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An Upper Bound for the Total Sum of the Baum-Bott Indexes of a Holomorphic Foliation and the Poincaré's Problem

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Abstract: An upper bound for the degree of an algebraic solution of a foliation in $\mathbb{CP}^2$ in terms of the degree of the foliation is given. This result follows from the existence of an upper bound for the total sum of the Baum-Bott indexes of a holomorphic foliation in a compact algebraic surface.

Abstract: Um limite superior para o grau de uma solução algébrica de um folheação em $\mathbb{CP}^2$ em termos do grau da folheação é dado. Este resultado segue da existência de um limite superior para a soma total dos indices de Baum-Bott de um folheação holomorfa em uma superficie algébrica compacta.

1. INTRODUCTION

In [P] Poincaré established that, a differential equation of the first order and of the first degree in $\mathbb{C}^2$ is algebraically integrable as soon an upper bound for the possible degrees of the algebraic solutions of the equation in terms of the degree of the differential equation is established. In general it is not possible to determine such upper bound as several simple examples show (see [CL]). However the problem can be solved if some restriction to the singularities of the differential equation are given. In [CL], for instance, a positive answer to this problem is given in the case that the singularities of the algebraic solution are ordinary nodes. Also, in [C], a positive answer is given in the case that the algebraic solution avoids the dicritical singularities of the differential equation. From then several authors have been studied this problem in a more general setting, [CCG], [Ba], [So].

The main purpose of this work is to show that an answer to the Poincaré's problem can be given as a consequence of a more general fact: the existence of an upper bound for the total sum of the Baum-Bott indexes of the singularities of a foliation.

More precisely, let $\{a\}^+$ be the maximum between 0 and the real number a and denote the intersection number between two divisors, say D and C, by DC and by D^2 the self-intersection number of D. We shall prove:

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Theorem 1.1. *Let $\mathcal{F}$ be a holomorphic foliation with isolated singularities in a smooth compact algebraic surface X and let E be an effective divisor such that $S := Supp\ E$ is an algebraic solution of $\mathcal{F}$. Assume that E has an irreducible component, say T with $T^2 \geq 0$ and such that $GSV(\mathcal{F}, T) \leq ET - T^2$. Then,*

$$BB(\mathcal{F}) \leq S^2 + 2GSV(\mathcal{F}, S) + (E-S)^2 + \sum_{i=1}^{n}(\ell_i - 1)^2\{-\tilde{S_i}^2\}^+$$

$$+\sum_{i=1}^{n} 2(\ell_i - 1)\{GSV(\mathcal{F}, S_i) + S_i^2 - ES_i\}^+ + \Delta_\pi(\mathcal{F}, S),$$

where $E = \ell_1 S_1 + \cdots + \ell_n S_n$, ℓ_i positive integers and $\{S_i\}_{i=1,\ldots,n}$ irreducible. π is a map adapted to the pair $(\mathcal{F}, S)$ (definition 3.3) and $\tilde{S}_i$ is the strict transform of S_i by π.

Here $BB(\mathcal{F})$ ($GSV(\mathcal{F}, S)$, respectively) is the sum of the Baum-Bott's indexes (Gomez Mont- Seade-Verjovsky's indexes, resp.) of all singularities of $\mathcal{F}$ in X (in S, resp.). The symbol $\Delta_\pi(\mathcal{F}, S)$ stands for a positive integer which depends on the desingularization of $\mathcal{F}$ along S (see definition 3.2).

Any differential equation in $\mathbb{C}^2$ induces a foliation in $\mathbb{CP}^2$ and the problem raised by Poincaré in this context is to find an upper bound for the possible degrees of the algebraic solutions of the foliation in terms of the degree of the foliation. From the above Theorem we obtain,

Theorem 1.2. *Let $\mathcal{F}$ be a holomorphic foliation in $\mathbb{CP}^2$ of degree n and let S be an algebraic solution of $\mathcal{F}$ of degree k. Then, $k \leq n + 2 + \sqrt{\Delta_\pi(\mathcal{F}, S)}$.*

In particular we recover the upper bounds obtained in [CL] and [C] (Corollaries 2.4 and 2.5).

This work is organized in the following form. In the next section we will recall the definition of Baum-Bott's indexes as well as Gomez Mont-Seade-Verjovsky's indexes and we will see how to obtain Theorem 1.2 from Theorem 1.1. In the last section we will prove Theorem 1.1.

2. The Poincaré's problem

This section is devoted to prove Theorem 1.2. Let us recall the definition of the Baum-Bott's indexes and Gomez Mont- Seade-Verjovsky's indexes.

2.1. Baum-Bott's index. Let X be a smooth compact algebraic surface and $\mathcal{F}$ a holomorphic foliation with isolated singularities in X. Let $N_\mathcal{F}$ be the co-normal bundle of $\mathcal{F}$ which is defined in the following way: let $\mathcal{U} = \{\mathcal{U}_i\}_{i\in I}$ be an open covering of X, if $\mathcal{F}$ is defined in each $\mathcal{U}_i$ by the holomorphic 1-form ω_i, there must exists $g_{ij} \in \mathcal{O}^*(\mathcal{U}_i \cap \mathcal{U}_j)$ such that $w_i = g_{ij} w_j$. Then $N_\mathcal{F}$ is the line bundle over X defined by the cocicle $\{g_{ij}\}_{ij} \in H^1(X, \mathcal{O}^*)$.

Denote Sing($\mathcal{F}$) the set of singular points of $\mathcal{F}$. For each $p \in$ Sing($\mathcal{F}$) denote by $BB_p(\mathcal{F})$ the Baum-Bott index of $\mathcal{F}$ at p which is defined as

$$BB_p(\mathcal{F}) := (\frac{1}{2\pi i})^2 \int_\Gamma \frac{(TrJ(x,y))^2}{P(x,y)Q(x,y)} dxdy,$$

where P and Q are holomorphic functions defined in a neighborhood of p with $gcd(P,Q) = 1$ such that $\mathcal{F}$ is defined by the holomorphic 1-form

$$\omega = P(x,y)dy - Q(x,y)dx,$$

and where $J(x,y)$ is the Jacobian matrix of (P,Q), $\Gamma = \{(x,y) \mid |P(x,y)| = |Q(x,y)| = \epsilon\}$ for sufficiently small number $\epsilon > 0$ and Γ is oriented in such a way that the form $d(argP) \wedge d(argQ)$ is positive. Baum-Bott formula asserts,

(BB): $\quad BB(\mathcal{F}) := \sum_{p \in \mathrm{Sing}(\mathcal{F})} BB_p(\mathcal{F}) = c_1^2(N_\mathcal{F}) \in H^4(X, \mathbb{Z}) \cong \mathbb{Z},$

where $c_1^2(N_\mathcal{F})$ denotes the first Chern's number of the line bundle $N_\mathcal{F}$ (cf. [BB]).

2.2. Gomez Mont- Seade-Verjovsky's index. Let X, $\mathcal{F}$ and Sing($\mathcal{F}$) be as in 2.1. Let S be an algebraic solution of $\mathcal{F}$ (that is, S is a reduced divisor in X such that $S \setminus$ Sing($\mathcal{F}$) is a leaf of $\mathcal{F}$). Given a singularity $p \in$ Sing($\mathcal{F}$) $\cap S$ in [GSV] it is introduced an index which is a kind of Poincaré-Hopf index for the restriction to S of the vector field which defines $\mathcal{F}$ in a neighborhood of p. This index has the particular importance that relate the restriction of the line bundle $N_\mathcal{F}$ to S to the self-intersection number of S. In fact, if $GSV_p(\mathcal{F}, S) \in \mathbb{Z}$ is the above index in $p \in S$, it was proved in [Br], that

(GSV): $\quad GSV(\mathcal{F}, S) := \sum_{p \in S \cap \mathrm{Sing}(\mathcal{F})} GSV_p(\mathcal{F}, S) = c_1(N_\mathcal{F})S - S^2.$

Remark 2.3. In particular, in the case that $\mathcal{F}$ is an holomorphic foliation in $\mathbb{CP}^2$ of degree n it is well known that $N_\mathcal{F} = \mathcal{O}(n+2)$ therefore, if T is a curve with degree t, we get $GSV(\mathcal{F}, T) = (n+2-t)t$.

We are now able to prove Theorem 1.2 using Theorem 1.1.

Proof. (of Theorem 1.2) Take $E := S$ and assume that $n + 2 \leq k$. Then for any irreducible curve T we have $GSV(\mathcal{F}, T) \leq ET - T^2$ in view of remark 2.3. Therefore, from Theorem 1.1, the fact that $BB(\mathcal{F}) = (n+2)^2$ and remark 2.3 we conclude that $(k-n-2)^2 \leq \Delta_\pi(\mathcal{F}, S)$. Hence the theorem follows.

□

Theorem 1.2 must be compared with [CC, Theorem 1] (see also [CCG]) where, from solving a problem of imposing singularities to a plane curve, it is obtained a bound for the degree of the curve. We observe that the terms that appear in the definition of Δ_π (definition 3.2) are the same that the coefficients of the divisor R in [CCG]. Nevertheless we stress that in the above result we do not need to establish the condition of virtually passage of some effective divisor A by the cluster of points of the resolution of the curve as it is needed in [CC] or [CCG] (see [CCG] for details). From the above theorem we recover the upper bounds obtained in [CL] and [C].

Corollary 2.4 (Cerveau-Lins Neto,[CL]). *Let $\mathcal{F}$ and S be as in Theorem 1.2. If S has only normal crossing singularities then, $k \leq n+2$.*

Recall that a singularity, q, is said non-dicritical if $\mathcal{F}$ has finite many separatrix through q.

Corollary 2.5 (Carnicer,[C]). *Let $\mathcal{F}$ and S be as in Theorem 1.2. If $\mathcal{F}$ has only non-dicritical singularities in S then, $k \leq n+2$.*

Proof. By [CLS, Theorem 3] we have the inequality $\mu_q + \varepsilon_q \leq v_q + 1$ at any infinitely near singularity of S, where μ_q, ε_q and v_q are as in 3.1. Therefore $\Delta_\pi(\mathcal{F}, S)$ is zero.

□

Remark 2.6. It is clear that all the above results hold (and with the same proof) if we replace $\mathbb{CP}^2$ with any compact algebraic surface X with Picard group $\mathbb{Z}$.

3. An upper bound for the total sum of the Baum-Bott indexes of a holomorphic foliation

As it was showed in the former section the result about the Poincaré's problem stated at the introduction is consequence of the existence of an upper bound for the total sum of the Baum-Bott indexes of $\mathcal{F}$. The aim of this section is to prove this upper bound.

Notation. We shall use the standard notation in algebraic geometry. In particular, if D is a divisor in the smooth compact algebraic surface X the symbol $\mathcal{O}_X(D)$ will denote the corresponding invertible sheaf induced by D in X and $\mathcal{O}_X(tD)$, or $\mathcal{O}(tD)$ if X is understood, will denote $\mathcal{O}_X(D)^{\otimes t}$. The intersection number between two divisors, say D and C, will be denoted by DC in particular D^2 will denote the self-intersection number of D. Let $\mathcal{L}$ be a holomorphic line bundle in X if C is an effective divisor in X, then $\mathcal{O}_C(\mathcal{L})$ will denote the restriction $\mathcal{L} \otimes \mathcal{O}_C$. Also $h^0(\mathcal{L}) := dim_{\mathbb{C}} H^0(X, \mathcal{L})$.

3.1. Infinitely near singularities. Let $\pi : \tilde{X} = X_n \xrightarrow{\pi_n} X_{n-1} \to \cdots \xrightarrow{\pi_1} X_0 = X$ be a sequence of blow ups at points such that the first blow up is made at p and the blow up π_{k+1} is made at a point that lies in the exceptional curve of the first kind, say D_k, introduced in X_k by π_k. Let S ($\mathcal{F}$, respectively) be an algebraic curve (a foliation, resp.) in X, for each k and each $q \in X_k$ the symbol $\mu_{q,k}$ ($v_{q,k}$, resp.) or μ_q (v_q, resp.) if X_k is understood will denote the algebraic multiplicity in q of the strict transform of S ($\mathcal{F}$, resp.), say S_k, ($\mathcal{F}_k$, resp.) by $\pi_1 \circ \cdots \circ \pi_k$.

For each $q \in X_k$ set ε_q as the number of irreducible components of the exceptional divisor of $\pi_1 \circ \cdots \circ \pi_k$ which contains q and are solutions of $\mathcal{F}_k$.

Definition 3.2. *With the notations of 3.1. We call the irregularity of $\mathcal{F}$ along S to the positive integer $\Delta_\pi(\mathcal{F}, S) := \sum(\mu_q + \varepsilon_q - v_q - 1)^2$, where the sum runs over all infinitely near singularities of S, say q, with $\mu_q + \varepsilon_q \geq v_q + 2$.*

Definition 3.3. *Let $\pi : \tilde{X} \to X$ be a sequence of blow ups with center at infinitely near singularities of S such that the strict transform of S by π, say $\tilde{S}$, has the following properties:*
1_ it is a disjoint union of smooth Riemann surfaces,
2_ all those irreducible components of the exceptional divisor of π which are algebraic solutions of the strict transform of $\mathcal{F}$ by π meets $\tilde{S}$ transversely.
A map π as above will be called a map adapted to the pair $(\mathcal{F}, S)$.

3.4. Let $\{S_i\}_{i=1,\ldots,n}$ be an ordered set of n irreducible algebraic solutions of $\mathcal{F}$. For each n-tuple $\overline{\ell} := (\ell_1, ..., \ell_n)$ of ordered positive integers we define

$$vol_{\overline{\ell}}(\mathcal{F}, S_1 + \cdots + S_n) := \limsup_{t \to +\infty} \frac{h^0([N_{\mathcal{F}}^* \otimes \mathcal{O}(\ell_1 S_1 + \cdots + \ell_n S_n)]^{\otimes t})}{t^2}.$$

Also denote S the curve whose irreducible components are the curves $S_1, \ldots, S_n$.

Proposition 3.5. *Let things be as in 3.4. If $\tilde{\pi} : \tilde{X} \to X$ is a map adapted to the pair $(\mathcal{F}, S)$ then,*

$$vol_{\overline{\ell}}(\mathcal{F}, S_1 + \cdots + S_n) \leq \frac{1}{2}\Delta_{\tilde{\pi}}(\mathcal{F}, S) + \frac{1}{2}\sum_{i=1}^{n}(\ell_i - 1)^2\{-\tilde{S}_i^2\}^+$$
$$+\sum_{i=1}^{n}(\ell_i - 1)\{-GSV(\mathcal{F}, S_i) + (\ell_i - 1)S_i^2 + \sum_{j \neq i}\ell_j S_j S_i\}^+,$$

where $\tilde{S}_i$ is the strict transform of S_i by $\tilde{\pi}$.

We postpone the proof of this result by a moment and we prove Theorem 1.1 using the above proposition.

Proof. (of Theorem 1.1) Let $E = \ell_1 S_1 + \cdots + \ell_n S_n$ with ℓ_i positive integers and S_i irreducible. Since $GSV(\mathcal{F}, T) \leq ET - T^2$ we have $c_1(N_{\mathcal{F}} \otimes \mathcal{O}(-E))T \leq 0$ in view of formula (GSV) in 2.2. On the other hand, since T is a numerically effective divisor the dimension $h^0(K_X \otimes (N_{\mathcal{F}} \otimes \mathcal{O}(-E))^{\otimes t})$ cannot grows like t^2, for otherwise we could write $m(N_{\mathcal{F}} \otimes \mathcal{O}(-E)) = H + L$ for some ample line bundle H, some effective divisor L and some $m \in \mathbb{N}$ (cf. [SS, pg. 143]) hence $c_1(N_{\mathcal{F}} \otimes \mathcal{O}(-E))T > 0$, contradiction. Therefore, by Riemann-Roch formula and Serre duality ([Har])

(1) $\quad c_1^2(N_{\mathcal{F}}^* \otimes \mathcal{O}(E)) \leq 2vol_{\overline{\ell}}(\mathcal{F}, S_1 + \cdots + S_n)$ where $\overline{\ell} := (\ell_1, \ldots, \ell_n)$.

On the other hand a simple algebraic manipulation using Baum-Bott's formula (BB) and formula (GSV) shows that

$$(2)\qquad c_1^2(N^*_{\mathcal{F}}\otimes \mathcal{O}(E)) = BB(\mathcal{F}) + \sum_{i=1}^{n} 2(\ell_i-1)c_1(N^*_{\mathcal{F}}\otimes \mathcal{O}(E))S_i +$$

$$+2\sum_{i<j}(\ell_i+\ell_j-\ell_i\ell_j)S_iS_j - \sum_{i=1}^{n} S_i^2 + 2GSV(\mathcal{F},S_i) + (\ell_i-1)^2S_i^2$$

Now, by proposition 3.5, inequality (1), equality (2) and the fact that

$$\sum_{i=1}^{n} GSV(\mathcal{F},S_i) = GSV(\mathcal{F},S) + 2\sum_{i<j} S_iS_j,$$

where S is as in 3.4, the Theorem follows.

□

The rest of this section will be devoted to prove proposition 3.5. The proof will follow as consequence of some technical results and Bogomolov-Sommese's vanishing theorem ([EV, Corollary 6.9 pag.58])

3.6. Let $\pi : \tilde{X} \to X$ be the blow up of X with center at p and let D be the exceptional curve of the first kind introduced by π. If $\tilde{\mathcal{F}}$ stands for the strict transform of $\mathcal{F}$ by π then,

$$N^*_{\tilde{\mathcal{F}}} = \pi^*(N^*_{\mathcal{F}}) \otimes \mathcal{O}_{\tilde{X}}((\upsilon_p+\delta_p)D),$$

where υ_p is as in 3.1 and δ_p is defined as 0 if D is an algebraic solution of $\tilde{\mathcal{F}}$ and 1 otherwise (see [Br, pag. 576]).

For the next result remind that the volume (also called degree) of a line bundle $\mathcal{L}$ is define by

$$vol(\mathcal{L}) := \limsup_{t\to+\infty} \frac{h^0(\mathcal{L}^{\otimes t})}{t^2}.$$

Lemma 3.7. *Let $vol(\mathcal{L})$ be as above, C an irreducible and reduced divisor in X and r an integer. If $\nu : \tilde{X} \to X$ is a sequence of blow ups such that the strict transform of C by ν, say $\tilde{C}$, is smooth then,*

$$vol(\mathcal{L}\otimes\mathcal{O}(rC)) - vol(\mathcal{L}\otimes\mathcal{O}(C)) \leq$$

$$\sum_{k=2}^{r} \limsup_{t\to+\infty}\frac{1}{t^2}\sum_{j=t(k-1)+1}^{tk} \{t(\mathcal{L}.C+rC^2)+(j-tr)\tilde{C}^2\}^+ \leq$$

$$\{(r-1)(\mathcal{L}.C+rC^2)\}^+ + \frac{(r-1)^2}{2}\{-\tilde{C}^2\}^+,$$

where $\{a\}^+$ is the maximum between 0 and the real number a.

Proof. We can assume that $r \geq 2$ since otherwise the inequalities are obvious. From the long exact sequences of cohomology induced by the short exact sequences of sheafs $0 \to \mathcal{L} \otimes \mathcal{O}_X(-(k+1)C) \to \mathcal{L} \otimes \mathcal{O}_X(-kC) \to \mathcal{L} \otimes \mathcal{O}_C(-kC) \to 0$, $k \in \mathbb{N}$ and induction we get

$$h^0((\mathcal{L} \otimes \mathcal{O}(rC))^{\otimes t}) - h^0((\mathcal{L} \otimes \mathcal{O}(C))^{\otimes t}) \leq$$

$$\sum_{k=2}^{r} \sum_{\jmath=t(k-1)+1}^{tk} h^0((\mathcal{L} \otimes \mathcal{O}_C(rC))^{\otimes t} \otimes \mathcal{O}_C((\jmath - tr)C)) \quad \text{for every} \quad t > 0. \tag{3}$$

Now, for each $\jmath = t(k-1)+1, \ldots, tk$, consider the line bundle over $\tilde{C}$ defined by $\mathcal{L}_\jmath := \mathcal{O}_{\tilde{C}}(\nu^*(\mathcal{L} \otimes \mathcal{O}(rC)))^{\otimes t} \otimes \mathcal{O}_{\tilde{C}}((\jmath - tr)\tilde{C})$ and the line bundle over X defined by $L_\jmath := (\mathcal{L} \otimes \mathcal{O}(rC))^{\otimes t} \otimes \mathcal{O}((\jmath - tr)C)$ then,

$$h^0(\mathcal{O}_{\tilde{C}}(\nu^*(L_\jmath) \otimes \mathcal{O}_{\tilde{C}}(E))) = h^0(\mathcal{L}_\jmath), \tag{4}$$

where E is an effective divisor in $\tilde{X}$ which is contracted to points by ν. In fact $E := \mathcal{O}((tr - \jmath)D)$ where D is the exceptional divisor of ν. On the other hand,

$$h^0(\mathcal{L}_\jmath) \leq 1 + \{deg(\mathcal{L}_\jmath)\}^+. \tag{5}$$

Meanwhile

$$h^0(\mathcal{O}_C(L_\jmath)) \leq h^0(\mathcal{O}_{\tilde{C}}(\nu^*(L_\jmath))). \tag{6}$$

Therefore, since $deg(\mathcal{L}_\jmath) = t(\mathcal{L}.C + rC^2) + (\jmath - tr)\tilde{C}^2$, the first inequality stated in the lemma follows by (3), (4), (5) and (6). The second inequality follows from the first one by observing that $\{deg(\mathcal{L}_\jmath)\}^+ \leq t\{\mathcal{L}.C + rC^2\}^+ + (tr - \jmath)\{-\tilde{C}^2\}^+$. □

Proposition 3.8. *Let things be as in 3.4 and $\tilde{\pi} : \tilde{X} \to X$ be a sequence of blow ups with the property that the strict transform of each S_i by $\tilde{\pi}$, say $\tilde{S}_i$, is smooth. Let $\tilde{\pi} : \tilde{X} \xrightarrow{\nu} \hat{X} \xrightarrow{\hat{\pi}} X$ be any decomposition of $\tilde{\pi}$, where ν and $\hat{\pi}$ are sequences of blow ups (possibly $\tilde{\pi} = \hat{\pi}$). Let $\hat{S}_i$ ($\hat{\mathcal{F}}$, resp.) be the strict transform of S_i (of $\mathcal{F}$, resp.) by $\hat{\pi}$. If D is the reduced divisor in $\hat{X}$ whose irreducible components are those irreducible components of the exceptional divisor of $\hat{\pi}$ which are algebraic solutions of $\hat{\mathcal{F}}$, then*

$$vol_{\bar{\ell}}(\mathcal{F}, S_1 + \cdots + S_n) \leq vol_{\bar{1}}(\hat{\mathcal{F}}, \hat{S}_1 + \cdots + \hat{S}_n + D) + \frac{1}{2}\sum_{i=1}^{n}(\ell_i - 1)^2\{-\tilde{S}_i^2\}^+$$

$$+\sum_{i=1}^{n}(\ell_i - 1)\{-GSV(\mathcal{F}, S_i) + (\ell_i - 1)S_i^2 + \sum_{j \neq i} \ell_j S_j S_i\}^+ + \frac{1}{2}\Delta_{\tilde{\pi}}(\mathcal{F}, S),$$

where $\bar{1}$ is the tuple with all entries equal to one.

Proof. We will prove this inequality by induction in the number of blow ups made to obtain $\hat{\pi}$. In order to simplify the notation, without any lost of generality, we will assume that $\mathcal{F}$ has only one singularity in S, say p. Now, we begin the induction process by assuming that $\hat{\pi}$ is a single blow up with center $p \in S$. Let D be the exceptional divisor of $\hat{\pi}$ then, using the identity in 3.6 and the projection formula (see [Har])

$$h^0([N^*_{\mathcal{F}} \otimes \mathcal{O}(\ell_1 S_1 + \cdots + \ell_n S_n)]^{\otimes t}) = h^0(\hat{\pi}^*([N^*_{\mathcal{F}} \otimes \mathcal{O}(\ell_1 S_1 + \cdots + \ell_n S_n)]^{\otimes t}))$$

$$= h^0([N^*_{\hat{\mathcal{F}}} \otimes \mathcal{O}(\ell_1 \hat{S}_1 + \cdots + \ell_n \hat{S}_n) \otimes \mathcal{O}((\overline{\ell}\mu_p - \upsilon_p - \delta_p)D]^{\otimes t}), \tag{7}$$

where we have used the notations $\overline{\ell}\mu_p := \ell_1\mu_p^1 + \cdots + \ell_n\mu_p^n$ and $\mu_p := \mu_p^1 + \cdots + \mu_p^n$ with μ_p^i the algebraic multiplicity of S_i in p.

In order to prove the proposition we must get rid of the coordinates greater than 1 in the n-tuple $\overline{\ell}$. To this end let us set $I = \{j \mid \ell_j \geq 2\}$ and let $\emptyset = I_0 \subset I_1 \subset \cdots \subset I_m = I$ be any sequence of subsets with the property that the number of elements of I_i is i. Without any loss of generality, by renumbering the indexes if necessary, we can (and shall) assume that $I_i = \{1, \ldots, i\}$. Now, for each $i = 1, \ldots, m$ we apply the second inequality in lemma 3.7 to the line bundle $\mathcal{L} := N^*_{\hat{\mathcal{F}}} \otimes \mathcal{O}(\ell_1\hat{S}_1 + \cdots + \ell_{i-1}\hat{S}_{i-1} + \hat{S}_{i+1} + \cdots + \hat{S}_n) \otimes \mathcal{O}((\overline{\ell}\mu_p - \upsilon_p - \delta_p)D)$, with $C := \hat{S}_i$, $\tilde{C} := \tilde{S}_i$ and $r := \ell_i$ and use formula (GSV) in 2.2, the identity in 3.6 and (7) to obtain

$$vol_{\overline{\ell}}(\mathcal{F}, S_1 + \cdots + S_n) \leq vol_{\overline{u}_0}(\hat{\mathcal{F}}, \hat{S}_1 + \cdots + \hat{S}_n + D) + \frac{1}{2}\sum_{i=1}^{n}(\ell_i - 1)^2\{-\tilde{S}_i^2\}^+$$

$$+ \sum_{i=1}^{n}(\ell_i - 1)\{-GSV(\mathcal{F}, S_i) + (\ell_i - 1)S_i^2 + \sum_{j \neq i}\ell_j S_j S_i\}^+, \tag{8}$$

where $\overline{u}_0$ is the tuple with all entries equal to one except the last one which is equal to $\overline{\ell}\mu_p - \upsilon_p - \delta_p$.

In order to compute $vol_{\overline{u}_0}(\hat{\mathcal{F}}, \hat{S}_1 + \cdots + \hat{S}_n + D)$ we must get rid of the coefficient $\overline{\ell}\mu_p - \upsilon_p - \delta_p$ in the $(n+1)$-tuple $\overline{u}_0$. To this end we use the first inequality stated in lemma 3.7 with $\mathcal{L} := N^*_{\hat{\mathcal{F}}} \otimes \mathcal{O}(\hat{S}_1 + \cdots + \hat{S}_n) \otimes \mathcal{O}(-\delta_p D)$, $C := D$ and $r := \overline{\ell}\mu_p - \upsilon_p$. Hence

$$vol_{\overline{u}_0}(\hat{\mathcal{F}}, \hat{S}_1 + \cdots + \hat{S}_n + D) \leq vol_{\overline{u}_1}(\hat{\mathcal{F}}, \hat{S}_1 + \cdots + \hat{S}_n + D) + \tag{9}$$

$$\sum_{k=2}^{\overline{\ell}\mu_p - \upsilon_p} \limsup_{t \to +\infty} \frac{1}{t^2} \sum_{\jmath = t(k-1)+1}^{tk} \{t\mathcal{L}.D - \jmath\}^+,$$

where $\overline{u}_1$ is the tuple with all entries equal to one except the last one which is equal to $1 - \delta_p$.

Let $\jmath \geq t(k-1)+1$ then, by the identity in 3.6, $t\mathcal{L}.D - \jmath < 0$ iff $k \geq \mu_p - \upsilon_p + 1$. Therefore the sum that appear in the right hand side of the inequality (9) is equal to $\frac{1}{2}(\mu_p - \upsilon_p - 1)^2$ if $\mu_p \geq \upsilon_p + 2$ and 0 otherwise. Thus, from (8) and (9) we get the proposition in the case that $\hat{\pi}$ is a single blow up.

To complete the induction process let us assume that $\hat{\pi}$ is a sequence of s blow ups. Let us decompose $\hat{\pi}$ as $\hat{X} \xrightarrow{\pi} Y \xrightarrow{\sigma} X$, where σ is a sequence of $s-1$ blow ups and π is a single blow up at the point $q \in Y$. Let D be the exceptional curve of the first kind introduced in $\hat{X}$ by π. Let us denote S_Y ($\mathcal{F}_Y$, resp.) the strict transform of S ($\mathcal{F}$, resp.) by σ and D_Y the algebraic curve in Y whose irreducible components are all those irreducible components of the exceptional divisor of σ

which are algebraic solution of $\mathcal{F}_Y$. Now, by inductive hypothesis applied to σ,

$$vol_{\overline{\ell}}(\mathcal{F}, S_1+\cdots+S_n) \leq vol_{\overline{1}}(\mathcal{F}_Y, S_Y + D_Y) + \frac{1}{2}\sum_{i=1}^{n}(\ell_i-1)^2\{-\tilde{S}_i^2\}^+$$

$$+\sum_{i=1}^{n}(\ell_i-1)\{-GSV(\mathcal{F},S_i)+(\ell_i-1)S_i^2+\sum_{j\neq i}\ell_j S_j S_i\}^+ + \frac{1}{2}\Delta_\sigma(\mathcal{F},S).$$

From the above inequality, by inductive hypothesis applied to π, the curve $S_Y + D_Y$ and the foliation $\mathcal{F}_Y$, we conclude

$$vol_{\overline{\ell}}(\mathcal{F}, S_1+\cdots+S_n) \leq vol_{\overline{u}_1}(\hat{\mathcal{F}}, \hat{S}_1+\cdots+\hat{S}_n+\hat{D}+D) + \frac{1}{2}\sum_{i=1}^{n}(\ell_i-1)^2\{-\tilde{S}_i^2\}^+$$

$$+\sum_{i=1}^{n}(\ell_i-1)\{-GSV(\mathcal{F},S_i)+(\ell_i-1)S_i^2+\sum_{j\neq i}\ell_j S_j S_i\}^+ + \frac{1}{2}\Delta_{\tilde{\pi}}(\mathcal{F},S).$$

where $\overline{u}_1$ is the tuple with all entries equal to one except the last one which is equal to $1-\delta_q$ and where $\hat{D}$ is the strict transform of D_Y by π. Thus the induction process is completed and the proposition follows.

□

Proof. (of proposition 3.5) Take $\tilde{\pi} = \hat{\pi}$ in the former proposition and observe that $vol_{\overline{1}}(\tilde{\mathcal{F}}, \tilde{S}+D) = 0$ where $\tilde{S}$ is the strict transform of S by $\tilde{\pi}$. In fact, since $\tilde{S}+D$ is an algebraic solution of $\tilde{\mathcal{F}}$ the sheaf $N^*_{\tilde{\mathcal{F}}} \otimes \mathcal{O}(\tilde{S}+D)$ is a subsheaf of $\Omega^1(\ln(\tilde{S}+D))$. Being $\tilde{S}+D$ a normal crossing divisor [EV, Corollary 6.9] applies to get $vol_{\overline{1}}(\tilde{\mathcal{F}}, \tilde{S}+D) = 0$. Thus, the proposition follows at once by the former proposition.

□

Remark 3.9. It is possible to obtain a similar inequality in Theorem 1.1 for any sequence of blow ups $\pi : \tilde{X} \to X$ as soon as both: 1_ in definition 3.3 holds and $\tilde{S}+D$ is a normal crossing divisor, where D is the exceptional divisor of π and $\tilde{S}$ the strict transform of S. In fact we can first decompose $\pi : \tilde{X} \xrightarrow{\nu} \hat{X} \xrightarrow{\hat{\pi}} X$, where $\hat{\pi}$ is adapted to $(\mathcal{F}, S)$ and ν is a sequence of blow ups at infinitely near points q of S with $\mu_q + \varepsilon_q \leq v_q + 1$ and then use proposition 3.8. However the upper bound in this way obtained could be greater than the one obtained using an adapted map since the self-intersection number of the strict transform of S could be negative.

References

[BB] Baum, P. and Bott, R. *On the zeros of meromorphic vector-fields* . 1970 Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham) pp. 29–47 Springer, New York

[Ba] Ballico, E. *Meromorphic singular foliations on complex projective surfaces*, Ann. Global Anal. Geom., 14, 3, 1996, 257–261.

[Br] Brunella, M. *Feuilletages holomorphes sur les surfaces complexes compactes.* Ann. Sci. École Norm. Sup. (4) 30 (1997), no. 5, 569–594.

[CLS] Camacho, C. Lins Neto and Sad, P. *Topological Invariants and Equidesingularization for Holomorphic Vector Fields*, J. Diff. Geometry, 20, 1984, 143-174.

[C] Carnicer, Manuel M. *The Poincaré problem in the nondicritical case.* Ann. of Math. (2) 140 (1994), no. 2, 289–294.

[CC] Campillo A. and Carnicer, M. *Proximity inequalities and bounds for the degree of invariant curves by foliations of* $\mathbf{P}^2_C$., Trans. Amer. Math. Soc., 349, 6, 1997, 2211-2228.

[CCG] Campillo, A., Carnicer, M. and García de la Fuente,J. *Invariant curves by vector fields on algebraic varieties.* J. London Math. Soc. (2), 62,1, 2000,56-70.

[CL] Cerveau, D and Lins Neto, A. *Holomorphic foliations in* $C\mathrm{P}(2)$ *having an invariant algebraic curve.* Ann. Inst. Fourier (Grenoble) 41 (1991), no. 4, 883–903.

[EV] Esnault, H. and Viehweg, E. *Lectures on vanishing theorems.* DMV Seminar, 20. Birkhäuser Verlag, Basel, 1992

[GSV] Gómez-Mont, X. and Seade, J. and Verjovsky, A. *The index of a holomorphic flow with an isolated singularity* . Math. Ann. 291 (1991), no. 4, 737–751.

[Har] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, 52. Springer Verlag, 1977.

[P] H. Poincare, Rend. Circ. Mat. Palermo 11 (1897), 193–239

[SS] B. Shiffman and A. J. Sommese, *Vanishing theorems on complex manifolds*, Progress in Mathematics, 56. Birkhäuser,1985.

[So] M. Soares, *The Poincaré problem for hypersurfaces invariant by one-dimensional foliations*, Invent. Math.,128, 3, 1997, 495–500.

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