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RESUMO

Mostra-se que operadores de Schrödinger com potenciais definidos pelos shifts assiociados a transformações de intercâmbio de intervalos minimais, num conjunto denso, possuem espectro singular contínuo puro para um conjunto de pontos do intervalo de medida de Lebesgue total. Esses potencias são generalizações naturais de seqüências Sturmianas.

ALMOST PERIODIC SCHRÖDINGER OPERATORS ALONG INTERVAL EXCHANGE TRANSFORMATIONS

CÉSAR R. DE OLIVEIRA AND CARLOS GUTIERREZ

ABSTRACT. It is shown that Schrödinger operators, with potentials along the shift embedding of irreducible interval exchange transformations in a dense set, have pure singular continuous spectrum for Lebesgue almost all points of the interval. Such potentials are natural generalizations of the Sturmian case.

1. Introduction and Main Results

In this work some techniques for the study of the spectrum of discrete Schrödinger operators $H_\omega : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$,

$$(H_\omega \psi)_j = \psi_{j+1} + \psi_{j-1} + \omega_j \psi_j, \tag{1}$$

with $\omega = (\omega_j)_{j\in\mathbb{Z}}$ a sequence of real numbers (usually called potential) taking a finite number of values, are used to show the presence of pure singular continuous spectrum for potentials along the shift embedding of some interval exchange transformations (briefly, iets) [8]. An iet preserves Lebesgue measure and our results apply for the shift associated with a dense set of iets and for a.e. points of the interval (the symbol a.e. with no specification means *almost everywhere* with respect to the corresponding Lebesgue measure). See ahead for precise formulations.

One of the main interests in the spectral type of such operators comes from its relation with the asymptotic temporal behavior of the solutions of Schrödinger equation (see, for instance, [7] and references therein)

$$i\frac{\partial \psi}{\partial t}(t) = H_\omega \psi(t), \qquad \psi(0) = \psi_0.$$

For example, assume that $||\psi_0|| = 1$, let $\psi(t) = \exp(-itH_\omega)\psi_0$ be the solution of this equation and denote by $p_\psi(T) = \frac{1}{T}\int_0^T |\langle\psi(t), \psi_0\rangle|^2 dt$ the average return probability, at time T, to the initial condition ψ_0; by Wiener theorem [1, 7] $\lim_{T\to\infty} p_\psi(T) = 0$ if, and only if, ψ_0 belongs to the continuous spectral subspace of H_ω; it is worth noting that for ψ_0 in the singular continuous subspace it is possible that $p_\psi(T)$ does not vanish for $T \to \infty$, which is sometimes called *exotic behavior* by physicists.

Here it will be considered the spectral type of operator (1) with sequences ω directly related to iets, so, in order to formulate such spectral results properly, it is convenient to introduce some notations and a description of the iets.

Key words: Schrödinger operator; interval exchange transformation; singular continuous spectrum.

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1.1. **iet: a brief account.** Fix $n \in \mathbb{N}$, an irreducible permutation

$$\pi : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$$

(i.e., $\pi\{1, 2, \cdots, j\} \neq \{1, 2, \cdots, j\}$ for all $1 \leq j < n$) and let

$$\Lambda_n = \{\mathbf{a} = (a_0, a_1, \cdots, a_n) \in \mathbb{R}^{n+1} : 0 = a_0 < a_1 < \cdots < a_n = 1\}$$

provided with the metric induced by the norm

$$|\mathbf{a} - \mathbf{b}| = \max\{|a_i - b_i| : i = 0, 1, \cdots, n\},$$

where $\mathbf{b} = (b_0, b_1, \cdots, b_n)$. To each $\mathbf{a} = (a_0, a_1, \cdots, a_n) \in \Lambda_n$ it is associated the iet $E = E_{\mathbf{a}} \colon [0, 1) \to [0, 1)$ defined by

$$E(x) = x + \sum_{k=1}^{\pi(i)-1} (a_{\pi^{-1}(k)} - a_{\pi^{-1}(k)-1}) - \sum_{k=1}^{i-1} (a_k - a_{k-1}), \qquad x \in [a_{i-1}, a_i).$$

Let $E(\pi) = \{E_{\mathbf{a}} \colon [0, 1) \to [0, 1) \colon \mathbf{a} \in \Lambda_n\}$, i.e., the collection of all iets associated to the given permutation π. The bijection $\Lambda_n \to E(\pi)$ is employed to transfer to $E(\pi)$ the metric of Λ_n.

Some simple properties of an iet are:

i) continuity, except at $\{a_1, a_2, \cdots, a_{n-1}\}$, where it is right continuous;
ii) invertibility;
iii) piecewise isometric.

In fact, the iets consist of the order-preserving piecewise isometries of $[0, 1)$.

An iet $E_{\mathbf{a}}$ is called *irrational* if the only rational relation between the lengths $\{a_1 - a_0, a_2 - a_1, \cdots, a_n - a_{n-1}\}$ is $(a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) = 1$, and *minimal* if for each $x \in [0, 1)$ its orbit $\mathcal{O}_{\mathbf{a}}(x) = \{(E_{\mathbf{a}})^k(x) : k \in \mathbb{Z}\}$ is dense in $[0, 1)$. $E_{\mathbf{a}}$ is called *rational* if $\mathbf{a} \in \mathbb{Q}^{n+1}$.

Lemma 1. [8] *(a) If $E_{\mathbf{a}}$ is irrational, then it is minimal.*

(b) If the orbits $\mathcal{O}_{\mathbf{a}}(a_j), 0 \leq j < n$, are infinite and pairwise disjoint, then $E_{\mathbf{a}}$ is minimal.

Given $\mathbf{a} = (a_0, a_1, \cdots, a_n) \in \Lambda_n$, let

$$\mathcal{A}_{\mathbf{a}} : [0, 1) \to \{1, 2, \cdots, n\}$$

be the map such that $\mathcal{A}_{\mathbf{a}}(x) = i$ if, and only if, $x \in [a_{i-1}, a_i)$, for some $i \in \{1, 2, \cdots, n\}$. Let $W_n = \{1, 2, \cdots, n\}^*$ be the set of finite sequences whose terms belong to $\{1, 2, \cdots, n\}$, i.e., *finite words* or *factors* in the alphabet $\{1, 2, \cdots, n\}$. Denote also by $\Sigma_n = \{1, 2, \cdots, n\}^{\mathbb{Z}}$, with the topology induced by the metric

$$\mathrm{dist}(\omega, \alpha) = \sum_{j \in \mathbb{Z}} \frac{d(\omega_j, \alpha_j)}{2^{|j|}},$$

($\omega = (\omega_j)_{j \in \mathbb{Z}}$ and $\alpha = (\alpha_j)_{j \in \mathbb{Z}}$) with d being the discrete metric, and by $S : \Sigma_n \to \Sigma_n$ the left shift $(S\omega)_j = \omega_{j+1}$. Extend naturally $\mathcal{A}_{\mathbf{a}}$ to $\mathcal{O}_{\mathbf{a}}(x)$ and define $\phi = \phi_{\mathbf{a}} : [0, 1) \to \Sigma_n$ by $\phi(x) = \mathcal{A}_{\mathbf{a}}(\mathcal{O}_{\mathbf{a}}(x))$; i.e., $\phi(x)$ is a natural coding of the orbit of x by assigning to each entry of this orbit the number of the interval which contains it. Set $\Omega_{\mathbf{a}} = \mathrm{closure}\{\phi([0, 1)\}$ in Σ_n.

Lemma 2. [8] *If $E_{\mathbf{a}}$ is minimal, then the dynamical system $(\Omega_{\mathbf{a}}, S)$ is a minimal subshift, i.e., the orbit $\{S^k(\omega)\}$ is dense in $\Omega_{\mathbf{a}}$ for all $\omega \in \Omega_{\mathbf{a}}$.*

Remark. [8, 9, 10, 13] For $n = 2, 3$ the iets reduce to the study of rotations of a circle and, therefore, minimality implies unique ergodicity; for $n \geq 4$ it is known the upper bound $n/2$ for the number of ergodic probability measures, and there are examples with $n = 4$ with exactly two ergodic probability measures; such results are transferred to the subshifts (Ω, S), i.e., the corresponding minimality and with a finite number of ergodic probability measures.

1.2. **Main Results.** As before fix an irreducible permutation

$$\pi : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$$

and let $E(\pi) = \{E_{\mathbf{a}} \colon [0, 1) \to [0, 1) \colon \mathbf{a} \in \Lambda_n\}$. Identify the metric spaces Λ_n and $E(\pi)$ by the homeomorphism $\mathbf{a} \in \Lambda_n \to E_{\mathbf{a}} \in E(\pi)$.

Given $\omega \in \Sigma_n$ and an injective map $V : \{1, 2, \cdots, n\} \to \mathbb{R}$, consider the potential $V(\omega) := (V(\omega_j))_{j \in \mathbb{Z}}$ and the operator $H_{V(\omega)}$ as in (1).

Theorem 1. *Given V as above, there is a dense subset $\mathcal{D} \subset E(\pi)$ such that:*

(i) *each $E_{\mathbf{a}} \in \mathcal{D}$ is minimal and aperiodic (i.e., no sequence in $\Omega_{\mathbf{a}}$ is periodic);*
(ii) *for each $E_{\mathbf{a}} \in \mathcal{D}$ the spectrum of $H_{V(\omega)}$ in (1) is the same for all $\omega \in \Omega_{\mathbf{a}}$.*
(iii) *for each $E_{\mathbf{a}} \in \mathcal{D}$ the corresponding Schrödinger operators (1) with potential $V(\phi(x))$ has pure singular continuous spectrum for a.e. $x \in [0, 1)$.*

Corollary 1. *Given V as above, there is a dense subset $\mathcal{D} \subset E(\pi)$ such that, for each $E_{\mathbf{a}} \in \mathcal{D}$, the set $\Gamma_{\mathbf{a}} \subset \Omega_{\mathbf{a}}$ for which $H_{V(\omega)}$ has pure singular continuous spectrum for any $\omega \in \Gamma_{\mathbf{a}}$ is a dense G_δ and $\nu_{\mathbf{a}}(\Gamma_{\mathbf{a}}) = 1$ for some ergodic probability measure $\nu_{\mathbf{a}}$ over $\Omega_{\mathbf{a}}$.*

Remark. According to Gottschalk's theorem [6, 14] the sequences in $\Omega_{\mathbf{a}}$ are almost periodic if, and only if, $\Omega_{\mathbf{a}}$ is minimal; therefore, the spectral results presented above refer to a (aperiodic) class of almost periodic Schrödinger operators.

Remark. For $n = 2$ and $\pi(1, 2) = (2, 1)$, there is only one discontinuity point $a_1 \in [0, 1)$, the system is reduced to rotations of the circle by the angle $(1 - a_1)$ and the potentials $\Omega_{\mathbf{a}}$ are the Sturmian sequences [2, 3], which take just two values and encompass the well-known Fibonacci substitution sequence [14, 17]; therefore, the potentials generated by iets are natural generalizations of Sturmian potentials—which have become standard models of quasicrystals.

Remark. From the proofs it is clear that V does not need to be injective; it is enough to require that the potentials $V(\omega)$, $\omega \in \Omega_{\mathbf{a}}$, are not periodic.

The main parts of the proof of this Theorem amount to exclude eigenvalues a.e. and absolutely continuous spectrum, and those are the contents of Sections 2 and 3, respectively; some arguments are well known, but a number of details is included for convenience of the reader. Before going into details, this section finishes with some related open problems:

(1) Does the complement of $\mathcal{D}$ above lie in a set of Lebesgue measure zero?
(2) Does $\mathcal{D}$ contain iets with more than one ergodic probability measure (Masur [13] has shown that a.e. $E_{\mathbf{a}}$ is uniquely ergodic)?
(3) What is the Lebesgue measure of the spectrum of such operators?
(4) Giving an explicit characterization of the elements of $\mathcal{D}$ in terms of the permutation π and arithmetic properties of $\{a_1, \cdots, a_{n-1}\}$.
(5) For which $E_{\mathbf{a}}$ the spectrum of H_ω is pure singular continuous for all $\omega \in \Omega_{\mathbf{a}}$? What about the spectrum of $H_{\mathcal{O}_{\mathbf{a}}(x)}$?

2. Absence of Point Spectrum

The discussions in this and next sections will be restricted to potentials $\omega \in \Sigma_n$ (or suitable subsets of it). This will cause no loss, since in Lemmas 3, 7 and 8 the exact values of the potentials are not relevant and the function $V : \{1, 2, \cdots, n\} \to \mathbb{R}$ is supposed to be injective.

An important tool to exclude eigenvalues for a given operator H_ω, $\omega \in \Sigma_n$, is the Delyon-Petritis version of an argument of Gordon [5], which utilizes local repetitions and can be stated as follows.

Lemma 3. [4] *If for given $\omega \in \Sigma_n$ there exists a sequence $k_i \to \infty$ such that*

$$\omega_{j-k_i} = \omega_j = \omega_{j+k_i},$$

for all $1 \leq j \leq k_i$, then the Schrödinger operator H_ω has no eigenvalues.

Given an irreducible permutation π, the idea is to construct a dense subset $\mathcal{D} \subset E(\pi)$ so that, for each $\mathbf{a} \in \mathcal{D}$, Lemma 3 applies to H_ω, $\omega = \phi_{\mathbf{a}}(x)$, with x in a set of total Lebesgue measure over $[0, 1)$.

Propositon 1. *There is a dense subset $\mathcal{D} \subset E(\pi)$ such that each $E_{\mathbf{a}} \in \mathcal{D}$ is (aperiodic) minimal and, for a.e. $x \in [0, 1)$, the coding $\phi_{\mathbf{a}}(x)$ satisfies the hypotheses of Lemma 3 and so, the operator $H_{\phi_{\mathbf{a}}(x)}$ has empty point spectrum.*

The proof of Proposition 1 will follow after a series of suitable remarks concerning iets. The length of a factor $B \in W_n$ will be denoted by $|B|$; the same U will designate open sets of both Λ_n and $E(\pi)$. It will also be convenient to use λ to indicate Lebesgue measure over $[0, 1)$.

Let $j, k \in \mathbb{Z}$ with $j \leq k$, and suppose that $I \subset [0, 1)$ is a nonempty interval (which may be reduced to a point) such that, for all integer $i \in [j, k]$, $E^i_{\mathbf{a}}|_I$ is continuous; then the sequence

$$\{\mathcal{A}_{\mathbf{a}}(E^j_{\mathbf{a}}(I)), \mathcal{A}_{\mathbf{a}}(E^{j+1}_{\mathbf{a}}(I)), \cdots, \mathcal{A}_{\mathbf{a}}(E^k_{\mathbf{a}}(I))\}$$

will be said to be the $E_{\mathbf{a}}$-*itinerary* of I associated to $[j, k]$.

Definition 1. *Given $\mathbf{a} \in \Lambda_n$ and $B \in W_n$, a nonempty interval $I \subset [0, 1)$ (which may be reduced to a point) is said to be of B-type for $E_{\mathbf{a}} \in E(\pi)$ if for all $i \in \{-k, -k+1, \cdots, 2k\}$, where $k = |B|$, $E^i_{\mathbf{a}}|_I$ is continuous, $\mathcal{A}_{\mathbf{a}}$ restricted to $E^i_{\mathbf{a}}(I)$ is constant and*

$$\begin{aligned} B &= \text{the } E_{\mathbf{a}}\text{-itinerary of } I \text{ associated to } [-k, 0] \\ &= \text{the } E_{\mathbf{a}}\text{-itinerary of } I \text{ associated to } [0, k] \\ &= \text{the } E_{\mathbf{a}}\text{-itinerary of } I \text{ associated to } [k, 2k]. \end{aligned}$$

For $\mathbf{a} = (a_0, a_1, \cdots, a_n) \in \Lambda_n$ a subinterval I of $[0, 1)$ is said to be $E_{\mathbf{a}}$-*periodic* of period $k \in \mathbb{N}$, $k \geq 1$, if

(a1) $E_{\mathbf{a}}|_I, E^2_{\mathbf{a}}|_I, \cdots, E^{k-1}_{\mathbf{a}}|_I$ are continuous and the intervals $I, E_{\mathbf{a}}(I), \cdots, E^{k-1}_{\mathbf{a}}(I)$ are pairwise disjoint, and

(a2) $E^k_{\mathbf{a}}|_I$ is the identity map of I; in particular, every $x \in I$ is $E_{\mathbf{a}}$-periodic with (minimum) period k.

Notice that if x is an $E_{\mathbf{a}}$-periodic point of period ℓ, then x is of B-type, where B is the $E_{\mathbf{a}}$-itinerary of x associated to $[0, \ell]$.

The following lemma collects some useful facts about rational iets and since its proof is rather simple it will be omitted.

Lemma 4. *If $\mathbf{a} = (a_0, a_1, \cdots, a_n) \in \Lambda_n \cap \mathbb{Q}^{n+1}$, i.e., $E_{\mathbf{a}}$ is rational, then:*

(b1) *the $E_{\mathbf{a}}$-saturated set of $\{a_0, a_1, \cdots, a_{n-1}\}$, that is*
$$S_{\mathbf{a}} := \cup_{k\in\mathbb{Z}} E^k_{\mathbf{a}}(\{a_0, a_1, \cdots, a_{n-1}\}),$$
is an $E_{\mathbf{a}}$-invariant finite set;

(b2) *every connected component of $[0,1)\backslash S_{\mathbf{a}}$ is an $E_{\mathbf{a}}$-periodic interval; therefore, there exist positive integers $m_{\mathbf{a}}, M_{\mathbf{a}}$ such that every $x \in [0,1)$ is $E_{\mathbf{a}}$-periodic and its period belongs to $[m_{\mathbf{a}}, M_{\mathbf{a}}]$;*

(b3) *for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\lambda(N_\delta(S_{\mathbf{a}})) < \epsilon$, where $N_\delta(S_{\mathbf{a}}) := \{x \in [0,1) : \exists y \in \{1\} \cup S_{\mathbf{a}}$ with $|x-y| < \delta\}$; notice that if $0 < \delta$ is small enough, then $N_\delta(S_{\mathbf{a}})$ is $E_{\mathbf{a}}-$invariant.*

Let $\mathbf{a} = (a_0, a_1, \cdots, a_n) \in \Lambda_n \cap \mathbb{Q}^{n+1}$. A positive integer s is said to be a *separating integer for* $\mathbf{a}$, if $\forall i \in \{0, 1, \cdots, n-1\}$, the $E_{\mathbf{a}}-$itinerary of a_i, associated to $[1, s]$, is disjoint of $\{a_0, a_1, \cdots, a_{n-1}\}$. Let
$$0 < \delta \leq \frac{1}{16} \min\{|x-y| : x, y \in \{1\} \cup S_{\mathbf{a}}, x \neq y\};$$
then $\bar{\delta} > 0$ is a *stability constant for the triple* $(\mathbf{a}, \delta, s)$ if $\bar{\delta} < \delta$ and, for all $\mathbf{b} \in \Lambda_n$, with $|\mathbf{a} - \mathbf{b}| < \bar{\delta}$, the following are satisfied:

- let I be an arbitrary connected component of $[0,1) \setminus N_\delta(S_{\mathbf{a}})$ and let τ be its $E_{\mathbf{a}}$-period; then I is of B-type for $E_{\mathbf{b}}$, where B denotes the $E_{\mathbf{a}}-$itinerary of I associated to $[0, \tau]$;
- s is a separating integer for $\mathbf{b}$.

Another simple and important approximation properties are:

Lemma 5. *Let $s \in \mathbb{N}$. Any $\mathbf{a} \in \Lambda_n$ can be arbitrarily approximated by $\mathbf{b} \in \Lambda_n$ and $\mathbf{c} \in \Lambda_n \cap \mathbb{Q}^{n+1}$ such that $E_{\mathbf{b}}$ is minimal and every orbit of $E_{\mathbf{c}}$ is periodic having period greater than s, which is also a separating integer for $\mathbf{c}$.*

Proof. By Lemma 1(a) any $\mathbf{a} \in \Lambda_n$ can be arbitrarily approximated by a $\mathbf{b} \in \Lambda_n$ such that $E_{\mathbf{b}}$ is minimal. This $\mathbf{b} \in \Lambda_n$ can be arbitrarily approximated by a $\mathbf{c} \in \Lambda_n \cap \mathbb{Q}^{n+1}$. As $E_{\mathbf{c}}$ is rational, all its orbits are periodic; the requirement on the period of its orbits is fulfilled by selecting $m_{\mathbf{c}} > s$. □

Lemma 6. *Let $\mathbf{a} = (a_0, a_1, \cdots, a_n) \in \Lambda_n \cap \mathbb{Q}^{n+1}$ and let s be a separating integer for $\mathbf{a}$. Let $0 < \delta \leq (1/16) \min\{|x-y| : x, y \in \{1\} \cup S_{\mathbf{a}}, x \neq y\}$. Then, there exists a stability constant for the triple $(\mathbf{a}, \delta, s)$.*

Proof. As s is a separating integer for $\mathbf{a}$, it follows from the definition of iet that if $\mathbf{b} = (b_0, b_1, \cdots, b_n) \in \Lambda_n$ is close enough to $\mathbf{a}$, then $\forall (i,j) \in \{1, 2, \cdots, n\} \times \{1, 2, \cdots, s\}$, $E^j_{\mathbf{b}}(b_i)$ depends continuously on $\mathbf{b} \in \Lambda_n$. We conclude that if $\mathbf{b} \in \Lambda_n$ is close enough to $\mathbf{a}$, then s is a separating integer for $\mathbf{b}$.

Now, let $[c, d]$ be a closed interval which is a connected component of $[0,1) \setminus N_\delta(S_{\mathbf{a}})$, then the $E_{\mathbf{a}}-$orbits of c and d are periodic and they are at a δ distance of $\{1\} \cup S_{\mathbf{a}}$. Therefore, we may apply the same argument above to the endpoints of the (finitely many) closed intervals which are connected component of $[0,1) \setminus N_\delta(S_{\mathbf{a}})$ so to obtain that there exists a stability constant $\bar{\delta}$ for the triple $(\mathbf{a}, \delta, s)$. □

Definition 2. *Consider factors $B_1, B_2 \in W_n$. Then B_1 precedes B_2, denoted by $B_1 \prec B_2$, if $|B_1| < |B_2|$ and the first $|B_1|$ entries of B_2 coincides with B_1.*

Proposition 1 follows immediately from the next one, where the existence of the set $\mathcal{D}$ will be proven.

Propositon 2. *Let U be an open subset of $E(\pi)$. There exists a minimal iet $E_{\mathbf{a_0}} \in U$ such that for a.e. $p \in [0,1)$ there exists a sequence $(B_i^p)_{i=1}^{\infty}$ in W_n such that*

(c1) $B_1^p \prec B_2^p \prec \cdots \prec B_i^p \prec \cdots$

(c2) *for all $i \in \mathbb{N}$, p is of B_i^p-type for $E_{\mathbf{a_0}}$.*

Proof. Fix, once for all, $0 < \epsilon < 1/9$. Let $\mathbf{a_1} \in \Lambda_n \cap \mathbb{Q}^{n+1} \cap U$. Select $\delta_1 > 0$ such that $\{\mathbf{b} \in \Lambda_n : |\mathbf{b} - \mathbf{a_1}| < 2\delta_1\} \subset U$ and

(1.1) $\lambda(N_{\delta_1}(S_{\mathbf{a_1}})) < \frac{\epsilon}{2}$

(1.2) $\delta_1 \leq (1/16)\min\{|x-y| : x, y \in \{1\} \cup S_{\mathbf{a_1}}, x \neq y\}$.

Let s_1 be a separating integer for $\mathbf{a_1}$ and choose $\bar{\delta}_1 > 0$ so that

(1.3) $\bar{\delta}_1 > 0$ is a stability constant for the triple $(\mathbf{a_1}, \delta_1, s_1)$.

In this proof, it will be selected inductively (among other sequences) a sequence $(\mathbf{a}_i)_{i=1}^{\infty}$ in $\Lambda_n \cap \mathbb{Q}^{n+1}$, a sequence of separating integers $(s_i)_{i=1}^{\infty}$, and sequences of positive numbers $(\delta_i)_{i=1}^{\infty}$ and $(\bar{\delta}_i)_{i=1}^{\infty}$. The elements $\mathbf{a_1}$, s_1, δ_1 and $\bar{\delta}_1$ have already been selected.

The following notation, related to the sequence $(\mathbf{a}_i)_{i=1}^{\infty}$, will be used throughout this proof:

- if $p \in [0,1)$ and τ denotes its $E_{\mathbf{a_k}}$-period, then

$$B_k^p = \text{the } E_{\mathbf{a_k}}\text{–itinerary of } p \text{ associated to } [0,\tau].$$

Now,

(2.1) select $\mathbf{a_2} \in \Lambda_n \cap \mathbb{Q}^{n+1}$ so that $m_{\mathbf{a_2}} > M_{\mathbf{a_1}}$ (see item (b2) of Lemma 4), the separating integer s_2 for $\mathbf{a_2}$ is greater than s_1 (see Lemma 5) and

$$|\mathbf{a_2} - \mathbf{a_1}| < \frac{\bar{\delta}_1}{2};$$

(2.2) select $0 < \delta_2 < \bar{\delta}_1$ so that $\lambda(N_{\delta_2}(S_{\mathbf{a_2}})) < \frac{\epsilon}{4}$ and

$$\delta_2 \leq \frac{1}{16}\min\{|x-y| : x, y \in \{1\} \cup S_{\mathbf{a_2}}, x \neq y\};$$

(2.3) select a stability constant $0 < \bar{\delta}_2 < \delta_2$ for the triple $(\mathbf{a_2}, \delta_2, s_2)$.

Using the fact that $\bar{\delta}_1$ is a stability constant for $(\mathbf{a_1}, \delta_1, s_1)$, one obtains that if $(\mathbf{a_i})_{i=3}^{\infty}$ is an arbitrary sequence in $\Lambda_n \cap \mathbb{Q}^{n+1}$ such that, for all $i = 2, 3, \cdots$,

$$|\mathbf{a_{i+1}} - \mathbf{a_i}| < \frac{\bar{\delta}_2}{2^i}$$

then, $(\mathbf{a}_i)_{i=1}^{\infty}$ is not only a convergent sequence but also (as $|\mathbf{a_0} - \mathbf{a_1}| < \delta_1$) its limit $\mathbf{a_0}$ belongs to U. Moreover, by (b2) of Lemma 4 and (2.1)-(2.3), it follows that:

(2.4) if $p \in [0,1) \setminus \{N_{\delta_1}(S_{\mathbf{a_1}}) \cup N_{\delta_2}(S_{\mathbf{a_2}})\}$ then
- $B_1^p \prec B_2^p$;
- for all $i = 0, 1, 2, \cdots$, p is of B_1^p–type for $E_{\mathbf{a_i}}$;
- for all $i = 0, 2, 3, \cdots$, $(i \neq 1)$, p is of B_2^p–type for $E_{\mathbf{a_i}}$;
- s_2 is a separating integer for $E_{\mathbf{a_0}}$;

(2.5) the following are true
- $\lambda([0,1) \setminus \{N_{\delta_1}(S_{\mathbf{a_1}}) \cup N_{\delta_2}(S_{\mathbf{a_2}})\}) \geq 1 - \epsilon/2 - \epsilon/4$;
- $\lambda([0,1) \setminus N_{\delta_2}(S_{\mathbf{a_2}})) \geq 1 - \epsilon/4$.

Suppose inductively that $\mathbf{a_{k-1}} \in \Lambda_n \cap \mathbb{Q}^{n+1}$, $s_{k-1} \in \mathbb{N}$, $\delta_{k-1} > 0$ and $\bar{\delta}_{k-1} > 0$ have been selected. Now proceed to

(k.1) select $\mathbf{a_k} \in \Lambda_n \cap \mathbb{Q}^{n+1}$ so that $m_{\mathbf{a_k}} > M_{\mathbf{a_{k-1}}}$, the separating integer s_k for $\mathbf{a_k}$ is greater than s_{k-1} and

$$|\mathbf{a_k} - \mathbf{a_{k-1}}| < \frac{\bar{\delta}_{k-1}}{2};$$

(k.2) select $0 < \delta_k < \bar{\delta}_{k-1}$ so that

$$\lambda(N_{\delta_k}(S_{\mathbf{a_k}})) < \frac{\epsilon}{2^k}$$

and

$$\delta_k \leq \frac{1}{16}\min\{|x-y| : x, y \in \{1\} \cup S_{\mathbf{a_k}}, x \neq y\};$$

(k.3) select a stability constant $0 < \bar{\delta}_k < \delta_k$ for the triple $(\mathbf{a_k}, \delta_k, s_k)$.

If $(\mathbf{a_i})_{i=k+1}^{\infty}$ is an arbitrary sequence in $\Lambda_n \cap \mathbb{Q}^{n+1}$ such that, for all $i = k, k+1, \cdots$,

$$|\mathbf{a_{i+1}} - \mathbf{a_i}| < \frac{\bar{\delta}_k}{2^i}$$

then, by what was said above, $(\mathbf{a}_i)_{i=1}^{\infty}$ is not only a convergent sequence but also its limit $\mathbf{a_0}$ belongs to U. Also, by (b2) of Lemma 4 and (k.1)–(k.3), one obtains that:

(k.4) if $p \in [0,1) \setminus \cup_{i=1}^{k} N_{\delta_i}(S_{\mathbf{a_i}})$ then
- $B_1^p \prec B_2^p < \cdots \prec B_k^p$;
- for all $i = 0, 1, 2, \cdots$, p is of B_1^p–type for $E_{\mathbf{a_i}}$;
- for all $i = 0, 2, 3, \cdots$, p is of B_2^p–type for $E_{\mathbf{a_i}}$;

  $\vdots$
- for all $i = 0, k, k+1, \cdots$, p is of B_k^p–type for $E_{\mathbf{a_i}}$;
- s_k is a separating integer for $E_{\mathbf{a_0}}$.

(k.5) the following are true
- $\lambda([0,1) \setminus \cup_{i=1}^{k} N_{\delta_i}(S_{\mathbf{a_i}})) \geq 1 - \epsilon/2 - \epsilon/4 - \cdots - \epsilon/2^k$;
- $\lambda([0,1) \setminus \cup_{i=2}^{k} N_{\delta_i}(S_{\mathbf{a_i}})) \geq 1 - \epsilon/4 - \cdots - \epsilon/2^k$;
- $\vdots$
- $\lambda([0,1) \setminus N_{\delta_k}(S_{\mathbf{a_k}})) \geq 1 - \epsilon/2^k$.

In this way it is completed the selection of the sequences $(\mathbf{a}_i)_{i=1}^{\infty}$, $(s_i)_{i=1}^{\infty}$, $(\delta_i)_{i=1}^{\infty}$ and $(\bar{\delta}_i)_{i=1}^{\infty}$. Recall that $\mathbf{a_k} \to \mathbf{a_0} \in U$ and $s_i \to \infty$. By using properties $\{(k.4)\}_{k=1}^{\infty}$ – $\{(k.5)\}_{k=1}^{\infty}$, it follows that, for all $p \in [0,1) \setminus \cup_{i=1}^{\infty} N_{\delta_i}(S_{\mathbf{a_i}})$ and for all $i = 1, 2, \cdots$,

- p is of B_i^p-type for $E_{\mathbf{a_0}}$;
- $B_1^p \prec B_2^p \prec \cdots \prec B_k^p \prec \cdots$;
- $\lambda([0,1) \setminus \cup_{i=1}^{\infty} N_{\delta_i}(S_{\mathbf{a_i}})) \geq 1 - \epsilon$;
- if $\mathbf{a_0} = (\alpha_0, \alpha_1, \cdots, \alpha_n)$, the $E_{\mathbf{a_0}}$ – orbits through $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$, are pairwise disjoint.

In other words, this proposition is true for a set of points having Lebesgue measure at least $1 - \epsilon$. Notice that by Lemma 1(b) $E_{\mathbf{a_0}}$ is minimal.

Now fix an arbitrary positive integer s. By using properties $\{(k.4)\}_{k=s}^{\infty}$ – $\{(k.5)\}_{k=s}^{\infty}$, one gets that, for all $p \in [0,1) \setminus \cup_{i=s}^{\infty} N_{\delta_i}(S_{\mathbf{a_i}})$ and for all $i = s, s+1, \cdots$,

- p is of B_i^p-type for $E_{\mathbf{a_0}}$.
- $B_s^p \prec B_{s+1}^p \prec \cdots$;

- $\lambda([0,1) \setminus \cup_{i=s}^{\infty} N_{\delta_i}(S_{\mathbf{a}_i})) \geq 1 - \epsilon/2^{s-1}$.

Therefore, this proposition is true for a set of points having Lebesgue measure at least $1 - \epsilon/2^{s-1}$, where s is an arbitrary positive integer. That is, the result holds for a.e. $p \in [0,1]$. □

3. Absence of Absolutely Continuous Spectrum

The absence of absolutely continuous spectrum will be gotten as a combination of two known results, gathered in what follows in the form of lemmas, and an observation from Section 2. In summary, ergodicity, minimality, aperiodicity and finite valued are the key ingredients to exclude absolutely continuous spectrum. This section aims at concluding:

Propositon 3. *Let $\mathcal{D}$ be as in Proposition 1. If $E_{\mathbf{a}} \in \mathcal{D}$, then H_ω has no absolutely continuous spectrum for all $\omega \in \Omega_{\mathbf{a}}$.*

Lemma 7. [11] *If μ is an ergodic probability measure over Σ_n, then the set of potentials ω for which H_ω has no absolutely continuous spectrum has full measure μ, unless the support of μ is periodic.*

Lemma 8. [12] *The absolutely continuous spectrum of (1) is constant over minimal subsets of Σ_n.*

Now the proof of Proposition 3 is immediate. Let $\mathcal{D}$ be as in Proposition 1. For each $E_{\mathbf{a}} \in \mathcal{D}$ the set $\Omega_{\mathbf{a}}$ is aperiodic and carries an ergodic measure, so by Lemma 7 there exists $\omega \in \Omega_{\mathbf{a}}$ such that H_ω has empty absolutely continuous spectrum; since $\Omega_{\mathbf{a}}$ is minimal (Proposition 1 and Lemma 2), the conclusion of Proposition 3 follows straight by Lemma 8.

4. Proof of the Main Results

First it will be discussed how Corollary 1 follows from Theorem 1. If $\mathcal{U}$ is the unitary operator representation of the shift in $l^2(\mathbb{Z})$, i.e., $(\mathcal{U}\psi)_j = \psi_{j-1}$, then the operator (1) satisfies the covariance relation

$$H_{S(\omega)} = \mathcal{U} H_\omega \mathcal{U}^*,$$

which implies

Lemma 9. *The spectrum of (1) is constant over orbits in Σ_n.*

Let $\mathcal{D}$ be as in Theorem 1 and for $E_{\mathbf{a}} \in \mathcal{D}$ let $\Gamma_{\mathbf{a}}$ be as in the Corollary, i.e., the subset of $\Omega_{\mathbf{a}}$ for which the operator (1) has pure singular continuous spectrum. According to Theorem 1(iii), $H_{\phi(x)}$ has pure singular continuous spectrum for a.e. $x \in [0,1)$; since the Lebesgue measure is a convex sum of finitely many (probability) ergodic measures, it follows that with respect to at least one of them $H_{\phi(x)}$ has pure singular continuous spectrum a.e.; to the latter measure corresponds an ergodic probability measure $\nu_{\mathbf{a}}$ over $\Omega_{\mathbf{a}}$, with $\nu_{\mathbf{a}}(\Gamma_{\mathbf{a}}) = 1$, that works for the Corollary.

It remains to show that such set $\Gamma_{\mathbf{a}}$ of potentials is a dense G_δ. For this purpose, another auxiliary result will be used; it belongs to the set of results known as "Wonderland Theorem."

Lemma 10. [16] *Consider a complete metric space of bounded self-adjoint operators whose convergence in the metric implies strong operator convergence. Then, the subset of such operators with empty point spectrum is a G_δ.*

By identifying the metric space Σ_n with the set of operators $\{H_\omega : \omega \in \Sigma_n\}$ one sees that two operators are at a small distance only if the corresponding potentials have equal terms around the zero position. Then, from

$$||H_{\omega^{(k)}}\psi - H_\omega\psi||^2 = \sum_{j\in\mathbb{Z}} |\omega_j^{(k)} - \alpha_j|^2|\psi_j|^2,$$

if a sequence of potentials $\omega^{(k)} \to \omega$ it follows the strong operator convergence $H_{\omega^{(k)}} \to H_\omega$. Hence, Lemma 10 is applicable to closed subsets of the metric space $\{H_\omega : \omega \in \Sigma_n\}$, in particular to $\Omega_{\mathbf{a}}$.

By Proposition 3 the set $\Gamma_{\mathbf{a}}$ coincides with the set of operators $H_\omega, \omega \in \Omega_{\mathbf{a}}$, with empty point spectrum; so, by Lemma 10, $\Gamma_{\mathbf{a}}$ is a G_δ, and it is left to show that it is also dense. By Theorem 1(iii) there is $\phi(x) \in \Omega_{\mathbf{a}}$ such that $H_{\phi(x)}$ has no point spectrum (hence, $\Gamma_{\mathbf{a}} \neq \emptyset$), and by Lemma 9 the spectrum is invariant over $\mathcal{O}_{\mathbf{a}}(x)$, which is dense in $\Omega_{\mathbf{a}}$ by Theorem 1(i). Corollary 1 is demonstrated.

Proof. (of Theorem 1)

Let $\mathcal{D}$ be the dense subset in $E(\pi)$ constructed in the proof of Proposition 2; from such construction each element of $\mathcal{D}$ is minimal and aperiodic, concluding (i) in the theorem. The assertion (iii) follows directly from Propositions 1 and 3.

Since the convergence in Σ_n implies strong operator convergence (as discussed above), the minimality and Lemma 9 imply (ii), as it is well known that the spectrum (as a set) does not increase under strong limits (see Theorem VIII.24 in [15]) and so it is constant over minimal sets of Σ_n (this last abstract result seems to be originally appeared in [17]). □

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