} r < \| \Gamma x_r \|_T &\leq M \| x_0 \| + N \| y_0 \| + NW(2r) \int_0^T m(s)ds \\ &+ \sum_{i=1}^{n} \left(ML_i^1 r + M \| I_i^1(0) \| + NL_i^2 r + N \| I_i^2(0) \| \right), \end{aligned}$$

hence,

$$1 \leq \liminf_{r \to \infty} \frac{MW(2r)}{r} \int_0^T m(s)ds + \sum_{i=1}^{n} (ML_i^1 + NL_i^2),$$

which is a absurd. Thus, there exists r_0 such that $\Gamma(B_{r_0}) \subset B_{r_0}$.

From the step 2 in the proof of Theorem 1, we know that $\overline{\Gamma}_2$ is completely continuous on B_{r_0}. Moreover, for $x, y \in B_{r_0}$ we have that

$$\| \overline{\Gamma}_1 x(t) - \overline{\Gamma}_1 y(t) \| \leq \sum_{i=1}^{n} (NL_i^1 + ML_i^2) \| x - y \|_T,$$

which show that $\overline{\gamma}_1$ is a contraction on B_{r_0}. These remarks prove that Γ is a condensing map on B_{r_0}. The existence of a mild solution follows from the Sadovskii point fixed Theorem. The proof is finished. ■

For functions f which are the perturbation of a linear operator, we can to establish the following modification of Theorem 1.

Theorem 3 *Assume that* **(H-1)** *and* **(H-2)** *hold. Suppose in addition that* $f = l + g$ *and that the followings conditions are satisfied.*

(a) *The functions* g, l *satisfies Caratheodory conditions as in Assumption* **A**.

(b) $l(t,\cdot) : X^2 \to X$ *is linear and* $L(t) = \| l(t,\cdot) \|_{L(X^2,X)}$ *is integrable on* I.

(c) *For each* $t \in I$, $t' \le t$ *and every constant* $r \ge 0$ *the set* $U(t,t',r) = \{S(t')g(s,x,y) : 0 \le s \le t, \|x\|, \|y\| \le r\}$ *is relatively compact in* X.

(d) *There exists a continuous function* $m_g : I \to [0,\infty)$ *and a continuous nondecreasing function* $W_g : [0,\infty) \to (0,\infty)$ *such that*

$$\|g(t,x,y)\| \le m_g(t) W_g(\| x \| + \| y \|), \qquad \forall\ (t,x,y) \in I \times X^2.$$

If $\liminf_{r\to\infty} \frac{MW_g(2r)}{r} \int_0^T m_g(s)ds + \int_0^T L(s)ds + \sum_{i=1}^{n}(Mc_i^1 + Nd_i^1) < 1$, *then there exists a mild solution of (5)-(8).*

Proof: Let $\widehat{\Gamma} = \Gamma + \Gamma_0$, where Γ is the map defined by (10), with g instead f and Γ_0 is the map

$$\Gamma_0 u(t) \;=\; \int_0^t S(t-s)l(s,u(s),u(a(s)))ds.$$

Using the ideas and estimates in the proofs of Theorem 2 and Theorem 1, follows that Γ is completely continuous and that there exists $r > 0$ such that $\Gamma(B_r) \subset B_r$. Moreover, it's clear that Γ_0 is a bounded linear operator and from the estimate

$$\begin{aligned} \| \Gamma_0 u(t) - \Gamma_0 v(t) \| &\le M \int_0^t (t-s)L(s)\left(\| u(s) - v(s) \| + \| u(a(s)) - v(a(s)) \|\right) ds \\ &\le 2M \int_0^t (t-s)L(s) \max_{0\le\xi\le s} \| u(\xi) - v(\xi) \| \, ds \end{aligned} \tag{15}$$

we infer that Γ_0^m is a contraction for m large enough so that the spectrum of Γ_0 is included in the open unit ball. Now, from Lemma 4.4.4 in [5] we see that the space $PC(I:X)$ can be renormed by a norm $|||\cdot|||$ such that $|||\Gamma_0||| < 1$. Consequently, $\widehat{\Gamma}$ is a condensing map on B_r which by the Sadovskii's theorem permit to conclude that $\widehat{\Gamma}$ has a fixed point in B_r. Thus there exists a mild solution of (5)-(8). The proof is finished. ■

It's clear that a mild solution of (5)-(8) verifies the impulsive condition (7). In relation to (8) we have

Proposition 2 *Let* $u(\cdot)$ *be a mild solution of (5)-(8). If* $x_0 \in E$ *and* $\mathcal{R}(I_i^1) \subset E$ *for every* $i = 1, 2, ...n$, *then* $u(\cdot)$ *verifies the impulsive condition (8).*

Proof: From the assumptions, we know that $u(\cdot)$ is continuously differentiable on each interval (t_i, t_{i+1}) and that

$$\begin{aligned} u'(t_i^-) &= AS(t_i)x_0 + C(t_i)y_0 + \int_0^{t_i} C(t_i - s)f(s,u(s),u(a(s)))ds \\ &\quad + \sum_{j=1}^{i-1} AS(t_i - t_j)I_j^1(u(t_j)) + \sum_{j=1}^{i-1} C(t_i - t_j)I_j^2(u(t_j)). \end{aligned}$$

Moreover, using that $AS(t)z \to 0$ as $t \to 0$ when $z \in E$, we find that

$$\begin{aligned} u'(t_i^+) &= AS(t_i)x_0 + C(t_i)y_0 + \int_0^{t_i} C(t_i - s)f(s, u(s), u(a(s)))ds \\ &+ \sum_{j=1}^{i-1} AS(t_i - t_j)I_j^1(u(t_j)) + \sum_{j=1}^{i-1} C(t_i - t_j)I_j^2(u(t_j)) + I_i^2(u(t_i)). \end{aligned}$$

Thus, $\triangle u'(t_i) = I_i^2(u(t_i))$, which complete the proof. ■

Next, we study differentiability of mild solutions of (5)-(8). Previously, we need some new definitions.

Definition 2 *A function $u(\cdot) : [a, b] \to X$ is a strong solution of*

$$x''(t) = Ax(t) + h(t), \qquad t \in (a, b], \tag{16}$$
$$x(a) = x_0, \quad x'(a) = x_1, \tag{17}$$

if $u(\cdot) \in W^{2,1}([a, b]; X)$, the equation (16) is satisfied a.e. on $[a, b]$ and (17) is verified.

In the followings definitions, $\tilde{u}_i$, $i = 1, 2...n$, are the functions defined in (1)-(2)-(3).

Definition 3 *We said that a function $u \in \mathcal{PC}$ is a strong solution of (5)-(8) if: $\tilde{u}_0 \in W^{2,1}([0, t_1] : X)$, $\tilde{u}_i \in W^{2,1}([t_i, t_{i+1}] : X)$, $i = 1, 2, .., n-1$, $\tilde{u}_n \in W^{2,1}([t_n, T] : X)$; the equation (5) is satisfied a.e. $t \in I$ and the conditions (7) and (8) are verified.*

Definition 4 *A function $u(\cdot) : I \to X$ is a classical solution of (5)-(8) if $u(\cdot)$ is a function of class C^2 on $I^* = I \setminus \{t_i, a(t_i) : i = 1, 2, ...n\}$ that satisfies the equation (5) on I^* and the initial conditions (6), (7) and (8).*

In order to reduce the notations and definitions, in the next results we consider understood the concept of mild solution for the impulsive initial value problem (18).

Lemma 1 *Let $g : [a, b] \to X$, $I^i : X \to X$, i=1,2, be a continuous functions and $\xi \in (a, b)$. Assume that $u : [a, b] \to X$ is a mild solution of the impulsive Cauchy problem*

$$\left\{ \begin{aligned} x''(t) &= Ax(t) + g(t), \qquad t \in [a, b],\ t \neq \xi, \\ x(0) &= x_0, \quad x'(0) = y_0, \\ \triangle x(\xi) &= I^1(x(\xi)), \quad \triangle x'(\xi) = I^2(x(\xi)). \end{aligned} \right. \tag{18}$$

Let $\tilde{u} : [\xi, b] \to X$ be the unique continuous extension of $u(\cdot)$ on $[\xi, b]$. If $x_0 \in E$, then $\tilde{u}(\cdot)$ is a mild solution of the abstract Cauchy problem

$$\begin{aligned} x''(t) &= Ax(t) + g(t), \qquad t \in [\xi, b], \\ x(\xi) &= u(\xi) + I^1(u(\xi)), \\ x'(\xi) &= u'(\xi^-) + I^2(u(\xi)). \end{aligned}$$

Proof: Since $x_0 \in E$, we have that

$$u'(\xi^-) = AS(\xi)x_0 + C(\xi)y_0 + \int_0^\xi C(\xi - s)g(s)ds. \tag{19}$$

On the other hand, for $t \in (\xi, b]$ we get

$$\begin{aligned}
u(t) &= C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)g(s)ds + C(t-\xi)I^1(u(\xi)) + S(t-\xi)I^2(u(\xi)), \\
&= C(t-\xi)C(\xi)x_0 + AS(t-\xi)S(\xi)x_0 + C(t-\xi)S(\xi)y_0 + S(t-\xi)C(\xi)y_0 \\
&\quad + C(t-\xi)\int_0^\xi S(\xi-s)g(s)ds + S(t-\xi)\int_0^\xi C(\xi-s)g(s)ds + \int_\xi^t S(t-s)g(s)ds \\
&\quad + C(t-\xi)I^1(u(\xi)) + S(t-\xi)I^2(u(\xi)) \\
&= C(t-\xi)[C(\xi)x_0 + S(\xi)y_0 + \int_0^\xi S(\xi-s)g(s)ds + I^1(u(\xi))] \\
&\quad + S(t-\xi)[AS(\xi)x_0 + C(\xi)y_0 + \int_0^\xi C(\xi-s)g(s)ds + I^2(u(\xi))] \\
&\quad + + \int_\xi^t S(t-s)g(s)ds,
\end{aligned}$$

thus,

$$u(t) = C(t-\xi)[\,u(\xi) + I^1(u(\xi))\,] + S(t-\xi)[u'(\xi^-) + I^2(u(\xi))] + \int_\xi^t S(t-s)g(s)ds,$$

which prove the assertion.■

In our results respect of strong and classical solutions, we consider Banach spaces that have the Radom Nikodym property, (abbreviated RNP). We refer to [3] for this matter.

Theorem 4 *Let $u(\cdot)$ be a mild solution of (5)-(8). Assume that X has the RNP and that the following conditions hold:*

(H-5) $x_0 \in D(A)$ *and* $u(t_i) + I_i^1(u(t_i)) \in D(A)$ *for every* $i = 1, 2, ...n$,

(H-6) $y_0 \in E$ *and* $u'(t_i^-) + I_i^2(u(t_i)) \in E$ *for every* $i = 1, 2, ...n$,

(H-7) *For each bounded set $D \subseteq X$, the functions $C(\cdot)f(t, x, y)$, $t \in I$, $x, y \in D$, are uniformly Lipschitz continuous.*

Then $u(\cdot)$ is a strong solution of (5)-(8).

Proof: Let $\tilde{u}_i(\cdot)$, $i = 0, 1, 2...n$, be the functions defined in (1)-(3) and f_u the function $f_u(t) = f(t, u(t), u(a(t)))$. From the preliminaries, for $t \in [0, t_1]$ and $s \in \mathbb{R}$ such that $t + s \in [0, t_1]$ we have that

$$\begin{aligned}
u'(t+s) - u'(t) &= (AS(t+s) - AS(t))x_0 + (C(t+s) - C(t))y_0 \\
&\quad + \int_0^t (C(t+s-\xi) - C(t-\xi))f_u(\xi)\,d\xi \\
&\quad + \int_t^{t+s} C(t+s-\xi)f_u(\xi)d\xi,
\end{aligned}$$

which implies that $u'(\cdot)$ is Lipchitz continuous on $[0, t_1]$. Since X has the RNP, it follows that $\tilde{u}_0(\cdot) \in W^{2,1}([0, t_1]; X)$ and from Proposition 3.3 in Henriquez [6] that $\tilde{u}_0(\cdot)$ is a strong solution of the abstract Cauchy problem

$$\begin{aligned} x''(t) &= Ax(t) + f_u(t), \qquad t \in [0, t_1], \\ x(0) &= x_0, \quad x'(0) = y_0. \end{aligned}$$

From Lemma 1, $\tilde{u}_1(\cdot)$ is a mild solution of

$$x''(t) = Ax(t) + f_u(t), \qquad t \in [t_1, t_2], \tag{20}$$
$$x(t_1) = u(t_1) + I_1^1(u(t_1)), \tag{21}$$
$$x'(t_1) = u'(t_1^-) + I_1^2(u(t_1)). \tag{22}$$

The same argument used in the first part of this proof, permit to conclude that $\tilde{u}_1(\cdot)$ is a strong solution of (20)-(22). Reiterating the previous arguments, we infer that each $\tilde{u}_i$, $i = 2, ...n$ is a strong solution of a second order differential equation of the form

$$\begin{aligned} x''(t) &= Ax(t) + f_u(t), \\ x(t_i) &= u(t_i) + I_i^1(u(t_i)), \\ x'(t_i) &= u'(t_i^-) + I_i^2(u(t_i)), \end{aligned}$$

which complete the proof. ■

Remark 1 *Related with the last result, we know from the results in [8] that condition* **H-7** *holds if $\mathcal{R}(f) \subseteq E$ and $f : I \times X \times X \to E$ takes bounded sets into the bounded sets.*

Theorem 5 *Assume that X has the RNP and that $(x_0, y_0) \in D(A) \times E$. If the following conditions hold;*

(H-8) *$\mathcal{R}(I_i^1) \subset D(A)$ and $\mathcal{R}(I_i^2) \subset E$ for every $i = 1, 2, ...n$,*

(H-9) *The function $a(\cdot)$ is Lipschitz continuous,*

(H-10) *For each bounded set $D \subseteq X$, the function $f : I \times D \times D \to X$ is Lipschitz continuous,*

then a mild solution, $u(\cdot)$, of (5)-(8) is a classical solution.

Proof: Let $U = \{\tilde{t}_1 < \tilde{t}_2 < ...\tilde{t}_k\} = \{t_i, a(t_i) : i = 1, 2, ...n\}$ and $f_u : [0, T] \to X$ the function $f_u(t) = f(t, u(t), u(a(t)))$. For every $i = 0, 1, 2...k$ we introduce the functions $v_0 : [0, \tilde{t}_1] \to X$, $v_i : [\tilde{t}_i, \tilde{t}_{i+1}] \to X$, $i = 1, 2, ...k-1$, and $v_k : [\tilde{t}_n, T] \to X$ defined by

$$v_i(t) = \begin{cases} u(t), & \text{for } t \in (\tilde{t}_i, \tilde{t}_{i+1}], \\ u(\tilde{t}_i^+), & \text{for } t = \tilde{t}_i, \end{cases}$$

$$\begin{aligned} v_0 &= u \text{ on } [0, \tilde{t}_1], \\ v_k &= \tilde{u}_n \text{ on } [t_n, T]. \end{aligned}$$

Clearly, $v_0(\cdot)$ is a mild solution of

$$\begin{cases} x''(t) &= Ax(t) + f_u(t), \qquad\qquad t \in [0, \tilde{t}_1], \\ x(0) &= x_0, \quad x'(0) = y_0. \end{cases} \tag{23}$$

From the assumptions, $v_0(\cdot)$ is a function of class C^1 and applying **H-9**, **H-10** it is easy to see that $f_u(\cdot)$ is Lipschitz continuous on $[0, t_1]$. We know from Theorem 3.1 in [6] that $v_0(\cdot)$ is a classical solution of (23) and that $v_0(\tilde{t}_1) = u(\tilde{t}_1) \in D(A)$ and $v_0'(\tilde{t}_1^-) = u'(\tilde{t}_1^-) \in E$.

From Lemma (1), the function $v_1(\cdot)$ is a mild solution of any of the following initial value problems on $[\tilde{t}_1, \tilde{t}_2]$

$$\begin{cases} x''(t) &= Ax(t) + f_u(t), \\ x(\tilde{t}_1) &= u(\tilde{t}_1) + I_1^1(u(\tilde{t}_1)), \\ x'(\tilde{t}_1) &= u'(\tilde{t}_1^-) + I_1^2(u(\tilde{t}_1)), \end{cases} \quad \text{or} \quad \begin{cases} x''(t) &= Ax(t) + f_u(t), \\ x(\tilde{t}_1) &= u(\tilde{t}_1), \\ x'(\tilde{t}_1) &= u'(\tilde{t}_1^-), \end{cases} \tag{24}$$

which from **H-8** and the same arguments used in the case of v_0, imply that $v_1(\cdot)$ is a classical solution of (24) and that $v_1(\tilde{t}_2) = u(\tilde{t}_2) \in D(A)$ and $v_1'(\tilde{t}_2^-) = u'(\tilde{t}_2^-) \in E$.

Using these remarks, we infer that each function $v_i(\cdot)$ is a classical solution of any of the problems

$$\begin{cases} x''(t) &= Ax(t) + f_u(t), \\ x(\tilde{t}_i) &= u(\tilde{t}_i) + I_i^1(u(\tilde{t}_i)), \\ x'(\tilde{t}_i) &= u'(\tilde{t}_i^-) + I_i^2(u(\tilde{t}_i)), \end{cases} \quad \text{or} \quad \begin{cases} x''(t) &= Ax(t) + f_u(t), \\ x(\tilde{t}_i) &= u(\tilde{t}_i), \\ x'(\tilde{t}_i) &= u'(\tilde{t}_i^-), \end{cases}$$

which permit to conclude that $u(\cdot)$ is a classical solution of (5)-(8). ∎

Next we discuss existence of solution for the abstract Cauchy problem (1)-(4). In general, the results and the hypotheses used in this case are similar to those in the first part of this section. For this reason, the proof of the next results will be summarized or simply omitted. At first, we introduce the following technical assumptions, which we will assume valid in the rest of this section.

Assumption $\mathbf{A_2}$

(a) The functions $a(\cdot), b(\cdot) : I \to I$ are continuous and $\max\{a(t), b(t)\} \le t,\ t \in I$.

(b) The function $f : I \times X^4 \to X$ satisfies the following Caratheodory conditions:

(i) $f(t, \cdot) : X^4 \to X$ is continuous a.e. $t \in I$;

(ii) for each $x \in X^4$, the function $f(\cdot, x) : I \to X$ is strongly measurable.

(c) There exists a continuous function $m : I \to [0, \infty)$ and a continuous nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that

$$\|f(t, x_1, x_2, x_3, x_4)\| \le m(t) W\Big(\sum_{i=1}^{4} \| x_i \|\Big), \qquad t \in I,\ x_i \in X.$$

(d) The functions $I_i^j : X^2 \to X$, $j = 1, 2$; $i = 1, 2, ...n$ are continuous.

Definition 5 *A function $u \in \mathcal{P}C^1(I : X)$ is a mild solution of the impulsive problem (1)-(4) if the impulsive conditions (3), (4) are satisfied and*

$$\begin{aligned} u(t) &= C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s,u(s),u(a(s)),u'(s),u'(b(s)))ds \\ &+ \sum_{t_i<t} C(t-t_i)I_i^1(u(t_i),u'(t_i)) + \sum_{t_i<t} S(t-t_i)I_i^2(u(t_i),u'(t_i)), \qquad t \in I. \end{aligned}$$

Theorem 6 *Assume that $x_0 \in E$ and that the following conditions hold:*

(H-11) *The functions $I_i^j, j = 1,2, i = 1,2,...n$, are completely continuous and there exist positive constants c_i^j, d_i^j,*

$$\begin{aligned} \| I_i^1(x,y) \| &\leq c_i^1 \| (x,y) \| + c_i^2, \qquad (x,y) \in X^2, \\ \| I_i^2(x,y) \| &\leq d_i^1 \| (x,y) \| + d_i^2, \qquad (x,y) \in X^2, \end{aligned}$$

for every $i = 1,2,...n$,

(H-12) *Any of the following properties hold:*

(a) *The functions I_i^1 are completely continuous with values in E and there exist positive constants $\tilde{c}_i^1, \tilde{c}_i^2$, $i = 1,2,...n$, such that*

$$\| I_i^1(x,y) \|_1 \leq \tilde{c}_i^1 \| (x,y) \| + \tilde{c}_i^2, \ (x,y) \in X^2,$$

(b) *$S(t)$ is compact for every $t \geq 0$, the functions I_i^1 are continuous with values in $[D(A)]$ and there exist positive constants $\tilde{c}_i^1, \tilde{c}_i^2$, $i = 1,2,...n$, such that*

$$\| AS(t)I_i^1(x,y) \| \leq \tilde{c}_i^1 \| (x,y) \| + \tilde{c}_i^2, \ (x,y) \in X^2, t \in I,$$

for every $i = 1,2,...n$.

(H-13) *For every $r > 0$, the set $f(I \times B_r(0,X^4))$ is relatively compact in X.*

If $\mu = \sum_{i=1}^n [\tilde{c}_i^1 + Mc_i^1 + (M+N)d_i^1] < 1$ and $\frac{2(M+N)}{1-\mu}\int_0^T m(s)ds < \int_{2c}^\infty \frac{ds}{W(s)}$ where $c = \frac{1}{1-\mu}[M \| x_0 \| + (N+M) \| y_0 \| + \| AS(\cdot)x_0 \|_T + \sum_{i=1}^n (Mc_i^2 + \tilde{c}_i^2 + (N+M)d_i^2)]$, then there exists a mild solution of (1)-(4).

Proof: Let $\Gamma : (\mathcal{P}C)^2 \to (\mathcal{P}C)^2$ the map defined by $\Gamma(u,v) = (\Gamma^1(u,v), \Gamma^2(u,v))$, where

$$\begin{aligned} \Gamma^1(u,v)(t) &= C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s,u(s),u(a(s)),v(s),v(b(s)))ds \\ &+ \sum_{t_i<t} C(t-t_i)I_i^1(u(t_i),v(t_i)) + \sum_{t_i<t} S(t-t_i)I_i^2(u(t_i),v(t_i)), \\ \Gamma^2(u,v)(t) &= AS(t)x_0 + C(t)y_0 + \int_0^t C(t-s)f(s,u(s),u(a(s)),v(s),v(b(s)))ds \\ &+ \sum_{t_i<t} AS(t-t_i)I_i^1(u(t_i),v(t_i)) + \sum_{t_i<t} C(t-t_i)I_i^2(u(t_i),v(t_i)). \end{aligned}$$

A straightforward estimation shows that Γ is well defined and continuous. If $z_\lambda = (x_\lambda, y_\lambda)$ is a solution of the integral equation $z_\lambda = \lambda\Gamma(z_\lambda), \lambda \in (0,1)$, then

$$\begin{aligned}\| x_\lambda(t) \| &\leq M \| x_0 \| + N \| y_0 \| + \sum_{t_i \leq t} [(Mc_i^1 + Nd_i^1) \| (x_\lambda(t_i), y_\lambda(t_i)) \| + Mc_i^2 + Nd_i^2] \\ &\quad + N \int_0^t m(s) W(\| x_\lambda(s) \| + \| x_\lambda(a(s)) \| + \| y_\lambda(s) \| + \| y_\lambda(b(s)) \|) ds \\ &:= \alpha_\lambda(t),\end{aligned}$$

and hence,

$$\begin{aligned}\alpha_\lambda(t) &\leq M \| x_0 \| + N \| y_0 \| + \sum_{t_i \leq t} [(Mc_i^1 + Nd_i^1)(\alpha_\lambda(t) + \| y_\lambda(t_i) \|) + Mc_i^2 + Nd_i^2] \\ &\quad + N \int_0^t m(s) W(2\alpha_\lambda(s) + \| y_\lambda(s) \| + \| y_\lambda(b(s)) \|) ds.\end{aligned}$$

Similarly, in both **(a)** or **(b)** in **(H-12)**, we find that

$$\begin{aligned}\| y_\lambda(t) \| &\leq \| AS(\cdot)x_0 \|_T + M \| y_0 \| + \sum_{t_i \leq t} [(\tilde{c}_i^1 + Md_i^1) \| (x_\lambda(t_i), y_\lambda(t_i)) \| + (\tilde{c}_i^2 + Md_i^2)] \\ &\quad + M \int_0^t m(s) W(2\alpha_\lambda(s) + \| y_\lambda(s) \| + \| y_\lambda(b(s)) \|) ds := \beta_\lambda(t),\end{aligned}$$

thus

$$\begin{aligned}\beta_\lambda(t) &\leq \| AS(\cdot)x_0 \|_T + M \| y_0 \| + \sum_{t_i \leq t} [(\tilde{c}_i^1 + Md_i^1)(\alpha_\lambda(t) + \beta_\lambda(t)) + \tilde{c}_i^2 + Md_i^2] \\ &\quad + N \int_0^t m(s) W(2\alpha_\lambda(s) + 2\beta_\lambda(s)) ds.\end{aligned}$$

From the assumption on μ and the previous estimates follows that

$$\begin{aligned}\alpha_\lambda(t) + \beta_\lambda(t) &\leq \frac{1}{1-\mu}[M \| x_0 \| + (N+M) \| y_0 \| + \| AS(\cdot)x_0 \|_T] \\ &\quad + \frac{1}{1-\mu} \sum_{i=1}^n (Mc_i^2 + \tilde{c}_i^2 + (N+M)d_i^2) \\ &\quad + \frac{M+N}{1-\mu} \int_0^t m(s) W(2(\alpha_\lambda(s) + \beta_\lambda(s))) ds,\end{aligned}$$

and so

$$\int_{2(\alpha_\lambda+\beta_\lambda)(0)}^{2(\alpha_\lambda+\beta_\lambda)(t)} \frac{ds}{W(s)} \leq \frac{2(M+N)}{1-\mu} \int_0^T m(s) ds < \int_{2c}^\infty \frac{ds}{W(s)}, \qquad t \in I,$$

which imply that the set $\{(x_\lambda, y_\lambda) : \lambda \in (0,1)\}$ is bounded in $(\mathcal{PC})^2$.

Repeating the arguments in the proof of Theorem 1, we can to prove that Γ^1 is completely continuous. Moreover, from the preliminaries and **(H-12)** we infer that the set of functions $U = \{t \to AS(t)I_i^1(x) : \| x \| \leq r, \ i = 1, 2, ...n\}$ is relatively compact in $C(I : X)$, which jointly with **(H-13)** permit to prove that Γ_2 is completely continuous.

Now, the existence of a mild solution of (1)-(4) is consequence of Theorem 1. ∎

The proof of the next results follows using the steps in the proofs of Theorems 2, 3 and 6.

Theorem 7 *Assume that* **(H-13)** *holds. Suppose, furthermore, that*

(H-14) $x_0 \in E$; *the maps* $I_i^1, i = 1, 2, ..n$, *are* E-*valued and that there exist positive constants* $L_i^j, i = 1, 2, ..n, j = 1, 2$, *such that*

$$\begin{aligned} \| I_i^1(x,y) - I_i^1(z,w) \|_1 &\leq L_i^1 \| (x,y) - (z,w) \|, \\ \| I_i^2(x,y) - I_i^2(z,w) \| &\leq L_i^2 \| (x,y) - (z,w) \|, \end{aligned}$$

for every $(x,y), (z,w) \in X^2$ *and every* $i = 1, 2, ...n$.

If

$$\sum_{i=1}^{n} \left((M+1)L_i^1 + (M+N)L_i^2\right) + (M+N) \liminf_{r\to\infty} \frac{W(4r)}{r} \int_0^T m(s)ds < 1,$$

then there exists a mild solution of (1)-(4).

Theorem 8 *Assume that* **(H-11)** *and* **(H-12)** *hold. Suppose, furthermore, that* $f = l + g$ *and that the following conditions are satisfied.*

(a) *The functions* g, l *satisfies Caratheodory conditions as in Assumption* $\mathbf{A_2}$.

(b) $l(t,\cdot) : X^4 \to X$ *is linear and* $L(t) = \| l(t,\cdot) \|_{L(X^4:X)}$ *is integrable on* I.

(c) *For every* $r > 0$, *the set* $g(I \times B_r(0, X^4))$ *is relatively compact in* X.

(d) *There exists a continuous function* $m_g : I \to [0,\infty)$ *and a continuous nondecreasing function* $W_g : [0,\infty) \to (0,\infty)$ *such that*

$$\|g(t, x_1, x_2, x_3, x_4)\| \leq m_g(t) W_g(\sum_{i=1}^{4} \| x_i \|), \qquad t \in I, x_i \in X.$$

If $(M+N)\left[\int_0^T L(s)ds + \liminf_{r\to\infty} \frac{W_g(4r)}{r} \int_0^T m_g(s)ds\right] + \sum_{i=1}^{n} \left[Mc_i^1 + \tilde{c}_i^1 + (M+N)d_i^1\right] < 1$, *then there exists a mild solution of (5)-(8).*

Definition 6 *We said that a function* $u(\cdot) \in \mathcal{PC}^\infty$ *is a strong solution of (1)-(4). if:* $\tilde{u}_0 \in W^{2,1}([0,t_1] : X)$, $\tilde{u}_i \in W^{2,1}([t_i, t_{i+1}] : X)$ $i = 1, 2, .., n-1$, $\tilde{u}_n \in W^{2,1}([t_n, T] : X)$; *the equation (1) is satisfied a.e.* $t \in I$ *and the conditions (2), (3) and (4) are verified.*

Definition 7 *A function* $u(\cdot) : I \to X$ *is a classical solution of (1)-(4) if* $u(\cdot)$ *is a function of class* C^2 *on* $\tilde{I} = I \setminus \{t_i, a(t_i), b(t_i) : i = 1, 2, ...n\}$ *that satisfies the equation (1) on* $\tilde{I}$ *and the conditions (2), (3) and (4) are verified.*

Theorem 9 *Let* $u(\cdot)$ *be a mild solution of (1)-(4) and assume that* X *has the RNP property. If the following conditions hold;*

(H-15) $x_0 \in D(A)$ *and* $u(t_i) + I_i^1(u(t_i), u'(t_i^-)) \in D(A)$ *for every* $i = 1, 2, ...n$,

(H-16) $y_0 \in E$ *and* $u'(t_i^-) + I_i^2(u(t_i), u'(t_i^-)) \in E$ *for every* $i = 1, 2, ...n$,

(H-17) *For each bounded set $D \subseteq X^4$, the functions $C(\cdot)f(t,x), t \in I, x \in D$, are uniformly Lipschitz continuous,*

then $u(\cdot)$ is a strong solution of (1)-(4).

Theorem 10 *Assume that $u(\cdot)$ is a mild solution of (1)-(4) and that X has the RNP property. If $x_0 \in D(A)$, $y_0 \in E$ and the following conditions hold;*

(H-18) *$\mathcal{R}(I_i^1) \subset D(A)$ and $\mathcal{R}(I_i^2) \subset E$ for every $i = 1, 2, ...n$,*

(H-19) *The function $a(\cdot), b(\cdot)$ are Lipschitz continuous,*

(H-20) *For each bounded set $D \subseteq X^4$, the function $f : I \times D \to X$ is Lipschitz continuous,*

then $u(\cdot)$ is a classical solution of (1)-(4).

3 Example

In this section we discuss existence of solutions for the " impulsive wave equation ". On the space $(X, \|\cdot\|_2) := (L^2([0,\pi]), \|\cdot\|_2)$ we consider the operator $Af(\xi) := f''(\xi)$ with domain

$$D(A) := \{f(\cdot) \in H^2(0,\pi) : f(0) = f(\pi) = 0\}.$$

It is well known that A is the generator of strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$ on X. Furthermore, A has discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with corresponding normalized eigenvectors $z_n(\xi) := \left(\frac{2}{\pi}\right)^{1/2} \sin(n\xi)$ and

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X.

(b) If $\varphi \in D(A)$ then $A\varphi = -\sum_{n=1}^{\infty} n^2 < \varphi, z_n > z_n$.

(c) For every $\varphi \in X$, $C(t)\varphi = \sum_{n=1}^{\infty} \cos(nt) < \varphi, z_n > z_n$ and $\| C(t) \| = 1$ for every $t \in I$.

(d) The associated sine function is given by $S(t)\varphi = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < \varphi, z_n > z_n$. Moreover, from this expression follow that $S(t)$ is compact and that $\| S(t) \| = 1$ for every $t \in I$.

(e) If G denotes the group of translations on X defined by $G(t)x(\xi) = \tilde{x}(\xi + t)$, where $\tilde{x}$ is the extension of x with period 2π, then $C(t) = \frac{1}{2}(G(t) + G(-t))$. Hence it follows, see [4], that $A = B^2$ where B is the infinitesimal generator of the group G and $E = \{x \in H^1(0,\pi) : x(0) = x(\pi) = 0\}$.

Now we consider the impulsive Cauchy problem

$$\begin{aligned}
\frac{\partial^2}{\partial^2 t}w(t,\xi) &= \frac{\partial^2}{\partial \xi^2}w(t,\xi) + F(t,\xi,w(t,\xi),w(a(t),\xi)), && (25)\\
w(t,0) &= w(t,\pi) = 0, \qquad t \in I, && (26)\\
\triangle w(t_i,\cdot)(\xi) &= P_i(w(t_i,\cdot))(\xi), \qquad \xi \in [0,\pi], && (27)\\
\triangle \frac{\partial w(t_i,\cdot)}{\partial t}(\xi) &= Q_i(w(t_i,\cdot))(\xi), \qquad \xi \in [0,\pi], && (28)
\end{aligned}$$

where $0 < t_1 < t_2 < .... < t_n < \pi$, $a(t) \le t$ for every $t \in I$ and

(1) The function $F : \mathbb{R}^4 \to \mathbb{R}$ satisfies the following Caratheodory conditions

(a) $F(t,\xi,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continuous *a.e.* $(t,\xi) \in \mathbb{R} \times [0,\pi]$.

(b) For every $w_1, w_2 \in \mathbb{R}$, the function $F(\cdot, w_1, w_2) : \mathbb{R} \times [0,\pi] \to \mathbb{R}$ is measurable.

(c) There exists a measurable, positive and bounded function $\eta(\cdot) : \mathbb{R} \times [0,\pi] \to \mathbb{R}^+$ such that

$$| F(t,\xi,w_1,w_2) | \le \eta(t,\xi)(| w_1 | + | w_2 |), \qquad w_1, w_2 \in \mathbb{R}.$$

(2) The functions $P_i, Q_i : X \to X$ are continuous and each P_i is completely continuous. In addition, we will assume that $\alpha_{i,r}$ and $\beta_{i,r}$ are positive constant such that $\|P_i(x)\|_2 \le \alpha_{i,r}$ and $\|Q_i(x)\|_2 \le \beta_{i,r}$ for every $\|x\|_2 \le r$. For examples of operators with these properties we refer to the reader to [10].

Let $f : [0,T] \times X^2 \to X$ be the substituting operator $f(t,x,y)(\xi) = F(t,\xi,x(\xi),y(\xi))$. Under the previous assumptions, for $x, y \in X$ we have that

$$\begin{aligned}
\| f(t,x,y) \|_2 &\le \left(\int_0^\pi | F(t,\xi,x(\xi),y(\xi)) |^2 d\xi\right)^{\frac{1}{2}}\\
&\le \left(\int_0^\pi \eta^2(t,\xi) (| x(\xi) | + | y(\xi) |)^2 d\xi\right)^{\frac{1}{2}}\\
&\le \eta(t,\cdot)_\pi (\| x \|_2 + \| y \|_2).
\end{aligned}$$

Theorem 11 *Assume that the previous conditions are verified and that*

$$2\int_0^T \eta(s,\cdot)_{0,\pi} ds + \liminf_{r\to\infty} \sum_{i=1}^n \frac{1}{r}(\alpha_{i,r} + \beta_{i,r}) < 1. \tag{29}$$

Then there exists a mild solution of the impulsive abstract Cauchy problem (25)-(28).

Proof: Let $\Gamma : \mathcal{PC} \to \mathcal{PC}$ the map defined in (10). We affirm that there exists $r > 0$ such that $\Gamma(B_r) \subset B_r$, where $B_r = B_r(0,\mathcal{PC})$. If the affirmation is false, for every $r > 0$ we can to fix $x_r \in B_r$ such that $r <\| \Gamma x_r \|_T$. Consequently

$$r <\| \Gamma x_r \|_T \le \| x_0 \| + \| y_0 \| + 2r \int_0^t \eta(s,\cdot)_\pi ds + \sum_{j=1}^n (\alpha_{i,r} + \beta_{i,r})$$

and hence

$$1 \leq 2 \int_0^T \eta(\cdot, s)_\pi ds + \liminf_{r \to \infty} \sum_{i=1}^{n} \frac{1}{r} \left(\alpha_{i,r} + \beta_{i,r} \right),$$

which is a absurd with the hypotheses.

Follows from the ideais in the proof of Theorem 1, that γ is completely continuous on B_r. Now, the existence of a mild solution of (25)-(28) is consequence of Schauder point fixed theorem. The proof is complete. ■

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