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Asymptotic Almost Periodic Solutions for a Second Order Differential Equation with Nonlocal Conditions

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Existência de Soluções Assintoticamente Quase Periódicas para uma Equação de Segunda Ordem com Condições Não Locais

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Resumo

Neste artigo mostramos a existência soluções assintoticamente quase periódicas para um problema de Cauchy não local modelado na forma

$$\frac{d}{dt}[x'(t) + g(t, x(t), x'(t))] = Ax(t) + f(t, x(t), x'(t)), \quad t \in I = [0, a], \tag{1}$$

$$x(0) = x_0 + p(x, x'), \tag{2}$$

$$x'(0) = y_0 + q(x, x'), \tag{3}$$

onde A é o gerador infinitesimal de um semigrupo fortemente contínuo de operadores lineares num espaço de Banach X e $f, g : [0, a] \times X \to X$, $p, q : C([0, a] : X) \to X$ são funções apropriadas.

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Abstract

In this work we establish some existence result of mild, strong and classical solutions for a class of semilinear second order partial differential equations with nonlocal conditions modelled in the form $\frac{d}{dt}(x'(t)+g(t,x(t),x'(t))) = Ax(t)+f(t,x(t),x'(t))$, where A is the the infinitesimal generator of a strongly continuous cosine function and f, g are appropriates functions.

1 Introduction

The purpose of this paper is study existence of mild, strong and classical solution for a second order partial differential equations with nonlocal conditions described in the form

$$\begin{aligned} \frac{d}{dt}(x'(t)+g(t,x(t),x'(t))) &= Ax(t)+f(t,x(t),x'(t)), \quad t \in I = [0,a], && (1)\\ x(0) &= x_0 + p(x,x'), && (2)\\ x'(0) &= y_0 + q(x,x'), && (3) \end{aligned}$$

where A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators on a Banach space X; $g(\cdot), f(\cdot) : I \times X^2 \to X$ and $p(\cdot), q(\cdot) : C(I : X) \to X$ are appropriates continuous function.

The system (1) is a generalization of the differential problem studied by Travis and Weeb in [10, 11] and more recently by Staněk in [6, 7, 8]. In the papers [10, 11] Travis and Weeb studied some important properties respect of strongly continuous cosine functions and applied their results in the study of the second order abstract Cauchy problem (7). On the other hand, Stanek studied in [6, 7, 8] different problems related to existence of solutions for a class of second order functional differential equations modelled in the form

$$\begin{aligned} (x'(t)+g(t,x(t),x'(t)))' &= f(t,x(t),x'(t)), \quad t \in I = [0,a], && (4)\\ r(x(0),x'(0),x(T)) &= \varphi(x), && (5)\\ \omega(x(0),x(T),x'(T)) &= \psi(x), && (6) \end{aligned}$$

where $r, \omega : \mathbb{R}^3 \to \mathbb{R}$, $f(\cdot), g(\cdot) \times \mathbb{R}^2 \to \mathbb{R}$, $\varphi, \psi : C^0(I : \mathbb{R}) \to \mathbb{R}$ are appropriates functions and $C^0(I : \mathbb{R})$ is a adequate subspace of $C(I : \mathbb{R})$.

It's important to observe that the problem (4)-(5) not consider partial evolution equations. This fact is part of the motivations of our paper.

Along of this section A is the infinitesimal generator of a strongly continuous cosine family, $(C(t))_{t\in\mathbb{R}}$, of bounded linear operators on X, $S(t)$ is the sine function associated with $(C(t))_{t\in\mathbb{R}}$, that is, $S(t)x = \int_0^t C(s)x ds$, $x \in X$, $t \in \mathbb{R}$, and $N \geq 1$ and $\tilde{N}$ are constants such that $\|C(t)\| \leq N$ and $\|S(t)\| \leq \tilde{N}$ for every $t \in I$. We refer the reader to [2] for the necessary concepts about cosine functions. Next we only mention a few results and notations needed to establish our results. Along of this section, $[D(A)]$ is the space

$$D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable}\},$$

endowed with the norm $\| x \|_A = \| x \| + \| Ax \|$, $x \in D(A)$. Moreover, the notation E stands for the space formed by the vectors $x \in X$ for which the function $C(\cdot)x$ is of class C^1. It was proved by Kisińsky [3] that E endowed with the norm

$$\|x\|_1 = \|x\| + \sup_{0\leq t\leq 1} \|AS(t)x\|, \qquad x \in E,$$

is a Banach space. The operator valued function $G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is a strongly continuous group of linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. From this it follows that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0$, $t \to 0$, for each $x \in E$. Furthermore, if $x : [0, \infty) \to X$ is locally integrable then $y(t) = \int_0^t S(t-s)x(s)ds$ defines an E-valued continuous function. This is an consequence of the fact that

$$\int_0^t G(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)\, ds \\ \int_0^t C(t-s)x(s)\, ds \end{bmatrix}$$

defines an $E \times X$-valued continuous function.

The existence of solutions of the second order abstract Cauchy problem

$$\begin{cases} x''(t) &= Ax(t) + h(t), \quad 0 \leq t \leq a, \\ x(0) &= x_0, \quad x'(0) = y_0, \end{cases} \tag{7}$$

where $h : [0, a] \to X$ is an integrable function, has been discussed in [10]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem it has been treated in [11]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)h(s)\, ds, \quad 0 \leq t \leq a, \tag{8}$$

is called, mild solution of (7) and that when $x_0 \in E$, $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)x_0 + C(t)y_0 + \int_0^t C(t-s)h(s)\, ds.$$

The paper has four sections. In section 2 we discuss existence of mild solution of some partial second order differential problems with nonlocal conditions. In section 3, we study the existence of global and quase-periodic solutions. In the section 4 a example is studied.

Finally, we remark that for the proof of some results in this paper we shall make use of the following well know result.

Theorem 1 *[5]* (**Leray Schauder Alternative**) *Let D a convex subset of a Banach space X and assume that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda F(x),\ 0 < \lambda < 1\}$ is unbounded or the map F has a fix point in D.*

2 Existence of mild solutions

Along of this paper, we always assume that N and $\tilde{N}$ are positive constants such that $\|C(t)\| \leq M$ and $\|S(t)\| \leq N$ for all $t \geq 0$. Initially we study the initial value problem

$$\begin{aligned} \frac{d}{dt}[x'(t) - g(t, x(t))] &= Ax(t) + f(t, x(t)), \qquad t \in I, & (9)\\ x(0) &= x_0 + p(x), & (10)\\ x'(0) &= y_0 + q(x), & (11) \end{aligned}$$

and to this end we introduce the followings conditions.

(H_1) The functions $f, g : I \times X \to X$ satisfy the following Caratheodory conditions:

(i) $f(t, \cdot),\ g(t, \cdot) : X \to X$ are continuous a.e. $t \in I$;

(ii) For each $x \in X$, the functions $f(\cdot, x), g(\cdot, x) : I \to X$ are strongly measurable.

(H_2) There exist continuous functions $m_f(\cdot), m_g(\cdot) : [0, \infty) \to [0, \infty)$ and continuous nondecreasing functions $W_f(\cdot), W_g(\cdot) : [0, \infty) \to (0, \infty)$ such that

$$\begin{aligned} \| f(t, x) \| &\leq m_f(t) W_f(\| x \|), \qquad (t, x) \in I \times X,\\ \| g(t, x) \| &\leq m_g(t) W_g(\| x \|), \qquad (t, x) \in I \times X. \end{aligned}$$

If $u(\cdot)$ is a solution of (9)-(11) and g is $D(A)$-valued with $Ag \in \mathrm{L}^1(\mathrm{I} : \mathrm{X})$ then

$$\frac{d}{dt}[u(t) - \int_0^t g(s, u(s))ds] = A(u(t) - \int_0^t g(s, u(s))ds) + A\int_0^t g(s, u(s))ds + f(t, u(t)),$$

which imply that

$$\begin{aligned} u(t) &= C(t)(x_0 + p(u)) + S(t)[y_0 - g(0, x_0) + q(u)] + \int_0^t g(s, u(s))ds\\ &\quad + \int_0^t AS(t-s)\int_0^s g(\tau, u(\tau))d\tau ds + \int_0^t S(t-s)f(s, u(s))ds. \qquad (12) \end{aligned}$$

The expression (12) is the motivation of the following definition.

Definition 1 *A function $x : I \to X$ is a mild solution of the problem (1)-(3) if condition (2) hold and*

$$\begin{aligned} x(t) &= C(t)(x_0 + p(x)) + S(t)[y_0 - g(0, x(0)) + q(x)] + \int_0^t C(t-s)g(s, x(s))ds \\ &\quad + \int_0^t S(t-s)f(s, x(s))ds, \qquad t \in I. \end{aligned}$$

Theorem 2 *Assume that conditions* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *are verified and that $g(\cdot)$ is completely continuous. Suppose, furthermore, that the followings conditions hold.*

(a) *For every $0 < t' \leq t$ and $r > 0$, the set $U(t, t', r) = \{S(t')f(s, \psi) : s \in [0, t], \| \psi \| \leq r\}$ is relatively compact in X.*

(b) *The function $p(\cdot)$ is completely continuous and there exist $N_p > 0$ such that $\| p(x) \| \leq N_p$ for every $x \in X$.*

(c) *For every $s \in I$ and every $r > 0$, the set $V(s, r) = \{S(s)q(x) : \| x \| \leq r\}$ is relatively compact in X and there exist $N_q > 0$ such that $\| q(x) \| \leq N_q$ for every $x \in X$.*

If

$$\int_0^a (Nm_g(s) + \tilde{N}m_f(s))ds < \int_c^\infty \frac{ds}{W_g(s) + W_f(s)}, \tag{13}$$

where $c = N(\| x_0 \| + N_p) + \tilde{N}(\| y_0 \| + N_q + \sup_{\|x\| \leq N_p} \| g(0, x_0 + x) \|)$, then there exists a mild solution of (9)-(11).

Proof. On the space $C(I : X)$ we define the map $\Gamma : C(I : X) \to C(I : X)$ by

$$\begin{aligned} \Gamma x(t) &= C(t)(x_0 + p(x)) + S(t)[y_0 + q(x) - g(0, x(0))] + \int_0^t C(t-s)g(s, x(s))ds \\ &\quad + \int_0^t S(t-s)f(s, x(s))ds, \qquad t \in I. \end{aligned} \tag{14}$$

In order to use Leray Schauder Alternative, we obtain an *a priori* bound for the solution of the integral equation $x = \lambda\Gamma(x)$, $\lambda \in (0, 1)$. Let x^λ be a solution of $x^\lambda = \lambda\Gamma(x)$, $\lambda \in (0, 1)$. Using the previous notations we get

$$\begin{aligned} \| x^\lambda(t) \| &\leq N(\| x_0 \| + N_p) + \tilde{N}(\| y_0 \| + N_q + \| g(0, x(0)) \|) \\ &\quad + N \int_0^t m_g(s)W_g(\| x^\lambda(s) \|)ds + \tilde{N} \int_0^t m_f(s)W_f(\| x^\lambda(s) \|)ds. \end{aligned}$$

Denoting by $\beta_\lambda(t)$ to the right hand side of the last inequality we find that

$$\beta'_\lambda(t) \leq (Nm_g(t) + \tilde{N}m_f(t))(W_g(\beta_\lambda(t)) + W_f(\beta_\lambda(t))),$$

and hence

$$\int_c^{\beta_\lambda(t)} \frac{ds}{W_g(s) + W_f(s)} \leq \int_0^t (Nm_g(s) + \tilde{N}m_f(s))ds < \int_c^\infty \frac{ds}{W_g(s) + W_f(s)},$$

which permit to conclude that the set $\{\beta_\lambda : \lambda \in (0,1)\}$ is bounded in $C(I : X)$ and hence that $\{x^\lambda : \lambda \in (0,1)\}$ is bounded in $C(I : X)$.

Next we prove that Γ is completely continuous. To this end, we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where

$$\begin{aligned} \Gamma_1 u(t) &= C(t)(x_0 + p(u)) + S(t)[y_0 + q(u) - g(0, u(0))], && t \in I, \\ \Gamma_2 u(t) &= \int_0^t C(t-s) g(s, u(s)) ds + \int_0^t S(t-s) f(s, u(s)) ds, && t \in I. \end{aligned}$$

It's easy to show that Γ_1 is completely continuous. To prove the property for Γ_2 we use Azcoli Arzela. In the sequel $B_r = B_r(0, C(I : X))$.

Step 1 The set $\Gamma_2 B_r = \{\Gamma_2 x : x \in B_r\}$ is equicontinuous on I.

Let $t \in I$ and $\epsilon > 0$. Since $C(\cdot)$ is strongly continuous and $g(\cdot)$ is completely continuous, there exists $\delta > 0$ such that

$$\| (C(t+h) - C(t)) g(t,x) \| < \epsilon, \qquad \| x \| \leq r,\ t \in I,$$

when $| h | \leq \delta$. Using these remarks; for $x \in B_r$ and $|h| \leq \delta$ with $t + h \in I$ we get

$$\begin{aligned} &\| \Gamma_2 x(t+h) - \Gamma_2 x(t) \| \\ &\quad \leq \int_0^t \| (C(t+h-s) - C(t-s)) g(s, x(s)) \| ds + N \int_t^{t+h} \| g(s, x(s)) \| ds \\ &\qquad + \int_0^t \| S(t+h-s) - S(t-s) \| \| f(s, x(s)) \| ds + \tilde{N} \int_t^{t+h} \| f(s, x(s)) \| ds \\ &\quad \leq \epsilon t + N W_g(r) \int_t^{t+h} m_g(s) ds + N h W_f(r) \int_0^t m_f(s) ds + \tilde{N} W_f(r) \int_t^{t+h} m_f(s) ds, \end{aligned}$$

which prove the assertion.

Step 2 The set $\Gamma_2 B_r(t) = \{\Gamma_2 x(t) : x \in B_r\}$ is relatively compact in X for every $t \in I$.

Let $t \in I$ and $\epsilon > 0$. If $x \in B_r$, from the estimate

$$\| f(\theta, x(\theta)) \| \leq m_f(\theta) W_f(\| x(\theta) \|) \leq m_f(\theta) W_f(r),$$

follows that the set $U = \{f(t-s, x(t-s)) : s \in [0,t], x \in B_r\}$ is bounded in X. Using that $S : I \to \mathcal{L}(X)$ is uniformly Lipschitz on I, we can chose $0 = s_1 < s_2 < ... < s_k = t$ such that for all $i = 1, 2, ...k-1$ and every $s, s' \in [s_i, s_{i+1}]$

$$\| S(s')y - S(s)y \| < \epsilon, \qquad y \in U.$$

Employing these remarks, the mean value theorem for the Bochner integral, see [5] Lemma 2.1.3, and the fact that $V = \{C(s) g(s', x) : s, s' \in I, \| x \| \leq r\}$ is relatively compact in X; for $x \in B_r$ we get

$$\begin{aligned} \Gamma_2 x(t) &= \int_0^t C(t-s) g(s, x(s)) ds + \sum_{i=1}^{k-1} \int_{s_i}^{s_{i+1}} (S(s) - S(s_i)) f(t-s, x(t-s)) ds \\ &\quad + \sum_{i=1}^{k-1} \int_{s_i}^{s_{i+1}} S(s_i) f(t-s, x(t-s)) ds \end{aligned}$$

and hence

$$\Gamma_2 x(t) \in \overline{tco(V)} + aB_\epsilon(0,X) + \sum_{i=1}^{k-1}(s_{i+1}-s_i)\overline{co(U(t,s_i,r))}, \tag{15}$$

where $co(V)$ denote the convex hull of V. Follows from (15) that $\Gamma_2 B_r(t)$ is totally bounded and consequently relatively compact in X.

From the steps 1 and 2, we infer that $\Gamma_2 B_r$ is relatively compact in $C(I;X)$. The continuity of Γ_2 follows from the assumptions and from the Lebesgue dominated convergence Theorem, we omit additional details. Thus, the operator Γ_2 and consequently Γ are completely continuous.

Finally, the Leray Schauder Alternative Theorem permit to conclude that Γ has a fixed in $C(I:X)$ which is, obviously, a mild solution of (9)-(11). The proof is complete. ■

In the most of the situation of practical interest the sine function is compact, it's the motivation of the next result.

Corollary 1 *Let assumptions* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *be satisfied. Assume that* $g(\cdot)$ *is completely continuous and that the followings conditions are fulfilled.*

(a) *The function* f *is takes bounded sets into the bounded sets.*

(b) *The function* p *is completely continuous and there exist positive constants* N_p, N_q *such that* $\| p(x) \| \leq N_p$ *and* $\| q(x) \| \leq N_q$ *for every* $x \in X$.

If inequality (13) holds, then there exists a mild solution (9)-(11).

Theorem 3 *Assume that the assumptions* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and condition* (a) *in Theorem 2 are verified and that there exist positive constants* l_g, l_p, l_q *such that*

$$\begin{aligned}
\| g(t,y_0) - g(t,x_2) \| &\leq l_g \| y_0 - x_2 \|, && (t,x_i) \in I \times X.\\
\| p(x) - p(y) \| &\leq l_p \| x - y \|, && x,y \in X.\\
\| q(x) - q(y) \| &\leq l_q \| x - y \|, && x,y \in X.
\end{aligned}$$

If $N(l_g a + l_p) + \tilde{N}(l_q + l_g) + \tilde{N}\liminf_{r\to\infty} W_f(r)\int_0^a m_f(s)ds < 1$, *then there exists a mild solution of (9)-(11).*

Proof. Let Γ the map defined in the proof of Theorem 2 and consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where

$$\begin{aligned}
\Gamma_1 x(t) &= C(t)(x_0 + p(x)) + S(t)[y_0 - g(0,x(0)) + q(x)] + \int_0^t C(t-s)g(s,x(s))ds,\\
\Gamma_2 x(t) &= \int_0^t S(t-s)f(s,x(s))ds, \quad t \in I.
\end{aligned}$$

We affirm that there exist $r > 0$ such that $\Gamma(B_r) \subset B_r$. In fact, if the affirmations is false, for each $r > 0$ there exist $x^r \in B_r$ such that $\| \Gamma x^r \|_a > r$ which imply that

$$\begin{aligned}
r &\leq \| \Gamma x^r \|_a\\
&\leq N[\| x(0) \| + l_p r + \| p(0) \|] + \tilde{N}(\| y_0 \| + l_g r + g(0,0) \| + l_q r + \| q(0) \|)\\
&\quad + N\int_0^a [l_g \| x^r(s) \| + \| g(s,0) \|]ds + \tilde{N}\int_0^a m_f(s)W_f(\| x^r(s) \|)ds
\end{aligned}$$

and so that

$$1 \leq N(l_g(a+l_p) + \tilde{N}(l_q+l_g) + \tilde{N} \liminf_{r\to\infty} W_f(r) \int_0^a m_f(s)ds,$$

which is an absurd.

Let $r_0 > 0$ such that $\Gamma(B_{r_0}) \subset B_{r_0}$. Using the steps in the proof of Theorem 2, follow that Γ_2 is completely continuous and from the estimate

$$\| \Gamma_2 u - \Gamma_2 v \|_a \;\leq\; \Big(N(l_g a + l_p) + \tilde{N}(l_q+l_g)\Big) \| u - v \|_a,$$

that Γ_1 is a contraction. Thus, Γ is a condensing map on B_{r_0} and from the Sadovskii point fixed Theorem there exists a mild solution of (9)-(11). The proof is finished. ■

Next we discuss existence of solution for the abstract Cauchy problem (1). In general, the results are similar to those in the first part of this section and for this reason, the proof of the next results will be summarized or simply omitted. To study these problem we introduce the next technical conditions.

($\tilde{\mathbf{H}}_1$) The function $f : I \times X \times X \to X$ satisfy the following conditions:

(i) $f(t,\cdot) : X \times X \to X$ is continuous a.e. $t \in I$;

(ii) For each $(x,y) \in X \times X$, the function $f(\cdot,x,y) : I \to X$ is strongly measurable;

(iii) There exists a continuous function $m_f : [0,\infty) \to [0,\infty)$ and a continuous nondecreasing function $W_f : [0,\infty) \to (0,\infty)$ such that

$$\| f(t,x,y) \| \leq m_f(t) W_f(\| x \| + \| y \|), \qquad (t,x,y) \in I \times X \times X.$$

($\tilde{\mathbf{H}}_2$) The function g is E-valued and verifies the following conditions

(i) $g(t,\cdot) : X^2 \to E$ is continuous a.e. $t \in I$;

(ii) For each $x \in X$, the function $g(\cdot,x) : I \to E$ is strongly measurable.

(iii) There exists a continuous function $\widetilde{m}_g : [0,\infty) \to [0,\infty)$ and a continuous nondecreasing function $\widetilde{W}_g : [0,\infty) \to (0,\infty)$ such that

$$\| g(t,x,y) \|_1 \;\leq\; \widetilde{m}_g(t) \widetilde{W}_g(\| x \| + \| y \|), \qquad (t,x,y) \in I \times X.$$

(iv) There exist positive constants c_1, c_2 such that

$$\| g(t,x,y) \| \leq c_1(\| x \| + \| y \|) + c_2, \qquad (t,x,y) \in I \times X.$$

($\mathbf{H_g}$) For every $r > 0$, the set of functions $\{t \to g(t,x(t),y(t)) : x,y \in B_r(0,C(I:X))\}$ is equicontinuos on I.

Theorem 4 *Let assumption ($\tilde{\mathbf{H}}_1$), ($\tilde{\mathbf{H}}_2$) and ($\mathbf{H_g}$) be satisfied. Assume in addition that the followings conditions hold.*

(a) *For each $r > 0$, the set $g(I \times B_r(0,X^2)) \times f(I \times B_r(0,X^2)$ is relatively compact in $E \times X$.*

(b) *The functions $p(\cdot), q(\cdot)$ are completely continuous with values in E and X respectively and there exist constants N_p, N_q such that $\| p(y) \|_1 \leq N_p$ and $\| q(x) \| \leq N_q$ for every $y \in E$ and every $x \in X$.*

$$\textit{If } \mu = 1 - c_1 > 0 \textit{ and } \mu \int_0^t \left[(N+1)\tilde{m}_g(s) + (\tilde{N}+N)m_f(s)\right] ds < \int_{c\mu^{-1}}^{\infty} \frac{ds}{\tilde{W}_g(s) + W_f(s)}$$

where

$$c = N(\| x_0 \| + N_p) + \| x_0 \|_1 + N_p + (\tilde{N}+N)[\| y_0 \| + c_1(\| x_0 \| + \| y_0 \| + N_p + N_q) + c_2] + c_2,$$

then there exists a mild solution of (1)-(3).

Proof. Let $Y = C(I : X)$ and $\Gamma(\cdot) : Y^2 \to Y^2$ the map $\Gamma(u,v) = (\Gamma_1(u,v), \Gamma_2(u,v))$ where

$$\begin{aligned}
\Gamma_1(u,v)(t) &= C(t)(x_0 + p(u,v)) + S(t)[y_0 + q(u,v) - g(0, x_0 + p(u,v), v(0) + q(u,v))] \\
&\quad + \int_0^t C(t-s)g(s,u(s),v(s))ds + \int_0^t S(t-s)f(s,u(s),v(s))ds, \\
\Gamma_2(u,v)(t) &= AS(t)(x_0 + p(u,v)) + C(t)[y_0 - g(0, x_0 + p(u,v), v(0) + q(u,v)) + q(u,v)] \\
&\quad + g(t,u(t),v(t)) + \int_0^t AS(t-s)g(s,u(s),v(s))ds \\
&\quad + \int_0^t C(t-s)f(s,u(s),v(s))ds.
\end{aligned}$$

Clearly Γ is well defined and continuous. If $(x^\lambda, y^\lambda) = \lambda\Gamma(x^\lambda, y^\lambda)$, $\lambda \in (0,1)$, by using the inequality $\| g(0, x_0 + p(u,v), v(0) + q(u,v)) \| \leq c_1(\| x_0 \| + \| y_0 \| + N_p + N_q) + c_2$ we get

$$\begin{aligned}
\| x^\lambda(t) \| &\leq N(\| x_0 \| + N_p) + \tilde{N}[\| y_0 \| + c_1(\| x_0 \| + \| y_0 \| + N_p + N_q) + c_2] \\
&\quad + N \int_0^t \widetilde{m}_g(s)\widetilde{W}_g(\| x^\lambda(s) \| + \| y^\lambda(s) \|)ds \\
&\quad + \tilde{N} \int_0^t m_f(s)W_f(\| x^\lambda(s) \| + \| y^\lambda(s) \|)ds
\end{aligned}$$

and that

$$\begin{aligned}
\| y^\lambda(t) \| &\leq \| x_0 \|_1 + N_p + N[\| y_0^\lambda \| + c_1(\| x_0 \| + \| y_0 \| + N_p + N_q) + c_2] \\
&\quad + c_1(\| x^\lambda(t) \| + \| y^\lambda(t) \|) + c_2 + \int_0^t \widetilde{m}_g(s)\widetilde{W}_g(\| x^\lambda(s) \| + \| y^\lambda(s) \|)ds \\
&\quad + N \int_0^t m_f(s)W_f(\| x^\lambda(s) + \| y^\lambda(s) \|)ds.
\end{aligned}$$

Using the notation $u^\lambda(t) = \| x^\lambda(t) \| + \| y^\lambda(t) \|$ and the previous inequalities, we find that

$$\begin{aligned}
&u^\lambda(t) \leq \\
&\quad N(\| x_0 \| + N_p) + \| x_0 \|_1 + N_p + (\tilde{N}+N)[\| y_0 \| + c_1(\| x_0 \| + \| y_0 \| + N_p + N_q) + c_2] \\
&\quad + c_1 u_\lambda(t) + c_2 + (N+1)\int_0^t \widetilde{m}_g(s)\widetilde{W}_g(u^\lambda(s))ds + (\tilde{N}+N)\int_0^t m_f(s)W_f(u^\lambda(s))ds
\end{aligned}$$

which imply that

$$u^\lambda(t) \le c\mu^{-1} + \mu^{-1}\left[(N+1)\int_0^t \widetilde{m}_g(s)\widetilde{W}_g(u_\lambda(s))ds + (\tilde{N}+N)\int_0^t m_f(s)W_f(u^\lambda(s))ds\right].$$

Defining as $\beta_\lambda(t)$ the right hand side of the last inequality, we see that

$$\beta_\lambda'(t) \le \mu^{-1}((N+1)\tilde{m}_g(t) + (\tilde{N}+N)m_f(t))(\tilde{W}_g(\beta_\lambda(t)) + W_f(\beta_\lambda(t)))$$

and hence

$$\begin{aligned}\int_{c\mu^{-1}}^{\beta_\lambda(t)} \frac{ds}{\tilde{W}_g(s)+W_f(s)} &\le \mu\int_0^t \left[(N+1)\tilde{m}_g(s) + (\tilde{N}+N)m_f(s)\right] ds \\ &< \int_{c\mu^{-1}}^{\infty} \frac{ds}{\tilde{W}_g(s)+W_f(s)},\end{aligned}$$

which prove that the set $\{z^\lambda = (x^\lambda, y^\lambda) : z^\lambda = \lambda\Gamma z^\lambda,\ \lambda \in (0,1)\}$ is bounded in Y^2.

Using the same arguments used in the proof of Theorem 2 follows that Γ_1 is completely continuous. Moreover, employing the compactness assumption on the functions g, f, the condition $(\mathbf{H_g})$ and the fact that the family $(AS(t))_{t\in\mathbb{R}}$ is a strongly continuous family of bounded linear operators from E into X, follows that Γ_2 is completely continuous and so that Γ is completely continuous. The assertion is now consequence of Theorem 1. ■

3 Quase-Periodic Solutions

In this section we discuss existence of quase periodic solutions for the abstract nonlocal Cauchy problem (9)-(11). In addition to $(\mathbf{H_1}), (\mathbf{H_2})$, along of this section we always assume that the next assumptions is verified.

$(\mathbf{H_3})$ The function $g(\cdot) : [0,\infty)$ is E-valued and the following properties

(i) $g(t,\cdot) : X \to E$ is continuous a.e. $t \in I$;

(ii) For each $x \in X$, the function $g(\cdot, x) : I \to E$ is strongly measurable.

(iii) there exist a continuous function $\tilde{m}_g(\cdot) : [0,\infty) \to [0,\infty)$ and a continuous nondecreasing function $\tilde{W}_g(\cdot) : [0,\infty) \to (0,\infty)$ such that

$$\| g(t,x) \|_1 \le \tilde{m}_g(t)W_g(\| x \|), \qquad (t,x) \in I \times X.$$

To begin we study the existence of global solutions.

3.1 Existence of global solutions

Next, we study existence of solutions in the space of continuous functions with weight. Let $h : [0,\infty) \to (0,\infty)$ be a continuous function increasing with $h(0) = 1$ and such that such $h(t) \to \infty$, as $t \to \infty$. In what follow, $C_h^0(X)$ is the space formed by the continuous functions $x : [0,\infty) \to X$ such that $\dfrac{\| x(t) \|}{h(t)} \to 0,\ t \to \infty,$ endowed with the norm $\|x\|_h = \sup_{t\geq 0} \dfrac{\|x(t)\|}{h(t)}$ and $C_0(X)$ is the space formed by the continuous functions $x : [0,\infty) \to X$ that vanish at infinity endowed with the uniform convergence topology. The following is well know.

Lemma 1 *A set $B \subseteq C_0(X)$ is relatively compact if, and only if, the following conditions are fulfilled:*

(a) *B is equicontinuous;*

(b) *$x(t) \to 0,\ t \to \infty,$ uniformly for $x \in B$;*

(c) *The orbits $B(t)$ are relatively compact in X, for all $t \geq 0$.*

Next, we establish existence of global solutions of (9)-(11).

Theorem 5 *Assume that the following conditions are verified.*

(a) *The functions $p,\ q : C_h^0(X) \to X$ are continuous and bounded, p is completely continuous and $S(t)q$ is completely continuous for each $t \geq 0$;*

(b) *For each $t \in I,\ t' \leq t$ and $L \geq 0$ the set $\{S(t')f(s,x,y) : 0 \leq s \leq t,\ \|x\|, \|y\| \leq L\}$ is relatively compact in X.*

(c) *For every $L \geq 0$, $\dfrac{1}{h(t)} \displaystyle\int_0^t (m_g(s)W_g(Lh(s)) + m_f(s)W_f(Lh(s)))\, ds \to 0$ as $t \to \infty$.*

(d) *The function $g(\cdot)$ is completely continuous with values in E.*

If $\displaystyle\int_0^\infty \left[N m_g(s) + \tilde{N} m_f(s)\right] ds < \int_c^\infty \frac{ds}{W_g(s) + W_f(s)}$ where $c = M(\|x_0\| + N_p) + N(\|y_0\| + N_q)$, then there exists a mild solution $x \in C_h^0(X)$ of (9)-(11).

Proof. For each $x \in C_h^0(X)$ we define $\Gamma x(t)$ by means of (14). Hence,

$$\begin{aligned} \|\Gamma x(t)\| \ &\leq\ M(\|x_0\| + N_p) + N(\|y_0\| + N_q + m_g(0)W_g(\| x_0 \| + N_q) \\ &\quad + \tilde{N} \int_0^t m_g(s)W_g(\|x(s)\|)\, ds + N \int_0^t m_f(s)W_f(\|x(s)\|)\, ds. \end{aligned}$$

Since $\|x(s)\| \leq \|x\|_h h(s)$, applying condition (c) in the above expression it follows that

$$\begin{aligned} \frac{\| \Gamma x(t) \|}{h(t)} \ &\leq\ \frac{N}{h(t)}(\|x_0\| + N_p)) + \frac{\tilde{N}}{h(t)}(\|y_0\| + N_q + m_g(0)W_g(\| x_0 \| + N_q)) \\ &\quad + \frac{N}{h(t)} \int_0^t m_g(s)W_g(\|x\|_h h(s))ds + \frac{\tilde{N}}{h(t)} \int_0^t m_f(s)W_f(\|x\|_h h(s))ds \quad (16) \end{aligned}$$

converges to zero as $t \to \infty$. This shows that Γ is a map well defined from $C_h^0(X)$ into $C_h^0(X)$. To prove the continuity of Γ, we select a sequence $(x_n)_n$ which converges to x in $C_h^0(X)$. Clearly, $g(s, u_n(s)) \to g(s, u(s))$, $f(s, u_n(s)) \to f(s, u(s))$ a.e. when $n \to \infty$, and

$$\|f(s, u_n(s))\| \leq m_g(s)W_g(Lh(s)), \qquad s \geq 0,$$

$$\|g(s, u_n(s))\| \leq m_f(s) W_f(Lh(s)), \qquad s \geq 0,$$

for some constant $L \geq 0$. Since the functions on the right hand side are integrable on $[0, t]$, we conclude that $\Gamma x_n(t) \to \Gamma x(t)$, when $n \to \infty$, uniformly for t in bounded intervals. Furthermore, given $\epsilon > 0$, applying **(a)** and **(c)** we can fix t_0 so that

$$\begin{aligned}
&\frac{1}{h(t)} \|\Gamma x_n(t) - \Gamma x(t)\| \\
&\leq \frac{N}{h(t)} \|p(x_n) - p(x)\| + \frac{\tilde{N}}{h(t)} (\|q(x_n) - q(x)\| + \| g(0, x_0 + p(x_n)) - g(0, x_0 + p(x_n)) \|) \\
&\quad \frac{N}{h(t)} \int_0^t m_g(s) W_g(Lh(s)) ds + \frac{\tilde{N}}{h(t)} \int_0^t m_f(s) W_f(Lh(s)) ds \\
&\leq \epsilon,
\end{aligned}$$

for $t \geq t_0$. These implies that $\Gamma x_n \to \Gamma x$ in $C_h^0(X)$ when $n \to \infty$. Thus, Γ is continuous.

On the other hand, for $0 < \lambda < 1$ and $x_\lambda \in C_h^0(X)$ such that $\lambda \mathbf{\Gamma}(\mathbf{x}_\lambda) = x_\lambda$ we find that

$$\begin{aligned}
\|x_\lambda(t)\| \leq\ & N(\|x_0\| + N_p) + \tilde{N}(\|y_0\| + N_q + m_g W_g(\| x_0 \| + N_p)) \\
& + N \int_0^t m_g(s) W_g(\|x_\lambda(s)\|) ds + \tilde{N} \int_0^t m_f(s) W_f(\|x_\lambda(s)\|) ds.
\end{aligned}$$

Let $\alpha_\lambda(t)$ be the right hand side of the last inequality. Repeating the argument used in the proof of Theorem 2 we conclude that the set $\{\alpha_\lambda(t) : 0 < \lambda < 1,\ t \geq 0\}$ is bounded and so that $\{\|x_\lambda\|_h : 0 < \lambda < 1\}$ is also bounded.

Next, by using Lemma 1 we show that $\Gamma(B_r)$, where $B_r = B_r(0, C_0(X))$, is relatively compact for each $r > 0$. At first, we prove that $\{\frac{\Gamma_2 x}{h} : x \in B_r\}$ is equicontinuous on $[0, \infty)$. Using condition **(c)** and the properties of f, g, follows that for $\epsilon > 0$ there is $t_0 > 0$ such that

$$\begin{aligned}
\| \frac{\Gamma x(t)}{h(t)} \| \leq\ & \frac{N}{h(t)} (\|x_0\| + N_p) + \frac{\tilde{N}}{h(t)} (\|y_0\| + N_q + m_g(0) W_g(\| x_0 \| + N_p)) + \\
& \frac{N}{h(t)} \int_0^t m_g(s) W_g(rh(s)) ds + \frac{\tilde{N}}{h(t)} \int_0^t m_f(s) W_f(rh(s)) ds \leq \epsilon, \qquad (17)
\end{aligned}$$

for every $t > t_0$ and every $x \in B_r$. On the other hand, using the **Step 1** in the proof Theorem 2 and that $\frac{1}{h}$ is uniformly continuous on $[0, t_0]$, we can to infer that the set of functions $\{\frac{\Gamma x}{h} : x \in B_r\}$ is equicontinuous on $[0, t_0]$, which jointly with the estimate (17) permit to conclude that $\{\frac{\Gamma x}{h} : x \in B_r\}$ is equicontinuous on $[0, \infty)$.

From the estimate (17), it's easy to see that the condition **(b)** in Lemma 1 is verified. On the other hand, applying the hypotheses **(a)**, **(c)** and proceeding as in the proof of Theorem 2, we derive that $\{\Gamma x(t) : x \in B\}$ is relatively compact in X, for each $t \geq 0$. Thus, $\{\frac{\Gamma x}{h} : x \in B\}$ is relatively compact in $C_0(X)$ and consequently $\Gamma(B)$ is relatively compact in $C_h^0(X)$.

Finally, the existence of a mild solution of (9)-(11) follows from Theorem 1. ∎

3.2 Existence of asymptotically almost periodic solutions

We study now the existence of asymptotically almost periodic solutions of (9)-(11). For the basic concepts about almost periodic and asymptotically almost periodic functions we refer to [13]. In what follows, $AP(X)$ (resp. $AAP(X)$) is space formed by the functions $x : [0,\infty) \rightarrow X$ which are almost periodic (resp. asymptotically almost periodic), endowed with the norm of the uniform convergence. We refer to [1] for the characterization of almost periodic cosine functions and to [4] for almost periodic sine functions. The next lemma is a Ascoli-Arzela type criterion in $AAP(X)$, see [13] for details.

Lemma 2 *Let $B \subseteq AAP(X)$ be a set with the following properties:*

(a) *B is uniformly equicontinuous on $[0,\infty)$;*

(b) *B is equi-asymptotically almost periodic. That is, for every $\epsilon > 0$ there are $T_\epsilon \geq 0$ and a relatively dense set $P_\epsilon \subset [0,\infty)$ such that*

$$\|x(t+\tau) - x(t)\| \leq \epsilon, \qquad x \in B,\, t \geq T_\epsilon,\, \tau \in P_\epsilon.$$

(c) *For each $t \geq 0$, $B(t)$ is relatively compact in X.*

Then B is relatively compact in $AAP(X)$.

To establish our result we need of the following result.

Proposition 1 *Let $(Z_i, \|\cdot\|_{Z_i}), i = 1,2$ be Banach spaces; $B \subset L^1([0,\infty) : Z_1)$ and $F_1 : [0,\infty) \rightarrow \mathcal{L}(Z_1, Z_2)$, $F_2 : [0,\infty) \rightarrow \mathcal{L}(Z_2, Z_2)$ be strongly continuous functions of bounded linear operators. Assume that the followings conditions are verified;*

(a) *$\int_L^\infty F(s)x(s)ds \rightarrow 0$ in Z_2 when $L \rightarrow \infty$, uniformly for $x \in B$;*

(b) *For each $t \geq 0$, the set $\{x(s) : x \in B,\ 0 \leq s \leq t\}$ is relatively compact in Z_1,*

(c) *The operator function $F(\cdot)$ is strongly continuous, that is, for every $z \in Z_1$ the function $t \rightarrow F(t)z$ is continuous with values in Z_2.*

then the following properties hold.

(i) *The set $W = \bigcup_{0\leq\alpha\leq t\leq\infty} W(t,\alpha) = \bigcup_{0\leq\alpha\leq t\leq\infty} \{\int_\alpha^t F(s)x(s)ds : x \in B\}$ is relatively compact in Z_2.*

(ii) *If $(F_2(t))_{t\geq 0}$ is uniformly bounded and $F_2(t)\int_t^{t+h} F_1(s)x(s)ds \rightarrow 0$ in Z_2 when $L \rightarrow \infty$, uniformly for $x \in B$, then the set of functions*

$$\tilde{B} = \{t \rightarrow F_2(t)\int_t^\infty F_1(s)x(s)ds : x \in B\}$$

is relatively compact in $C_0(Z_2)$.

Proof. For each $\epsilon > 0$ we select $L \geq 0$ such that $\| \int_L^\infty F(s)x(s)ds\|_{Z_2} \leq \epsilon$, for all $x \in B$. From **(b)**, there is a compact subset of Z_1, $U_0(L)$, such that $\{x(s) : x \in B, s \in [0, L]\} \subset U_0(L)$, which in view of that the map $F(\cdot)$ is strongly continuous imply that $U_1(L) = \{F(s)x : x \in U_0(L), 0 \leq s \leq L\}$ is relatively compact in Z_2. Using the mean value Theorem for the Bochner integral, see [5] Lemma 2.1.3, we get

$$\begin{aligned}
\bigcup_{0\leq\alpha\leq t\leq L} W(t,\alpha) &= \bigcup_{0\leq\alpha\leq t\leq L} \{\int_\alpha^t F(s)x(s)ds : x \in B\} \\
&\subset \bigcup_{t\in[0,L]} t\,\overline{con(\{F(s)x(s) : s \in [0,t],\, x \in B\})} \\
&\subset \bigcup_{t\in[0,L]} t\,\overline{con(U_1(L))},
\end{aligned}$$

which imply that $\bigcup_{0\leq\alpha\leq t\leq L} W(t,\alpha)$ is relatively compact in Z_2. Now, the assertion **(i)** is consequence of the fact that

$$W \subset \bigcup_{0\leq\alpha\leq t\leq L} W(t,\alpha) + \bigcup_{L\leq\alpha\leq t\leq\infty} W(t,\alpha) \subset \bigcup_{0\leq\alpha\leq t\leq L} W(t,\alpha) + \epsilon B_1(0, Z_2).$$

To prove **(ii)** we will use Lemma 1. From **(ii)** it's obvious that for every $t \geq 0$, the set $\{F_2(t) \int_t^\infty F_1(s)x(s)ds : x \in B\}$ is relatively compact in Z_2. The condition **(b)** in Lemma 1 is consequence of the assumptions.

Let be $L > 0$ as in the first part of this proof. Since $(F_2(t))_{t\geq 0}$ is uniformly bounded and the set $U = \{\int_\alpha^\infty F_1(s)x(s)ds : x \in B,\ \alpha \geq 0\}$ is relatively compact; for $\epsilon > 0$ we can fix $\delta > 0$ such that

$$\| F_1(s')x - F_1(s)x \| \leq \epsilon, \qquad x \in U,$$

when $| s' - s | < \delta$ and $s, s' \in [0, L+1]$. If $t \geq 0$, $x \in B$ and $0 < h < \delta$ we find that

$$\begin{aligned}
&\| F_2(t+h) \int_{t+h}^\infty F_1(s)x(s)ds - F_2(t) \int_t^\infty F_1(s)x(s)ds \| \\
&\quad\leq \| (F_2(t+h) - F_2(t)) \int_{t+h}^\infty F_1(s)x(s)ds \| + \| F_2(t) \int_t^{t+h} F_1(s)x(s)ds \| \\
&\quad\leq \epsilon + \| F_2(t) \int_t^{t+h} F_1(s)x(s)ds \|
\end{aligned}$$

which prove that $\tilde{B}$ is equicontinuous at the right hand side at t. Similarly, we can prove that B is equicontinuous at the left hand side at t. Thus $\tilde{B}$ is equicontinuous and consequently relatively compact in $C_0(Z_2)$. The proof is now complete. ■

Corollary 2 *Let the assumptions in Lemma 1 be satisfied for $Z_1 = E, Z_2 = X$ and the operator family $F_1(s) = AS(s)$ and $F_2(t) = S(t)$. Assume in addition that for each $t \geq 0$, $\int_t^{t+h} \|x(s)\|ds \to 0$ as $h \to 0$ uniformly for $x \in B$. Then the set of functions*

$$B_1 = \{t \to S(t) \int_t^\infty AS(s)x(s)ds : x \in B\};$$

is relatively compact in $C_0(X)$.

Proof. The result is consequence of the Proposition 1. We only remark that the assumptions in **(ii)** of Proposition 1, follows from the relation $AS(s)S(t) = C(t+s) - C(t-s)$. ∎

By using $Z_1 = Z_2 = X$ in Proposition 1, we can prove the followings result.

Corollary 3 *Let be $B \subset \mathrm{L}^1([0,\infty) : X)$ and assume that the next conditions hold.*

(a) *For each $t \geq 0$, the set $\{x(s) : x \in B,\ 0 \leq s \leq t\}$ is relatively compact in X,*

(b) $\int_L^\infty \| x(s) \| \, ds \to 0$ *in X when $L \to \infty$ and* $\int_t^{t+h} \|x(s)\| ds \to 0$ *ad $h \to 0$ uniformly for $x \in B$.*

Then the sets of functions

$$B_{i,j} = \{t \to F_i(t) \int_t^\infty F_j(s)x(s)ds : x \in B\}, \qquad i,j \in \{1,2\},$$

where $F_1 = C(t)$ and $F_2 t) = S(t)$, are relatively compact in $C_0(X)$.

In the next results we use the notation $z_x, y_x : [0,\infty) \to X$ for the functions

$$z_x(t) = \int_0^t C(t-s)x(s)ds \quad \text{and} \quad y_x(t) = \int_0^t S(t-s)x(s)ds, \qquad t \geq 0. \tag{18}$$

Proposition 2 *Let the assumptions of Corollary 2 be satisfied. If $\int_L^\infty \| x(s) \| \, ds \to 0$ when $L \to \infty$, uniformly for $x \in B$ then the set $\Lambda = \{z_x : x \in B\}$ is relatively compact in $AAP(X)$.*

Proof. From the relation $C(t+s)x = C(t)C(s)x + AS(t)S(s)x$, $x \in X$, we can write

$$\begin{aligned} z_x(t) &= C(t)\int_0^\infty C(s)x(s)ds - C(t)\int_t^\infty C(s)x(s)ds \\ &\quad -S(t)\int_0^\infty AS(s)x(s)ds + S(t)\int_t^\infty AS(s)x(s)ds. \end{aligned}$$

From the properties of almost periodic sine functions, see [4], we deduce that the first and third term on the right hand side define almost periodic functions and from the hypothesis that the second and fourth term are functions that vanish at ∞. We infer from this that $z_x \in AAP(X)$ if $x \in B$. From Proposition 1, we infer that the sets of functions $\{C(\cdot)\int_0^\infty C(s)x(s)ds : x \in B\}$ and $\{S(\cdot)\int_0^\infty AS(s)x(s)ds : x \in B\}$ are relatively compact in $AP(X)$ since the sine function is almost periodic. On the other hand, from corollaries 2 and 3 we know that the set of functions $\{t \to C(t)\int_t^\infty C(s)x(s)ds + S(t)\int_t^\infty AS(s)x(s)ds : x \in B\}$ is relatively compact in $C_0(X)$, which proved that Λ is relatively compact in AAP. The proof is finished. ∎

Proposition 3 *Let $B \subseteq L^1([0,\infty); X)$ be a set that satisfies the hypotheses of Corollary 3. If the sine function $S(\cdot)$ is almost periodic, then the set $\Lambda' = \{y_x : x \in B\}$ is relatively compact in $AAP(X)$.*

Proof. For $x \in B$, we can write

$$\begin{aligned} y_x(t) &= S(t)\int_0^t C(s)x(s)\,ds \;-\; C(t)\int_0^t S(s)x(s)\,ds \\ &= S(t)\int_0^\infty C(s)x(s)\,ds \;-\; S(t)\int_t^\infty C(s)x(s)\,ds \\ &\quad - C(t)\int_0^\infty S(s)x(s)\,ds \;+\; C(t)\int_t^\infty S(s)x(s)\,ds. \end{aligned}$$

The first and third term on the right hand side define almost periodic functions while the second and fourth term are functions that vanish at ∞. Thus, $y_x \in AAP(X)$. From Proposition 1 we know that the integrals $\int_0^\infty C(s)x(s)ds$ and $\int_0^\infty S(s)x(s)ds$, $x \in B$, are included in a compact set of X which implies that the set formed by the functions $S(\cdot)\int_0^\infty C(s)x(s)ds - C(\cdot)\int_0^\infty S(s)x(s)ds$, $x \in B$, is relatively compact in $AP(X)$. Moreover, from Corollary 3, the set of functions $C(t)\int_t^\infty S(s)x(s)ds - S(t)\int_t^\infty C(s)x(s)ds$, $x \in B$, is relatively compact in $C_0(X)$. Collecting these remarks follows that Λ' is relatively compact in $AAP(X)$. ■

When the function $[0,\infty) \to \mathcal{L}(X)$; $t \to S(t)$ is almost periodic, in this case we said the the sine function is uniformly almost periodic, we can obtain the following variation of the previous result.

Proposition 4 *Let $B \subseteq L^1([0,\infty);X)$. Assume that the sine function is uniformly periodic, that $S(t)$ is compact for each $t > 0$ and that the next conditions hold.*

(a) $\int_L^\infty \| x(s) \| \, ds \to 0$ *in X when $L \to \infty$, uniformly for $x \in B$.*

(b) $\int_t^{t+h} \|x(s)\| ds \to 0$, $h \to 0$, *uniformly for $x \in B$.*

Then the set $\{y_x : x \in B\}$ is relatively compact in $AAP(X)$.

Now we are in conditions to establish the main result of this work.

Theorem 6 *Assume that $S(\cdot)$ is almost periodic and that the following conditions hold.*

(a) *The functions p, $q : AAP(X) \to X$ are completely continuous and bounded;*

(b) *For each $t > 0$ and each constant $L \geq 0$ the sets $\{f(s,x) : 0 \leq s \leq t,\ \|x\| \leq L\}$ and $\{g(s,x) : 0 \leq s \leq t,\ \|x\| \leq L\}$ are relatively compacts in X and E respectively.*

(c) *There exists a continuous function $\hat{g} \in \mathrm{L}^1([0,\infty))$ and a continuous nondecreasing function $\hat{W}_g : [0,\infty) \to (0,\infty)$ such that*

$$\| AS(s)g(s,x) \| \;\leq\; \hat{g}(s)\hat{W}(\| x \|), \qquad s \geq 0,\ x \in X. \tag{19}$$

If

$$\int_0^\infty (Nm_g(s) + \tilde{N}m_f(s))ds < \int_c^\infty \frac{ds}{W_g(s) + W_f(s)}, \tag{20}$$

where $c = N(\| x_0 \| + N_p) + \tilde{N}(\| y_0 \| + N_q + \sup_{\|x\| \leq N_p} \| g(0, x_0 + x) \|)$, then there exists a mild solution $x(\cdot) \in AAP(X)$ of (9)-(11).

Proof. For each $x \in AAP(X)$ we define $\Gamma x(t)$ by means of (14). From our hypotheses and proceeding as in the proofs of Propositions 2 and 3 follows that $\Gamma(x) \in AAP(X)$. If $(x_n)_n$ is a sequence in $AAP(X)$ that converges to x, then $S(t-s)f(s,x_n(s)) \to S(t-s)f(s,x(s))$ and $C(t-s)g(s,x_n(s)) \to C(t-s)g(s,x(s))$, as $n \to \infty$, a.e. on $[0,t]$, which from the inequalities

$$\begin{aligned} \|C(t-s)g(s,x_n(s)) - C(t-s)g(s,x(s))\| &\leq 2Nm_g(s)W_g(L), & (21)\\ \|S(t-s)f(s,x_n(s)) - S(t-s)f(s,x(s))\| &\leq 2\tilde{N}m_f(s)W_f(L), & (22)\end{aligned}$$

for some constant $L \geq 0$, permit to conclude that $\Gamma x_n(t) \to \Gamma x(t)$, $n \to \infty$, and that the convergence is uniform on $[0,\infty)$ since the right hand side of (21) and (22) are integrable function on $[0,\infty)$. Thus, Γ is a continuous map.

On the other hand, proceeding as in the proof of Theorem 2, we conclude that the set of functions $\{x_\lambda :\in AAP(X) : \lambda\Gamma(x_\lambda) = x_\lambda, 0 < \lambda < 1\}$ is uniformly bounded on $[0,\infty)$.

Finally, we show that Γ is completely continuous. We take a bounded set $B \subseteq AAP(X)$. Since p and q are completely continuous, the set of functions $\{C(\cdot)(x_0+p(x)) + S(\cdot)(y_0+q(x));\ x \in B\}$ is relatively compact in $AAP(X)$. In addition, since the set of functions $\Lambda = \{g(s,x(s)) : x \in B\}$ and $\Lambda' = \{f(s,x(s)) : x \in B\}$ satisfies the hypotheses of Propositions 2 and 3, we infer that $\{z_x, y_x : x \in B\}$, where z_x, y_x are defined in (18), is relatively compact in $AAP(X)$. The assertion is now consequence of Theorem 1. ■

The proof of the next result will be omitted.

Theorem 7 *Assume that $t \to S(\cdot)$ is uniformly almost periodic, that $S(t)$ is compact for each $t > 0$ and that the condition* **(c)** *and the inequality 20 of Theorem 6 hold. If*

(a) *The function p, q are bounded and $p : AAP(X) \to X$ is completely continuous.*

(b) *The function $f(\cdot)$ takes bounded sets into the bounded sets and $g(\cdot)$ is completely continuous with values in E.*

then there exists a mild solution $x(\cdot) \in AAP(X)$ of (9)-(11).

Next we establish conditions under which the derivative of a asymptotically almost periodic mild solution of (9)-(11) is also a asymptotically almost periodic. To this end, we need the following result.

Lemma 3 *Assume that $S(\cdot)$ is almost periodic. Then, for each $x \in X$ the function $S(\cdot)x$ is almost periodic for the norm in E. Furthermore, if $x : [0,\infty) \to X$ is integrable then the function $y : [0,\infty) \to E$ given by $y(t) = \int_0^t S(s)x(s)\,ds$ is continuous with range relatively compact in E.*

Proof. Since $C(\cdot)x$ and $S(\cdot)x$ are almost periodic functions it follows that $(C(\cdot)x,\ S(\cdot)x)$ is almost periodic in $X \times X$. Thus, given $\epsilon > 0$, there is a relatively dense set P_ϵ such that

$$\|C(t+\tau)x - C(t)x\| + \|S(t+\tau)x - S(t)x\| \leq \epsilon, \quad t \geq 0, \quad \tau \in P_\epsilon.$$

Using that $C(\cdot)$ is uniformly bounded and

$$C(t+s)x = C(t)C(s)x \ + \ AS(t)S(s)x \tag{23}$$

we obtain

$$
\begin{aligned}
&\| S(t+\tau)x - S(t)x \|_1 \\
&\quad = \|S(t+\tau)x - S(t)x\| + \sup_{0\le h\le 1} \|AS(h)[S(t+\tau)x - S(t)x]\| \\
&\quad \le \epsilon + \sup_{0\le h\le 1} \|C(t+h+\tau)x - C(h)C(t+\tau)x + C(t)C(h)x - C(t+h)x\| \\
&\quad \le \epsilon + \sup_{0\le h\le 1} \|C(h)\|\|C(t+\tau)x - C(t)x\| + \sup_{0\le h\le 1} \|C(t+h+\tau)x - C(t+h)x\| \\
&\quad \le (N+2)\epsilon,
\end{aligned}
$$

for all $t \ge 0$ and $\tau \in P_\epsilon$, which proves the first assertion.

On the other hand, from the preliminaries we know that $y(\cdot)$ is a continuous E-valued function. Moreover, from (23) we infer that $S(\cdot)x(\cdot) \in L^1([0,\infty); E)$ which allows us to complete the demonstration with the same argument in the proof of Proposition 1. ■

Proposition 5 *Assume that the assumptions of Theorem 6 hold and let $u(\cdot)$ be a asymptotically almost periodic mild solution of (9)-(11). If the next conditions are satisfied,*

(a) *There exist $\tilde{m}_g \in L^1([0,\infty))$ and a continuous function $\tilde{W}_g(\cdot) : [0,\infty) \to [0,\infty)$ such that*

$$\| g(t,x) \|_1 \;\le\; \hat{m}_g(t)\hat{W}_g(\| x \|), \qquad (t,x) \in I \times X; \tag{24}$$

(b) *The function $t \to g(t,u(t))$ is asymptotically almost periodic;*

(c) *$x_0 - p(x) \in E$ and the operator family $(AS(t))_{t\ge 0}$ is bounded in $\mathcal{L}(E : X)$;*

then $u'(\cdot)$ is a asymptotically almost periodic.

Proof. Since $u(\cdot)$ is a mild solution and $g(\cdot)$ is E-valued we have that

$$
\begin{aligned}
u'(t) &= AS(t)(x_0 + p(u)) + C(t)(y_0 + q(u)) - g(t,u(t)) - \int_0^t AS(t-s)g(s,u(s)) \\
&\quad + \int_0^t C(t-s)f(s,u(s))ds, \qquad t \ge 0.
\end{aligned}
$$

From the assumptions and the inequality

$$\|AS(t+h)y - AS(t)y\| \le \|AS(t)\|\|(C(h)-I)y\| + \|C(t)\|\|AS(h)y\|, \quad y \in E,\, t \ge 0,$$

follows that for every $y \in E$, the function $AS(\cdot)y$ is uniformly continuous on $[0,\infty)$ which for the Bochner Theorem, see [13] pp.24, implies that $AS(\cdot)(x_0 + p(u))$ is $a.p$ and in turn that $AS(\cdot)(x_0 + p(u)) + C(t)(y_0 + q(u))$ is $a.p$.

To prove that $z(t) = \int_0^t AS(t-s)g(s,u(s))ds$ and $y(t) = \int_0^t C(t-s)f(s,u(s))ds$ are $a.a.p$, one more time, we use the Bochner Theorem. For $t \ge 0$ and $0 < h < 1$ we get

$$
\begin{aligned}
&\| z(t+h) - z(t) \| \\
&\quad = \| \int_0^{t+h} AS(t+h-s)g(s,u(s)) - \int_0^t AS(t-s)g(s,u(s))ds \|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \ \| \int_0^{t+h} (AS(t-s)C(h) + AS(h)C(t-s))g(s,u(s))ds - \int_0^t AS(t-s)g(s,u(s))ds \| \\
&\leq \ \| (C(h)-I) \int_0^t AS(t-s)g(s,u(s)) \| + \| \int_t^{t+h} AS(t-s)C(h)g(s,u(s))ds \| \\
&\quad + \| AS(h) \int_0^{t+h} C(t-s)g(s,u(s))ds \| \\
&= \ \sum_{i=1}^{3} I_i(t,h).
\end{aligned}
$$

From the integrability condition (24) and the fact the $AS(\cdot)$ is uniformly continuous on $[0,\infty)$, follows that $I_2(t,h) + I_3(t,h) \to 0$ uniformly for $t \geq 0$ when $h \to 0$. On the other hand, using that $s \to g(s,u(s)) \in \mathrm{L}^1([0,\infty) : \mathrm{E})$, the relation $S(t-s) = S(-s)C(t) + S(t)C(-s)$ and the fact the operator family $AS(t), t \geq 0$, is uniformly bounded in $\mathcal{L}(E : X)$; follows that the set $\{\int_0^t AS(t-s)g(s,u(s))ds : t \geq 0\}$ is relatively compact in X and so that the set of functions $\{s \to C(s)x : x \in U\}$ is equicontinuous at $t = 0$. Consequently $I_1(t,h) \to 0$ uniformly for $t \geq 0$ when $h \to 0$, and hence $z(\cdot)$ is *a.a.p.*

Finally we study the function $y(\cdot)$. To this purpose we decompose

$$
\begin{aligned}
y(t+h) - y(t) &= \int_0^t [C(t+h-s) - C(t-s)] f(s,u(s))\, ds \\
&\quad + \int_t^{t+h} C(t+h-s) f(s,u(s))\, ds \\
&= (C(h) - I)y(t) + AS(h) \int_0^t S(t-s) f(s,u(s))ds \\
&\quad + \int_t^{t+h} C(t+h-s) f(s,u(s))ds.
\end{aligned}
$$

Since $f(s,u(s))$ is integrable on $[0,\infty)$, the range of $y(\cdot)$ is relatively compact, which implies that $(C(h) - I)y(t) \to 0,\ h \to 0,$ uniformly for $t \geq 0$. The same argument serves to show that the third term of the above expression converges to zero when $h \to 0$, uniformly with respect to $t \geq 0$. From the assumptions on $f(\cdot)$, for $\epsilon > 0$ we can to fix positive numbers $L, \delta > 0$ such that

$$
\begin{aligned}
\int_L^\infty \| f(s,u(s)) \| \, ds &< \frac{\epsilon}{2N}, \\
\| C(s)f(t,u(t)) - C(s')f(t,u(t)) \| &\leq \frac{\epsilon}{2L}, \qquad t \in [0,L]
\end{aligned}
$$

when $s, s' \in [0, 2L+1]$ and $| s - s' | \leq \delta$. Under these conditions, the relation $C(t+s) - C(t-s) = AS(t)S(s)$, for $t \in [0,L]$ we see that

$$
\| AS(h) \int_0^t S(t-s)f(s,u(s))ds \| \leq \int_0^t \| (C(t-s+h) - C(t-s-h))f(s,u(s)) \| \, ds \leq \frac{\epsilon}{2},
$$

which permit to conclude that $AS(h) \int_0^t S(t-s)f(s,u(s))ds \to 0,\ h \to 0$, uniformly with respect to $t \geq 0$. The proof is complete. ■

4 Applications

The one dimensional wave equation modelled as an abstract Cauchy problem it has been studied extensively, see for example [12]. In this section we illustrate some of results in this work with a variant of the wave equation.

On the space $X = L^2([0,\pi])$ we consider the operator $Af(\xi) = f''(\xi)$ with domain $D(A) = \{f(\cdot) \in H^2(0,\pi) : f(0) = f(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a strongly continuous cosine function, $(C(t))_{t\in\mathbb{R}}$, on X. Furthermore, A has discrete spectrum, the eigenvalues are $-n^2$, $n \in N$, with corresponding normalized eigenvectors $z_n(\xi) := \left(\frac{2}{\pi}\right)^{1/2} \sin(n\xi)$ and

(a) $\{z_n : n \in N\}$ is an orthonormal basis of X.

(b) If $\varphi \in D(A)$ then $A\varphi = -\sum_{n=1}^{\infty} n^2 < \varphi, z_n > z_n$.

(c) For $\varphi \in X$, $C(t)\varphi = \sum_{n=1}^{\infty} \cos(nt) < \varphi, z_n > z_n$. Moreover, it follow from this expression that $S(t)\varphi = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < \varphi, z_n > z_n$, that $S(t)$ is compact for $t > 0$ and that $\| C(t) \| \leq 1$ and $\| S(t) \| \leq 1$ for every $t \in [0,\pi]$.

(d) If Φ denotes the group of translations on X defined by $\Phi(t)x(\xi) = \tilde{x}(\xi + t)$, where $\tilde{x}$ is the extension of x with period 2π, then $C(t) = \frac{1}{2}(\Phi(t) + \Phi(-t))$; $A = B^2$ where B is the infinitesimal generator of the group Φ and $E = \{x \in H^1(0,\pi) : x(0) = x(\pi) = 0\}$, see [2] for details.

At first we consider the following problem

$$\begin{aligned} \frac{\partial}{\partial t}\left[\frac{\partial u(t,\xi)}{\partial t} + G(t,\xi,u(t,\xi))\right] &= \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + F(t,\xi,u(t,\xi)), \quad t \in I, \xi \in [0,\pi], && (25) \\ u(t,0) &= u(t,\pi) = 0, \quad t \in I, && (26) \end{aligned}$$

where $G, F : I \times [0,\pi] \times \mathbb{R} \to \mathbb{R}$ are appropriates functions verifying the following conditions.

(H$_3$) There exist functions $\eta_1^F \in \mathrm{L}^2(\mathrm{I} \times [0,\pi])$ and $\eta_2^F \in \mathrm{L}^1(\mathrm{I} : \mathrm{L}^\infty([0,\pi] : \mathbb{R}))$ such that

$$| F(t,\xi,w) | \leq \eta_1^F(t,\xi) + \eta_2^F(t,\xi) \mid w \mid, \qquad t,\xi \in I, w \in \mathbb{R}.$$

(H$_4$) There exists a continuous function $\eta^G : I \times [0,\pi] \to: \mathrm{L}^\infty([0,\pi] : \mathbb{R}))$ such that

$$| G(t,\xi,w_1) - G(t,\xi,w_2) | \leq \eta^G(t,\xi) \mid w_1 - w_2 \mid, \qquad t,\xi \in I, w_i \in \mathbb{R}.$$

Associated to problem (25)-(26), we consider the nonlocal conditions

$$\begin{aligned} w(0,\xi) &= x_0(\xi) + \sum_{i=1}^{n} \alpha_i w(t_i,\xi), \quad \xi \in I, && (27) \\ \frac{\partial w(0,\xi)}{\partial t} &= y_0(\xi) + \sum_{i=1}^{k} \beta_i w(s_i,\xi), \quad \xi \in I, && (28) \end{aligned}$$

where $0 < t_i, s_j < \pi$ and $\alpha_i, \beta_j \in \mathbb{R}$ are prefixed numbers.

By using the substituting operators $g(t,x)(\xi) = G(t,\xi,x(\xi))$, $f(t,x)(\xi) = F(t,\xi,x(\xi))$, $x \in X$, and

$$p(u)(\xi) = \sum_{i=1}^{n} \alpha_i u(t_i,\xi) \quad \text{and} \quad q(u)(\xi) = \sum_{i=1}^{k} \beta_i u(s_i,\xi), \tag{29}$$

the problem (25)-(28) can be modelled as the abstract nonlocal Cauchy problem

$$\frac{d}{dt}[u'(t) + g(t,u(t))] = Au(t) + f(t,u(t)), \qquad t \in I, \tag{30}$$

$$u(0) = x_0 + q(u), \tag{31}$$

$$u'(0) = y_0 + p(u). \tag{32}$$

It is not difficult to see that

$$\|f(t,x)\|_2 \leq \left(\int_0^\pi \eta_1^F(t,\xi)^2 d\xi\right)^{1/2} + \eta_2^F(t,\cdot)_\pi \parallel x \parallel_2, \qquad t \in I, x \in X,$$

$$\|g(t,x) - g(t,y)\|_2 \leq \eta^G(t,\cdot)_\pi \parallel x - y \parallel_2, \qquad t \in I, x,y \in B_r(0,X).$$

Proposition 6 *Assume that the previous conditions are satisfied. If*

$$\int_0^a \left[\left(\int_0^\pi \eta_1^F(s,\xi)^2 d\xi\right)^{\frac{1}{2}} + \eta_2^F(s,\cdot)_\pi\right] ds + \sup_{s\in I} \eta^G(s,\cdot)_\pi + \sum_{i=1}^{n} \mid \alpha_i \mid + \sum_{i=1}^{k} \mid \beta_i \mid < 1,$$

then there exists a mid solution of (30)-(32).

Proof. The existence of a mild solution of (30)-(32) follows directly of Theorem 3. since $S(t)$ is compact for all $t \in \mathbb{R}$. ∎

To finish this section, we consider the differential system

$$[u'(t,\xi) + \int_0^\pi b(t,\eta,\xi)u(t,\eta)d\eta]' = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + F(t,\xi,u(t,\xi)), \ t \geq 0, \xi \in I = [0,\pi], \tag{33}$$

$$u(t,0) = u(t,\pi) = 0, \tag{34}$$

$$u(0) = x_0 + \int_0^\infty P(u(s))(\xi)d\mu, \tag{35}$$

$$u'(0) = y_0 + \int_0^\infty Q(u(s))(\xi)d\mu. \tag{36}$$

To study this problem, we introduce some new technical conditions.

(H$_5$) μ, ν are real functions of bounded variation on $[0,\infty)$ and $P : X \to E$, $Q : X \to X$ are completely continuous. (see [5] for operators with these properties.)

(H$_6$) There exist $\eta_2^F \in L^1(\mathbb{R} : L^\infty(I : \mathbb{R}))$ such that

$$\mid F(t,\xi,w) \mid \leq \eta_2^F(t,\xi) \mid w \mid, \qquad t,\xi \in I, w \in \mathbb{R}.$$

(H$_7$) $b(s,\eta,\xi)$, $\dfrac{\partial}{\partial\xi}b(s,\eta,\xi)$, $\dfrac{\partial^2}{\partial\xi^2}b(s,\eta,\xi)$ are appropriate, $b(s,\eta,\pi)=b(s,\eta,0)=0$ for all $(\eta,s)\in I\times[0,\infty)$ and

$$N(s):=\max\left\{\int_0^\pi\int_0^\pi b(s,\eta,\xi)^2 d\eta d\xi,\ \int_0^\pi\int_0^\pi\left(\frac{\partial^j b(s,\eta,\xi)}{\partial\xi^j}\right)^2 d\eta d\xi : j=1,2\right\}<\infty,$$

for every $s\geq 0$.

Defining the maps f,g,p,q by $f(t,x)(\xi)=F(t,\xi,x(\xi))$ and

$$\begin{aligned}
g(t,x)(\xi) &= \int_0^\pi b(t,\eta,\xi)x(\eta)d\eta, \quad x,y\in X,\\
p(u,v)(\xi) &= \int_0^\infty P(u(s))(\xi)d\mu(s), \quad u,v\in C_b([0,\infty):X),\\
q(u,v)(\xi) &= \int_0^\infty Q(u(s))(\xi)d\nu(s), \quad u,v\in C_b([0,\infty):X),
\end{aligned}$$

the problem (33)-(36) can be modelled as the abstract Cauchy problem (30)-(32).

In the next result $h(\cdot)$ is a function verifying the assumption in section 3.1.

Proposition 7 *Assume that the previous conditions are verified. If*

(a) $\dfrac{1}{h(t)}\displaystyle\int_0^t(N^{\frac{1}{2}}(s)+\eta^F(s,\cdot)_\pi)ds\to 0$ *as* $t\to\infty$ *and* $\displaystyle\int_0^\infty(N^{\frac{1}{2}}(s)+\eta^F(s,\cdot)_\pi)ds<\infty$, *then there exist a global solution solutions of (33)-(36).*

(b) *if* $\displaystyle\int_0^\infty(N^{\frac{1}{2}}(s)+\eta^F(s,\cdot)_\pi)ds<\infty$ *then there exist a asymptotic almost periodic solution of (33)-(36).*

Proof. From $\mathbf{H_7}$ we infer that $g(\cdot)$ is a $[D(A)]$-valued linear continuous operator, that $\| g(t,\cdot)\|_{\mathcal{L}(X)}\leq N^{\frac{1}{2}}(t)$ for every $t\geq 0$ and that $g(\cdot)$ is completely continuous with values in E. The assertions **(a)** and **(b)** are consequence of the Theorems 5 and 7 respectively.

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