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Existence Results for a Second Order Partial Differential Equation with Nonlocal Conditions

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Resultados de Existência para uma Equação de Segunda Ordem com Condições Não Locais

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Resumo

Neste artigo mostramos a existência de soluções fracas, fortes e clássicas para um problema de Cauchy não local modelado na forma

$$\frac{d}{dt}[x'(t) + g(t, x(t), x'(t))] = Ax(t) + f(t, x(t), x'(t)), \quad t \in I = [0, a], \tag{1}$$

$$x(0) = x_0 + p(x, x'), \tag{2}$$

$$x'(0) = y_0 + q(x, x'), \tag{3}$$

onde A é o gerador infinitesimal de um semigrupo fortemente contínuo de operadores lineares num espaço de Banach X e $f, g : [0, a] \times X \to X$, $p, q : C([0, a] : X) \to X$ são funções apropriadas.

Existence Results for a Second Order Partial Differential Equation with Nonlocal Conditions.

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Abstract

By using the cosine function theory, we establish existence of mild, strong and classical solutions for a partial second order differential equations with nonlocal conditions.

1 Introduction

The purpose of this paper is to study the existence of mild, strong and classical solutions for a class of second order partial differential equations with nonlocal conditions described in the form

$$\frac{d}{dt}[x'(t) + g(t, x(t), x'(t))] = Ax(t) + f(t, x(t), x'(t)), \quad t \in I = [0, a], \tag{1}$$

$$x(0) = x_0 + p(x, x'), \tag{2}$$

$$x'(0) = y_0 + q(x, x'), \tag{3}$$

where A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators, $C(\cdot) = (C(t))_{t\in\mathbb{R}}$, on a Banach space X and $g(\cdot), f(\cdot) : I \times X^2 \to X$; $p(\cdot), q(\cdot) : C(I : X)^2 \to X$ are appropriate functions.

The system (1)-(3) is a generalization of the differential problems studied by Travis and Weeb in [16, 17] and more recently by Staněk in [12, 13, 14]. In the papers [16, 17], Travis and Weeb studied some important properties of strongly continuous cosine functions and applied their results in the study of partial second order differential equations. On the other hand, Staněk studied in [12, 13, 14] different problems related to the existence of solutions of some second order differential problems modelled in the form

$$\begin{aligned}
\frac{d}{dt}(x'(t) + g(t, x(t), x'(t))) &= f(t, x(t), x'(t)), \quad t \in I = [0, a],\\
r(x(0), x'(0)) &= \varphi(x, x'),\\
\omega(x(0), x'(0)) &= \psi(x, x'),
\end{aligned}$$

where the functions $r, \omega : \mathbb{R}^2 \to \mathbb{R}$; $f(\cdot), g(\cdot) \times \mathbb{R}^3 \to \mathbb{R}$, $\varphi, \psi : C^0(I : \mathbb{R})^2 \to \mathbb{R}$ are appropriate and $C^0(I : \mathbb{R})$ is a subspace of $C(I : \mathbb{R})$. It's relevant to observe that the abstract framework used by Staněk in [12, 13, 14] does not permit to consider " partial " second order differential equations, which is the motivation of our work.

On the other hand, it's convenient to comment on some similar problems studied recently by another authors. Specifically, Balachandran and Benchohra studied in [1, 2, 3] some problems associated to the neutral functional differential equation with finite delay

$$\begin{aligned} \frac{d}{dt}[y'(t) + g(t, y_t)] &= Ay(t) + f(t, y_t), \qquad t \in [0, a], \\ y_0 &= \phi, \quad \phi \in C([-r, 0] : X), \end{aligned}$$

where A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$. In all these works, the authors assumed that the cosine function generated by A is such that $C(t)$ is compact for every $t > 0$ which implies, see [16] pp. 557, that X is a finite dimensional space. Consequently, these models do not include partial differential problems.

The principal motivation of this work is the generalization of the Staněk's papers in the context of partial second order evolution equation. We remark that our results are not consequence of the results in [1, 2, 3] and that the ideas and techniques in our paper permit the reformulation of these results in order to consider " partial " neutral differential problems.

Along this paper, A is the infinitesimal generator of a strongly continuous cosine family $C(\cdot) = (C(t))_{t\in\mathbb{R}}$ of bounded linear operators on a Banach space X. We refer the reader to [6] for the necessary concepts about cosine functions. Next, we only mention a few results and notations needed to establish our results. We denote by $S(t)$ the sine function associated with $C(\cdot)$ which is defined by

$$S(t)x := \int_0^t C(s)x ds, \quad x \in X, \quad t \in \mathbb{R}.$$

We reserve the notation $[D(A)]$ for the space

$$D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable}\},$$

endowed with the graph norm $\| x \|_A = \| x \| + \| Ax \|$, $x \in D(A)$. Moreover, the notation E stands for the space formed by the vectors $x \in X$ for which the function $C(\cdot)x$ is of class C^1. We know from Kisińsky [7], that E endowed with the norm

$$\|x\|_1 = \|x\| + \sup_{0\le t\le 1} \|AS(t)x\|, \quad x \in E, \tag{4}$$

is a Banach space. The operator valued function $g(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is a strongly continuous group of linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. From this, it follows that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0$, $t \to 0$, for each $x \in E$. Furthermore, if

$x : [0, \infty) \to X$ is locally integrable, then $y(t) = \int_0^t S(t-s)x(s)ds$ defines an E-valued continuous function which is a consequence of the fact that

$$\int_0^t g(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)\,ds \\ \int_0^t C(t-s)x(s)\,ds \end{bmatrix}$$

defines an $E \times X$-valued continuous function.

The existence of solutions of the second order abstract Cauchy problem

$$x''(t) = Ax(t) + h(t), \qquad t \in [0, a], \tag{5}$$
$$x(0) = x_0, \tag{6}$$
$$x'(0) = x_1, \tag{7}$$

where $h : [0, a] \to X$ is an integrable function has been discussed in [16]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem has been treated in [17]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s)\,ds, \qquad t \in [0, a], \tag{8}$$

is called mild solution of (5) and that when $x_0 \in E$, $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)x_0 + C(t)x_1 + \int_0^t C(t-s)h(s)\,ds.$$

Regularity of mild solutions of the problem (5)-(7) was treated by Travis & Weeb in [17], by Bochenek in [4] and more recently, by Henrquez & Vasquez in [8]. In this work, we consider the next concepts of strong and classical solutions of (5)-(7), see [8] for details.

Definition 1 *A function $u : I \to X$ is a classical solution of (5)-(7) if $u(\cdot)$ is a function of class C^2 that satisfies (5) and the initial conditions (6) and (7).*

Definition 2 *A function $u : I \to X$ is a strong solution of (5)-(7) if $u(\cdot) \in W^{2,1}(I; X)$, the equation (5) is satisfied a.e. $t \in I$ and the conditions (6) and (7) are verified.*

This paper has four sections. In section 2, we discuss the existence of mild solutions for some second order partial differential problems. Using the ideas and concepts in Travis & Weeb [16, 17] and Henrquez & Vazquez [8], in the section 3, we study the regularity of the mild solutions of the problems studied in section 2. The section 4 is reserved for examples.

The terminologies and notations are those generally used in functional analysis. In particular, if $(Z, \|\cdot\|_Z)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, we indicate by $\mathcal{L}(Z : Y)$ the Banach space of the bounded linear operators of Z in Y and we abbreviate this notation to $\mathcal{L}(Z)$ whenever $Z = Y$. Along this paper, $B_r(x : Z)$ denotes the closed ball with center at x and radius $r > 0$ in $(Z, \|\cdot\|_Z)$. Additionally, for a bounded function $\xi : [0, a] \to Z$ and $0 \le t \le a$ we will employ the notation $\xi_{Z,t}$ for

$$\xi_{Z,t} = \sup\{\| \xi(s) \|_Z : s \in [0, t]\},$$

and we will write simply ξ_t in the place of $\xi_{Z,t}$ when no confusion arises. Finally, we remark that the prefix $\mathcal{R}$ is used to indicate the image of a map.

2 Existence of mild solutions

In this section, we discuss the existence of mild solutions for the abstract nonlocal Cauchy problem (1)-(3). We always assume that $N \geq 1$ and $\tilde{N} \geq 1$ are constants such that $\|C(t)\| \leq N$ and $\|S(t)\| \leq \tilde{N}$ for every $t \in I$.

Initially, we study the second order system

$$\begin{aligned} \frac{d}{dt}[x'(t) + g(t, x(t))] &= Ax(t) + f(t, x(t)), \qquad t \in I, &(9)\\ x(0) &= x_0 + p(x), &(10)\\ x'(0) &= y_0 + q(x), &(11) \end{aligned}$$

Definition 3 *A function $u : I \to X$ is a mild solution of (9)-(11) if the condition (10) is verified and*

$$\begin{aligned} u(t) &= C(t)(x_0 + p(u)) + S(t)(y_0 + q(u) + g(0, u(0))) - \int_0^t C(t-s)g(s, u(s))ds \\ &+ \int_0^t S(t-s)f(s, u(s))ds, \qquad t \in I. \end{aligned}$$

Now we establish our first existence result.

Theorem 1 *Let $x_0, y_0 \in X$ and assume that the following conditions are satisfied.*

(a) *The functions p, q are continuous and there exist positive constants l_p, l_q such that*

$$\| J(u) - J(v) \| \leq l_J \| u - v \|_a, \qquad J = p, q; \ u, v \in C(I : X).$$

(b) *The function $g(\cdot)$ is completely continuous and for every $0 < t' < t$ and all $r > 0$, the set*

$$U(t, t', r) = \{S(t')f(s, x) : s \in [0, t], x \in B_r(0, X)\}$$

is relatively compact in X.

(c) *For every $r > 0$ there exist positive constants α_r^g, α_r^f such that $\| g(t, x) \| \leq \alpha_r^g$ and $\| f(t, x) \| \leq \alpha_r^f$ for every $(t, x) \in I \times B_r(0, X)$.*

If

$$(Nl_p + \tilde{N}l_q) + \liminf_{r \to +\infty} \left(\frac{\tilde{N}\alpha_r^g + (N\alpha_r^g + \tilde{N}\alpha_r^f)a}{r} \right) < 1, \tag{12}$$

then there exists a mild solution of (9)-(11).

Proof. On the space $Y = C(I : X)$ endowed with the norm $\| \cdot \|_a$, we define the operator $\Gamma : C(I : X) \to C(I : X)$ by

$$\begin{aligned} \Gamma u(t) &= C(t)(x_0 + p(u)) + S(t)(y_0 + q(u) + g(0, u(0))) - \int_0^t C(t-s)g(s, u(s))ds \\ &+ \int_0^t S(t-s)f(s, u(s))ds. \end{aligned}$$

We affirm that there exists $r > 0$ such that $\Gamma(B_r(0,Y)) \subset B_r(0,Y)$. In fact, if the affirmation is false, for every $r > 0$ there exists $x^r \in C(I : X)$ such that $\| \Gamma x^r \|_a \geq r$. Consequently,

$$\begin{aligned} r \leq \| x^r \| &\leq N(\| x_0 \| + l_p r + \| p(0) \|) + \tilde{N}(\| y_0 \| + l_q r + \| q(0) \| + \alpha_r^g) \\ &\quad + N \int_0^a \alpha_r^g ds + \tilde{N} \int_0^a \alpha_r^f ds, \end{aligned}$$

and then

$$1 \leq (N l_p + \tilde{N} l_q) + \liminf_{r \to +\infty} \left(\frac{\tilde{N}\alpha_r^g + (N\alpha_r^g + \tilde{N}\alpha_r^f)a}{r} \right)$$

which is an absurd.

Let $r > 0$ be such that $\Gamma(B_r(0,Y)) \subset B_r(0,Y)$. Next, we prove that Γ is a condensing operator on B_r and to this end we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where

$$\Gamma_2 u(t) = S(t)g(0,u(0)) - \int_0^t C(t-s)g(s,u(s))ds + \int_0^t S(t-s)f(s,u(s))ds.$$

If $u, v \in C(I : X)$, from the assumptions we find that

$$\| \Gamma_1 u(t) - \Gamma_1 v(t) \| \leq \left(N l_p + \tilde{N} l_q \right) \| u - v \|_a, \qquad t \in I,$$

which from (11) imply that Γ_1 is a contraction on $B_r(0,Y)$.

In the sequel, by using the Ascoli-Arzela criterion, we prove that Γ_2 is completely continuous on $B_r(0,Y)$. We divide this part of the proof in two steps.

Step 1 The set $\Gamma_2(B_r(0,Y))(t) = \{\Gamma_2 u(t) : u \in B_r(0,Y)\}$ is relatively compact for all $t \in I$.

Let $t \in I$. Since $C(\cdot)$ is strongly continuous and $g(\cdot)$ is completely continuous, the sets $U_r^1 = \{S(t)g(0,u(0)) : u \in B_r(0,Y)\}$ and $U_r^2 = \{-C(s)g(\theta,x) : s, \theta \in I, x \in B_r(0,X)\}$ are relatively compact in X. On the other hand, using that $s \to S(s)$ is uniformly continuous with values in $\mathcal{L}(X)$, we fix $0 = s_1 < s_2 < \ldots < s_k = t$ such that

$$\| S(s) - S(s') \|_{\mathcal{L}(X)} \leq \epsilon,$$

when $s, s' \in [s_i, s_{i+1}]$ for some $i = 1, 2, ..k-1$. From these remarks and the mean value theorem for the Bochner integral, see [9] Lemma 2.1.3; for $u \in B_r(0,Y)$ we get

$$\begin{aligned} \Gamma_2 u(t) &= S(t)g(0,u(0)) - \int_0^t C(t-s)g(s,u(s))ds + \sum_{i=1}^{k-1} \int_{s_i}^{s_{i+1}} S(s_i)f(t-s,u(t-s))ds \\ &\quad + \sum_{i=1}^{k-1} \int_{s_i}^{s_{i+1}} (S(s) - S(s_i))f(t-s,u(t-s))ds \\ &\in U_r^1 + t\overline{conv(U_r^2)} + \sum_{i=1}^{k-1} (s_{i+1} - s_i)\overline{conv(U(t,s_i,r))} + \epsilon a \alpha_r^f B_1(0,X), \end{aligned}$$

where $conv(U)$ denotes the convex hull of U. Thus, $\Gamma_2(B_r(0,Y))(t)$ is totally bounded and consequently relatively compact in X.

Step 2 The set $\Gamma_2(B_r(0,Y))$ is equicontinuous on I.

Let $t \in I$ and $0 < \epsilon$. Using that $g(\cdot)$ is completely continuous, we fix $\delta > 0$ such that

$$\| C(s)g(\theta,x) - C(s')g(\theta,x) \| < \epsilon, \qquad \theta \in I,\ \| x \| \leq r,$$

when $|s - s'| \leq \delta$ and $s, s' \in I$. Under the previous condition; for $u \in B_r(0,Y)$ and $0 < |h| < \delta$ with $t + h \in I$ we get

$$\begin{aligned}
&\| \Gamma_2 u(t+h) - \Gamma_2 u(t) \| \\
&\leq\ hN(\| y_0 \| + \| q(u) \| + \| g(0,u(0)) \|) \\
&\quad + \int_0^t \| (C(t+h-s) - C(t-s))g(s,u(s)) \| \, ds + N \int_t^{t+h} \| g(s,u(s)) \| \, ds \\
&\quad + \int_0^t \| (S(t+h-s) - S(t-s))f(s,u(s)) \| \, ds + \int_t^{t+h} \| S(t+h-s)f(s,u(s)) \| \, ds \\
&\leq\ hN(\| y_0 \| + l_q r + q(0) + \alpha_r^g) + \epsilon t + N\alpha_r^g h + N\alpha_r^f a h + \tilde{N}\alpha_r^f h,
\end{aligned}$$

which proves the equicontinuity at t.

It follows from **Step 1** and **2** that Γ_2 is completely continuous on B_r.

The previous remarks and the Sadovskii point fixed theorem permit to conclude that Γ has a fixed point in $B_r(0,Y)$. This point is a mild solution of (9)-(11). The proof is complete. ■

Using similar arguments we can prove the next result.

Proposition 1 *Let x_0, $y_0 \in X$ and assume that the following conditions are satisfied.*

(a) *The functions p, q, g are continuous and there exist positive constants l_p, l_q, l_g such that*

$$\begin{aligned}
\| J(u) - J(v) \| &\leq l_J \| u - v \|_a, \qquad J = p, q;\ u, v \in C(I : X). \\
\| g(t,x) - g(t,y) \| &\leq l_g \| x - y \|, \qquad x, y \in X.
\end{aligned} \tag{13}$$

(b) *For every $0 < t' < t$ and all $r > 0$, the set $U(t,t',r) = \{S(t')f(s,x) : s \in [0,t], x \in B_r(0,X)\}$ is relatively compact in X.*

(c) *For every $r > 0$ there exist positive constants α_r^f such that $\| f(t,x) \| \leq \alpha_r^f$ for every $(t,x) \in I \times B_r(0,X)$.*

If

$$(Nl_p + \tilde{N}l_q) + (\tilde{N} + Na)l_g + \tilde{N}a \liminf_{r \to +\infty} \frac{\alpha_r^f}{r} < 1, \tag{14}$$

then there exists a mild solution of (9)-(11).

Theorem 2 *Let $x_0, y_0 \in X$ and assume that there exist positive constants l_f, l_g, l_q, l_p such that*

$$\| J(t,x) - J(t,y) \| \leq l_J \| x - y \|, \qquad J = f, g;\ x, y \in X, \tag{15}$$

$$\| J(u) - J(v) \| \leq l_J \| u - v \|_a, \qquad J = p, q;\ u, v \in C(I : X). \tag{16}$$

If $[N(l_p + a l_g) + \tilde{N}(l_q + l_g + a l_f)] < 1$, then there exists a unique mild solution of (9)-(11).

Proof. Let $Y = C(I : X)$ and $\Gamma : Y \to Y$ be the map defined in the proof of Theorem 1. If $u, v \in C(I : X)$; from the definition of Γ we get

$$\begin{aligned} \| \Gamma u(t) - \Gamma v(t) \| &\leq N l_p \| u - v \|_a + \tilde{N} (l_q \| u - v \|_a + l_g \| u - v \|_a) \\ &\quad + N l_g \int_0^t \| u(s) - v(s) \| ds + \tilde{N} l_f \int_0^t \| u(s) - v(s) \| ds, \end{aligned}$$

and hence

$$\| \Gamma u - \Gamma v \|_a \leq \left[N(l_p + l_g a) + \tilde{N}(l_q + l_g + a l_f) \right] \| u - v \|_a .$$

The assertion is now a consequence of the contraction mapping principle. ■

Next, we study the abstract Cauchy problem (1)-(3). In general, the results are similar at those in the first part of this section and for this reason, the proof will be summarized or simply omitted.

Definition 4 *A function $u : I \to X$ is a mild solution of (1)-(3) if $u' \in C(I : X)$; the conditions (2) and (3) are verified and*

$$\begin{aligned} u(t) &= C(t)(x_0 + p(u, \dot{u})) + S(t)(y_0 + q(u, \dot{u}) + g(0, u(0), u'(0))) \\ &\quad - \int_0^t C(t-s) g(s, u(s), \dot{u}(s)) ds + \int_0^t S(t-s) f(s, u(s), \dot{u}(s)) ds, \qquad t \in I. \end{aligned}$$

Theorem 3 *Let $(x_0, y_0) \in E \times X$ and assume that the following conditions are verified.*

(a) *The function $f : I \times X^2 \to X$ is completely continuous and for every $r > 0$ there is a constant α_r^f such that $\| f(s, x, y) \| \leq \alpha_r^f$ for every $(t, x, y) \in I \times B_r(0, X)^2$.*

(b) *The function $p(\cdot)$ is E-valued continuous and there exist constants l_p, l_q such that*

$$\begin{aligned} \| q(u_1, v_1) - q(u_2, v_2) \| &\leq l_q(\| u_1 - u_2 \|_a + \| v_1 - v_2 \|_a), \qquad u_i, v_i \in C(I : X), \\ \| p(u_1, v_1) - p(u_2, v_2) \|_1 &\leq l_p(\| u_1 - u_2 \|_a + \| v_1 - v_2 \|_a), \qquad u_i, v_i \in C(I : X). \end{aligned}$$

(c) *The function $g(\cdot)$ is completely continuous with values in E and for every $r > 0$, there exists a constant α_r^g such that $\| g(s, x, y) \|_1 \leq \alpha_r^g$ for every $(t, x, y) \in I \times B_r(0, X)^2$.*

(d) *For each $r > 0$, the set of functions*

$$\{ s \to g(s, u(s), u'(s)) : u \in C^1(I : X), \| u \|_a + \| u' \|_a \leq r \}$$

is equicontinuous on I.

If

$$(N+1) l_p + (N + \tilde{N}) l_q + \liminf_{r \to \infty} \frac{1}{r} [(N + \tilde{N}) \alpha_r^g + a(\alpha_r^g (N+1) + \alpha_r^f (N + \tilde{N}))] < 1,$$

then there exists a mild solution of (1)-(3).

Proof. Let $Y = C^1(I : X)$ be endowed with the norm $\| u \|_Y = \| u \|_a + \| u' \|_a$ and $\Gamma : Y \to Y$, the map defined by

$$\begin{aligned} \Gamma u(t) &= C(t)(x_0 + p(u,u')) + S(t)(y_0 + q(u,u') + g(0,u(0),u'(0))) \\ &\quad - \int_0^t C(t-s)g(s,u(s),u'(s))ds + \int_0^t S(t-s)f(s,u(s),u'(s))ds. \end{aligned}$$

Since $g(\cdot)$ and $p(\cdot)$ are continuous with values in E, it follows from the preliminaries that

$$\begin{aligned} (\Gamma u)'(t) &= AS(t)(x_0 + p(u,u')) + C(t)(y_0 + q(u,u') + g(0,u(0),u'(0))) \\ &\quad -g(t,u(t),u'(t)) - \int_0^t AS(t-s)g(s,u(s),u'(s))ds \\ &\quad + \int_0^t C(t-s)f(s,u(s),u'(s))ds, \end{aligned}$$

which proves that Γ is well defined and with values in Y.

Next, we prove that there exists $r > 0$ such that $\Gamma(B_r(0,Y)) \subset B_r(0,Y)$. We assume, by absurd, that for every $r > 0$ there exists $u^r \in B_r(0,Y)$ such that $r \leq \| \Gamma u^r \|_Y$. From the assumptions, we find that

$$\begin{aligned} \| \Gamma u^r \| &\leq N\left(\| x_0 \|_1 + l_p r + \| p(0,0) \|\right) + \tilde{N}(\| y_0 \| + l_q r + \| q(0,0) \| + \alpha_r^g) \\ &\quad + a(N\alpha_r^g + \tilde{N}\alpha_r^f) \end{aligned}$$

and that

$$\begin{aligned} \| (\Gamma u^r)' \| &\leq (\| x_0 \|_1 + l_p r + \| p(0,0) \|_r) + N(\| y_0 \| + l_q r + \| q(0,0) \| + \alpha_r^g) \\ &\quad + (\alpha_r^g + N\alpha_r^f)a. \end{aligned}$$

Consequently,

$$\begin{aligned} r &\leq (N+1)\left(\| x_0 \|_1 + l_p r + \| p(0,0) \|_1\right) + (N+\tilde{N})\left(\| y_0 \| + l_q r + \| q(0,0) \| + \alpha_r^g\right) \\ &\quad + a\left(\alpha_r^g(N+1) + \alpha_r^f(N+\tilde{N})\right) \end{aligned}$$

and then

$$1 \leq (N+1)l_p + (N+\tilde{N})l_q + \liminf_{r\to\infty}\left[\frac{(N+\tilde{N})\alpha_r^g + a(\alpha_r^g(N+1) + \alpha_r^f(N+\tilde{N}))}{r}\right],$$

which is contrary to the hypotheses.

Let $r > 0$ be such that $\Gamma(B_r(0,Y)) \subset B_r(0,Y)$ and consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where

$$\begin{aligned} \Gamma_2 u(t) &= S(t)g(0,u(0),u'(0)) - \int_0^t C(t-s)g(s,u(s),u'(s))ds \\ &\quad + \int_0^t S(t-s)f(s,u(s),u'(s))ds. \end{aligned}$$

From the Lipschitz assumption on p, q and the fact that $(N+1)l_p + (N+\tilde{N})l_q < 1$, it follows that Γ_1 is a contraction on $B_r(0,Y)$. On the other hand, proceeding as in the Steps 1 and 2 of the proof of Theorem 1, we infer that $\Gamma_2 : B_r(0,Y) \to C(I : X)$ is completely continuous. Moreover, using the conditions (a), (b) and (d), the ideas in the proof of Theorem 1 and the fact that the operator family $AS(\cdot)$ is strongly continuous from E into X (that is, for every $x \in E$ the function $s \to AS(s)x$ is continuous in X), we can prove that the map $\widetilde{\Gamma}_2 : B_r(0,Y) \to C(I : X)$ defined by

$$\begin{aligned}\widetilde{\Gamma}_2 u(t) &= C(t)g(0,u(0),u'(0)) - g(t,u(t),u'(t)) - \int_0^t AS(t-s)g(s,u(s),u'(s))ds \\ &\quad + \int_0^t C(t-s)f(s,u(s),u'(s))ds,\end{aligned}$$

is completely continuous. Thus, $\Gamma_2 : B_r(0,Y) \to Y$ is completely continuous.

From the previous remark, it follows that Γ is a condensing operator from $B_r(0,Y)$ into $B_r(0,Y)$ which from the Sadovskii point fixed theorem permit to conclude that there exists a mild solution of (1)-(3). The proof is finished. ■

Theorem 4 *Assume that the following conditions hold.*

(a) *The function $q(\cdot), f(\cdot)$ are continuous and there exist constants l_f, l_q such that*

$$\begin{aligned}\| f(t,x_1,y_1) - f(t,x_2,y_2) \| &\leq l_f(\| x_1 - x_2 \| + \| y_1 - y_2 \|), \\ \| q(u_1,v_1) - q(u_2,v_2) \| &\leq l_q(\| u_1 - u_2 \|_a + \| v_1 - v_2 \|_a),\end{aligned}$$

for all $x_i, y_i \in X$ and every $u_i, v_i \in C(I : X)$.

(b) *The functions $p(\cdot), g(\cdot)$ are E-valued continuous and there exist positive constants l_p, l_g such that*

$$\begin{aligned}\| g(t,x_1,y_1) - g(t,x_2,y_2) \|_1 &\leq l_g(\| x_1 - x_2 \| + \| y_1 - y_2 \|), \\ \| p(u_1,v_1) - p(u_2,v_2) \|_1 &\leq l_p(\| u_1 - u_2 \|_a + \| v_1 - v_2 \|_a),\end{aligned}$$

for all $x_i, y_i \in X$ and every $u_i, v_i \in C(I : X)$.

If $\max\{Nl_p + \tilde{N}(l_q + l_p) + a(Nl_g + \tilde{N}l_f),\ l_p + l_g a + N(l_q + l_g) + l_q + aNl_f\} < 1$ *and* $(x_0, y_0) \in E \times X$*, then there exists a unique mild solution of (1)-(3).*

Proof. Let $Y = C(I : X)^2$ be provided with the norm $\| (u,v) \|_a = \| u \|_a + \| v \|_a$ and $\Gamma : Y \to Y$, the map $\Gamma(u,v) = (\Gamma_1(u,v), \Gamma_2(u,v))$ where

$$\begin{aligned}\Gamma_1(u,v)(t) &= C(t)(x_0 + p(u,v)) + S(t)[y_0 + q(u,v) + g(0,u(0),v(0))] \\ &\quad - \int_0^t C(t-s)g(s,u(s),v(s))ds + \int_0^t S(t-s)f(s,u(s),v(s))ds, \\ \Gamma_2(u,v)(t) &= AS(t)(x_0 + p(u,v)) + C(t)[y_0 + q(u,v) + g(0,u(0),v(0))] \\ &\quad -g(t,u(t),v(t)) - \int_0^t AS(t-s)g(s,u(s),v(s))ds \\ &\quad + \int_0^t C(t-s)f(s,u(s),v(s))ds.\end{aligned}$$

It's clear that Γ is well defined and with values in Y. Moreover, using the assumptions, it's easy to see that for $u_i, v_i \in C(I : X)$, $i = 1, 2$,

$$\| \Gamma_1(u_1,v_1) - \Gamma_1(u_2,v_2) \| \leq [Nl_p + \tilde{N}(l_q + l_g) + a(Nl_g + \tilde{N}l_f)] \| (u_1,v_1) - (u_2,v_2) \|_a . \quad (17)$$

On the other hand, from condition (b) we get

$$\begin{aligned}
&\| \Gamma_2(u_1, v_1)(t) - \Gamma_2(u_2, v_2)(t) \| \\
&\quad \leq \ \| p(u_1, v_1) - p(u_2, v_2) \|_1 + \int_0^t \| AS(t-s)g(s, u_1(s), v_1(s)) - g(s, u_2(s), v_2(s)) \| ds \\
&\qquad +(N(l_q + l_g) + l_g + aNl_f) \| (u_1, v_1) - (u_2, v_2) \|_a,
\end{aligned}$$

and hence

$$\| \Gamma_2(u_1, v_1) - \Gamma_2(u_2, v_2) \|_a \leq (l_p + al_g + N(l_q + l_g) + l_g + aNl_f) \| (u_1, v_1) - (u_2, v_2) \|_a . \quad (18)$$

The assertion is now a consequence of (17), (18) and the contraction mapping principle. The proof is complete. ■

3 Regularity of mild solutions

In this section, we study the regularity of the mild solutions of the problems studied in the previous section. At first, we study regularity of mild solutions of (9)-(11).

Definition 5 *A function $u(\cdot) : I \to X$ is a strong solution of (9)-(11) if $u' \in C(I : X)$; the function $t \to u(t) + \int_0^t g(s, u(s))ds \in W^{2,1}(I : X)$; $u(t) \in D(A)$ a.e. $t \in I$; the equation (9) is satisfied a.e. $t \in I$ and the initial conditions (10)-(11) are verified.*

Definition 6 *A function $u(\cdot) : I \to X$ is a classical solution of the problem (9)-(11) if $u' \in C(I : X)$; the function $t \to u(t) + \int_0^t g(s, u(s))ds$ is of class C^2 on I; $u(t) \in D(A)$ for every $t \in I$ and the initial conditions (10)-(11) are verified.*

In the rest of the paper, $\mathbf{H_g}$ is the following condition.

$\mathbf{H_g}$ *The function $g(\cdot)$ is continuous with values in $[D(A)]$.*

In some of our next results, we consider Banach spaces that have the Radon-Nikodym property (abbreviated, *RNP*). We refer to [10] for a discussion about this matter.

At first, we observe that if g is $D(A)$-valued continuous, a function $u(\cdot)$ is a mild solution of (9)-(11) if and only if

$$\begin{aligned}
u(t) + \int_0^t g(s, u(s))ds \ &= \ C(t)(x_0 + p(u)) + S(t)[y_0 + q(u) + g(0, u(0))] \\
&\quad - \int_0^t S(t-s)A \int_0^s g(\tau, u(\tau))d\tau ds + \int_0^t S(t-s)f(s, u(s))ds.
\end{aligned} \quad (19)$$

Moreover, if $u(\cdot)$ is a mild solution of (9)-(11) and $(x_0 + p(u), y_0 + q(u)) \in E \times X$, then $u(\cdot)$ is differentiable and

$$\begin{aligned}
u'(t) + g(t, u(t)) = \ &AS(t)(x_0 + p(u)) + C(t)[y_0 + q(u) + g(0, u(0))] \\
&- \int_0^t S(t-s)Ag(s, u(s))ds + \int_0^t C(t-s)f(s, u(s))ds.
\end{aligned} \quad (20)$$

Theorem 5 *Assume that X has the RNP property and that condition* $\mathbf{H_g}$ *holds. Suppose, furthermore, that* $(x_0 + \mathcal{R}_{\surd}, \dagger, + \mathcal{R}\Pi) \subset \mathcal{D}(A) \times \mathcal{E}$ *and that for every bounded set* $D \subset I \times X$ *the set of functions* $\{\theta \to C(\theta)f(t,x) : (t,x) \in D\}$ *is uniformly Lipschitz continuous. Then a mild solution of (10)-(11) is a strong solution.*

Proof. Let $u(\cdot)$ be a mild solution of (10)-(11) and $y(\cdot)$ be the unique mild solution of the abstract Cauchy problem

$$\left\{ \begin{array}{ll} x''(t) & = Ax(t) + F(t) - A\int_0^t G(s)ds, \qquad t \in I, \\ x(0) & = x_0 + p(u), \ x'(0) = y_0 + q(u) + g(0, u(0)). \end{array} \right. \tag{21}$$

where $F(\cdot), G(\cdot) : I \to X$ are the functions defined by $G(t) = g(t, u(t))$, $F(t) = f(t, u(t))$. Using that $g(\cdot)$ is continuous with values in $[D(A)]$, we find that

$$\begin{aligned} y'(t) \ &= AS(t)(x_0 + p(u)) + C(t)(y_0 + q(u) + g(0, u(0))) + \int_0^t C(t-s)F(s)ds \\ &\quad - \int_0^t C(t-s)\int_0^s AG(\xi)d\xi ds, \end{aligned} \tag{22}$$

and so that

$$\begin{aligned} &\| y'(t+h) - y'(t) \| \\ &\leq \| (AS(t+h) - AS(t))(x_0 + p(u)) \| + \| (C(t+h) - C(t))(y_0 + q(u) + g(0, u(0)) \| \\ &\quad + \int_0^t \| (C(t+h-s) - C(t-s))F(s) \| ds + \int_t^{t+h} \| C(t+h-s)F(s) \| ds \\ &\quad + \int_0^h \| C(t+h-s) \| \int_0^s \| AG(\xi)d\xi \| ds + \int_0^t \| C(t-s) \| \int_s^{s+h} \| AG(\xi) \| d\xi ds, \end{aligned}$$

which implies that $y'(\cdot)$ is Lipschitz continuous on I and so that $y(\cdot) \in W^{2,1}(I : X)$ since X has the *RNP*. Moreover, from Proposition 3.3 in [8], we know that $y(\cdot)$ is a strong solution of (21). Finally, by the uniqueness of mild solutions of (21) we infer that $y(t) = u(t) + \int_0^t g(s, u(s))ds$ and that $u(t) \in D(A)$ *a.e.* $t \in I$ since $g(\cdot)$ is $D(A)$-valued. Thus, $u(\cdot)$ is a classical solution. ■

Remark 1 *Related with the last result, we know from [7] that the functions $C(\cdot)f(t,x)$, $(t,x) \in D$, are uniformly Lipschitz continuous on I if, for instance, $\{f(t,x) : (t,x) \in D\}$ is bounded in E.*

Theorem 6 *Assume that X has the RNP property and that condition* $\mathbf{H_g}$ *holds. If in addition,* $(x_0 + \mathcal{R}_{\surd}, \dagger, + \mathcal{R}\Pi) \subset \mathcal{D}(A) \times \mathcal{E}$ *and $f(\cdot)$ is Lipschitz continuous on bounded subsets of $I \times X$, then a mild solution of (9)-(11) is a classical solution.*

Proof. Let $u(\cdot)$ be a mild solution of (9)-(11). Using that X has the *RNP* and that $\xi(t) = -\int_0^t Ag(s, u(s))ds + f(t, u(t))$ is Lipschitz on I, it follows that $\xi(\cdot) \in W^{1,1}(I : X)$. From Theorem 3.1 in [8] we know that the mild solution, $y(\cdot)$, of

$$\left\{ \begin{array}{ll} x''(t) & = Ax(t) + \xi(t), \qquad t \in I, \\ x(0) & = x_0 + p(u), \ x'(0) = y_0 + q(u) + g(0, u(0)), \end{array} \right. \tag{23}$$

is a classical solution which by the uniqueness of solution of (23) permit to conclude that $y(t) = u(t) + \int_0^t g(s, u(s))ds$ is a function of class C^2 and that $u(t) \in D(A)$ for every $t \in I$ since $g(\cdot)$ is $D(A)$-valued. Thus, $u(\cdot)$ is a classical solution. ■

Using the ideas and concepts in Theorems 5 and 6, it's possible to establish similar regularity results for the mild solutions of (1)-(3), we will omit additional details. Another interesting and classical approach is by assuming that f, g are differentiable functions. In what follows, for a function $j : I \to X$ and $h \in \mathbb{R}$ we use the notation $\partial_h j(\cdot)$ for the function

$$\partial_h j(t) := \frac{j(t+h) - j(t)}{h}.$$

If $(Z, \|\cdot\|_Z)$ and $(Y, \|\cdot\|_Y)$ are Banach space and $j(\cdot) : I \times Y^2 \to Z$ is differentiable, we use the decomposition

$$\begin{aligned} j(t+s, y+y_1, w+w_1) - j(t, y, w) &= (D_1 j(t,y,w), D_2 j(t,y,w), D_3 j(t,y,w))(s, y_1, w_1) \\ &\quad + \| (s, y_1, w_1) \|_{I \times Y^2} R_Y^Z(j(t,y,w), s, y_1, w_1), \end{aligned} \tag{24}$$

where $\| R_Y^Z(j(t,y,w), s, y_1, w_1) \|_Z \to 0$ if $\| (s, y_1, w_1) \|_{I\times Y^2} \to 0$.

Theorem 7 *Let assumptions in Theorem 4 and condition* $\mathbf{H_g}$ *be satisfied. Let $u(\cdot)$ be a mild solution of (1)-(3) and assume that $f(\cdot) : I \times X^2 \to X$, $g(\cdot) : I \times X^2 \to E$ are continuously differentiable and that $(x_0 + p(u, u'), y_0 + q(u, u')) \in D(A) \times E$. If*

$$\| D_3 g(w(\cdot)) \|_{\mathcal{L}(X), a} + \int_0^a \left(\| D_3 g(w(s)) \|_{\mathcal{L}(X,E)} + \| D_3 f(w(s)) \|_{\mathcal{L}(X)} \right) ds < 1, \tag{25}$$

where $w(t) = (t, u(t), u'(t))$, then $u(\cdot)$ is a classical solution.

Proof. Let $v : I \to X$ be the unique solution of the integral equation

$$\begin{aligned} v(t) &= P(t) - D_3 g(w(t))(v(t)) - \int_0^t AS(t-s) D_3 g(w(s))(v(s))ds \\ &\quad + \int_0^t C(t-s) D_3 f(w(s))(v(s))ds, \qquad t \in I, \end{aligned} \tag{26}$$

where

$$\begin{aligned} P(t) &= C(t)A\tilde{x}_0 + AS(t)\tilde{y}_0 - D_1 g(w(t)) - D_2 g(w(t))(u'(t)) \\ &\quad - \int_0^t AS(t-s)[D_1 g(w(s)) + D_2 g(w(s))(u'(s))]ds \\ &\quad + C(t) f(0, \tilde{x}_0, \tilde{y}_0) + \int_0^t C(t-s)\left(D_1 f(w(s)) + D_2 f(w(s))(u'(s))\right) ds, \end{aligned}$$

and

$$\tilde{x}_0 = x_0 + p(u, u'), \quad \tilde{y}_0 = y_0 + q(u, u') + g(0, u(0), u'(0)).$$

The existence and uniqueness of a solution of (26) follows from the contraction mapping principle and (25), we omit additional details. Next, we prove that $u''(\cdot) = v(\cdot)$ on I. Let be $t \in I$ and $h \in \mathbb{R}$ with $t + h \in I$. From (26) we get

$$
\begin{aligned}
&\| \partial_h u'(t) - v(t) \| \\
&\quad \leq \xi_1(h,t) + \| [\partial_h C(t)] g(0, \tilde{x}_0, \tilde{y}_0) - \frac{1}{h}\int_0^h AS(t+h-s) g(w(s)) ds \| \\
&\qquad + \| -\partial_h g(w(t)) + D_1 g(w(t)) + D_2 g(w(t))(u'(t)) + D_3 g(w(t))(v(t)) \| \\
&\qquad + \int_0^t \| -\partial_h g(w(s)) + D_1 g(w(s)) + D_2 g(w(s))(u'(s)) + D_3 g(w(s))(v(s)) \|_1 \, ds \\
&\qquad + \| \frac{1}{h}\int_0^h C(t+h-s) f(w(s)) ds - C(t) f(0, \tilde{x}_0, \tilde{y}_0) \| \\
&\qquad + N \int_0^t \| \partial_h f(w(s)) - D_1 f(w(s)) - D_2 f(w(s))(u'(s)) - D_3 f(w(s))(v(s)) \| \, ds \\
&\quad \leq \xi_2(h,t) + \frac{1}{h}\int_0^h \| S(t+h-s)(Ag(w(0)) - Ag(w(s)) \| \, ds \\
&\qquad + \| D_3 g(w(t)) \|_{\mathcal{L}(X;E)} \| \partial_h u'(t) - v(t) \| \\
&\qquad + \| (1, \partial_h u(t), \partial_h u'(t)) \|_{I \times X^2} \| R_X^X(g(w(t)), h, h\partial_h u(t), h\partial_h u'(t)) \| \\
&\qquad \int_0^t \left[\| D_3 g(w(s)) \|_{\mathcal{L}(X;E)} + N \| D_3 f(w(s)) \|_{\mathcal{L}(X)} \right] \| \partial_h u'(s) - v(s) \| \, ds \\
&\qquad + \int_0^t \| (1, \partial_h u(s), \partial_h u'(s)) \|_{I \times X^2} \| R_X^E(g(w(s)), h, h\partial_h u(s), h\partial_h u'(s) \| \, ds \\
&\qquad + N \int_0^t \| (1, \partial_h u(s), \partial_h u'(s)) \|_{I \times X^2} \| R_X^X(f(w(s)), h, h\partial_h u(s), h\partial_h u'(s)) \| \, ds,
\end{aligned}
$$

where $\xi_2(h,t) \to 0$ as $h \to 0$. Since $\mu = 1 - \| D_3 g(w(t)) \|_{\mathcal{L}(X),a} > 0$, it follows that

$$
\begin{aligned}
&\| \partial_h u'(t) - v(t) \| \\
&\quad \leq \xi_3(h,t) + \mu^{-1} \| (1, \partial_h u(t), \partial_h u'(t)) \| \| R_X^X(g(w(t)), h, h\partial_h u(t), h\partial_h u'(t)) \| \\
&\qquad + \mu^{-1} \int_0^t \left[\| D_3 g(w(s)) \|_{\mathcal{L}(X;E)} + N \| D_3 f(w(s)) \|_{\mathcal{L}(X)} \right] \| \partial_h u'(s) - v(s) \| \, ds \\
&\qquad + \mu^{-1} \int_0^t \| (1, \partial_h u(s), \partial_h u'(s)) \|_{I \times X^2} \| R_X^E(g(w(s)), h, h\partial_h u(s), h\partial_h u'(s)) \| \, ds \\
&\qquad + \mu^{-1} \int_0^t \| (1, \partial_h u(s), \partial_h u'(s)) \|_{I \times X^2} \| R_X^X(f(w(s)), h, h\partial_h u(s), h\partial_h u'(s)) \| \, ds,
\end{aligned}
$$

which from the Lemma 1 and the Gronwall Bellman inequality implies that $u''(\cdot)$ exists and that $u''(\cdot) = v(\cdot)$ on I.

From these remarks and Proposition 2.4 in [17], it follows that the mild solution, $y(\cdot)$, of the abstract Cauchy problem

$$
\begin{cases}
x''(t) = Ax(t) + f(t, u(t), u'(t)) - A\int_0^t g(s, u(s), u'(s)) ds, & t \in I, \\
x(0) = \tilde{x}_0, \quad x'(0) = \tilde{y}_0,
\end{cases}
\tag{27}
$$

is a classical solution. Consequently, $y(t) = u(t) + \int_0^t g(s, u(s), u'(s)) ds$ is a function of class C^2 and $u(t) \in D(A)$ for every $t \in I$ since $g(\cdot)$ is continuous with values in $[D(A)]$. Thus, $u(\cdot)$ is a classical solution of (1)-(3). ■

Lemma 1 *Under the conditions of Theorem 7, $u'(\cdot)$ is Lipschitz on I.*

Proof. Let be $t \in I$ and $h \in \mathbb{R}$ with $t+h \in I$. Using that $s \to u(s)$ is Lipschitz continuous, we get

$$\begin{aligned}
\| u'(t+h) - u'(t) \| \;\leq\;& C_1 h + l_g \| u(t+h) - u(t) \| + l_g \| u'(t+h) - u'(t) \| \\
&+ \int_t^{t+h} \| S(t+h-s) A g(s, u(s), u'(s)) \| ds \\
&+ \int_0^t \| (S(t+h-s) - S(t-s)) A g(s, u(s), u'(s)) \| ds \\
&+ \int_0^h \| C(t+h-s) f(s, u(s), u'(s)) \| ds \\
&+ N \int_0^t [l_f \| u(s+h) - u(s) \| + l_f \| u'(s+h) - u'(s) \| + C_2 h] ds \\
\leq\;& C_3 h + l_g \| u'(t+h) - u'(t) \| + N l_f \int_0^t \| u'(s+h) - u'(s) \| ds,
\end{aligned}$$

where the constants C_i are independent of t and h. Finally, since $l_g < 1$ it follows that there exist constants C_4, C_5 independent of t and h such that

$$\| u'(t+h) - u'(t) \| \leq C_4 h + C_5 \int_0^t \| u'(s+h) - u'(s) \| ds,$$

which permit to conclude the proof. ■

4 Examples

In this section we consider some applications of our results. On the space $X = L^2([0,\pi])$ we consider the operator $Af(\xi) = f''(\xi)$ with domain $D(A) = \{f(\cdot) \in H^2(0,\pi) : f(0) = f(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a strongly continuous cosine function, $(C(t))_{t\in\mathbb{R}}$, on X. Furthermore, A has discrete spectrum, the eigenvalues are $-n^2$, $n \in N$, with corresponding normalized eigenvectors $z_n(\xi) := \left(\frac{2}{\pi}\right)^{1/2} \sin(n\xi)$ and

(a) $\{z_n : n \in N\}$ is an orthonormal basis of X.

(b) If $\varphi \in D(A)$ then $A\varphi = -\sum_{n=1}^{\infty} n^2 < \varphi, z_n > z_n$.

(c) For $\varphi \in X$, $C(t)\varphi = \sum_{n=1}^{\infty} \cos(nt) < \varphi, z_n > z_n$. Moreover, it follow from this expression that $S(t)\varphi = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < \varphi, z_n > z_n$, that $S(t)$ is compact for $t > 0$ and that $\| C(t) \| \leq 1$ and $\| S(t) \| \leq 1$ for every $t \in [0,\pi]$.

(d) If Φ denotes the group of translations on X defined by $\Phi(t)x(\xi) = \tilde{x}(\xi + t)$, where $\tilde{x}$ is the extension of x with period 2π, then $C(t) = \frac{1}{2}(\Phi(t) + \Phi(-t))$; $A = B^2$ where B is the infinitesimal generator of the group Φ and $E = \{x \in H^1(0,\pi) : x(0) = x(\pi) = 0\}$, see [6] for details.

To begin this section we consider the following problem

$$\begin{aligned}
\frac{\partial}{\partial t}\left[\frac{\partial u(t,\xi)}{\partial t}+G(t,\xi,u(t,\xi))\right] &= \frac{\partial^2 u(t,\xi)}{\partial \xi^2}+F(t,\xi,u(t,\xi)), \quad t\in I=[0,\pi], &(28)\\
u(t,0) &= u(t,\pi)=0, \qquad t\in I, &(29)
\end{aligned}$$

where $G, F : I^2\times \mathbb{R}\to\mathbb{R}$ are appropriate functions verifying the following conditions.

$(\mathbf{H_1})$ There exist functions $\eta_1^F\in \mathrm{L}^2(\mathrm{I}^2)$ and $\eta_2^F\in \mathrm{L}^1(\mathrm{I}:\mathrm{L}^\infty(\mathrm{I}:\mathbb{R}))$ such that

$$\mid F(t,\xi,w)\mid\leq \eta_1^F(t,\xi)+\eta_2^F(t,\xi)\mid w\mid, \qquad t,\xi\in I, w\in\mathbb{R}.$$

$(\mathbf{H_2})$ There exists $\eta^G\in \mathrm{L}^1(\mathrm{I}:\mathrm{L}^\infty(\mathrm{I}:\mathbb{R}))$ such that

$$\mid G(t,\xi,w_1)-G(t,\xi,w_2)\mid\leq \eta^G(t,\xi)\mid w_1-w_2\mid, \qquad t,\xi\in I, w_i\in X.$$

Associated to problem (28)-(29), we consider the nonlocal conditions

$$\begin{aligned}
w(0,\xi) &= x_0(\xi)+\int_0^\pi P(w(s,\cdot))(\xi)\,d\mu(s), \quad \xi\in I, &(30)\\
\frac{\partial w(0,\xi)}{\partial t} &= y_0(\xi)+\int_0^\pi Q(w(s,\cdot))(\xi)\,d\nu(s) \quad \xi\in I, &(31)
\end{aligned}$$

where $x_0, y_0\in X$; μ,ν are real functions of bounded variation on I and $P, Q: X\to X$ are completely continuous. In addition, we will assume that there exist positive constants α_r^P and α_r^Q such that $\|P(x)\|_2\leq \alpha_r^P$ and $\|Q(x)\|_2\leq\alpha_r^Q$ when $\|x\|_2\leq r$. We refer to [9] for examples of operators with these properties. Under the previous conditions, the problem (28)-(31) can be modelled as the abstract nonlocal Cauchy problem

$$\begin{aligned}
\frac{d}{dt}[u'(t)+g(t,u(t))] &= Au(t)+f(t,u(t)), \qquad t\in I, &(32)\\
u(0) &= x_0+q(u), &(33)\\
u'(0) &= y_0+p(u), &(34)
\end{aligned}$$

where the substituting operators $f, g: I\times X\to X$ and $p, q: C(I;X)\to X$ are defined by $g(t,x)(\xi)=G(t,\xi,x(\xi))$, $f(t,x)(\xi)=F(t,\xi,x(\xi))$, $x\in X$, and

$$p(u) = \int_0^\pi P(u(s))(\xi)d\mu(s), \quad q(u)=\int_0^\pi Q(u(s))(\xi)d\nu(s), \quad u\in C(I:X). \quad (35)$$

It is not difficult to see that the maps p, q are completely continuous, that $\|p(x)\|_2\leq \alpha_r^P V(\mu)$ and that $\|q(x)\|_2\leq\alpha_r^Q V(\nu)$ when $\|x\|_2\leq r$. Moreover, for $f(\cdot), g(\cdot)$ we find that

$$\begin{aligned}
\|f(t,x)\|_2 &\leq \left(\int_0^\pi \eta_1^F(t,\xi)^2 d\xi\right)^{1/2}+\eta_2^F(t,\cdot)_\pi\parallel x\parallel_2, \qquad t\in I, x\in X,\\
\|g(t,x)-g(t,y)\|_2 &\leq \eta^G(t,\cdot)_\pi\parallel x-y\parallel_2, \qquad t\in I, x,y\in B_r(0,X).
\end{aligned}$$

Proposition 2 *Assume that the previous conditions are satisfied. If*

$$\int_0^\pi[\eta_2^F(s,\cdot)_\pi+\eta^G(s,\cdot)_\pi]ds+\liminf_{r\to\infty}\frac{1}{r}\left[\alpha_r^P V(\mu)+\alpha_r^Q V(\nu)\right]<1,$$

then there exists a mild solution of (32)-(34).

Proof. Let $Y = C(I : X)$ and $\Gamma = \Gamma_1 + \Gamma_2 : Y \to Y$ be the maps defined in the proof of Theorem 1. Using that $S(t)$ is compact for all $t \in \mathbb{R}$ and the ideas and estimates in the cited proof, it follows that there exists $r > 0$ such that $\Gamma(B_r(0, Y)) \subset B_r(0, Y)$ and that Γ_2 is completely continuous on $B_r(0, Y)$. Finally, the existence of a mild solution follows from the Schauders point fixed theorem since Γ_1 is also completely continuous. ∎

Now we consider the problem (28)-(29) submitted to the nonlocal conditions,

$$w(0,\xi) = x_0(\xi) + \sum_{i=1}^{n} \alpha_i w(t_i, \xi), \qquad \xi \in I, \tag{36}$$

$$\frac{\partial w(0,\xi)}{\partial t} = y_0(\xi) + \sum_{i=1}^{k} \beta_i w(s_i, \xi), \qquad \xi \in I, \tag{37}$$

where $0 < t_i, s_j < \pi$ and $\alpha_i, \beta_j \in \mathbb{R}$ are prefixed numbers. In this case, the nonlocal Cauchy problem (28), (29), (36), (37) can be modelled as the abstract nonlocal Cauchy problem (32)-(34) using

$$p(u)(\xi) = \sum_{i=1}^{n} \alpha_i u(t_i, \xi) \quad \text{and} \quad q(u)(\xi) = \sum_{i=1}^{k} \beta_i u(s_i, \xi). \tag{38}$$

The next result is a consequence of Theorem 2.

Proposition 3 *Assume that there exist continuous functions $\eta^G, \eta^F : I^2 \to \mathbb{R}$ such that*

$$\begin{aligned} | F(t,\xi,x_1) - F(t,\xi,x_2) | &\le \eta^F(t) \mid x_1 - x_2 \mid, \quad t,\xi \in I, x_i \in \mathbb{R}, \\ | G(t,\xi,x_1) - G(t,\xi,x_2) | &\le \eta^G(t,\xi) \mid x_1 - x_2 \mid, \quad t,\xi \in I, x_i \in \mathbb{R}. \end{aligned}$$

If $\Theta = \sum_{i=1}^{n} \mid \alpha_i \mid + \sum_{i=1}^{k} \mid \beta_i \mid + \sup\{\eta^F(t,s) + \eta^G(t,s) : (t,s) \in I^2\} < 1$, then there exists a unique mild solution of (32)-(33).

To finish this section, we consider the differential system

$$[u'(t,\xi) + \int_0^{\pi} b(t,\eta,\xi)[u(t,\eta) + u'(t,\eta)]d\eta]' = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + F(t,\xi,u(t,\xi),u'(t,\xi)), \ t \in I, \tag{39}$$

$$u(t,0) = u(t,\pi) = 0, \qquad t \in I, \tag{40}$$

$$u(0) = x_0 + \int_0^{\pi} P(u(s),u'(s))(\xi) d\mu \tag{41}$$

$$u'(0) = y_0 + \int_0^{\pi} Q(u(s),u'(s))(\xi) d\nu(s), \tag{42}$$

To study this problem, we assume that $b(s,\eta,\xi)$, $\frac{\partial}{\partial \xi} b(s,\eta,\xi)$, $\frac{\partial^2}{\partial \xi^2} b(s,\eta,\xi)$ are functions in $C([0,\pi] : \mathrm{L}^2(\mathrm{I}^2))$ and that $b(s,\eta,\pi) = b(s,\eta,0) = 0$ for all $\eta, s \in I$. In particular, we see that

$$N := \sup_{s\in[0,\pi]} \left\{ \int_0^{\pi}\int_0^{\pi} b(s,\eta,\xi)^2 d\eta d\xi, \int_0^{\pi}\int_0^{\pi} \left(\frac{\partial^j b(s,\eta,\xi)}{\partial \xi^j}\right)^2 d\eta d\xi : j = 1,2 \right\} < \infty.$$

Moreover, we suppose that μ, ν are real functions of bounded variation on I and that $P : X \to E$ and $Q : X \to X$ are completely continuous. In addition, α_r^P, α_r^Q will be positive constants such that $\|P(x)\|_E \leq \alpha_r^P$, ($\|\cdot\|_E$ the norm in E, see (4)) and $\|Q(x)\|_2 \leq \alpha_r^Q$ for every $\|x\|_2 \leq r$. See [9] for examples of operators with these properties. Defining the maps f, g, p, q by $f(t,x,y)(\xi) = F(t, \xi, x(\xi), y(\xi))$ and

$$\begin{aligned} g(t,x,y)(\xi) &= \int_0^\pi b(\eta, \xi, t)(x(\eta) + y(\eta))d\eta, \quad x, y \in X, \\ p(u,v) &= \int_0^\pi P(u(s), v(s))(\xi) d\mu(s), \quad u, v \in C(I : X), \\ q(u,v) &= \int_0^\pi Q(u(s), v(s))(\xi) d\nu(s), \quad u, v \in C(I : X), \end{aligned}$$

the problem (39)-(42) can be modelled as the abstract Cauchy problem (1)-(3).

Theorem 8 *Assume that the previous conditions on the function $b(\cdot), P, Q$ are verified and that there exist a continuous function $\eta^F : I \to \mathbb{R}$ such that*

$$| F(t, \xi, x_1, y_1) - F(t, \xi, x_2, y_2) | \leq \eta^F(t) (| x_1 - x_2 | + | y_1 - y_2 |), \quad t, \xi \in I, x_i, y_i \in \mathbb{R}.$$

If

$$N(3 + 2\pi) + 2N_\pi^F + 2 \liminf_{r \to \infty} \frac{(\alpha_r^p + \alpha_r^q)}{r} < 1, \tag{43}$$

then there exists a mild solution of (39)-(42). Moreover, if $u(\cdot)$ is a mild solution of (39)-(42), $N < 1$ and $(x_0 + p(u,u'), y_0 + q(u,u')) \in D(A) \times E$, then $u(\cdot)$ is a classical solution.

Proof. At first, we observe that a straightforward estimation using the assumptions on the function $b(\cdot)$ shows that $g(\cdot)$ is continuous with values in $[D(A)]$ and that $\| g(t) \|_{\mathcal{L}(X^2, [D(A)])} \leq N$ for every $t \in [0, \pi]$. Let Y and $\Gamma(u,v) = (\Gamma_1(u,v), \Gamma_2(u,v))$ be as in the proof of Theorem 3. Using (43) and similar estimates at those in the proof of Theorem 3, it follows that there exists $r > 0$ such that $\Gamma(B_r(0,Y)) \subset B_r(0,Y)$. From the assumptions on $P(\cdot), Q(\cdot)$, it's easy to prove that Γ_1 is completely continuous and an easy computation prove that

$$\| \Gamma_2(u,v) - \Gamma_2(w,z) \| \left(2\pi + 1)N + 2\pi\eta_\pi^F \right) \| (u,v) - (w,z) \|, \qquad u, v, w, z \in X,$$

which implies that Γ_2 is a contraction and so that Γ is a condensing map on B_r. Thus, there exists a mild solution of (39)-(42).

Finally, we discuss the regularity assertion. From the estimates in the proof of Lemma 1 we can prove that u' is Lipschitz on I which implies that $\xi(t) = -\int_0^t Ag(s, u(s), u'(s))ds + f(t, u(t), u'(s))$ is Lipschitz and so that $\xi \in W^{1,1}(I : X)$. Since the mild solution of

$$\left\{ \begin{aligned} x''(t) &= Ax(t) + f(t, u(t), u'(t)) - A \int_0^t g(s, u(s), u'(s))ds, \qquad t \in I, \\ x(0) &= x_0 + p(u,u'), \ x'(0) = y_0 + q(u,u') + g(0, u(0), u'(0)). \end{aligned} \right. \tag{44}$$

is a classical solution, see Theorem 3.1 in [8], we conclude that $u(\cdot)$ is a classical solution. The proof is complete.

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