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Existence Results for Second a Order Partial Neutral Functional Differential Equation

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Resultados de Existência para uma Equação Funcional de Segunda ordem do Tipo Neutro.

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Resumo

Neste artigo mostramos a existência de soluções fracas, fortes e clássicas para uma equação de segunda ordem do tipo neutro modelada na forma

$$\frac{d}{dt}[x'(t) - g(t, x_t)] = Ax(t) + f(t, x_t, x'(t)), \qquad t \in I = [0, a], \tag{1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{2}$$

$$x'(0) = y_0 \in X, \tag{3}$$

onde A é o gerador infinitesimal de uma função cosseno fortemente contínua de operadores lineares num espaço de Banach X, as historias $x_t : (-\infty, 0] \rightarrow X$, $x_t(\theta) = x(t + \theta)$, estão em um espaço de fase, $\mathcal{B}$, definido de maneira axiomática e f, g são funções apropriadas.

Existence Results for Second a Order Partial Neutral Functional Differential Equation.

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Abstract

By using the theory of strongly continuous cosine functions, we study the existence of mild, strong and classical solutions for a second order partial neutral functional differential equation with unbounded delay. The results in this paper, permits to establish a new a correct version of the results reported by Balachandran [1],[2] and Benchorra [3]

1 Introduction

In this paper we study existence of mild, strong and classical solutions for a class of second order partial neutral functional differential equation with unbounded delay modelled in the form

$$\frac{d}{dt}[x'(t) - g(t, x_t)] = Ax(t) + f(t, x_t, x'(t)), \qquad t \in I = [0, a], \tag{1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{2}$$

$$x'(0) = y_0 \in X, \tag{3}$$

where A is the infinitesimal generator of a strongly cosine family of bounded linear operators, $(C(t))_{t\in\mathbb{R}}$, on a Banach space X; the history $x_t : (\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically and f, g are appropriated functions.

Neutral differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature for first order neutral functional differential equations is the Hale book [9] and the references therein. The work in first order partial neutral functional differential equations with unbounded delay was initiated by Hernández & Henríquez in [11, 12]. In these papers Hernández & Henríquez proved existence of mild, strong and periodic solutions for a neutral equation modeled in the form

$$\frac{d}{dt}[x(t) + G(t, x_t)] = Ax(t) + F(t, x_t), \qquad t \geq \sigma,$$

$$x_\sigma = \varphi \in \mathcal{B},$$

where A is the infinitesimal generator of an analytic semigroup of linear operators and $\mathcal{B}$ is a phase space defined axiomatically. In general, the results was obtained using the semigroup theory and the Sadovskii fixed point theorem. Another related papers are [1, 3, 5].

On the other hand, some second order partial neutral differential systems was studied recently by Balachandran [1, 2] and Benchohra [3]. In these works the authors proved existence of mild solution for some neutral differential problems similar to (1)-(3) using Leray Schauder Alternative and the cosine function Theory. To obtain their results, Balachandran and Benchohra used standard growth restrictions on g, f and a severe assumption on the cosine function, $(C(t))_{t\in \mathbb{R}}$, generates by A. Specifically, the authors assumed that $C(t)$ is compact for every $t \in (0, a]$; which is only possible if X is finite dimensional, see Travis [16] pp. 557. Obvioulsly, this fact is relevant, since under this conditions the problems studied by Balachandran and Benchohra not included " partial " neutral differential equations. Moreover, it's necessary to observe that under these assumption the examples in Balachandran [1], [2] aren't consequence of the results reported for this author.

Our objective in this paper is to establish some existence results of mild, strong and classical solution, concepts to be specified later, for some neutral problems similar to (1)-(3). Evidently, to obtain our results we will not assume some direct compactness assumption on the operators $C(t)$, $t \in I$.

In this paper, henceforth, $C(\cdot) = (C(t))_{t\in \mathbb{R}}$ is a strongly continuous cosine function of bounded linear operators on a Banach space X with infinitesimal generator A. We refer the reader to [7, 16, 17] for the necessary concepts about cosine functions. Next we only mention a few results and notations needed to establish our results. We denote by $S(t)$ the sine function associated with $C(t)$, which is defined by

$$S(t)x := \int_0^t C(s)x\, ds, \quad x \in X, \quad t \in \mathbb{R}.$$

For a closed operator $B : D(B) \to X$ we denote by $[D(B)]$ the space $D(B)$ endowed with the graph norm $\| \cdot \|_B$. In particular $[D(A)]$ is the space

$$D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable}\},$$

endowed with the norm $\| x \|_A = \| x \| + \| Ax \|$, $x \in D(A)$. Moreover, in this work the notation E stands for the space formed by the vectors $x \in X$ for which the function $C(\cdot)x$ is of class C^1. It was proved by Kisińsky [14] that E endowed with the norm

$$\|x\|_1 = \|x\| + \sup_{0\leq t\leq 1} \|AS(t)x\|, \quad x \in E,$$

is a Banach space. The operator valued function $G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is a strongly continuous group of linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. From this it follows that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0$, $t \to 0$, for each $x \in E$. Furthermore, if $x : [0, \infty) \to X$ is a locally integrable function then

$$y(t) = \int_0^t S(t-s)x(s)ds$$

defines an E-valued continuous function. This is an consequence of the fact that

$$\int_0^t G(t-s)\begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)\,ds \\ \int_0^t C(t-s)x(s)\,ds \end{bmatrix}$$

defines an $E \times X$-valued continuous function.

The existence of solutions of the second order abstract Cauchy problem

$$\begin{cases} x''(t) &= Ax(t) + h(t), \;\; 0 \le t \le a, \\ x(0) &= x_0, \;\; x'(0) = x_1, \end{cases} \tag{4}$$

where $h : [0,T] \to X$ is an integrable function, has been discussed in [16]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem it has been treated in [17]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s)\,ds, \;\; 0 \le t \le a, \tag{5}$$

is called a mild solution of (4). When $x_0 \in E$, $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)x_0 + C(t)x_1 + \int_0^t C(t-s)h(s)\,ds. \tag{6}$$

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato [8]. To establish the axioms of the space $\mathcal{B}$ we follows the terminology used in Hino-Murakami-Naito [13], and thus, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty,0]$ into X, endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$. We will assume that $\mathcal{B}$ satisfies the following axioms:

(A) If $x : (-\infty, \sigma+b) \to X$, $b > 0$, is continuous on $[\sigma, \sigma+b)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma+b)$, the following conditions hold:

i) x_t is in $\mathcal{B}$.

ii) $\| x(t) \| \le H \| x_t \|_{\mathcal{B}}$.

iii) $\| x_t \|_{\mathcal{B}} \le K(t-\sigma)\sup\{\| x(s) \| : \sigma \le s \le t\} + M(t-\sigma) \| x_\sigma \|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0,\infty) \to [0,\infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in **(A)**, x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma+b)$.

(B) The space $\mathcal{B}$ is complete.

Example 1 *We consider the phase space $\mathcal{B} := C_r \times L^p(g;X), r \ge 0, 1 \le p < \infty$ in [13], which consists of all classes of functions $\varphi : (-\infty,0] \to X$ such that φ is continuous on $[-r,0]$, Lebesgue-measurable and $g \mid \varphi(\cdot) \mid^p$ is Lebesgue integrable on $(-\infty,-r)$, where $g : (-\infty,-r) \to \mathbb{R}$ is a positive Lebesgue integrable function. The seminorm in $\|\cdot\|_{\mathcal{B}}$ is defined by*

$$\| \varphi \|_{\mathcal{B}} := \sup\{\| \varphi(\theta) \| : -r \le \theta \le 0\} + \left(\int_{-\infty}^{-r} g(\theta) \| \varphi(\theta) \|^p \, d\theta \right)^{1/p}.$$

We will assume that g satisfies conditions (g-6) and (g-7) in the terminology of [13]. This means that g is integrable on $(-\infty, -r)$ and that there exists a non-negative and locally bounded function γ on $(-\infty, 0]$ such that

$$g(\xi + \theta) \leq \gamma(\xi)\, g(\theta),$$

for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. In this case, $\mathcal{B}$ is a phase space which verifies axioms (A), (A-1) and (B) ([13], Theorem 1.3.8).

To study regularity of solutions, we need introduce some additional notations. In the rest of the paper, for $t \geq 0$, $W_1(t) : \mathcal{B} \to \mathcal{B}$ and $W_2(t) : X \to \mathcal{B}$ are the operators defined by

$$[W_1(t)\varphi](\theta) = \begin{cases} C(t+\theta)\varphi(0), & \text{for } \ \theta \in [-t, 0], \\ \varphi(\theta + t), & \text{for } \ \theta < -t, \end{cases}$$

and

$$[W_2(t)x](\theta) = \begin{cases} S(t+\theta)x, & \text{for } \ \theta \in [-t, 0], \\ 0, & \text{for } \ \theta < -t. \end{cases}$$

For the literature on phase space, we refer the reader to [13].

This paper has three sections. In Section 2, we establish existence results of mild solution for some second order partial neutral functional differential equations with unbounded delay. In section 3, we study regularity of mild solutions using resents results reported by Henriquez in [10] and the ideas in Travis [17].

The terminologies and notations are those generally used in functional analysis. In particular, $\mathcal{L}(\mathcal{X})$ stands for the Banach space of bounded linear operators from X into X. The prefix $\mathcal{R}$ is used to indicate the image of a map and $B_r(x : Z)$ denotes the closed ball with center at x and radius $r > 0$ in an metric space Z. Additionally, for a bounded function $\xi : [0, a] \to [0, \infty)$ and $0 \leq t \leq a$ we will employ the notation ξ_t for

$$\xi_t = \sup\{\xi(s) : s \in [0, t]\}.$$

To finish this section and for completeness, we include the followings well known result.

Theorem 1 *(Leray Schauder Alternative) Let D be a convex subset of a Banach space X and assume that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda F(x),\ 0 < \lambda < 1\}$ is unbounded or the map F has a fix point in D.*

2 Existence Results.

In this section we study existence of mild solutions of some second order neutral differential equations. In the rest of the paper we assume that $N \geq 1$ and $\tilde{N}$ are constants such that $\|C(t)\| \leq N$ and $\|S(t)\| \leq \tilde{N}$, for every $t \in I$. Initially, we consider the following partial second order neutral differential equation

$$\frac{d}{dt}[x'(t) - g(t, x_t))] = Ax(t) + f(t, x_t), \qquad t \in I = [0, a], \tag{7}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{8}$$

$$x'(0) = y_0 \in X. \tag{9}$$

To study this problem, we introduce the following conditions.

(H₁) The functions $f, g : I \times \mathcal{B} \to X$ satisfy the following Caratheodory conditions:

(i) the functions $f(t, \cdot)$, $g(t, \cdot) : \mathcal{B} \to X$ are continuous a.e. $t \in I$;

(ii) for each $\phi \in \mathcal{B}$, the functions $f(\cdot, \phi), g(\cdot, \phi) : I \to X$ are strongly measurable.

(H₂) There exist continuous functions $m_f(\cdot), m_g(\cdot) : [0, \infty) \to [0, \infty)$ and continuous nondecreasing functions $W_f(\cdot), W_g(\cdot) : [0, \infty) \to (0, \infty)$ such that

$$\begin{aligned} \| f(t, \phi) \| &\leq m_f(t) W_f(\| \phi \|_{\mathcal{B}}), \\ \| g(t, \phi) \| &\leq m_g(t) W_g(\| \phi \|_{\mathcal{B}}), \end{aligned}$$

for every $\phi \in \mathcal{B}$ and every $t \in I$.

To begin the study of the equation (7)-(9) we consider some definitions.

Definition 1 *A function $u(\cdot) : (-\infty, a] \to X$ is a classical solution of the problem (7)-(8)-(9) if $u_0 = \varphi$, the function $s \to (u'(s) + g(s, u_s))$ is of class C^1 on I; $u(t) \in D(A)$ for every $t \in I$ and (7)-(8)-(9) are verified.*

If $u(\cdot)$ is a classical solution of (7)-(9), g is $D(A)$ valued and $Ag \in \mathrm{L}^1(\mathrm{I} : \mathrm{X})$ then

$$\frac{d}{dt}[u(t) - \int_0^t g(s, x_s)ds] = A(u(t) - \int_0^t g(s, x_s))ds + A\int_0^t g(s, x_s)ds + f(t, x_t),$$

which implies that

$$\begin{aligned} u(t) &= C(t)\varphi(0) + S(t)[y_0 - g(0, \varphi)] + \int_0^t g(s, x_s)ds + \int_0^t AS(t-s)\int_0^s g(\tau, x_\tau)d\tau ds \\ &\quad + \int_0^t S(t-s)f(s, x_s)ds, \end{aligned} \tag{10}$$

for every $t \in I$. The expression (10) motivates the following definition.

Definition 2 *A function $x : (-\infty, a] \to X$ is a mild solution of the problem (7)-(8)-(9) if $x_0 = \varphi$; $x(\cdot)$ is continuous on I and*

$$\begin{aligned} x(t) &= C(t)\varphi(0) + S(t)[y_0 - g(0, \varphi)] + \int_0^t C(t-s)g(s, x_s)ds \\ &\quad + \int_0^t S(t-s)f(s, x_s)ds, \qquad t \in I. \end{aligned}$$

Now, we establish our first existence result.

Theorem 2 *Assume that conditions* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *are verified and that* $g(\cdot)$ *is completely continuous. Suppose, furthermore, that the followings conditions hold.*

(a) *For every* $0 < t' \leq t$ *and* $r > 0$*, the set* $U(t,t',r) = \{S(t')f(s,\psi) : s \in [0,t], \| \psi \|_{\mathcal{B}} \leq r\}$ *is relatively compact in* X.

If

$$K_a \int_0^a (Nm_g(s) + \tilde{N}m_f(s))ds < \int_c^\infty \frac{ds}{W_f(s) + W_g(s)}, \tag{11}$$

where $c = (M_a + K_a NH) \| \varphi \|_{\mathcal{B}} + K_a \tilde{N}(\| y_0 \| + \| g(0,\varphi) \|)$*, then there exists a mild solution of (7)-(9).*

Proof. On the space $S(\varphi) = \{x \in C([0,a] : X) : x(0) = \varphi(0)\}$ endowed with the uniform convergence norm, $\| \cdot \|_a$, we define the map $\Gamma : S(\varphi) \to S(\varphi)$ by

$$\begin{aligned} \Gamma x(t) &= C(t)\varphi(0) + S(t)[y_0 - g(0,\varphi)] + \int_0^t C(t-s)g(s,\tilde{x}_s)ds \\ &\quad + \int_0^t S(t-s)f(s,\tilde{x}_s)ds, \quad t \in I, \end{aligned} \tag{12}$$

where $\tilde{x}$ denote the function $\tilde{x} : (-\infty, a] \to X$ defined by $\tilde{x}(\theta) = \varphi(\theta)$, for $\theta \leq 0$, and $\tilde{x}(t) = x(t)$, for $0 \leq t \leq a$. In order to use the Leray Schauder Alternative, we obtain an *a priori* bound for the solution of the integral equation $x = \lambda\Gamma(x)$, $\lambda \in (0,1)$. If x^λ is a solution of $x = \lambda\Gamma(x)$, $\lambda \in (0,1)$; from $(\mathbf{H_2})$ and the axioms of the phase space $\mathcal{B}$, we get

$$\begin{aligned} &\| x^\lambda(t) \| \\ &\leq NH \| \varphi \|_{\mathcal{B}} + \tilde{N}(\| y_0 \| + \| g(0,\varphi) \|) + N \int_0^t m_g(s) W_g(\| \tilde{x}_s^\lambda \|_{\mathcal{B}})ds \\ &\quad + \tilde{N} \int_0^t m_f(s) W_f(\| \tilde{x}_s^\lambda \|_{\mathcal{B}})ds \\ &\leq NH \| \varphi \|_{\mathcal{B}} + \tilde{N}(\| y_0 \| + \| g(0,\varphi) \|) + N \int_0^t m_g(s) W_g(K_a \| \tilde{x}^\lambda \|_s + M_a \| \varphi \|_{\mathcal{B}})ds \\ &\quad + \tilde{N} \int_0^t m_f(s) W_f(K_a \| \tilde{x}^\lambda \|_s + M_a \| \varphi \|_{\mathcal{B}})ds. \end{aligned}$$

Using he notation $u_\lambda(t) = K_a \| \tilde{x}^\lambda \|_t + M_a \| \varphi \|_{\mathcal{B}}$ follows that

$$\begin{aligned} u_\lambda(t) &\leq (M_a + K_a NH) \| \varphi \|_{\mathcal{B}} + K_a \tilde{N}(\| y_0 \| + \| g(0,\varphi) \|) \\ &\quad + K_a N \int_0^t m_g(s) W_g(u_\lambda(s))ds + K_a \tilde{N} \int_0^t m_f(s) W_f(u_\lambda(s))ds, \end{aligned}$$

and so

$$\begin{aligned} u_\lambda'(t) &\leq K_a N m_g(t) W_g(u_\lambda(t)) + K_a \tilde{N} m_f(t) W_f(u_\lambda(t)) \\ &\leq (K_a N m_g(t) + K_a \tilde{N} m_f(t))(W_g(u_\lambda(t)) + W_f(u_\lambda(t))) \end{aligned}$$

which from the assumptions implies that

$$\int_{u_\lambda(0)}^{u_\lambda(t)} \frac{ds}{W_g(s) + W_f(s)} \leq K_a \int_0^t (Nm_g(s) + \tilde{N}m_f(s))ds < \int_c^\infty \frac{ds}{W_g(s) + W_f(s)}.$$

These estimates permit to conclude that $\{u_\lambda : \lambda \in (0,1)\}$ is bounded in $C([0,a]:X)$, which in turn implies that $\{x^\lambda : \lambda \in (0,1)\}$ is bounded in $C([0,a]:X)$.

Next, we prove that Γ is completely continuous. To this end we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where

$$\Gamma_2 x(t) \;=\; \int_0^t C(t-s)g(s,\tilde{x}_s)ds + \int_0^t S(t-s)f(s,\tilde{x}_s)ds.$$

In order to use Ascoli-Arzela Theorem we will divide this part of proof in two steps. In what follows, we denote by B_r, $r>0$, the set $B_r = \{x \in S(\varphi) : \sup_{\theta\in I} \| x(\theta) - \varphi(0) \| \leq r\}$.

Step 1 The set $\Gamma_2 B_r = \{\Gamma_2 x : x \in B_r\}$ is equicontinuous on I.

Let $t \in I$ and $\epsilon > 0$. At first, we observe that $\| \tilde{x}_s \|_{\mathcal{B}} \leq (M_a + K_a H) \| \varphi \|_{\mathcal{B}} + K_a r := r^*$, for every $x \in B_r$ and all $s \in I$. Since $C(\cdot)$ is strongly continuous and $g(\cdot)$ is completely continuous, there exists $\delta > 0$ such that

$$\| (C(t+h) - C(t))g(t,\psi) \| < \epsilon, \qquad \| \psi \|_{\mathcal{B}} \leq r^*,\ t \in I,$$

when $| h | \leq \delta$. Using these remarks; for $x \in B_r$ and $|h| \leq \delta$ with $t+h \in I$ we get

$$\begin{aligned}
&\| \Gamma_2 x(t+h) - \Gamma_2 x(t) \| \\
&\leq \int_0^t \| (C(t+h-s) - C(t-s))g(s,\tilde{x}_s) \| ds + N \int_t^{t+h} \| g(s,\tilde{x}_s) \| ds \\
&\quad + \int_0^t \| S(t+h-s) - S(t-s) \| \| f(s,\tilde{x}_s) \| ds + \tilde{N} \int_t^{t+h} \| f(s,\tilde{x}_s) \| ds \\
&\leq \epsilon t + N W_g(r^*) \int_t^{t+h} m_g(s)ds + N h W_f(r^*) \int_0^t m_f(s)ds + \tilde{N} W_f(r^*) \int_t^{t+h} m_f(s)ds,
\end{aligned}$$

which prove the assertion.

Step 2 The set $\Gamma_2 B_r(t) = \{\Gamma_2 x(t) : x \in B_r\}$ is relatively compact in X for every $t \in I$.

Let $t \in I$, $\epsilon > 0$ and $r^* = (M_a + K_a H) \| \varphi \| + K_a r$. If $x \in B_r$, from the estimate

$$\| f(\theta, x_\theta) \| \;\leq\; m_f(\theta) W_f(\| x_\theta \|_{\mathcal{B}}) \leq m_f(\theta) W_f(r^*),$$

follows that the set $U = \{f(t-s, \tilde{x}_{t-s}) : s \in [0,t], x \in B_r\}$ is bounded in X. Using that $S : I \to \mathcal{L}(\mathcal{X})$ is uniformly Lipschitz on I, we can chose $0 = t_1 < t_2 < \ldots < t_n = t$ such that for every $s \in [0,t]$ there exists t_i verifying

$$\| S(s)y \;-\; S(t_i)y \| < \epsilon, \qquad y \in U.$$

Employing these remark, the mean value theorem for the Bochner integral, see [15] Lemma 2.1.3, and the fact that $V = \{C(s)g(\theta,\psi) : s,\theta \in I, \| \psi \| \leq r^*\}$ is relatively compact in X; follows that for $x \in B_r$

$$\begin{aligned}
\Gamma_2 x(t) \;&=\; \int_0^t C(t-s)g(s,\tilde{x}_s)ds + \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} (S(s) - S(t_i))f(t-s,\tilde{x}_{t-s})ds \\
&\quad + \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} S(t_i)f(t-s,\tilde{x}_{t-s})ds \\
&\in\; t\overline{co(V)} + 2aB_\epsilon(0,X) + \sum_{i=1}^{n-1} (t_{i+1} - t_i)\overline{co(U(t,t_i,r^*))}
\end{aligned}$$

where *co* denote the convex hull. Thus, $\Gamma_2 B_r(t)$ is totally bounded and consequently relatively compact in X.

From the steps 1 and 2, we infer that $\Gamma_2 B_r$ is relatively compact in $C(I;X)$. The continuity of Γ_2 follows from the axioms of the phase space and the Lebesgue dominated convergence Theorem. Thus, Γ_2 and consequently Γ, are completely continuous.

These remark and the Leray Schauder Alternative Theorem permit to conclude that Γ has a fixed point in $S(\varphi)$. This fixed point is a mild solution of (7)-(9). The proof is complete. ■

In the most of the situation of practical interest the sine function is compact, it's the motivation of the next result.

Corollary 1 *Let conditions* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *be satisfied and assume that* $g(\cdot)$ *is completely continuous. If* $S(t)$ *is compact for every* $t \geq 0$ *and inequality (11) hold, then there exists a mild solution of (7)-(9).*

Theorem 3 *Assume that* $f(\cdot)$ *verifies the assumptions in Theorem 2 and that there exists* $L_g > 0$ *such that*

$$\| g(t,\psi_1) - g(t,\psi_2) \| \;\leq\; L_g \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \qquad \forall\ (t,\psi) \in I \times \mathcal{B}.$$

If $NL_gK_a a + \tilde{N} \liminf_{r\to\infty} W_f(K_a r + (M_a + K_a H) \| \varphi \|_{\mathcal{B}}) \int_0^a m_f(s)ds < 1$, *then there exists a mild solution of (7)-(9).*

Proof. Let $\Gamma : S(\varphi) \to S(\varphi)$ the map defined in the proof of Theorem 2 and consider the decomposition $\Gamma = \overline{\Gamma}_1 + \overline{\Gamma}_2$ where

$$\overline{\Gamma}_1 x \;=\; \int_0^t C(t-s)g(s,\tilde{x}_s)ds.$$

Let $B_r = \{x \in S(\varphi) : \sup_{\theta\in I} \| x(\theta) - \varphi(0) \| \leq r\}$. We affirm that there exist $r > 0$ such that $\Gamma(B_r) \subset B_r$. In fact, if the affirmation is false, for each $r > 0$ there exist $x^r \in B_r$ such $\| \Gamma x^r \| > r$. On the other hand, from the assumptions on f, g we get

$$\begin{aligned}
r &\leq \| \Gamma x^r \|_a \\
&\leq N \| \varphi(0) \| + N(\| y_0 \| + \| g(0,\varphi) \|) + N \int_0^a [L_g \| \tilde{x}_s^r - W_1(s)\varphi \| + \| W_1(s)\varphi \|]ds \\
&\quad + \tilde{N} \int_0^a m_f(s) W_f(\| \tilde{x}_s^r \|_{\mathcal{B}})ds \\
&\leq NH \| \varphi \| + N(\| y_0 \| + \| g(0,\varphi) \|) + NL_gK_a a r + N \int_0^a \| W_1(s)\varphi \|_{\mathcal{B}}\, ds \\
&\quad + \tilde{N} \int_0^a m_f(s) W_f(K_a r + \| W_1(s)\varphi \|_{\mathcal{B}})ds
\end{aligned}$$

and so

$$1 \leq NL_gK_a a + \tilde{N} \liminf_{r\to\infty} W_f(K_a r + (M_a + K_a H) \| \varphi \|_{\mathcal{B}}) \int_0^a m_f(s)ds,$$

which is an absurd. Let $r_0 > 0$ such that $\Gamma(B_{r_0}) \subset B_{r_0}$. Follows, from the proof of Theorem 2 that $\overline{\Gamma}_2$ is completely continuous on B_{r_0}. Moreover, for $u, v \in B_{r_0}$ we have that

$$\begin{aligned} \| \overline{\Gamma}_1 u - \overline{\Gamma}_1 v \|_a &\leq N \int_0^a \| g(s, \tilde{u}_s) - g(s, \tilde{v}_s) \| ds \\ &\leq N K_a L_g \int_0^t \| u - v \|_s ds \end{aligned} \tag{13}$$

and hence,

$$\| \overline{\Gamma}_1 u - \overline{\Gamma}_1 v \|_a \leq N K_a L_g a \| u - v \|_a, \tag{14}$$

which show that $\overline{\Gamma}_1$ is a contraction on B_{r_0}. These remarks prove that Γ is a condensing map on B_{r_0}. Now, the existence of a mild solution is consequence of the Sadovskii point fixed Theorem. The proof is finished. ■

In what follows we study existence of solutions for (1)-(3). In this case, the results and their proofs are similar at those in the first part of this section, and for this reason, the proofs will be summarized or simply omitted. In first time, we consider the following definition.

Definition 3 *A function $x : (-\infty, a] \to X$ is a mild solution of the neutral problem (1)-(3) if $x_0 = \varphi$; $x(\cdot)$ is continuously differentiable on I and*

$$\begin{aligned} x(t) &= C(t)\varphi(0) + S(t)[y_0 - g(0, \varphi)] + \int_0^t C(t-s) g(s, x_s) ds \\ &\quad + \int_0^t S(t-s) f(s, x_s, x'(s)) ds, \qquad t \in I. \end{aligned}$$

To study the neutral system, we introduce the next conditions.

($\tilde{\mathbf{H}}_1$) The function $f : I \times \mathcal{B} \times X \to X$ satisfy the following conditions:

(i) $f(t, \cdot) : \mathcal{B} \times X \to X$ is continuous a.e. $t \in I$;

(ii) For each $(\psi, x) \in \mathcal{B} \times \mathcal{X}$, the function $f(\cdot, \psi, x) : I \to X$ is strongly measurable;

(iii) There exists a continuous function $m_f : [0, \infty) \to [0, \infty)$ and a continuous nondecreasing function $W_f : [0, \infty) \to (0, \infty)$ such that

$$\| f(t, \psi, x) \| \leq m_f(t) W_f(\| \psi \|_{\mathcal{B}} + \| x \|), \qquad \forall\ (t, \psi, x) \in I \times \mathcal{B} \times \mathcal{X}.$$

($\tilde{\mathbf{H}}_2$) The function g is E-valued and verifies the following conditions

(i) $g(t, \cdot) : \mathcal{B} \to E$ is continuous a.e. $t \in I$;

(ii) For each $\psi \in \mathcal{B}$, the function $g(\cdot, \psi) : I \to E$ is strongly measurable.

(iii) There exists a continuous function $\widetilde{m}_g : [0, \infty) \to [0, \infty)$ and a continuous nondecreasing function $\widetilde{W}_g : [0, \infty) \to (0, \infty)$ such that

$$\| g(t, \psi) \|_1 \leq \widetilde{m}_g(t) \widetilde{W}_g(\| \psi \|_{\mathcal{B}}), \qquad (t, \psi) \in I \times \mathcal{B}.$$

(iv) There exist positive constants c_1, c_2 such that

$$\| g(t,\psi) \| \leq c_1 \| \psi \|_{\mathcal{B}} + c_2, \qquad (t,\psi) \in I \times \mathcal{B}.$$

($\mathbf{g}_\varphi$) For every $r>0$, the set of functions $\{t \to g(t,\tilde{x}_t) : x_0 \in S(\varphi), \| x \|_a \leq r\}$, where $S(\varphi)$ and $\tilde{x}$ are as in the proof of Theorem 2, is equicontinuos on $[0,a]$.

Theorem 4 *Let conditions* $(\tilde{\mathbf{H}}_1), (\tilde{\mathbf{H}}_2)$, $(\mathbf{g}_\varphi)$ *be satisfied and assume that the functions* $f(\cdot) : I \times \mathcal{B} \times \mathcal{X} \to \mathcal{X}$ *and* $g(\cdot) : I \times \mathcal{B} \to \mathcal{E}$ *are completely continuous. If* $\varphi(0) \in E$, $\mu = 1 - c_1 > 0$ *and*

$$\mu^{-1} \int_0^a (K_a N + 1)\widetilde{m}_g(s) + (\tilde{N} K_a + N) m_f(s)) ds < \int_c^\infty \frac{ds}{\widetilde{W}_g(s) + W_f(s)}$$

where $c = \mu^{-1}[(NHK_a + M_a) \| \varphi \|_{\mathcal{B}} + \| AS(\cdot)\varphi(0) \|_a + (K_a\tilde{N} + N)(\| y_0 \| + \| g(0,\varphi) \|) + c_2]$, *then there exists a mild solution of the neutral system (1)-(3).*

Proof. Let $Y = C([0,a] : X)$ and $S(\varphi)$ the space introduced in the proof of Theorem 2. On the space $S = S(\varphi) \times Y$ we define the map $\Gamma(u,v) = (\Gamma_1(u,v), \Gamma_2(u,v))$ where

$$\begin{aligned} \Gamma_1(u,v)(t) &= C(t)\varphi(0) + S(t)[y_0 - g(0,\varphi)] + \int_0^t C(t-s) g(s,\tilde{u}_s) ds \\ &\quad + \int_0^t S(t-s) f(s,\tilde{u}_s, v(s)) ds, \quad t \in I, \\ \Gamma_2(u,v)(t) &= AS(t)\varphi(0) + C(t)[y_0 - g(0,\varphi)] + g(t,\tilde{u}_t) + \int_0^t AS(t-s) g(s,\tilde{u}_s) ds \\ &\quad + \int_0^t C(t-s) f(s,\tilde{u}_s, v(s)) ds \quad t \in I, \end{aligned}$$

and $\tilde{u} : (-\infty, a] \to X$ is defined by $\tilde{u}_0 = \varphi$ and $\tilde{u}(t) = u(t)$, for $t \in I$. Clearly Γ is well defined and continuous. If $(x^\lambda, y^\lambda) = \lambda \Gamma(x^\lambda, y^\lambda)$, $\lambda \in (0,1)$, we get

$$\begin{aligned} &\| x^\lambda(t) \| \\ &\leq NH \| \varphi \|_{\mathcal{B}} + \tilde{N}[\| y_0 \| + \| g(0,\varphi) \|] + N \int_0^t \widetilde{m}_g(s) \widetilde{W}_g(M_a \| \varphi \|_{\mathcal{B}} + K_a \| \tilde{x}^\lambda \|_s) ds \\ &\quad + \tilde{N} \int_0^t m_f(s) W_f(M_a \| \varphi \|_{\mathcal{B}} + K_a \| \tilde{x}^\lambda \|_s + \| y^\lambda(s) \|) ds \end{aligned}$$

and

$$\begin{aligned} \| y^\lambda(t) \| &\leq \| AS(\cdot)\varphi(0) \|_a + N[\| y_0 \| + \| g(0,\varphi) \|] + c_1 \| \tilde{x}^\lambda_t \| + c_2 \\ &\quad + \int_0^t \widetilde{m}_g(s) \widetilde{W}_g(M_a \| \varphi \|_{\mathcal{B}} + K_a \| \tilde{x}^\lambda \|_s) ds \\ &\quad + \tilde{N} \int_0^t m_f(s) W_f(M_a \| \varphi \|_{\mathcal{B}} + K_a \| \tilde{x}^\lambda \|_s + \| y^\lambda(s) \|) ds. \end{aligned}$$

Using the notation $u_\lambda(t) = M_a \| \varphi \|_{\mathcal{B}} + K_a \| x^\lambda \|_t + \| y^\lambda(t) \|$, we find that

$$\begin{aligned} u_\lambda(t) &\leq \mu^{-1}((M_a + HNK_a) \| \varphi \|_{\mathcal{B}} + \| AS(\cdot)\varphi(0) \|_a + c_2) \\ &\quad + \mu^{-1}(\tilde{N} K_a + N)[\| y_0 \| + \| g(0,\varphi) \|] + \mu^{-1}(NK_a + 1) \int_0^t \widetilde{m}_g(s) \widetilde{W}_g(u_\lambda(s)) ds \\ &\quad + \mu^{-1}(\tilde{N} K_a + N) \int_0^t m_f(s) W_f(u_\lambda(s)) ds := \beta_\lambda(t) \end{aligned}$$

and so

$$\beta_\lambda'(t) \;\leq\; \mu^{-1}\left((NK_a+1)\widetilde{m}_g(t)+(\tilde{N}K_a+N)m_f(t)\right)\left(\widetilde{W}_g(\beta_\lambda(t))+W_f(\beta_\lambda(t))\right)$$

and hence

$$\begin{aligned}\int_c^{\beta_\lambda(t)}\frac{ds}{\widetilde{W}_g(s)+W_f(s)} &< \mu^{-1}\int_0^a[(NK_a+1)\widetilde{m}_g(s)+(\tilde{N}K_a+N)m_f(s)]ds\\ &< \int_c^\infty\frac{ds}{\widetilde{W}_g(s)+W_f(s)},\end{aligned}$$

which prove that $\{z^\lambda=(x^\lambda,y^\lambda):z^\lambda=\lambda\Gamma z^\lambda,\ \lambda\in(0,1)\}$ is bounded in S.

Using that $f:I\times\mathcal{B}\times\mathcal{X}\to\mathcal{X}$, $g:I\times\mathcal{B}\to\mathcal{E}$ are completely continuous and condition $\mathbf{g}_\varphi$, we infer from the proof of Theorem 2 that each Γ_i, $i=1,2$, is completely continuous. The existence of a mild solution of the neutral system (1)-(3) is now consequence Theorem 1. The proof is complete. ■

The proof of Theorem 5, below, is analogous to the proof of Theorem 3 and will be omitted.

Theorem 5 *Assume that condition* $(\mathbf{g}_\varphi)$ *holds and that* f *and* $\varphi(0)$ *verifies the assumptions in Theorem 4. Suppose in addition that there exist* $L_g>0$ *such that*

$$\|\, g(t,\psi_1)-g(s,\psi_2)\,\|_1\leq L_g\,\|\,\psi_1-\psi_2\,\|_{\mathcal{B}},\qquad\forall\,(t,\psi)\in I\times\mathcal{B}.$$

If $L_g(N_a+1+a)+(N+\tilde{N})\liminf_{r\to\infty}W_f((K_a+1)r+(M_a+K_a+1)\,\|\,\varphi\,\|)_{\mathcal{B}}<1$, *then there exists a mild solution of the neutral system (1)-(3).*

When f,g are Lipschitz continuous, the existence of mild solution can be established using the contraction mapping principle.

Proposition 1 *Assume that* $g(\cdot)$ *is continuous with values in* E, *that* $\varphi(0)\in E$ *and that there exist positive constants* L_g^i, L_f^i, $i=1,2$, *such that*

$$\begin{aligned}\|\, f(t,\psi_1,x)-f(s,\psi_2,y)\,\| &\leq L_f^1\mid t-s\mid+L_f^2\,\|\,\psi_1-\psi_2\,\|_{\mathcal{B}}+L_f^3\,\|\,x-y\,\|,\\ \|\, g(t,\psi_1)-g(s,\psi_2)\,\|_1 &\leq L_g^1\mid t-s\mid+L_g^2\,\|\,\psi_1-\psi_2\,\|_{\mathcal{B}},\end{aligned}$$

for every $t,s\in I$, $\psi_i\in\mathcal{B}$ *and* $x,y\in X$. *If* $\max\{\Theta_1,\Theta_2\}<1$ *where*

$$\Theta_1=K_a[L_g^2+a(L_g^2++N(L_f^1+L_g^2)+\tilde{N}L_f^2],\;\;\Theta_2=aL_f^3(N+\tilde{N}),$$

then there exists a unique mild solution of (1)-(3).

Proof. Let $\Gamma:S\to S$ the map defined in the proof of Theorem 4. For $(u,v),(w,z)\in S$ we get

$$\begin{aligned}\|\,\Gamma_1(u,v)(t)-\Gamma_1(w,z)(t)\,\| &\leq N\int_0^t\|\, g(s,\tilde{u}_s)-g(s,\tilde{w}_s)\,\|\,ds\\ &\quad+\tilde{N}\int_0^t\|\, f(s,\tilde{u}_s,v(s))-f(s,\tilde{w}_s,z(s))\,\|\,ds\end{aligned}$$

and hence

$$\| u - v \|_a \leq K_a(NL_g + \tilde{N}L_f^2)a \| u - w \|_a + \tilde{N}L_f^3 a \| v - z \|_a . \tag{15}$$

On the other hand, for Γ_2 we have

$$\begin{aligned}
&\| \Gamma_2(u,v)(t) - \Gamma_2(w,z)(t) \| \\
&\qquad \leq \| g(t,u_t) - g(t,w_t) \| + \int_0^t \| AS(t-s)(g(s,\tilde{u}_s) - g(s,\tilde{w}_s)) \| ds \\
&\qquad\quad + N \int_0^t \| f(s,\tilde{u}_s, v(s)) - f(s,\tilde{w}_s, z(s)) \| ds \\
&\qquad \leq K_a(L_g + a(L_g + NL_f^2)) \| u - w \|_a + NL_f^3 a \| v - z \|_a,
\end{aligned}$$

which from (15) implies that

$$\| \Gamma(u,v) - \Gamma(w,z) \|_a \leq \Theta_1 \| u - w \|_a + \Theta_2 \| v - z \|_a .$$

The existence of a unique mild solution of (1)-(3) is now consequence of the contraction mapping principle. ■

3 Regularity of mild solutions

In this section, we study regularity of mild solutions of the neutral problems studied in the previous section. To obtain our results we use the concepts and ideas in Henriquez [10] and Travis [17]. At first, we study briefly the abstract Cauchy problem

$$\begin{aligned}
x''(t) &= Ax(t) + F(t) + A\int_0^t G(s)ds, \qquad t \in I = [0,a], && (1)\\
x(0) &= x_0, && (2)\\
x'(0) &= y_0, && (3)
\end{aligned}$$

where $G(\cdot) : I \to [D(A)]$ and $F(\cdot) : I \to X$ are continuous.

Definition 1 *A function $u(\cdot) : I \to X$ is a strong solution of (1)-(2)-(3) if $u(\cdot) \in W^{2,1}(I : X)$; the equation (1) is satisfied a.e. $t \in I$ and the conditions (2) and (3) are verified.*

Definition 2 *A function $u(\cdot) : I \to X$ is a classical solution of the problem (1)-(2)-(3) if $u(\cdot)$ is a function of class C^2 that satisfies the equation (1) and the initial conditions (2) and (3).*

In some of our next results we consider Banach spaces that have the Radon-Nikodym property (abbreviated, RNP). We refer to [6] for a discussion about this matter.

Lemma 1 *Assume that X has the RNP property and that the functions $C(\cdot)F(t)$, $t \in I$, are uniformly Lipschitz continuous on I. If $(x_0, y_0) \in D(A) \times E$, then a mild solution of (1)-(3) is a strong solution.*

Proof. If $y(\cdot)$ is the mild solution of (1)-(3), then

$$y(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)F(s)ds + \int_0^t S(t-s)\int_0^s AG(\xi)d\xi ds,$$

which from the assumptions implies that

$$y'(t) = AS(t)x_0 + C(t)y_0 + \int_0^t C(t-s)F(s)ds + \int_0^t C(t-s)\int_0^s AG(\xi)d\xi ds. \quad (4)$$

Employing the expression (4); for $t \in I$ and $h \in \mathbb{R}$ such that $t+h \in I$ we get

$$\begin{aligned} \| y'(t+h) - y'(t) \| \leq\ & \| (AS(t+h) - AS(t))x_0 \| + \| (C(t+h) - C(t))y_0 \| \\ & + \int_0^t \| (C(t+h-s) - C(t-s))F(s) \| ds \\ & + \int_t^{t+h} \| C(t+h-s)F(s) \| ds \\ & + \int_0^h \| C(t+h-s) \| (\int_0^s \| AG(\xi)d\xi \|)ds \\ & + \int_0^t \| C(t-s) \| \int_s^{s+h} \| AG(\xi) \| d\xi)ds \end{aligned}$$

which implies that $y'(\cdot)$ is Lipschitz continuous on I and that $y(\cdot) \in W^{2,1}([0,a] : X)$ since X has the RNP. Now, the assertion is consequence of Proposition 3.3 in [10]. ■

Lemma 2 *If X has the RNP property, $(x_0, y_0) \in D(A) \times E$ and F is Lipschitz continuous on I, then a mild solution of (1)-(3) is a classical solution.*

Proof. It is easy to see that the function $p(t) = \int_0^t AG(s)ds + F(t)$ is Lipschitz continuous on I. Now, the result follows from Theorem 3.1 in [10].

In what follow, for a mild solution $u(\cdot)$ of (7)-(9) we consider the decomposition $u(t) = W_1(t)\varphi + W_2(t)y_0 + z^1(t) + z^2(t)$, where $z_0^i \equiv 0$, $i = 1, 2$, and

$$\begin{aligned} z^1(t) &= -S(t)g(0,\varphi) + \int_0^t C(t-s)g(s,u_s)ds, \qquad t \in I, \\ z^2(t) &= \int_0^t S(t-s)f(s,u_s)ds, \qquad t \in I. \end{aligned}$$

In rest of the paper, we refer as **Assumption g** to the next condition

Assumption g : The function $g(\cdot)$ is continuous with values in $[D(A)]$.

Lemma 3 *Let* **Assumption g** *be satified and assume that $t \to W_1(t)\varphi$ is Lipschitz continuous on I. If $u(\cdot)$ is a mild solution of (7)-(9), then $t \to u_t$ is Lipschitz on I.*

Proof. From **Assumption g** we find that

$$\begin{aligned} u'(t) =\ & AS(t)\varphi(0) + C(t)[y_0 - g(0,\varphi)] + g(t,x_t) + \int_0^t S(t-s)Ag(s,x_s)ds \\ & + \int_0^t C(t-s)f(s,x_s)ds, \qquad t \in I, \end{aligned}$$

which implies that $u(\cdot)$ is of class C^1 on I and consequently Lipschitz continuous on I. Using axiom **A** referent to the phase space $\mathcal{B}$, we get

$$\begin{aligned}
\| u_{t+h} - u_t \|_{\mathcal{B}} &\leq M_a \| u_h - \varphi \|_{\mathcal{B}} + K_a \| u(\theta + h) - u(\theta) \|_t \\
&\leq M_a \| W_1(h)\varphi - \varphi \|_{\mathcal{B}} + M_a \| W_2(h)y_0 \|_{\mathcal{B}} + M_a \| z_h^1 \|_{\mathcal{B}} \\
&\quad + M_a \| z_h^2 \|_{\mathcal{B}} + K_a \| u(\theta + h) - u(\theta) \|_t \\
&\leq C_1 h + K_a M_a \| S(\cdot)y_0 \|_h + M_a K_a (\| z^1 \|_h + \| z^2 \|_h) \\
&\leq C_2 h + K_a M_a \left(2N \| g(s, u_s) \|_a + \tilde{N} \| f(s, u_s) \|_a\right) h,
\end{aligned}$$

which prove the assertion. ■

Remark 1 *Let $\varphi \in \mathcal{B}$ and assume that $\mathcal{B}$ verifies the axiom C_2 in the nomenclature of Hino [13]. If $\varphi' \in C_b((-\infty, 0] : X)$ and $t \to C(t)\varphi(0)$ is differentiable in $t = 0$, from the phase space axioms we get*

$$\begin{aligned}
\| W_1(h)\varphi - \varphi \|_{\mathcal{B}} &\leq \| W_1(h)\varphi - W_0(h)\varphi \|_{\mathcal{B}} + \| W_0(h)\varphi - \varphi \|_{\mathcal{B}} \\
&\leq K_a \sup_{\theta \in [0,h]} \| C(h + \theta)\varphi(0) - \varphi(0) \| + \| W_0(h)\varphi - \varphi \|_{\mathcal{B}} \\
&\leq K_a C_1 h + \| W_0(h)\varphi - \varphi \|_{\mathcal{B}}
\end{aligned}$$

which prove that $t \to W(t)\varphi$ verifies a Lipschitz conditions on I, since under these conditions $\varphi' \in \mathcal{B}$. We observe that in particular, $t \to W(t)\varphi$ is Lipschitz for every $\varphi \in C_{00}^{\infty}((-\infty, 0] : X)$.

In order to use the precedent lemmas, we observe that under **Assumption g**, a function $u(\cdot)$ is a mild solution of (7)-(9) if, and only if,

$$\begin{aligned}
u(t) - \int_0^t g(s, x_s) ds &= C(t)\varphi(0) + S(t)[y_0 - g(0, \varphi)] + \int_0^t S(t - s)A \int_0^s g(\tau, x_\tau) d\tau ds \\
&\quad + \int_0^t S(t - s) f(s, x_s) ds, \qquad t \in I. \qquad (5)
\end{aligned}$$

Now, we discuss existence of strong solutions of (7)-(9).

Definition 3 *A function $u(\cdot) : (-\infty, a] \to X$ is a strong solution of (7)-(9) if $u_0 = \varphi$; the function $t \to u'(t) + g(t, u_t) \in W^{1,1}(I : X)$; $u(t) \in D(A)$ a.e. $t \in I$; the equation (7) is satisfied a.e. $t \in I$ and the initial conditions (8) and (9) are verified.*

Theorem 1 *Assume that X has the RNP property and that* **Assumption g** *holds. If $(\varphi(0), y_0) \in D(A) \times E$, $t \to W(t)\varphi$ verifies a Lipschitz conditions on I and*

(a) *For every bounded set $D \subset \mathcal{B}$, the functions $C(\cdot)f(t, \phi)$, $(t, \phi) \in I \times D$, are uniformly Lipschitz on I,*

then a mild solution, $u(\cdot)$, of (7)-(9) is a strong solution. If in addition, $g(\cdot)$ verifies a Lipschitz condition on $I \times \mathcal{B}$ then $u(\cdot) \in W^{2,1}(I; X)$.

Proof. Let $u(\cdot)$ a mild solution of (7)-(9) and $G(\cdot), F(\cdot) : I \to X$ the functions defined by $G(t) = g(t, u_t)$ and $F(t) = f(t, u_t)$. We know from Lemma 1 that a mild solution, $w(\cdot)$, of the abstract Cauchy problem

$$\begin{cases} x''(t) &= Ax(t) + F(t) + A\int_0^t G(s)ds, \qquad t \in I = [0, a], \\ x(0) &= \varphi(0), \ \ x'(0) = y_0 - g(0, \varphi), \end{cases} \tag{6}$$

is a strong solution, which from the uniqueness of solutions of (6) imply that $w(t) = u(t) - \int_0^t g(s, u_s)ds \in W^{2,1}(I, X)$ and that $u(t) \in D(A)$ *a.e.* $t \in I$ since $g(\cdot)$ is $D(A)$-valued. Thus, $u(\cdot)$ is a strong solution of (7)-(9).

When $g(\cdot)$ is Lipschitz, from Lemma 3 follows that $s \to g(s, u_s)$ is Lipschitz on I and so that $t \to \int_0^t g(s, u_s)ds \in W^{2,1}(I, X)$. Thus $u(\cdot) \in W^{2,1}(I; X)$. ■

In the next result we discuss the existence of classical solutions of (7)-(9).

Theorem 2 *Assume that X has the RNP property and that $(\varphi(0), y_0) \in D(A) \times E$. Suppose in addition that* **Assumption g** *hold, that $W_1(\cdot)\varphi$ verifies a Lipschitz condition on I and that*

(a) *For every bounded set $D \subset \mathcal{B}$, the function $f : I \times D \to X$ is Lipschitz continuous.*

Then a mild solution of (7)-(9) is a classical solution.

Proof. Let $u(\cdot)$ be a mild solution of (7)-(9), $G(t) = g(t, u_t)$ and $F(t) = f(t, u_t)$. Follows from Lemmas 2 and 3, that the mild solution, $w(\cdot)$, of

$$\begin{cases} x''(t) &= Ax(t) + F(t) + A\int_0^t G(s)ds, \qquad t \in I, \\ x(0) &= \varphi(0), \ \ x'(0) = y_0 - g(0, \varphi), \end{cases} \tag{7}$$

is a classical solution. From the uniqueness of solutions of (7) and (5), we infer that $w(t) = u(t) - \int_0^t g(s, u_s)ds$ and that $u(t) \in D(A)$ for every $t \in I$, since $g(\cdot)$ is $D(A)$-valued. Thus, $u(\cdot)$ is a classical solution. ■

To finish this section we study regularity of solutions of the neutral system (1)-(3), assuming that f, g are differentiable functions. In what follows, for a function $j : [0, a] \to X$ and $h \in \mathbb{R}$ we use the notation $\partial_h j$ for the function

$$\partial_h j(t) := \frac{j(t+h) - j(t)}{h}. \tag{8}$$

If Y, Z are normed space and $j(\cdot) : I \times Y \to Z$ is differentiable, we will employ the decomposition

$$j(t+s, y+y_1) - j(t, y) = (D_1 j(t, y), D_2 j(t, y))(s, y_1) + \parallel (s, y_1) \parallel R(j, t, y, s, y_1), \tag{9}$$

where $R(j, t, y, s, y_1) \to 0$ when $\parallel (s, y_1) \parallel \to 0$.

Lemma 4 *Let $u(\cdot) : (-\infty, a] \to X$ be a mild solution of (1)-(3). If $\varphi(0) \in E$, $g(\cdot)$ is continuous with values in $E$and the function $t \to W_1(t)\varphi + W_2(t)y_0$ is differentiable at $t = 0$, then $s \to u_s$ is differentiable on I and $(u_t)'(0) = u'(t)$ for every $t \in I$.*

Proof. Let $B(\varphi, y_0) = \frac{d}{dt}(W_1(t)\varphi + W_2(t)y_0)\mid_{t=0}$ and $w(\cdot) : (-\infty, a] \to X$ the function defined by

$$w(\theta) = \begin{cases} B(\varphi, y_0)(\theta), & \text{for } \theta \in (-\infty, 0], \\ u'(\theta), & \text{for } \theta \in [0, a]. \end{cases} \tag{10}$$

From Axiom (**A**), relative to the phase space, follows that $B(\varphi, y_0)(0) = y_0$ and that $s \to w_s$ is a continuous function with values in $\mathcal{B}$. Moreover, employing that $g(\cdot)$ is continuous with values in E, we find that

$$\begin{aligned} u'(t) &= AS(t)\varphi(0) + C(t)[y_0 - g(0, \varphi)] + g(t, u_t) + \int_0^t AS(t-s)g(s, u_s)ds \\ &+ \int_0^t C(t-s)f(s, u_s, u'(s))ds, \qquad t \in I, \end{aligned} \tag{11}$$

and a straightforward estimation shows that $\partial_h u(s) \to u'(s)$ uniformly on I. Considering these remarks; for $t \in I$ and $h > 0$ with $t + h \in I$ we get

$$\begin{aligned} \| \frac{u_{t+h} - u_t}{h} - w_t \|_{\mathcal{B}} &\leq M_a \| \frac{u_h - \varphi}{h} - w_0 \|_{\mathcal{B}} + K_a \| \frac{u(s+h) - u(s)}{h} - u'(\cdot) \|_t \\ &\leq M_a \| \frac{W_1(h)\varphi + W_2(h)y_0 - \varphi}{h} - w_0 \|_{\mathcal{B}} + M_a \| \frac{z_h^2}{h} \|_{\mathcal{B}} \\ &\quad + M_a \| \frac{z_h^3}{h} \|_{\mathcal{B}} + \| \frac{u(s+h) - u(s)}{h} - u'(s) \|_{0,t}, \\ &\leq \xi(h) + M_a K_a \| \frac{z^2(\theta)}{h} \|_h + M_a K_a \| \frac{z^3(\theta)}{h} \|_h \\ &\leq \xi(h) + M_a K_a \| \frac{1}{h} \int_0^\theta C(\theta - s)(g(0, \varphi) - g(s, u_s)ds \|_h \\ &\quad + M_a K_a \| \frac{1}{h} \int_0^\theta S(\theta - s)f(s, u_s, u'(s))ds \|_h \\ &\leq \xi(h) + M_a K_a N(\| g(0, \varphi) - g(s, u_s) \|_h + \| f(s, u_s, u'(s)) \|_h h), \end{aligned}$$

where $\xi(h) \to 0$ as $h \to 0$. The previous inequality prove that $t \to u_t$ is differentiable and that $(u_t)' = w_t$ on I. ■

Now we turn our attention to the problem of existence of classical solutions.

Theorem 3 *Let* **Assumption g** *be satisfied and assume that the hypotheses in Proposition 1 and Lemma 4 are verified. If $(\varphi(0), y_0) \in D(A) \times E$ and $f(\cdot) : I \times \mathcal{B} \times \mathcal{X} \to \mathcal{X}$, $g(\cdot) : I \times \mathcal{B} \to \mathcal{E}$ are continuously differentiable, then the mild solution u of (1)-(3) is a classical solution.*

Proof. In order to facilitate the reading of the text, we abbreviate the notations using $w(t) = (t, u_t, u'(t))$, $\tilde{f}(t) = f(t, u_t, u'(t))$ and $\tilde{g}(t) = g(t, u_t)$. Let $v : I \to X$ the solution of the integral equation

$$\begin{cases} v(t) &= P(t) + \int_0^t C(t-s)(D_3 f)(w(s))(v(s))ds, \qquad t \in I, \\ v(0) &= A\varphi(0), \end{cases} \tag{12}$$

where

$$\begin{aligned}
P(t) &= C(t)A\varphi(0) + AS(t)y_0 + D_1g(t,u_t) + D_2g(t,u_t)(u_t)' \\
&\quad + \int_0^t AS(t-s)[D_1g(s,u_s) + D_2g(s,u_s)(u_s)']ds \\
&\quad + C(t)f(0,\varphi,y_0) + \int_0^t C(t-s)\left(D_1f(w(s)) + D_2f(w(s))(u_s)'\right)ds.
\end{aligned}$$

The existence and uniqueness of a solution of (12) is clear and we will omit additional details. Next, we prove that $u''(\cdot) = v(\cdot)$ on I. Using (11); for $t \in I$ and $h \in \mathbb{R}$ with $t+h \in I$ we find that

$$\begin{aligned}
&\| \partial_h u'(t) - v(t) \| \\
&\quad \leq \xi_1(h,t) + \| -[\partial_h C(t)]g(0,\varphi) + \frac{1}{h}\int_0^h AS(t+h-s)\tilde{g}(s)ds \| \\
&\qquad + \| \partial_h \tilde{g}(t) - D_1g(t,u_t) - D_2g(t,u_t)((u_t)') \| \\
&\qquad + \int_0^t \| AS(t-s)(\partial_h \tilde{g}(s) - D_1g(s,u_s) - D_2g(s,u_s))((u_s)') \| ds \\
&\qquad + \| \frac{1}{h}\int_0^h C(t+h-s)f(w(s))ds - C(t)f(0,\varphi,y_0) \| \\
&\qquad + N\int_0^t \| \partial_h \tilde{f}(s) - D_1f(w(s)) - D_2f(w(s))(u_s)' - D_3f(w(s))(v(s)) \| ds, \\
&\quad \leq \xi_2(h,t) + \| \partial_h C(t)g(0,\varphi) - \frac{1}{h}\int_0^h AS(t+h-s)\tilde{g}(s)ds \| \\
&\qquad + N\int_0^t \| \partial_h \tilde{f}(s) - D_1f(w(s)) - D_2f(w(s))(u_s)' - D_3f(w(s))(v(s)) \| ds,
\end{aligned}$$

where $\xi_i(h,t) \to 0, i = 1,2,$ when $h \to 0$. Employing the notations introduced in (9), we get

$$\begin{aligned}
&\| \partial_h u'(t) - v(t) \| \\
&\quad \leq \xi_3(h) + \frac{1}{h}\int_0^h \| AS(t+h-s)(\tilde{g}(0) - \tilde{g}(s)) \| ds \\
&\qquad + N\int_0^t \| D_3f(w(s))(\partial_h u'(s) - v(s)) \| ds \\
&\qquad + N\int_0^t \| (1, \partial_h u_s, \partial_h u'(s))R(f,s,u_s,u'(s),h,u_{s+h} - u_s, u'(s+h) - u'(s)) \| ds,
\end{aligned}$$

which from Lemma 5 below, permit to conclude $u'' = v$.

From these remarks, Lemma 4 and Proposition 2.4 in [17] follows that the mild solution, $y(\cdot)$, of the abstract Cauchy problem

$$\left\{\begin{aligned}
x''(t) &= Ax(t) + f(t,u_t,u'(t)) + A\int_0^t g(t,u_t)ds, \qquad t \in I = [0,a], \\
x(0) &= \varphi(0), \ x'(0) = y_0 - g(0,\varphi),
\end{aligned}\right. \tag{13}$$

is a classical solution. Consequently, $y(t) = u(t) - \int_0^t g(s,u_s)ds$ is a function of class C^2 and $u(t) \in D(A)$ for every $t \in I$ since $g(\cdot)$ is continuous with values in $[D(A)]$. Thus, $u(\cdot)$ is a classical solution. ∎

Lemma 5 *Assume that the assumptions of Theorem 3 hold. If $u(\cdot)$ is a mild solution of (1)-(3), then $u'(\cdot)$ is Lipschitz on I.*

Proof. Employing Lemma 3; for $t \in I$ and $h \in \mathbb{R}$ with $t+h \in I$ we get

$$
\begin{aligned}
&\| u'(t+h) - u'(t) \| \\
&\quad \leq \; C_1 h + L_g^2 \| u_{t+h} - u_t \|_{\mathcal{B}} + \int_t^{t+h} \| S(t+h-s)Ag(s,u_s) \| \, ds \\
&\qquad \int_0^t \| S(t+h-s) - S(t-s))Ag(s,u_s) \| \, ds + \int_0^h \| C(t-s)f(s,u_s,u'(s)) \| \, ds \\
&\qquad + N \int_0^t (L_f^1 h + L_f^2 \| u_{s+h} - u_s \|_{\mathcal{B}} + L_f^3 \| u'(s+h) - u'(s) \|) ds \\
&\quad \leq \; C_2 h + N L_f^3 \int_0^t \| u'(s+h) - u'(s) \| \, ds
\end{aligned}
$$

where $C_2 > 0$ is a constant independent of h and $t \in I$. Now, the assertion follows directly from the Gronwall Bellman Inequality. The proof is finished. ■

4 Example

In this section we consider a application of our results. Along of this section, $X = (L^2([0,\pi]), \| \cdot \|_2)$ and A the operator $Af(\xi) := f''(\xi)$ with domain

$$D(A) := \{ f(\cdot) \in H^2(0,\pi) : f(0) = f(\pi) = 0 \}.$$

It is well known that A is the generator of strongly continuous cosine function $(C(t))_{t \in \mathbb{R}}$ on X. Furthermore, A has discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with corresponding normalized eigenvectors $z_n(\xi) := \left(\frac{2}{\pi}\right)^{1/2} \sin(n\xi)$ and

(a) $\{ z_n : n \in \mathbb{N} \}$ is an orthonormal basis of X.

(b) If $\varphi \in D(A)$ then $A\varphi = -\sum_{n=1}^{\infty} n^2 < \varphi, z_n > z_n$.

(c) For every $\varphi \in X$, $C(t)\varphi = \sum_{n=1}^{\infty} \cos(nt) < \varphi, z_n > z_n$ and $\| C(t) \| = 1$ for every $t \in I$.

(d) The associated sine function is given by $S(t)\varphi = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < \varphi, z_n > z_n$. Moreover, from this expression follow that $S(t)$ is compact and that $\| S(t) \| = 1$ for every $t \in I$.

(e) If G denotes the group of translations on X defined by $G(t)x(\xi) = \tilde{x}(\xi + t)$, where $\tilde{x}$ is the extension of x with period 2π, then $C(t) = \frac{1}{2}(G(t) + G(-t))$. Hence it follows, see [7], that $A = B^2$ where B is the infinitesimal generator of the group G and $E = \{ x \in H^1(0,\pi) : x(0) = x(\pi) = 0 \}$.

Let $\mathcal{B} = \mathcal{C}_\nabla \times \mathcal{L}^\in(\}; \mathcal{X})$, $\nabla = \prime$, the space defined in Example 1. In this case (see [13]) $H = 1$; $K(t) = 1 + \left(\int_{-t}^{0} g(\theta) \, d\theta \right)^{1/2}$ and $M(t) = \gamma(-t)^{1/2}$ for all $t \geq 0$.

4.1 Example 1

Now we consider the neutral differential equation

$$\frac{\partial}{\partial \xi}[\frac{\partial}{\partial \xi}u(t,\xi)+\int_{-\infty}^{0}\int_{0}^{\pi} b(s,\eta,\xi)u(t+s,\eta)d\eta ds] = \frac{\partial^2}{\partial \xi^2}u(t,\xi)$$
$$+\int_{-r}^{0}\int_{0}^{\pi} a_0(s)u(t+s,\eta)d\eta ds + a_1(t,\xi,u(t,\xi)), \quad t\in I=[0,\pi] \qquad (14)$$

submitted to the initial conditions

$$\begin{aligned} u(t,0) &= u(t,\pi)=0, \qquad t\in I, & (15)\\ u_0 &= \varphi, & \\ \frac{\partial}{\partial \xi}u(0,\cdot) &= y_0, & (16)\end{aligned}$$

where $\varphi\in\mathcal{B}$, $y_0\in X$ and

(i) The functions $\frac{\partial}{\partial \zeta}b(\theta,\eta,\zeta)$, $\frac{\partial^2}{\partial \zeta^2}b(\theta,\eta,\zeta)$ are measurable; $b(\theta,\ \eta,\pi)=b(\theta,\ \eta,\ 0)=0$ and

$$N_1 := \max\left\{\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\frac{1}{g(\theta)}(\frac{\partial^j}{\partial \zeta^j}b(\theta,\eta,\zeta))^2 d\eta d\theta d\zeta : j=1,2\right\} < \infty.$$

(ii) The function $a_0(\cdot)$ is measurable with $\int_{-\infty}^{0}\frac{a_0^2(\theta)}{g(\theta)}d\theta<\infty$, $a_1(\cdot)$ is continuous and there exists a continuous function $\eta: I^2\times \mathbb{R}\to[0,\infty)$ such that $\mid a_1(t,\xi,x)\mid\leq \eta(t,\xi)\mid x\mid$ for every $(t,\xi,x)\in I^2\times X$.

Under these conditions the neutral problem (14) can be modelled in the form

$$\begin{aligned} \frac{d}{dt}[x'(t)-g(t,x_t))] &= Ax(t)+f(t,x_t)+h(t,x(t)), \qquad t\in I=[0,a], & (17)\\ x_0 &= \varphi\in\mathcal{B}, & (18)\\ x'(0) &= y_0\in X. & (19)\end{aligned}$$

where $f,g:[0,\infty)\times\mathcal{B}\to\mathcal{X}$, $h: I\times X\to X$ are defined by

$$\begin{aligned} g(\psi)(\xi) &:= \int_{-\infty}^{0}\int_0^{\pi} b(\theta,\eta,\xi)\psi(\theta,\eta)\,d\eta\,d\theta,\\ f(\psi)(\xi) &:= \int_{-\infty}^{0} a_0(\theta)\psi(\theta,\xi)\,d\theta,\\ h(t,x)(\xi) &:= a_1(t,\xi,x(\xi)).\end{aligned}$$

Follows from the assumptions, that f,g are bounded linear operators, that $\parallel g(t,\cdot)\parallel_{\mathcal{L}(\mathcal{X})}\leq N_1$, $\parallel f(t,\cdot)\parallel_{\mathcal{L}(\mathcal{X})}\leq N_2$ where

$$N_2 = \max\{\parallel a_0\parallel_\infty, \left(\int_{-\infty}^{0}\frac{a^2(\theta)}{g(\theta)}d\theta\right)^{\frac{1}{2}}\}$$

and that $\| h(t,x) \| \leq \eta(t,\cdot)_\pi \| x \|$ for every $(t,x) \in I \times X$. Moreover, a straithforward estimation using **(i)** shows that $g(\cdot)$ is continuous with values in $[D(A)]$ and that $\| Ag(t,\cdot) \|_{\mathcal{L}(X)} \leq N_1$.

Proposition 1 *Under the previous conditions, there exists a mild solution, $u(\cdot)$, of the abstract neutral problem (14), (15), (16). If in addition, $(\varphi, y_0) \in D(A) \times E$, $t \to W_1(t)\varphi$ is Lipschitz continuous on I and*

(a) *for every $r > 0$ there exist positive constants $L_{i,r}$, $i = 1, 2$, such that*

$$| a_1(t,\xi,x) - a_1(t,\xi,y) | \leq L_{1,r} \mid t - s \mid + L_{2,r} \mid x - y \mid$$

for every $t, s, \xi \in I$ and every $| x | \leq r, | y | \leq r$,

then $u(\cdot)$ is a classical solution.

Proof. The existence of a mild solution of (14), (15), (16), follows directly of Theorem 2. On the other hand, the assertion respect regularity, follows from Theorem 2, since X has the *RNP*.

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