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Polar Multiplicities, Equisingularity and Euler Obstruction of Map Germs from C^n to C^n

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Polar multiplicities, equisingularity and Euler obstruction of map germs from $\mathbb{C}^n$ to $\mathbb{C}^n$

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Resumo

Neste trabalho é estudada a questão de se determinar o menor número de invariantes suficientes para a Whitney equisingularidade de uma deformação a um parâmetro de um germe $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ de corank um e finitamente determinado. De acordo com um resultado de Gaffney, estes são os invariantes 0-estáveis e as multiplicidades polares que aparecem em uma deformação estável do germe f. Inicialmente são determinados os tipos estáveis na fonte e na meta e são determinadas as relações entre os invariantes. Com isto é possível diminuir o número de invariantes que garantem a Whitney equisingularidade. Além disto é determinada um fórmula algébrica para a obstrução local de Euler para os tipos estáveis das multiplicidades polares. Como consequência é mostrado que a obstrução de Euler é um invariante para a Whitney equisingularidade.

Abstract

We study how to determine the minimal number of invariants that garantee the Whitney equisingularidade of a one parameter deformation of corank one finitely determined holomorphic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. According to a result of Gaffney, these are the 0-stable invariants and all polar multiplicities which appear in the stable types of a stable deformation of the germ. First we describe all stable types, then we show how the invariants in the source and the target are related and reduce the number using these relations. We also investigate the relationship between the local Euler obstruction for nonsingular varieties and the polar multiplicities of the stable types. We show an algebraic formula for the local Euler obstruction in terms of the polar multiplicities and show that the Euler obstruction is an invariant for the Whitney equisingularity

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1 Introduction

Gaffney describes in [3] the following problem: "Given a 1-parameter family of map germs $F(x,t)\colon (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^p \times \mathbb{C}, 0)$, find analytic invariants whose constancy in the family implies the family is Whitney equisingular." He shows that for the class of finitely determined map germs of discrete stable type, the Whitney equisingularity (hence the topological triviality) of such a family is guaranted by the zero stable invariants and the polar multiplicities of the polar varieties associated to all the stable types.

The number of invariants depends on the dimensions (n,p) and this number can be very big according to n and p are big. Then a natural question arises:

"For a fixed pair of dimensions (n,p), what is the minimum number of invariants in Gaffney's theorem that are necessary to guarantee the Whitney equisigularity of the family?"

When $p = 1$, Teissier showed that the invariants for the Whitney equisingularity are the sequence of invariants $\mu^* = (\mu^0, \mu^1, \dots, \mu^{n+1})$, where μ^j denotes the Milnor number of the intersection of $F^{-1}(0)$ with a generic j-plane, $j = 0, \dots, n+1$.

The known cases for map germs germs of corank one, are $n = p = 2$, and $n = 2$, $p = 3$ studied by Gaffney in [3]. More recently, the first named author dealt with the cases $n = p = 3$ in [21] and $n = 3$, $p = 4$ in [22], Vohra in [23] also used Gaffney's approach to study map germs from n-space ($n \geq 3$) to the plane.

In particular, the number of invariants required for Whitney equisingularity in each of these situations was shown to be smaller than the a priori number given by the general result of Gaffney [3].

In this paper we deal with the case $n = p$ and consider germs of corank one. We reduce the number of invariants neeeded finding relations among then and using the fact that these are upper semi-continuous.

Another invariant that is associated to the polar invariants is the local Euler obstruction for nonsingular varieties, introduced by R. MacPherson in [16] in a purely obstructional way. Gonzales-Sprinberg gave in [7] an algebraic interpretation of the local Euler obstruction and Lê and Teissier used this to show that the local Euler obstruction is an alternate sum of the polar multiplicities of the local polar variety, see [11].

Here we apply these results to obtain explicit and algebraic formulae for the Euler obstruction in the stable type of mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$.

In section 2 we recall the basic definitions and results needed to prove our main result, which is shown in section 3, in the section 4 we show the formulae for the local Euler obstruction and in section 5 we show an algorithm that allow us to compute the ideals that define the stable types in the source.

2 Notation and preliminaries

We follow the notation used by Gaffney in [3] and denote by $\mathcal{O}(n,p)$ the set of origin preserving germs of holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^p$, $\mathcal{O}_e(n,p)$ denotes the set of germs at the origin but not necessarily origin preserving.

For a germ $f \in \mathcal{O}_e(n,p)$, $J(f)$ denotes the ideal generated by the set of $p \times p$ minors of the derivative of f. The critical set of f, denoted by $\Sigma(f)$ is the set of points $x \in \mathbb{C}^n$ such that $J(f)(x) = 0$. The discriminant of f is the image of the critical set by f, $\Delta(f) = f(\Sigma(f))$. The determinant of the derivative of $f \in \mathcal{O}_e(n,n)$ is denoted by $J[f]$.

Our interest is primarily in corank one $\mathcal{A}$-finitely determined map-germs ($\mathcal{A}$ is the Mather group) $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$. We denote by F a versal unfolding of such an f.

Definition 2.1. *We say that a stable type $\mathcal{Q}$ appears in F if for any representative $F = (id, f_u(x))$ of F, there exists a point $(s,y) \in \mathbb{C}^s \times \mathbb{C}^p$ such that the germ $f_u : \mathbb{C}^n, S \to \mathbb{C}^p, y$ is a stable germ of type $\mathcal{Q}$ where $S = f^{-1}(y) \cap \Sigma(f_u)$. The points (s,y) and (s,x) with $x \in S$ are called points of stable type $\mathcal{Q}$ in the target and in the source, respectively.*

If f is stable, we denote the set of points in $\mathbb{C}^s \times \mathbb{C}^p$ of type $\mathcal{Q}$ by $\mathcal{Q}(f)$ and the set $\mathcal{Q}_S(f) = f^{-1}(\mathcal{Q}(f)) - \mathcal{Q}_\Sigma(f)$, where $\mathcal{Q}_\Sigma(f)$ denotes $f^{-1}(\mathcal{Q}(f)) \cap \Sigma(f)$.

If f is finitely determined, we denote by $\overline{\mathcal{Q}(f)} = (\{0\} \times \mathbb{C}^p) \cap \overline{\mathcal{Q}(F)}$ and $\overline{\mathcal{Q}_S(f)} = (\{0\} \times \mathbb{C}^n) \cap \overline{\mathcal{Q}_S(F)}$, $\overline{\mathcal{Q}_\Sigma(f)} = (\{0\} \times \mathbb{C}^n) \cap \overline{\mathcal{Q}_\Sigma(F)}$, here the bar over a set means the closure of this set.

Definition 2.2. *We say that $\mathcal{Q}$ is a zero-dimensional stable type for the pair (n,p) if $\mathcal{Q}(f)$ has dimension 0 where f is a representative of the stable type $\mathcal{Q}$.*

We observe that the set $\overline{\mathcal{Q}(F)} = \cap F(j^{(p+1)}F^{-1}(\overline{\mathcal{A}z_i}))$ is closed and analytic, where z_i is the $p+1$ jet of the stable type $\mathcal{Q}$ and $\mathcal{A}z_i$ is the $\mathcal{A}$-orbit of z_i.

A finitely determined germ f has discrete stable type if there exist a versal unfolding F of f in which appears only a finite number of stable types. If (n,p) is in the nice dimensions, any finitely determined germ f has a discrete stable type (see [17]).

It is shown in [3] p. 208, that the stable types in the source and in the target form a regular stratification, i.e. satisfy the Whitney conditions, and therefore the family F is Whitney equisingular.

Suppose that $\mathcal{Q}(F) = \{p_1, \cdots, p_r\}$ is the set of points of zero-dimensional type, where F is a versal unfolding of f. The 0-stable invariant of type $\mathcal{Q}$ of f, denoted by $m(f; \mathcal{Q})$ is the multiplicity of the ideal $m_s \mathcal{O}_{\overline{\mathcal{Q}(F)},(0,0)}$ in $\mathcal{O}_{\overline{\mathcal{Q}(F)},(0,0)}$.

We consider now $F = (u, \overline{f}(u, x))$ be a 1-parameter unfolding of a finitely determined germ f, such that $\overline{f}(u, -)$ preserves the origin for all u. We say that F is a *good unfolding* of f if there exist neighborhoods U, W of the origin in $\mathbb{C} \times \mathbb{C}^n$ and $\mathbb{C} \times \mathbb{C}^p$ respectively such that $F^{-1}(W) = U$, F maps $U \cap \Sigma(F) - T$ to $W - T$ and if $(t_0, y_0) \in W - T$ where $S = F^{-1}(t_0, y_0) \cap \Sigma(F)$ and $T = \mathbb{C} \times \{0\}$, then the germ $f_{t_0} : \mathbb{C}^n, S \to \mathbb{C}^p, y_0$ is stable.

A good unfolding is said to be *excellent* if all the 0-stable invariants are constant in the unfolding and f is of discrete type. In the equidimensional case $(n = p)$, it is also assumed that the degree of f, $\delta(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{f^*(m_n)\mathcal{O}_n}$, is constant in the unfolding. Using the polar multiplicities of the polar varieties of the stable types (defined by Teissier in [20])and Thom's isotopy lemmas, Gaffney showed the following principal result.

Theorem 2.3. ([3], p. 207) *Suppose that $F : \mathbb{C} \times \mathbb{C}^n, (0,0) \to \mathbb{C} \times \mathbb{C}^p, (0,0)$ is an excellent unfolding of a finitely determined germ $f \in \mathcal{O}(n,p)$. Also suppose that the polar invariants of all the stable types defined in:*

1. *the discriminant $\Delta(f_t) = f_t(\Sigma(f_t))$,*
2. *the singular set $\Sigma(f_t)$ and also in the set*
3. *$X(f_t) = \overline{(f_t^{-1}(\Delta(f_t)) - \Sigma(f_t))}$,*

are constant at the origin for all t. Then the unfolding is Whitney equisingular.

Remark 2.4. 1. The theorem also implies that such unfolding is topologically trivial; for the proof of this result Gaffney uses Thom's second isotopy lemma for complex analytic mappings, see [3] page 204.

2. The theorem remains valid if we replace the term "an excellent unfolding" in the hypothesis by "a 1-parameter unfolding which, when stratified by stable types and by the parameter axis T, has only the parameter axis T as 1-dimensional stratum at the origin" ([23]). We shall apply this version of the Theorem in order to give sufficient conditions for the Whitney equisingularity of 1-parameter unfolding.

In the next section we shall need the following definition and results.

Theorem 2.5. (Lê-Greuel, [10], page 47) *Let X_1 be an I.C.I.S., with a singularity at $0 \in \mathbb{C}^n$. Let X be an I.C.I.S. defined in X_1 by $f_k = 0$, and let $f_1, ..., f_{k-1}$ be the generators of the ideal that defines X_1 at 0 in $\mathbb{C}^n$. Then*

$$\mu(X_1, 0) + \mu(X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(f_1, ..., f_{k-1}, J(f_1, ..., f_k))}.$$

In the case of a zero-dimensional I.C.I.S. we can use the following simpler formula. Let $f : \mathbb{C}^k, 0 \to \mathbb{C}^k, 0$ be a germ such that $X = f^{-1}(0)$ is an I.C.I.S. Then $\mu(X, 0) = \delta(f) - 1$; see [12] p. 78.

Another elementary result that we use here is the following. Let $f : \mathbb{C}^n, 0 \to \mathbb{C}^n, 0$ be a finitely determined germ. Then $f : \Sigma(f) \subset \mathbb{C}^n, 0 \to \Delta(f) \subset \mathbb{C}^n, 0$ is bimeromorphic; see [2] p.154.

3 Equisingularity of map germs in $\mathcal{O}(n, n)$

3.1 The stable types in $\mathcal{O}(n, n)$

As heiglighted in the introduction, our aim is to minimize the number of invariants defined in the stable types of f whose constancy in the family f_t implies the family is Whitney equisingular.

The strategy is to apply Theorem 2.3 and the techniques used by Gaffney in [3], that is, stratify the source and the target by the stable types and establish relations among the invariants on the strata. As these invariants are upper semi-continuous, the relations will allow us to reduce the number of invariants required in Gaffney's theorem.

We show here an explicity description of all the stable types in the source and target for corank one germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^n, 0)$. This description is done in terms of subschemes of multiple points of a germ f. The description of the 0-stable types is shown in [14], we generalise here this description for all r-stable types, with $0 \leq r \leq n-1$.

For this we first give the following preliminary definition. Given a continuous mapping $f : X \to Y$ on analytic spaces, we define the k^{tk} multiple point space of f as

$$D^k(f) = \text{closure}\left\{(x_1, x_2, ..., x_k) \in X^k : f(x_1) = ... = f(x_k) \text{ for } x_i \neq x_j, i \neq j\right\}.$$

Let $f \in \mathcal{O}(n,n)$ be a finitely determined germ of corank 1, we write $f(x,z) = (x_1, \dots, x_{n-1}, g(x,z))$, with $x = (x_1, \dots, x_{n-1}) \in C^{n-1}$ and $z \in \mathbb{C}$. For each partition $\mathcal{P} = (r_1, \cdots, r_\ell)$ of an $m \leq n$, i. e, $r_1 + \cdots + r_\ell = m$ we consider the subscheme $D^\ell(f, \mathcal{P})$ of $\mathbb{C}^{n-1} \times \mathbb{C}^\ell$, with $\ell = \text{length}(\mathcal{P})$, defined by

$$D^\ell(f,\mathcal{P}) = clos \left\{ (x, z_1, \cdots, z_\ell) \in \mathbb{C}^{n-1} \times \mathbb{C}^\ell : \begin{array}{l} z_i \neq z_j, \\ f(x,z_i) = f(x,z_j) \text{ and} \\ f \text{ has a singularity of type} \\ A_{r_j} \text{ at } (x, y_j) \end{array} \right\}$$

where 'clos ' means the analytic closure in $\mathbb{C}^{n-1} \times \mathbb{C}^\ell$. We remark that when $m = n$, the subschemes $D^n(f,\mathcal{P})$ are called zero-schemes and are related to the 0-stable types.

Nearby the $(A_{r_1} + \cdots + A_{r_\ell}) = A_{r_1, \cdots, r_\ell}$ multi-germs, there are points in the target with $(r_1 + 1) + (r_2 + 1) + \cdots + (r_{\ell+1})$ preimages (i.e. $m + \ell$ preimages). We define an $(m+\ell)-$tuple scheme in $\mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell}$, which on the appropriate diagonal specializes to the ideal defining $A_{r_1, \cdots, r_\ell}$ multi-germs, see Lemma 3.1.

We denote the coordinates of $\mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell}$ by

$$(x, \mathbf{z}) = (x, z_0^1, \cdots, z_{r_1}^1, z_0^2, \cdots, z_{r_2}^2, \cdots, z_0^\ell, \cdots, z_{r_\ell}^\ell)$$

and define the sheaf of ideals $\mathcal{J}(f,\mathcal{P}) = \langle h_1, h_2, \cdots, h_{m+\ell-1} \rangle \subset \mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^{n+\ell}}$, with

$$h_i = V^{-1}. \begin{vmatrix} 1 & z_0^1 & \cdots & (z_0^1)^{i-1} & g_0^1 & (z_0^1)^{i+1} & \cdots & (z_0^1)^{m+\ell-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_{r_1}^1 & \cdots & (z_{r_1}^1)^{i-1} & g_{r_1}^1 & (z_{r_1}^1)^{i+1} & \cdots & (z_{r_1}^1)^{m+\ell-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_{r_0}^\ell & \cdots & (z_{r_0}^\ell)^{i-1} & g_{r_0}^\ell & (z_0^\ell)^{i+1} & \cdots & (z_{r_0}^\ell)^{m+\ell-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_{r_\ell}^\ell & \cdots & (z_{r_\ell}^\ell)^{i-1} & g_{r_\ell}^\ell & (z_{r_\ell}^\ell)^{i+1} & \cdots & (z_{r_\ell}^\ell)^{m+\ell-1} \end{vmatrix}$$

where $V = V(z_0^1, \cdots, z_{r_1}^1, \cdots, z_0^\ell, \cdots, z_{r_\ell}^\ell)$ is the Vandermonde determinant and $g_i^k = g(x, z_i^k)$. In $\mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell}$ there is a diagonal of particular interest, namely,

$$\Delta(\mathcal{P}) = \{(x,\mathbf{z}) \in \mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell} | z_i^k = z_j^k, \ \forall i,j = 1, \cdots, r_k, \ \forall k = 1, \cdots, \ell\}$$

which can be parametrized by $(x, z^1, \cdots, z^\ell)$:

$$(x, \mathbf{z}) = (x, z^1, \cdots, z^1, z^2, \cdots, z^2, \cdots, z^\ell, \cdots, z^\ell) \tag{3.1}$$

with z^i repeated r_i+1 times. This corresponds to an embedding of $\mathbb{C}^{n-1}\times\mathbb{C}^\ell$ in $\mathbb{C}^{n-1}\times\mathbb{C}^{m+\ell}$.

Let $\mathcal{I}_{\Delta(\mathcal{P})}=\langle z_i^k-z_0^k, \forall k=1,\cdots,\ell\rangle$ be the ideal defining $\Delta(\mathcal{P})$. A generic point of $\mathcal{I}_{\Delta(\mathcal{P})}$ is one of the form (3.1) with $z^i\neq z^j$, for $i\neq j$.

Let $\mathcal{J}_\Delta(f,\mathcal{P})$ be the ideal in $\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^\ell}$ defined by $\mathcal{J}_\Delta(f,\mathcal{P})=\mathcal{J}(f,\mathcal{P})+\mathcal{I}_{\Delta(\mathcal{P})}$.

We see in the next lemma that at a generic point of $D^\ell(f,\mathcal{P})=V(\mathcal{J}_\Delta(f,\mathcal{P}))$, f has a singularity of type A_{r_j} at (x,z^j) and $f(x,z^1)=\cdots=f(x,y^\ell)$.

Lemma 3.1. *([[13], lemma 2.7]) At a generic point of $\Delta(\mathcal{P})$ we have,*

$$\mathcal{J}_\Delta(f,\mathcal{P}) = \langle\tfrac{\partial g}{\partial z}(x,z^1),\cdots,\tfrac{\partial^{r_1}g}{\partial z^{r_1}}(x,z^1),\cdots,\tfrac{\partial g}{\partial z}(x,z^\ell),\cdots,\tfrac{\partial^{r_\ell}g}{\partial z^{r_\ell}}(x,z^\ell)\rangle$$

$$+\langle g(x,z^i)-g(x,z^1);\ 2\leq i\leq\ell\rangle+\mathcal{I}_{\Delta(\mathcal{P})}$$

We remark that for the partition $\mathcal{P}=(1,\cdots,1)$ of m with $1+\cdots+1=m=\ell$, we have $D^\ell(f,\mathcal{P})=D^\ell(f)$, where $D^\ell(f)$ denotes the set of ℓ-multiple points of f.

For the partition $\mathcal{P}=(r_i)$, then $r_i=m$, $\ell=1$, and here $D^1(f,\mathcal{P})=\Sigma^{1,\cdots,1}(f)$, where $\Sigma^{1,\cdots,1}(f)$ is the set of singularities of type $\Sigma^{1,\cdots,1}$ of f with $1,\cdots,1$ repeated r_i-times, i.e, f has an singularity of type A_{r_i}. We know by the Lemma 3.1, that the ideal that defines $\Sigma^{1,\cdots,1}(f)$ is generated by the system $\{\frac{\partial g}{\partial z}(x,z^1),\cdots,\frac{\partial^{r_i}g}{\partial z^{r_1}}(x,z^1)\}$.

The structure of these sets is described below.

Proposition 3.2. *Let $j_\ell:\mathbb{C}^{n-1}\times\mathbb{C}^\ell\to\mathbb{C}^{n-1}\times\mathbb{C}^{m+\ell}$ be the embedding with image $\Delta(\mathcal{P})$. Then the surjection $j_\ell^*:\mathcal{O}_{n-1+m+\ell}\to\mathcal{O}_{n-1+\ell}$ satisfies $j_\ell^*(\mathcal{J}_\Delta(f,\mathcal{P}))=\mathcal{I}_\Delta(f,\mathcal{P})$ and consequently induces an isomorphism*

$$j_\ell^*:\frac{\mathcal{O}_{n-1+m+\ell}}{\mathcal{J}_\Delta(f,\mathcal{P})}\to\frac{\mathcal{O}_{n-1+\ell}}{\mathcal{I}_\Delta(f,\mathcal{P})}$$

Proof Since j_ℓ^* is a sujection, we show that it is also an immersion. Suppose that it is not, hence there exists a non zero $g\in\frac{\mathcal{O}_{n-1+m+\ell}}{\mathcal{J}_\Delta(f,\mathcal{P})}$ such that $j_\ell^*(g)=0$, therefore $g_{|j(D^\ell(f,\mathcal{P}))}=0$; and $j(D^\ell(f,\mathcal{P}))$ is in the subvariety of $V(\mathcal{I}_\Delta(f,\mathcal{P}))$, defined by the zero set of g. Then $j(D^\ell(f,\mathcal{P}))\subset V(\mathcal{I}_\Delta(f,\mathcal{P}))$, and this implies that $j_\ell^*(\mathcal{J}_\Delta(f,\mathcal{P}))\subset\mathcal{I}_\Delta(f,\mathcal{P})$. ■

Remark 3.3. Since from the Proposition 3.2 we know that $D^\ell(f,\mathcal{P})$ is embedded in $\mathbb{C}^{n-1}\times\mathbb{C}^\ell$, from now on we shall denote the ideal that defines this set in the space $\mathbb{C}^{n-1}\times\mathbb{C}^\ell$ also by $\mathcal{I}_\Delta(f,\mathcal{P})$.

Proposition 3.4. *Let $F = (u, \overline{f})$ be a versal unfolding of a finitely determined map germ f of corank 1. Then, for any partition $\mathcal{P} = (r_1, \dots, r_\ell)$ of each $m \le n$,*

1. *The ideal $\mathcal{J}_\Delta(\overline{f}, \mathcal{P})$ is reduced, and the multiple point variety*
$$V(\mathcal{J}_\Delta(\overline{f}, \mathcal{P})) \subset \mathbb{C}^{n-1} \times \mathbb{C}^\ell \times \mathbb{C}^s$$
is smooth of dimension $n - m + s$ (or empty);
2. *for each partition $\mathcal{P} = (r_1, \cdots, r_\ell)$ of m satisfying $n - m + s \ge 0$, the germ of $D^\ell(F, \mathcal{P})$ at 0 is either an ICIS of dimension $n - m + s$, or is empty;*
3. *$\pi : D^\ell(F, \mathcal{P}) \to \mathbb{C}^{n-m} \times \mathbb{C}^s$ is finite and $\pi^{-1}(\pi(0)) = \{0\}$*

Proof The item 1. follows from the Lemma 3.1. To prove the item 2. we observe that a system generators of the ideal $\mathcal{J}^\ell(f, \mathcal{P})$ which defines $D^\ell(f, \mathcal{P})$ is given by the equations

$$H^\ell(\Delta)(x, z^1, ..., z^\ell, t) = \left(\frac{\partial g}{\partial z}(x, z^1, t), \cdots, \frac{\partial^{r_1} g}{\partial z^{r_1}}(x, z^1, t), \cdots, \frac{\partial g}{\partial z}(x, z^\ell),\right.$$

$$\left.\cdots, \frac{\partial^{r_\ell} g}{\partial z^{r_\ell}}(x, z^\ell, t),\ g(x, z^2, t) - g(x, z^1, t), ...,\ g(x, z^\ell, t) - g(x, z^1, t)\right),$$

as $\mathcal{I}_{\Delta(\mathcal{P})})$ is a submersion (see [13]) if we calculate $\mathcal{I}_{\Delta(\mathcal{P})}$ in $\mathcal{J}_\Delta(f, \mathcal{P})$ we obtain that each $H^\ell(\Delta)(x, z^1, ..., z^\ell, t) = \left(\frac{\partial g}{\partial z}(x, z^1, t), \cdots, \frac{\partial^{r_1} g}{\partial z^{r_1}}(x, z^1, t), \cdots, \frac{\partial g}{\partial z}(x, z^\ell, t),\right.$

$$\left.\cdots, \frac{\partial^{r_\ell} g}{\partial z^{r_\ell}}(x, z^\ell, t),\ g(x, z^2, t) - g(x, z^1, t), ...,\ g(x, z^\ell, t) - g(x, z^1, t)\right)$$

is a submersion, therefore $(H^\ell(\Delta))^{-1}(0) = D^\ell(f_s, \mathcal{P}) \subset \mathbb{C}^{n-1} \times \mathbb{C}^\ell \times \mathbb{C}^s$ is an ICIS of dimension $n - m + s$. The proof of the item 3. is immediate from the definition of the sets $D^\ell(f, \mathcal{P})$. ∎

Now we give an explicit description of the stable types in the source and the target of any germ f with corank one in $\mathcal{O}(n, n)$ which is finitely determined.

For each partition $\mathcal{P} = (r_1, \dots, r_\ell)$ of an $m \le n$, we denote by $D_1^\ell(f, \mathcal{P})$ the projection of $D^\ell(f, \mathcal{P})$ to the (x, z)-space, we remember that each set $D_1^\ell(f, \mathcal{P})$ is a subset of $\Sigma(f)$.

Example 3.5. For finitely determined map germs $f \in \mathcal{O}(2,2)$ of corank 1, the possible partitions of 1 and 2 are (1), (1,1) and (2), then $D_1^1(f,(1)) = \Sigma(f)$, $D_1^2(f,(1,1)) = D^2(f)$ is the set of double points of f and $D_1^1(f,(2)) = \Sigma^{1,1}(f)$ is the cusp of f.

For finitely determined map germs $f \in \mathcal{O}(3,3)$ of corank 1, the possible partitions of 1, 2 and 3 are (1), (1,1) (2), (2,1), (1,1,1) and (3), hence $D_1^1(f,(1)) = \Sigma(f)$ is the singular set of f, $D_1^2(f,(1,1)) = D^2(f)$ is the curve of double points of f, $D_1^1(f,(2)) = \Sigma^{1,1}(f)$ is a cuspidal curve, $D^3(f,(1,1,1)) = D^3(f)$ is the set of triple points and $D_1^1(f,(3)) = \Sigma^{1,1,1}(f)$ is the Swallowtail of f.

For each $\mathcal{P}$, we define the sets $X_1^\ell(f,\mathcal{P})$ in the set $X(f) = \overline{(f^{-1}(\Delta(f)) - \Sigma(f))}$ by,

$$X_1^\ell(f,\mathcal{P}) = f^{-1}(f(D_1^\ell(f,\mathcal{P}))) - (D_1^\ell(f,\mathcal{P}) \cap \Sigma(f))\,.$$

Theorem 3.6. *Let $f \in \mathcal{O}(n,n)$ be a finitely determined germ of corank 1.*

a. *The stables types in* **the source** *are $D_1^\ell(f,\mathcal{P})$ in $\Sigma(f)$ and $X_1^\ell(f,\mathcal{P})$ in $X(f)$, in* **the target** *the stable types are $f(D_1^\ell(f,\mathcal{P})) \subset \Delta(f)$ for all partitions $\mathcal{P} = (r_1,\dots,r_\ell)$ of each $m \leq n$.*

b. *The dimension of $X_1^\ell(f,\mathcal{P})$ and of $f(D_1^\ell(f,\mathcal{P}))$ is $n-m$.*

Proof of the Theorem 3.6 **a.** A stable map germ $f \in \mathcal{O}(n,n)$ has an A_k singularity if it is $\mathcal{A}$-equivalent to the germ

$$(x_1, \dots, x_{n-1}, z) \to (x_1, \dots, x_{n-1}, z^{k+1} + x_1 z^{n-1} + \dots + x_{k-1} z).$$

Moreover, any stable corank 1 map germ is an A_k singularity for some natural number k, hence the set of points in $\mathbb{C}^n$ where a stable map has an A_k singularity is a smooth submanifold of codimension k. The image of this set by f is also a smooth submanifold of codimension k. Since f is finitely determined, it follows by the Geometric criterion of Mather-Gaffney [17] that there exist neighborhoods U and V of 0 in $\mathbb{C}^n$ such that $f^{-1}(0) \cap U \cap \Sigma(f) = 0$ and for each $y \in V$, $y \neq 0$, the germ $f : (\mathbb{C}^n, S) \to \mathbb{C}^n, y$ is stable ($S = f^{-1}(y) \cap U \cap \Sigma(f)$), hence for each $x \in S$, the germ $f : \mathbb{C}^n, x \to \mathbb{C}^n, y$ is an A_k for some k and these submanifolds in the discriminant are in general position. But this occurs if and only if $r_1 + r_2 + \dots + r_j = m \leq n$. We call such multigerm and $A_\mathcal{P}$-singularity and the result follows from the Lemma 3.1.

b. From the corollary given in the page 19 of [6], we know that there exist neighbourhoods of the origin U_1 in $\mathbb{C}^{n-1} \times \mathbb{C}^\ell$ and U_2 in $\mathbb{C}^n$ such that the map

$p : D^m(f,\mathcal{P}) \to U_2$ induced by the projection $U_1 \to U_2$ is proper and finite. Since f is proper and finite, the map $f \circ p$ is also proper and finite, then $V = (f \circ p)(D^m(f,\mathcal{P}))$ is an analytic subvariety $n-m$-dimensional, in particular, $f(D_1^\ell(f,\mathcal{P}))$ is $n-m$-dimensional. Since $D_1^\ell(f,\mathcal{P})$ is $n-m$-dimensional and f is proper and finite, we obtain that $X_1^\ell(f,\mathcal{P})$ is an analytic space of dimension $n-m$. ■

To describe the structure of the sets $D_1^\ell(f,\mathcal{P})$ in $\mathbb{C}^n$ we have the following

Proposition 3.7. a. *Let $j_\ell : \mathbb{C}^n \to \mathbb{C}^{n-1} \times \mathbb{C}^\ell$ be the embedding with image $p^{-1}(0)$ where $p : \mathbb{C}^{n-1} \times \mathbb{C}^\ell \to \mathbb{C}^n$ is a generic linear projection. Then the surjection $j_\ell^* : \mathcal{O}_{n-1+\ell} \to \mathcal{O}_n$ with $j_\ell^*(\mathcal{I}_\Delta(f,\mathcal{P})) = \mathcal{I}_\Delta^1(f,\mathcal{P})$ induces an isomorphism*

$$j_\ell^* : \frac{\mathcal{O}_{n-1+\ell}}{\mathcal{I}_\Delta(f,\mathcal{P})} \to \frac{\mathcal{O}_n}{j_\ell^*(\mathcal{I}_\Delta(f,\mathcal{P}))}.$$

b. *If f is finitely determined of corank 1, for each partition $\mathcal{P} = (r_1, \ldots r_\ell)$ of $m \leq n$, the set $D_1^\ell(f,\mathcal{P})$ is an ICIS.*

Proof (a) Since j_ℓ^* is surjective, we show that it is injective. Suppose that it is not injective, there exists a non zero element $g \in \dfrac{\mathcal{O}_{n-1+\ell}}{\mathcal{J}_\Delta(f,\mathcal{P})}$ such that $j_\ell^*(g) = 0$, therefore $g_{|j(D^\ell(f,\mathcal{P}))} = 0$ and $j(D^\ell(f,\mathcal{P}))$ is contained in the submanifold of the set $V(\mathcal{I}_\Delta(f,\mathcal{P}))$ defined by the zero set of g, hence $j_\ell(D_1^\ell(f,\mathcal{P})) \subset V(\mathcal{I}_\Delta(f,\mathcal{P}))$, and this implies that $j_\ell^*(\mathcal{I}_\Delta(f,\mathcal{P})) \subset \mathcal{I}_\Delta^1(f,\mathcal{P})$.

(b) From the item above we have that

$$j_\ell^* : \frac{\mathcal{O}_{n-1+\ell}}{\mathcal{I}_\Delta(f,\mathcal{P})} \to \frac{\mathcal{O}_n}{j_\ell^*(\mathcal{I}(f,\mathcal{P}))}$$

is an isomorphism, then we conclude from the lemma 1.8 of [12] that if

$$d(\mathcal{I}_\Delta(f,\mathcal{P})) \text{ and } d(j_\ell^*(\mathcal{I}_\Delta(f,\mathcal{P})))$$

is the minimal number of generators of $\mathcal{I}_\Delta(f,\mathcal{P})$ and $j_\ell^*\mathcal{I}_\Delta(f,\mathcal{P})$ respectively, hence

$$n - 1 - \ell - d(\mathcal{I}_\Delta(f,\mathcal{P})) = n - d(j_\ell^*(\mathcal{I}_\Delta(f,\mathcal{P}))),$$

and we conclude that $d(j_\ell^*(\mathcal{I}_\Delta(f,\mathcal{P}))) = m$.

We choose a projection p that is finite (this is possible from [6]), then

$$p(V((\mathcal{I}_\Delta(f,\mathcal{P}))) = V(j_\ell^*(\mathcal{I}_\Delta(f,\mathcal{P}))) = D_1^\ell(f,\mathcal{P})$$

is of dimension $n-m$, since the ideal $j_\ell^*(\mathcal{I}_\Delta(f,\mathcal{P})) = \mathcal{I}_\Delta^1(f,\mathcal{P})$ which defines $D_1^\ell(f,\mathcal{P})$ has m generators, we obtain that $D_1^\ell(f,\mathcal{P})$ is an ICIS. ■

In the following we shall show how the polar multiplicities in the target and in the source are related. We first give this relation for the polar invariants in the target.

3.2 The polar multiplicities in the discriminant $\Delta(f)$

To show how the polar multiplicities of the stable types in the target are related we describe the polar varieties of the strata $f(D_1^\ell(f,\mathcal{P}))$ for each partition $\mathcal{P}$ of $m \le n$.

For a map germ $f \in \mathcal{O}(n,n)$ with corank 1, each set $f(D_1^\ell(f,\mathcal{P}))$ is of dimension $n-m$ and the polar varieties with codimension j ($0 \le j \le n-m$), of this stratum are obtained in the following way:

We choose generic projections $p : \mathbb{C}^{n-1} \times \mathbb{C}^\ell \to \mathbb{C}^n$ and $p_{n-m-j+1} : \mathbb{C}^n \to \mathbb{C}^{n-m-j+1}$, each polar variety is defined as

$$P_j(f \circ p(D^\ell(f,\mathcal{P}))) = \overline{\Sigma(p_{n-m-j+1}|f \circ p(D^\ell(f,\mathcal{P})^0)}$$

where $0 \le j \le n-m$.

We see in [20] that the codimension of each polar variety is of codimension j in $\mathbb{C}^n$. The polar invariants associate to each polar variety are the multiplicities $m_j(P_j(f \circ p(D^\ell(f,\mathcal{P}))))$, $0 \le j \le n-m$ for each partition $\mathcal{P}$ of all $m \le n$.

To compute the multiplicites $m_j(P_j(f \circ p(D^\ell(f,\mathcal{P}))))$, we need to consider the sets

$$\overline{\Sigma(p_{n-m-j+1}|f \circ p(D^\ell(f,\mathcal{P})^0)},$$

however it is better to work with the sets

$$X_j(\mathcal{P}) = \overline{\Sigma(p_{n-m-j+1} \circ f \circ p|D^\ell(f,\mathcal{P}))},$$

which are nothing more than

$$X_j(\mathcal{P}) = V\left(\mathcal{I}^\ell(f,\mathcal{P}),\ J(p_{n-m-j+1} \circ f \circ p,\ \mathcal{I}_\Delta^1(f,\mathcal{P})\right).$$

The advantages to work with these sets is that they are in the source and the equations that define the associate polar varieties are computable. Our strategy is to compute the polar invariants, choosing generic projections, for these sets and considering that the f is bimeromorphic we compute the polar invariants in the target.

Let $\mathcal{P} = (r_1, \cdots, r_\ell)$ be a partition of $m \le n$ with $r_1 \ge r_2 \ge \cdots r_\ell \ge 1$. Define $N(\mathcal{P})$ to be the order of the subgroup of S_ℓ which fixes $\mathcal{P}$. Here S_ℓ acts on $\mathbb{R}^\ell$

by permuting the coordinates, for example, if $\mathcal{P} = (4,4,4,2,2,2,1,1)$ we have $N(\mathcal{P}) = (3!)^2.2!$, we remark that if $\mathcal{P} \neq (r_i)$, then $N(\mathcal{P}) \neq 1$.

Now we show the main theorem of this section.

Theorem 3.8. *Let $f \in \mathcal{O}(n,n)$ be a finitely determined germ of corank 1. Then*

$$\begin{aligned}\Sigma_{i=0}^{n-m-1}(-1)^i N(\mathcal{P}) m_i(f \circ p(D^\ell(f,\mathcal{P}))) &= (-1)^{n-m}\mu(D^\ell(f,\mathcal{P})) + 1 + \\ & (-1)^{n-m+1} dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(\mathcal{I}^\ell(f,\mathcal{P}), J(p_1 \circ f \circ p, \mathcal{I}^\ell(f,\mathcal{P})))}.\end{aligned}$$

Proposition 3.9. *Let $\mathcal{P}$ be a partition of $m \leq n$, then*

$$m_j(P_j(f(D^\ell(f,\mathcal{P})))) = \frac{1}{N(\mathcal{P})} deg((p_{n-m-j} \circ f \circ p)|X_j(\mathcal{P}))$$

Proof For each $j = 0,...,n-m$, we have $X_j(\mathcal{P}) \subset D^\ell(f,\mathcal{P})$, let $\mathbf{y} = (x, z_1, z_2, \ldots, z_\ell) \in X_j(\mathcal{P})$ and $\sigma \in S_\ell$, then we have

$$\mathbf{y}_\sigma = (x, z_{\sigma(1)}, z_{\sigma(2)}, ..., z_{\sigma(\ell)}) \in X_j(\mathcal{P})$$

if and only if $r_{\sigma(k)} = r_i$ for all $k, i = 1,...,\ell$. There exist $N(\mathcal{P})$ of these σ, such that the points $\mathbf{y}$, $\mathbf{y}_\sigma$ are different, but the corresponding sets $\{z_1, z_2, ..., z_\ell\}$ are equal, i.e., the germ f in $\{(x,z_1),(x,z_2),...,(x,z_\ell)\}$ has an ordinary ℓ-multiple point in $X_j(\mathcal{P})$, we observe that each of these points point gives $N(\mathcal{P})$ points in $(p_{n-m-j} \circ f \circ p)^{-1}(z)$, and these point are all computable in $X_j(\mathcal{P})$. ∎

Remark 3.10. Since the projections p_{n-m-j} and p are generic and the germ f is finitely determined, the variety $X_j(\mathcal{P})$ is an ICIS, therefore it is Cohen Macaulay, then

$$deg(K_{n-m-j}|X_j(\mathcal{P})) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(K_{n-m-j}, \mathcal{I}^\ell(f,\mathcal{P}), J(K_{n-m-j+1}, \mathcal{I}^\ell(f,\mathcal{P}))}$$

with $K_{n-m-j} = p_{n-m-j} \circ f \circ p$

Proof of the Theorem 3.8 We choose a projections $p_{n-m} : \mathbb{C}^n \rightarrow \mathbb{C}^{n-m}$ such that the degree of $p_{n-m}|f \circ p(D^\ell(f,\mathcal{P}))$ is equal to the multiplicity of $f \circ p(D^\ell(f,\mathcal{P}))$ at the origin and a generic linear projection $p_{n-m-1}|P_1(f \circ p(D^\ell(f,\mathcal{P}) : \mathbb{C}^n \rightarrow \mathbb{C}^{n-m-1}$ such that its degree is the multiplicity of the polar variety $P_1(f \circ p(D^\ell(f,\mathcal{P})))$, denoted by $m_1(f \circ p(D^\ell(f,\mathcal{P})))$.

Let

$$X_1 = V(p_{n-m-1} \circ f \circ p, \mathcal{I}^\ell(f,\mathcal{P})),$$

$$X = V(p_{n-m} \circ f \circ p, \mathcal{I}^\ell(f, \mathcal{P})).$$

As these varieties are I.C.I.S., we apply the Theorem of Lê-Greuel and obtain

$$\mu(X_1) + \mu(X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(p_{n-m-1} \circ f \circ p, \mathcal{I}^\ell(f, \mathcal{P}), J[p_{n-m} \circ f \circ p, \mathcal{I}^\ell(f, \mathcal{P})])}$$

Since the germ f is bimeromorphic and the projections are generic, we have

$$deg((p_{n-m-1} \circ f \circ p)|X_1(\mathcal{P})) = \deg(p_{n-m-1}|P_1(f \circ (D^\ell(f, \mathcal{P}))))$$

from the Proposition 3.9 and the Remark 3.10, it follows that

$$\mu(X_1) + \mu(X) = N(\mathcal{P})m_1(f \circ P(D^\ell(f, \mathcal{P}))) \tag{3.2}$$

Choosing the projection $p_{n-m-2} : \mathbb{C}^n \to \mathbb{C}^{n-m-2}$ such that the degree of $p_{n-m-2}|P_2(f \circ p(D^\ell(f, \mathcal{P})))$ is $m_2(f \circ p(D^\ell(f, \mathcal{P})))$, let $X_2 = V(p_{n-m-2} \circ f, \mathcal{I}^\ell(f, \mathcal{P}))$ hence

$$\mu(X_1) + \mu(X_2) = dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(p_{n-m-2} \circ f \circ p, \mathcal{I}^\ell(f, \mathcal{P}), J(p_{n-m-1} \circ f \circ, \mathcal{I}^\ell(f, \mathcal{P})))} \tag{3.3}$$

Again, as f is bimeromorphic and the projection p_{n-m-2} is generic, we obtain

$$deg((p_{n-m-2} \circ f)|X_2(\mathcal{P})) = \deg(p_{n-m-2}|P_2(f \circ p(D^\ell(f, \mathcal{P}))))$$

applying the Proposition 3.9 to the equality (3.3), it follows from the Remark 3.10 that

$$\mu(X_1) + \mu(X_2) = N(\mathcal{P})m_2(f \circ p(D^\ell(f, \mathcal{P}))).$$

This equality and the equality (3.2) give

$$N(\mathcal{P})m_1(f \circ p(D^\ell(f, \mathcal{P}))) - \mu(X) + \mu(X_2) = N(\mathcal{P})m_2(f \circ p(D^\ell(f, \mathcal{P})))$$

Now, for all $3 < s < n-m-2$ we choose the projections p_{n-m-s} in a successive way and construct the sets X_s and X_{s-1} analogously than above, we obtain the sets

$$X_{n-m-1} = V(p_1 \circ f \circ p, \mathcal{I}^\ell(f, \mathcal{P}))$$
$$X_{n-m} = D^\ell(f, \mathcal{P})$$

and the equality

$$\Sigma_{i=1}^{n-m-1}(-1)^i N(\mathcal{P}) m_i(f \circ p(D^\ell(f,\mathcal{P}))) - \mu(X_{n-m}) + \mu(X) =$$

$$= dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(\mathcal{I}^\ell(f,\mathcal{P}), J(p_1 \circ f \circ p, \mathcal{I}^\ell(f,\mathcal{P})))},$$

therefore we have $\mu(X) = deg(p_{n-m} \circ f \circ p, \mathcal{I}^\ell(f,\mathcal{P})) - 1$. Since $f \circ p : D^\ell(f,\mathcal{P}) \to f(D_1^\ell(f,\mathcal{P}))$ is finite and bimeromorphic,

$$deg(p_{n-m}|f(D_1^\ell(f,\mathcal{P}))) = deg(p_{n-m} \circ f, \mathcal{I}^\ell(f,\mathcal{P}))$$

and $deg(p_{n-m} \circ f \circ p, \mathcal{I}^\ell(f,\mathcal{P})) = N(\mathcal{P}) m_0(f(D_1^\ell(f,\mathcal{P})))$. ∎

In the next theorem we show how the number of points of type $A_\mathcal{P}$ are related to the polar multiplicities of a stable unfolding.

Theorem 3.11. *Let $f \in \mathcal{O}(n,n)$ be a finitely determined germ of corank 1. Then*

$$dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(\mathcal{I}_\Delta^1(f,\mathcal{P}), J(p_1 \circ f, \mathcal{I}_\Delta^1(f,\mathcal{P})))} = N(\mathcal{P}) m_{n-m}(f(D_1^\ell(f,\mathcal{P}))) + \Sigma_\mathcal{P} c_\mathcal{P} \sharp A_\mathcal{P}.$$

Proof Choose a versal unfolding F of f and consider $F : D_1^\ell(f,\mathcal{P}) \subset \mathbb{C}^n \times \mathbb{C}^s \to f(D_1^\ell(f,\mathcal{P})) \subset \mathbb{C}^n \times \mathbb{C}^s$. We know that $D_1^\ell(f,\mathcal{P})$ is an ICIS in $\mathbb{C}^n \times \mathbb{C}^s$. As $(p_1, \pi_s) : \mathbb{C}^n \times \mathbb{C}^s \to \mathbb{C} \times \mathbb{C}^s$ is a generic linear projection, we have $\Sigma((p_1,\pi_s) \circ F)|D_1^\ell(f,\mathcal{P})) = V(\mathcal{I}_\Delta^1(f,\mathcal{P}), J((p_1,\pi_s) \circ F, \mathcal{I}_\Delta^1(f,\mathcal{P})))$.

The invariant $m_{n-m}(f(D_1^\ell(f,\mathcal{P})))$ is controlled by the degree of the projection $\pi_s|V(\mathcal{I}_\Delta^1(f,\mathcal{P}), J((p_1,\pi_s) \circ F, \mathcal{I}_\Delta^1(f,\mathcal{P})))$, that is, by the colength $e_J(f)$ of the maximal ideal m_s in $\mathcal{O}_n$ of the source. This is given by

$$e_J(f) = dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(\mathcal{I}_\Delta^1(f,\mathcal{P}), J(p_1 \circ f, \mathcal{I}_\Delta^1(f,\mathcal{P})))}.$$

The possible components of $V(\mathcal{I}_\Delta^1(f,\mathcal{P}), J((p_1,\pi_s) \circ F, \mathcal{I}_\Delta^1(f,\mathcal{P})))$ are the closure of the following sets $F^{-1}(P_{n-m}(D_1^\ell(f,\mathcal{P}),\pi_s))$, $\cup_\mathcal{P} F^{-1}(A_\mathcal{P})$.

To count the contribution of π_s restricted to these components, we use the normal forms of the stable types. We choose a generic parameter s and neighborhoods $U_2 \subset \mathbb{C}^n \times \mathbb{C}^s$ and $U_1 \subset \mathbb{C}^s$ such that for every point in U_1 we have $e_J(f)$ pre-images in $V \cap U_2$ counting multiplicity. Then

$$\begin{aligned} e_J(f) &= \textstyle\sum_{x \in S} dim_{\mathbb{C}} \frac{\mathcal{O}_{s+n,x}}{(m_s, \mathcal{I}_\Delta^1(f,\mathcal{P}), J((p_1,\pi_s)\circ F, \mathcal{I}_\Delta^1(f,\mathcal{P})))} \\ &= \textstyle\sum_{x \in S} dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{(\mathcal{I}_1^k(f_s,\mathcal{P}), J(p_1 \circ f_s, \mathcal{I}_1^k(f_s,\mathcal{P})))}. \end{aligned}$$

where $S = {\pi_s}^{-1}(0) \cap V$. As the parameter s is generic, we can suppose that f_s is a stable multigerm of type $A_{\mathcal{P}}$, with $\mathcal{P}$ being a partition of n and we count its contribution in the variety V.

Therefore the contribution of the points $A_{\mathcal{P}}$ in $e_J(f)$ is a constant $C_{\mathcal{P}} \in \mathbb{Z}^+$, this means that these points can or not appear in V.

Since F restricted to each component is finite and bimeromorphic, we have

$$deg(\pi_s|V) = N(\mathcal{P})m_{n-m}(f(D_1^2(f))) + \Sigma_{\mathcal{P}} C_{\mathcal{P}} \sharp A_{\mathcal{P}}.$$

The theorem follows now by joining all the above equalities and remark 3.10.

■

3.3 The polar multiplicities in $\Sigma(f)$

We know from [19] that the absolute polar multiplicities of a hypersurface X with isolated singularity are related to the Milnor numbers μ^k, of the plane sections by the following equalities

$$m_k(X) = \mu^{k+1}(X) + \mu^k(X)$$

for $0 \leq k \leq d-1$, where $d = dim(X)$. This result is also valid for I.C.I.S. (see [10],[4]). The absolute polar multiplicities are defined when the dimension of X is ≥ 1. The multiplicity $m_d(X)$ cannot be defined directly like the other m_k, $0 \leq k \leq d-1$, because the singularities of $p_1|X$ are isolated points. However, Gaffney [4] defines this multiplicity for spaces that are I.C.I.S as follows.

Definition 3.12. *The $d-th$ polar multiplicity at 0, denoted by $m_d(X^d)$, of a d-dimensional I.C.I.S. X^d is defined as*

$$m_d(X^d) = dim_{\mathbb{C}} \frac{\mathcal{O}_X}{J(p_1, f)}$$

where $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-d}, 0)$, $f^{-1}(0) = X^d$ and $p_1 : \mathbb{C}^n \to \mathbb{C}$ is a generic linear projection.

Remark 3.13. *As $V(p_1, f)$ is I.C.I.S., then by the Theorem of Lê-Greuel, we have*

$$m_d(X^d) = \mu(X^d) + \mu(X^d \cap p_1^{-1}(0)).$$

When $f \in \mathcal{O}(n,n)$ is finitely determined, $\Sigma(f)$ is a hypersurface with an isolated singularity. If f is of corank 1, we apply Definition 3.12 and all the properties above to obtain the following.

Proposition 3.14. *Let $f \in \mathcal{O}(n,n)$ be a finitely determined germ of corank 1. Then for each partition $\mathcal{P}$ of $m \leq n$*

$$\begin{aligned}\Sigma_{i=0}^{n-m-1}(-1)^i m_i(D^\ell(f,\mathcal{P})) &= (-1)^{n-m}\mu(D^\ell(f,\mathcal{P}))+1+\\ &\quad (-1)^{n-m+1} dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(\mathcal{I}^\ell(f,\mathcal{P}), J(p_1 \circ f \circ p, \mathcal{I}^\ell(f,\mathcal{P})))}.\\ \Sigma_{i=0}^{n-m-1}(-1)^i m_i(D_1^\ell(f,\mathcal{P})) &= (-1)^{n-m}\mu(D_1^\ell(f,\mathcal{P}))+1+\\ &\quad (-1)^{n-m+1} dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(\mathcal{I}_\Delta^1(f,\mathcal{P}), J(p_1 \circ f, \mathcal{I}_\Delta^1(f,\mathcal{P})))}.\end{aligned}$$

We can now deduce the following

Theorem 3.15. *Suppose that $f \in \mathcal{O}(n,n)$ is a finitely determined germ of corank 1 and $F = (t, f_t)$ is a good $1-$parameter unfolding. Then F is Whitney equisingular along $T = \mathbb{C} \times \{0\}$ if, and only if, $m_{2i-1}(D_1^\ell(f_t,\mathcal{P}))$, $m_{2i-1}(f_t(D_1^\ell(f_t,\mathcal{P})))$, are constant for t close to the origin.*

3.4 The Lê numbers of $X(f_t)$

We consider now the set $X(f_t) = \overline{(f_t^{-1}(\Delta(f_t)) - \Sigma(f_t))}$. Acording to Gaffney's Theorem 2.3, we need the constancy of the polar invariants of all the stable types defined in $X(f_t)$, from the theorem 3.6, we see that these stable types are the sets $X_1^\ell(f_t,\mathcal{P}) = f_t^{-1}(f_t(D_1^\ell(f,\mathcal{P}))) - \left(D_1^\ell(f_t,\mathcal{P}) \cap \Sigma(f)\right)$, for each partition $\mathcal{P}$ of all $m \leq n$.

Unfortunately, we do not know how the polar invariants of all the stable types $X_1^\ell(f_t,\mathcal{P})$ are related, since these sets are not hypersurfaces.

On the other side, the set $X(f_t) = \overline{(f_t^{-1}(\Delta(f_t)) - \Sigma(f_t))}$, is an hypersurface in $\mathbb{C}^n \times C$, possibly with non isolated singularities. In this case, we can apply the following result of Lê and Teissier given in [11], which shows that the Whitney equisingularity of the hypersurface $X(f_t) \subset \mathbb{C} \times \mathbb{C}^n$ along the parameter space is obtained in terms of the Lê numbers, see [15]. These numbers are the generalization of the Milnor numbers for hipersurfaces with isolated singularities.

We denote by h_t the function that defines the set $X(f_t)$ and call $X_0(f_t) = V(f_t(x, h_t(x,z))/J[f_t])$.

Theorem 3.16. *([11], pag. 95.) The pair $(X(f_t) - \Sigma(X_0(f)), T)$ is Whitney equisingular iff the Lê numbers of $X(f_t)$ and the Lê numbers of all generic planar sections of $X(f_t)$ are constant on T, i.e., $\lambda^i(X_0(f_t))$ and $\lambda^i(X_0(f_t)/H^k)$ are constant on T, for all $i = 0, ..., k-1$, $k = 1, ..., n-2$, where H^k is a generic plane of $\{t\} \times \mathbb{C}^n$.*

Using the relations of Gafffney and Gassler in [5], p. 710 we obtain that:

$$\lambda^k(X_0(f_t)/H^{n-k}) = \lambda^k(X_0(f_t)) + m_k(X_0(f_t)),$$

and

$$\lambda^i(X_0(f_t)/H^{n-k}) = \lambda^i(X_0(f_t))$$

for $i = 1, \dots, k-1$.

Since $\lambda^i(X_0(f_t))$ is lexicographically upper semi continuous and $m_i(X_0(f_t))$ is also upper semi continuous, we obtain the following

Proposition 3.17. *The set $(X(f_t) - \Sigma(X_0(f))$ is Whitney equisingular along the parameter space if and only if, for all $i = 1, \dots, k-1$ and all $\lambda^i(X_0(f_t)/H^{n-k})$ and $\lambda^k(X_0(f_t))$ are constants on T.*

From the results shown here, we obtain the following:

Theorem 3.18. *Let $F : (\mathbb{C} \times \mathbb{C}^n, (0,0)) \to (\mathbb{C} \times \mathbb{C}^n, (0,0))$ be an unfolding of a finite germ $f \in \mathcal{O}(n,n)$ with corank 1. Then F is Whitney equisingular if and only if the polar multiplicities $m_{2i+1}(Y^m)$, $m_{2i+1}(D_1^\ell(f_t, \mathcal{P}))$, the Lê numbers $\lambda^i(X_0(f_t)/H^{n-k})$ and $\lambda^k(X_0(f_t))$, and the polar multiplicities of $X_1^\ell(f_t, \mathcal{P})$ for all partition $\mathcal{P}$ of each $m \leq n$ are constants at the origin for f_t.*

4 The Euler obstruction

The local Euler obstruction for nonsingular varieties, introduced by R. MacPherson in [16], in a purely obstructional way is an invariant that is also associated to the polar invariants. The local Euler obstruction plays an important role in his affirmative response to a conjecture of Deligne and Grothendieck on the existence of Chern class for singular complex algebraic varieties (see [8],[16],[11].)

In [11], Lê and Teissier proved a formula for the multiplicity of the local polar varieties and, with the aid of Gonzales-Sprinberg's purely algebraic interpretation of the local Euler obstruction, they showed that the local Euler obstruction is an alternate sum of the multiplicity of the local polar variety.

Here we apply these results to obtain explicit and algebraic formulae for the Euler obstruction in the stable type of mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$.

The local Euler obstruction (see [16] or [7] for the definition and more details). Suppose that $X \subset \mathbb{C}^n$ is an space analytic of dimension d, ν the transformation of Nash of X. Let $p \in X$ and $z = (z_1, \cdots, z_n)$ be local coordinates in $\mathbb{C}^n$ such that $z_i(p) = 0$.

Let $\| z \|^2 = \Sigma z_i \overline{z_i}$. Since $\| z \|^2$ is a real-valued function, $d\| z \|^2$ may be considered as a section of $(T\mathbb{C}^n)^*$ where $*$ denotes the real dual bundle retaining only its orientation from the complex structure. We can also consider $d\| z \|^2$ as a restriction to a section r of $(TX)^*$. In [1] it is showed that for small ϵ, the section r is non zero over ν^{-1} where $0 \leq \| z \| \leq \epsilon$. Therefore let $B_\epsilon = \{z/\| z \| \leq \epsilon\}$ and $S_\epsilon = \{z/\| z \| = \epsilon\}$. The obstruction to extending r as a non zero section of TX^* from $\nu^{-1}(S_\epsilon)$ to $\nu^{-1}(B_\epsilon)$, which we denoted by $Eu(TX^*, r)$, lies in $H^d(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon); \mathbb{Z})$. If $O_{(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon))}$ denotes the orientation class in $H_d(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon); \mathbb{Z})$, then we define the local Euler obstruction of X at p to be $Eu(TX^*, r)$ evaluated on $O_{(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon))}$ or symbolically

$$Eu_p(X) = \left\langle Eu(TX^*, r), O_{(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon))} \right\rangle$$

to $\nu^{-1}(B_\epsilon)$

The following result show how the local Euler obstruction and the polar multiplicities are related.

Theorem 4.1. (Lê Dũng Trang et B. Teissier, [11]) *Let X be a space analytic reduced at $0 \in \mathbb{C}^{n+1}$ of dimension d. Then*

$$Eu_0(X) = \Sigma_{i=0}^{d-1} (-1)^{d-i-1} m_i(X)$$

were $m_i(X)$ is the polar multiplicity polar of the polar variety $P_i(X)$.

We can now deduce the formulae to the Euler obtructions of the types stables at $\Sigma(f)$.

Corollary 4.2. *Let $f \in \mathcal{O}(n, n)$ be a finitely determined germ of corank 1. Then for each partition $\mathcal{P}$ of $m \leq n$*

$$\begin{aligned} \Sigma_{i=0}^{n-m-1} (-1)^i m_i(D^\ell(f, \mathcal{P})) &= (-1)^{n-m} \mu(D^\ell(f, \mathcal{P})) + 1 + \\ &\quad (-1)^{n-m+1} dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(\mathcal{I}^\ell(f, \mathcal{P}), J(p_1 \circ f \circ p, \mathcal{I}^\ell(f, \mathcal{P})))}. \end{aligned}$$

$$\begin{aligned} \Sigma_{i=0}^{n-m-1} (-1)^i m_i(D_1^\ell(f, \mathcal{P})) &= (-1)^{n-m} \mu(D_1^\ell(f, \mathcal{P})) + 1 + \\ &\quad (-1)^{n-m+1} dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(\mathcal{I}_\Delta^1(f, \mathcal{P}), J(p_1 \circ f, \mathcal{I}_\Delta^1(f, \mathcal{P})))}. \end{aligned}$$

$$\begin{aligned} Eu_0(f(D^\ell(f, \mathcal{P}))) &= (-1)^{n-m} \mu(D^\ell(f, \mathcal{P})) + 1 \\ &\quad + (-1)^{n-m+1} dim_{\mathbb{C}} \frac{\mathcal{O}_{n+\ell-1}}{(\mathcal{I}_\Delta(f, \mathcal{P}), J(p_1 \circ f \circ p, \mathcal{I}_\Delta(f, \mathcal{P})))} \end{aligned}$$

From this formula we can deduce the following special cases:

1. Case $\mathcal{P} = (1)$

$$Eu_0(\Sigma(f)) = (-1)^n m_i(\Sigma(f)) + (-1)^{n+1}\mu(\Sigma(f)) + 1$$

$$Eu_0(\Delta(f)) = \mu(\Sigma(f)) + (-1)^n dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(J[f], J(p_1 \circ f, J[f]))} + 1$$

2. Case $\mathcal{P} = (r_i)$. Then we obtain the stable type 1-germ, i.e, all the stable types contained in the source of f. Let the ideal $\mathcal{I}^I(f)$ that define $\Sigma^I(f)$, where $I = (1, \cdots, 1)$, and the number 1 is repetead r_i times.

$$Eu_0(f(\Sigma^I(f))) - (-1)^{d+1}\mu(\Sigma^I(f)) - 1 = (-1)^{d+1} dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(\mathcal{I}^I(f), J(p_1 \circ f, \mathcal{I}^I(f)))}$$

As a consequence of these results and the Theorems 2.3, 4.1 we show that the Euler obstruction is an invariant for the Whitney equisingularity.

Theorem 4.3. *Suppose that $f \in \mathcal{O}(n,n)$ is a finitely determined germ of corank 1 and $F = (t, f_t)$ is a good $1-$parameter unfolding. If F is Whitney equisingular along $T = \mathbb{C} \times \{0\}$, then $Eu_0(D_1^\ell(f_t, \mathcal{P}))$, $Eu_0(f_t \circ p(D^\ell(f_t, \mathcal{P})))$, are constant for t close to the origin, for all partition $\mathcal{P}$ of $m \leq n$.*

5 Algorithm to compute the ideals of the stable types in the source

We show here a generalization, for all $m \leq n$, of the algorithm given in [18] for the case $m = n$. This algorithm computes the ideals of the stable types in the source for all partition $\mathcal{P}$ of $m < n$. We implemented this algorithm in the Maple software

>restart;

>with(linalg);

Defines the germ f and a partition $\mathcal{P}$ of $m \leq n$:

>$f_1(\mathbf{x}, z) = (\mathbf{x}, f(\mathbf{x}, z))$

Find the dimension of the "space" and the length of the partition

>

```
>test:=proc(f, L, j)
local nL, A, V, i, dv, da;
nL := nops(L);
with(linalg):
V :=vandermonde(L);
dv :=factor(det(V));
for i from 1 to nL do
V[i, j] :=subs(y = V[i, 2], f);
od;
A :=convert(V, matrix); da :=factor(det(A));
print(V);
print(factor(da)/factor(dv));
```

(here it is choosen the partition of m and its the length, therefore we are choosing the variables z[i,j], for instance, if $\mathcal{P} = (r_1, ..., r_\ell)$ we choose the variables

$$z[1,0], ...z[1,r_1], z[2,0], ...z[2,r_2], ..., z[\ell,0], ...z[\ell,r_\ell])$$

```
RETURN(subs(z[1, 1] = z[1, 0], z[2, 1] = z[2, 0], da/dv));
```

(here we exchange the variables $z[1, r_1] = z[1, 0]$)

```
> end:
> r := test(y^5 + x[1] * y + x[2]^2 * y^2 + x[2] * y^3, [z[1, 0], z[1, 1], z[2, 0], z[2, 1]], 2);
```

Example 5.1. Using this algorithm we find the ideals $I^\ell(f, \mathcal{P})$ for all partition $\mathcal{P}$ of 3 for the following germs

$$\begin{array}{ll} I & (x, y, z^4 + (xy + 1y^3 + y^4)z + xz^2) \\ II & (x, y, z^4 + (xy + y^3)z + xz^2) \\ III & (x, y, z^4 + (x^2 + xy^2 + y^4)z + xz^2) \end{array}$$

Case I:

$$I^2(f, (1, 1)) = (2z_1z^2 + 2zz_1^2 + xy + y^3 + y^4, -z^2 - 4z_1z - z_1^2 + x, 2z + 2z_1)$$

$$I^2(f, (1, 1, 1)) = (y(y^3 + y^2 + x), x, 0, 1, 0)$$

$$I^2(f, (2, 1)) = (y(y^3 + y^2 + x), x, 0, 1)$$

$$I^2(f, 2)) = (-8z^3 + xy + y^3 + y^4, 6z^2 + x)$$

Analogously we compute for the cases II and III.

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References

[1] J.P. Brasselet et M. H. Schwartz. Sur les classes de Chern d'un ensemble analytique par. *Asterisque.* **82-83**, 93-148 (1981).

[2] T. Gaffney, Properties of finitely determined germs. Ph.D. Thessis, Brandeis University (1975).

[3] T. Gaffney, Polar multiplicities and equisingularity of map germs, *Topology* vol.**32** No.1 (1993) 185 – 223.

[4] T. Gaffney, Multiplicities and equisingularity of I.C.I.S. germs, *Invent. Math.* **123** (1996) 209 – 220.

[5] T. Gaffney and Robert Gassler, Segre numbers and hypersurface singularities. *Journal of Algebraic Geometry* vol.**32** No.8 (1999) 695-736.

[6] R. T. Gunning , Lectures on complex analytic varieties, *Princeton, New Jersey* 1970.

[7] G. Gonzalez-Sprinberg, L'obstruction locale d'Euler et le théorème de MacPherson, *Astérisque* **82-83** (1978 – 1979) 7 – 33.

[8] Kennedy, G., MacPherson's Chern classes of singular algebraic varieties, Preprint.

[9] G.M. Greuel. Der Gauss-Manin Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten. Dissertation. Göttingen, 1973.

[10] Lê Dũng Tráng, Calculation of Milnor number of isolated singularity of complete intersection. *Funktsional' ny: Analizi Ego Prilozheniya*, vol.**8**, no. 2 (1974) 45 – 49.

[11] Lê Dũng Tráng-B. Teissier, Variétés polaires locales et classes de Chern de variétés singulères, *Annals of Mhatematics* , 114 (1981) 457 – 491.

[12] E.J.N. Looijenga, Isolated singular points on complete intersections, *London Mathematical Soc. Lecture Note Series* **77** (1984).

[13] W.L. Marar and D. Mond, Multiple point schemes for corank 1 maps. *J. London Math. Soc.* **39** 2 553–567, (1989).

[14] W. L. Marar, J. Montaldi and M.A.S. Ruas, Multiplicities of zero-schemes in quasihomogeneous corank 1 singularities. Notas ICMC. USP - São Carlos. (1997)

[15] D. Massey, Lê cycles and hypersurface singularities, *Springer Lecture Notes of Mathematics*, **1615** 1995.

[16] R.D. MacPherson, Chern class for singular algebraic varieties, *Ann. of Math.*. 1974(423-432).

[17] J. Mather, Stability of C^∞ mappings: IV classification of stable germs by $\mathbb{R}$ algebras, *Publ. Math. IMES*, **37** (1970) $223-248$.

[18] M.A.S. Ruas, On the equisingularity of families of corank 1 generic germs, *Contemporary Mathematics*, vol. **161** (1994) $113-121$.

[19] B. Teissier, Cycles évanescents, sections planes et conditions de Whitney, Singularités à Cargèse 1972, *Asterisques*, (1972) (1973) $7-8$.

[20] B. Teissier, Variétés polaires 2: Multiplicités polaires, sections planes, et conditions de Whitney, *Actes de la conference de géometrie algébique à la Rábida, Springer Lecture Notes*, **961** (1981) $314-491$.

[21] V.H. Jorge Pérez, Polar multiplicities and equisingularity of map germs from $\mathbb{C}^3$ to $\mathbb{C}^3$, *Cadernos de Matemática ICMC* **1** (1999) 261–280.

[22] V.H. Jorge Pérez. Polar multiplicities and equisingularity of map germ froms $\mathbb{C}^3$ to $\mathbb{C}^4$, To appear in *Real and Complex Singularities, Proceedings of the VI Workshop at São Carlos*, Eds. D. Mond and M. J. Saia, Lecture Notes Series in Pure and Applied Maths, Marcell-Decker (2002).

[23] R. Vohra. Equisingularity of map germs from n-space,($n \geq 3$), to the plane, PhD. Thesis, Northeastern University, 2000.

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