UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2577

Are Partial Neutral Functional Differential Equations!.

Eduardo Hernández M.

Nº 163

NOTAS

Série Matemática

São Carlos – SP
Abr./2003

São Equações Parciais do Tipo Neutro !!!.

Eduardo Hernández M.

Departamento de Matemática - ICMC - USP.

Resumo

Neste artigo estabelecemos vários resultados de existência de soluções para equações de evolução em derivadas parciais do tipo neutro com retardamento não limitado modeladas na forma

$$\begin{aligned} \frac{d}{dt}(x(t) + g(t, x_t)) &= Ax(t) + f(t, x_t), \qquad t \in I = [0, a], && (1) \\ x_0 &= \varphi \in \mathcal{B}, && (2) \\ \triangle x(t_i) &= I_i(x_{t_i}) && (3) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}(x'(t) + g(t, x_t, x'(t))) &= Ax(t) + f(t, x_t, x'(t)), \qquad t \in I = [0, a], && (4) \\ x_0 &= \varphi \in \mathcal{B},\ x'(0) = y_0, && (5) \\ \triangle x(t_i) &= I_i^1(x_{t_i}, x'(t_i)) && (6) \\ \triangle x'(t_i) &= I_i^2(x_{t_i}, x'(t_i)) && (7) \end{aligned}$$

sendo A, dependendo do caso, o gerador infinitesimal de um semigrupo ou de uma função cosseno de operadores lineares limitados em um espaço de Banach X; f, g, I_i^j funções apropriadas e $\mathcal{B}$ um espaço de fase definido en forma axiomática.

Are Partial Neutral Functional Differential Equations!.

Eduardo Hernández M.

AMS-Subject Classification: Primary 35F25, 34G10, 34K40.

Keywords: Neutral equations, semigroup of linear operators, cosine function.

Abstract

Recently, some erroneous existence results of mild solution for first and second order " partial " neutral functional differential equations with delay was reported. In this paper we comment briefly respect these results and by studying the existence of mild solutions for some impulsive neutral systems we give simples ideas to correct these works.

1 Introduction

The purpose of this paper is study existence of mild solutions for some impulsive partial neutral functional differential equations modelled in the form

$$\frac{d}{dt}(x(t) + g(t, x_t)) = Ax(t) + f(t, x_t), \quad t \in I = [0, a], \tag{1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{2}$$

$$\triangle x(t_i) = I_i(x_{t_i}) \tag{3}$$

and

$$\frac{d}{dt}(x'(t) + g(t, x_t, x'(t))) = Ax(t) + f(t, x_t, x'(t)), \quad t \in I = [0, a], \tag{4}$$

$$x_0 = \varphi \in \mathcal{B}, \ x'(0) = y_0, \tag{5}$$

$$\triangle x(t_i) = I_i^1(x_{t_i}, x'(t_i)) \tag{6}$$

$$\triangle x'(t_i) = I_i^2(x_{t_i}, x'(t_i)) \tag{7}$$

where, depending of the case, A is the infinitesimal generator of an analytic semigroup of linear operators or of an strongly continuous cosine function of bounded linear operators on a Banach space X; the history $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; g, f, I_j^i are appropriate functions; $0 < t_1 < t_2 < .... < t_n < a$ are fixed points and the symbol $\triangle\xi(t_i)$ represent the jump of a function ξ at t_i, that is, $\triangle\xi(t_i) = \xi(t_i^+) - \xi(t_i^-)$.

Recently several works reporting existence results of mild solutions for first and second order partial neutral differential problems similar to (1)-(3) and (4)-(7) was published, see for example [1], [2], [5], [6], [7], [9] for the first order case and [3], [4], [7], [9] and [19] for the second order. Inexplicably, in all theses works some severes compactness assumptions on the operator family generates by A, which are valid only if the space X is a finite dimensional, was used. The fact that X be finite dimensional is relevant, since under these conditions the models studies don't permit to describe " partial " neutral differential equations, the real motivations of theses papers. Our objetive in this work is alert respect of these erroneous papers and by mean of the study of the initial value problems (1)-(3) and (4)-(7), give some simples ideas for the correction of these papers.

To precise the previous comment, we specificate, briefly, these " severe assumptions. "

Assumption in the first order case: *" A is the infinitesimal generator of a compact C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on X and there exist positive constants M_1, M_2 such that*

$$\| T(t) \| \leq M_1 \quad \textit{and} \quad \| AT(t) \| \leq M_2, \quad t \in [0, a].$$

It's easy to prove that the condition on $\| AT(t) \|$ imply that A is a bounded operator, which in turn imply that the function $[0, a) \to \mathcal{L}(\mathcal{X})$; $t \to T(t)$ is continuous on $[0, a)$ and so that X is a finite dimensional space since the semigroup is compact.

Assumption in the second order case: *" A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators on X and $C(t)$ is compact for every $t > 0$."*

It's well known that the compactness assumption on $C(t)$ imply that X is finite dimensional, see [26] pp. 557.

We want to remark that the the semigroup and the cosine function theory are elegant and sophisticated techniques which permit, in some sense, the study of partial evolution equations as been " ordinary " differential equations in Banach space.

Neutral differential equations arises in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature for "ordinary" neutral functional differential equations is the Hale book [13] and the references therein. The work in partial neutral functional differential equations with unbounded delay was initiated by Hernández & Henríquez in [14, 15]. In these papers, Hernández & Henríquez proved the existence of mild, strong and periodic solutions for the neutral problem (1)-(3). In general, the results was obtained using the semigroup theory and the Sadovskii fixed point theorem.

On the other hand, the theory of impulsive differential equations has become an important area of investigation in recent years stimulated by their numerous applications to problems arising in mechanics, electrical engineering, medicine, biology, ecology, etc. Relative to this matter, we refer to the reader to Rogovchenko [23, 24], Liu [20] and Hernndez [16].

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$ which is similar to those introduced by Hale and Kato [12]. Consequently, along of this paper, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X, endowed with a seminorm $\| \cdot \|_{\mathcal{B}}$ and we will assume that $\mathcal{B}$ satisfies the following axioms:

(A) If $x : (-\infty, a) \to X$, $a > 0$, is such that $x|_{[0,a]} \in \mathcal{PC}$ and $x_0 \in \mathcal{B}$, then for every $t \in [0, a)$ the following conditions hold:

(i) x_t is in $\mathcal{B}$,

ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$,

iii) $\| x_t \|_{\mathcal{B}} \leq K(t-\sigma) \sup\{\| x(s) \| : \sigma \leq s \leq t\} + M(t-\sigma) \| x_\sigma \|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \to [0, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in **(A)**, $t \to x_t$ is a $\mathcal{B}$-valued integrable on $[0, a)$.

(B) The space $\mathcal{B}$ is complete.

To consider the impulsive conditions, it's necessary introduce some additional notations and results. In what follow, $\mathcal{PC}$ is the space

$$\mathcal{PC} = \left\{ u : [0, a] \to X; \quad \begin{array}{l} u \text{ is continuous in } t \neq t_i,\ u(t_i) = u(t_i^-) \text{ for all } i = 1, .., n, \\ \text{and}\, u(t_i^+) \text{ exists for each } i = 1, .., n \end{array} \right\},$$

endowed with the uniform convergence topology. For $u \in \mathcal{PC}$, we introduce the functions $\tilde{u}_0 \in C([0, t_1] : X)$, $\tilde{u}_i \in C([t_i, t_{i+1}] : X)$, $i = 1, 2, ...n-1$, and $\tilde{u}_n \in C([t_n, a] : X)$ defined by

$$\tilde{u}_0 = u \text{ on } [0, t_1], \tag{8}$$

$$\tilde{u}_i(t) = \begin{cases} u(t), & \text{for } t \in (t_i, t_{i+1}], \\ u(t_i^+), & \text{for } t = t_i, \end{cases} \tag{9}$$

$$\tilde{u}_n(t) = \begin{cases} u(t), & \text{for } t \in (t_n, a], \\ u(t_n^+), & \text{for } t = t_n. \end{cases} \tag{10}$$

Moreover, for $B \subset \mathcal{PC}$ we employ the notation $\widetilde{B}_i$, $i = 0, 1, ...n$, for the sets

$$\widetilde{B}_i = \{\tilde{u}_i : u \in B\}. \tag{11}$$

The proof of Lemma 1, below, follows essentially from the Ascoli-Arzela Theorem.

Lemma 1 *$B \subset \mathcal{PC}$ is relatively compact if and only if each set $\widetilde{B}_i$ is relatively compact.*

The paper has three sections. In section 2 we discuss existence of mild solutions for the first order impulsive neutral problem (1)-(3) and a similar work is development in section 3 for the second order system (4)-(7). In each section we mentions a few results and notations referents to semigroup and cosine function of bounded linear operators which are needed to establish our results. The section 3 is reserved for examples.

The terminologies and notations are those generally used in functional analysis. In particular, $\mathcal{L}(\mathcal{X})$ stands for the Banach space of bounded linear operators from X into X and $B_r(x : Z)$ denotes the closed ball with center at x and radius $r > 0$ in an metric space Z. Additionally, for a bounded function $\xi : [0, a] \to [0, \infty)$ and $0 \leq t \leq a$ we will employ the notation ξ_t for

$$\xi_t = \sup\{\xi(s) : s \in [0, t]\}.$$

and the prefix $\mathcal{R}$ will be used to indicate the image of a map.

Some of our results will proved using the well known Leray Schauder Alternative Theorem, which we include by completeness.

Theorem 1 *[21] (* **Leray Schauder Alternative** *) Let D be a convex subset of a Banach space X and assume that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda F(x),\ 0 < \lambda < 1\}$ is unbounded or the map F has a fix point in D.*

2 The first order problem

In this section we study the existence of mild solutions for the neutral problem (1)-(3). Throughout of this section, $A : D(A) \to X$ is the infinitesimal generator of an uniformly bounded analytic semigroup of linear operators, $(T(t))_{t\geq 0}$, on X such that $0 \in \rho(A)$ and we always assume that $\tilde{M}$ is a constant such that $\| T(t) \| \leq \tilde{M}$ for every $t \in I$. Under these conditions, it's possible to define the fractional power $(-A)^{\alpha}$, $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^{\alpha}$. Furthermore, $D(-A)^{\alpha}$ is dense in X and the expression $\| x \|_{\alpha} = \| (-A)^{\alpha} x \|$ defines a norm in $D(-A)^{\alpha}$. If X_{α} is the space $D(-A)^{\alpha}$ endowed with the norm $\| \cdot \|_{\alpha}$, then the following properties hold ([22], pp. 74).

Lemma 2 *Under the previous conditions, the following properties hold:*

(1) *Let $0 < \alpha \leq 1$. Then X_{α} is a Banach space.*

(2) *If $0 < \beta \leq \alpha$ then $X_{\alpha} \to X_{\beta}$ is continuous.*

(3) *For every $0 < \alpha \leq 1$ there exists $C_{\alpha} > 0$ such that*

$$\| (-A)^{\alpha} T(t) \| \leq \frac{C_{\alpha}}{t^{\alpha}}, \qquad 0 < t \leq a.$$

For a complementary literature we refer to the reader to Pazy [22].

Definition 1 *A function $x : (-\infty, a] \to X$ is a mild solution of the neutral problem (1)-(3), if $x_0 = \varphi$; $x(\cdot)_{|I} \in \mathcal{PC}$; the function $s \to AT(t-s)g(s, x_s)$ is integrable on $[0, t)$ for each $0 \leq t < a$ and*

$$\begin{aligned} x(t) &= T(t)(\varphi(0) + g(0, \varphi)) - g(t, x_t) - \int_0^t AT(t-s)g(s, x_s)ds \\ &\quad + \int_0^t T(t-s)f(s, x_s)ds + \sum_{t_i < t} T(t-t_i)I_i(x_{t_i}), \qquad t \in I. \end{aligned}$$

Remark 1 *Our definitions of mild solution are different at those introduced by Benchohra in [10]. Unfortunately the concept adopt in [10] are erroneous.*

To study the neutral system (1)-(3), we introduce the following conditions.

$\mathbf{H_1}$ The function $f : I \times \mathcal{B} \to X$ satisfies the following conditions:

(i) For every $t \in I$, the function $f(t, \cdot) : \mathcal{B} \times X \to X$ is continuous.

(ii) For each $\psi \in \mathcal{B}$, the function $f(\cdot, \psi) : I \to X$ is strongly measurable.

(iii) There exist a continuous function $m_f : I \to [0, \infty)$ and a continuous nondecreasing function $W_f : [0, \infty) \to (0, \infty)$ such that

$$\| f(t, \psi) \| \leq m_f(t) W_f(\| \psi \|_{\mathcal{B}}), \qquad (t, \psi) \in I \times \mathcal{B}.$$

H₂ There exist constants $0 < \beta < 1$, c_1, c_2 such that g is X_β-valued, $(-A)^\beta g : I \times \mathcal{B} \to X$ is uniformly continuous on bounded sets and

$$\| (-A)^\beta g(t, \psi) \| \leq c_1 \| \psi \|_{\mathcal{B}} + c_2, \qquad (t, \psi) \in I \times \mathcal{B}.$$

H₃ If $y : (-\infty, a] \to X$ is the extension of φ such that $y(t) = T(t)\varphi(0)$ on I and $S(a)$ is the space $S(a) = \{x : (-\infty, a] \to X : x_0 = 0, \ x \in \mathcal{PC}(\mathcal{I} : \mathcal{X})\}$ endowed with uniform convergence topology on I, then the set of functions $\{t \to g(t, x_t + y_t) : x \in Q\}$ is equicontinuous on I, for every bounded $Q \subset S$.

H₄ The function $g : I \times \mathcal{B} \to X$ satisfy the following conditions:

(i) $g(t, \cdot) : \mathcal{B} \to X$ are continuous a.e. $t \in I$;

(ii) For each $\phi \in \mathcal{B}$, the function $g(\cdot, \phi) : I \to X$ is strongly measurable;

(iii) There exists a continuous function $m_g(\cdot) : [0, \infty) \to [0, \infty)$ and a continuous nondecreasing function $W_g(\cdot) : [0, \infty) \to (0, \infty)$ such that

$$\| g(t, \phi) \| \leq m_g(t) W_g(\| \phi \|_{\mathcal{B}}), \qquad (t, \phi) \in I \times \mathcal{B}.$$

Theorem 2 *Let* **H₁**-**H₃** *be satisfied and suppose that the following conditions hold.*

(a) *For every $r > 0$ and all $\epsilon > 0$ there are compact sets $W^i_{\epsilon,r} \subset X$, $i = 1, 2$, such that $T(\epsilon)(-A)^\beta g(s, \psi) \in W^1_{\epsilon,r}$ and $T(\epsilon) f(s, \psi) \in W^2_{\epsilon,r}$ for every $(s, \psi) \in I \times B_r(0, \mathcal{B})$.*

(b) *The maps I_i are completely continuous and there are constants c^j_i, $i = 1, 2, ..n, j = 1, 2$ such that $\| I_i(\psi) \| \leq c^1_i \| \psi \|_{\mathcal{B}} + c^2_i$, for every $\psi \in \mathcal{B}$.*

If $\mu = K_a[c_1 + \dfrac{C_{1-\beta} c_1 a^\beta}{\beta} + \tilde{M} \sum_{i=1}^{n} c_i] < 1$ *and* $\dfrac{K_a \tilde{M}}{1 - \mu} \int_0^t m_f(s) ds < \int_c^\infty \dfrac{1}{W_f(s)} ds$ *where*

$$c = \frac{1}{1 - \mu} \left[(M_a + \tilde{M}(K_a c_1 + H)) \| \varphi \|_{\mathcal{B}} + K_a c_2 (\tilde{M} + \| (-A)^\beta \| + \frac{a^\beta C_{1-\beta}}{\beta} + \tilde{M} \sum_{i=1}^{n} d_i) \right]$$

, then there exists a mild solution of the initial value problem (1).

Proof: Let $y(\cdot)$ and $S(a)$ as in condition **H₃** and $\Gamma : S(a) \to S(a)$ the map defined by $\Gamma x(t) = 0$ for $t \leq 0$ and

$$\Gamma x(t) = \begin{cases} 0, & \text{for } t \leq 0, \\ T(t) g(0, \varphi) - g(t, x_t + y_t) - \displaystyle\int_0^t A T(t - s) g(s, x_s + y_s) ds & \\ \quad + \displaystyle\int_0^t T(t - s) f(s, x_s + y_s) ds + \sum_{t_i < t} T(t - t_i) I_i(x_{t_i} + y_{t_i}), & \text{for } t \in I. \end{cases}$$

From the assumptions, we know that $s \to (-A)^{\beta} g(s, x_s + y_s)$ and $s \to g(s, x_s + y_s)$ are integrable. In addition, in view of the fact that $T(\cdot)$ is an analytic semigroups (see [22]), the operator function $s \to AT(t-s)$ is continuous in the uniform operator topology on $[0,t)$ which from the estimate

$$\| (-A)T(t-s)g(s, x_s + y_s) \| = \| (-A)^{1-\beta} T(t-s)(-A)^{\beta} g(s, x_s + y_s) \| \leq \frac{C_{1-\beta} Cte}{(t-s)^{1-\beta}},$$

and the Bochner's Theorem imply that $\| AT(t-s)g(s, x_s + y_s) \|$ is integrable on $[0,t)$. Now, it's clear that Γ is well defined and with values in $S(a)$. Moreover, from the phase space axioms, the Lebesgue dominated convergence theorem and $\mathbf{H_2}$, we infer that Γ is continuous.

In order to use the Leray Schauder Alternative, we obtain *a priori* estimates for the solutions of the integral equation $x = \lambda \Gamma x$, $\lambda \in (0,1)$. Let $\lambda \in (0,1)$ and $x^{\lambda}(\cdot)$ be a solution of $x = \lambda \Gamma x$. Using $\mathbf{H_1}$ and $\mathbf{H_2}$, the notation $v^{\lambda}(t) := (M_a + K_a \tilde{M} H) \| \varphi \|_{\mathcal{B}} + K_a \| x^{\lambda} \|_t$ and the fact that $\| x_t + y_t \| \leq v^{\lambda}(t)$ on I, we find that

$$\begin{aligned} \| x^{\lambda}(t) \| \; &\leq \; \tilde{M} \| g(0,\varphi) \| + \| (-A)^{-\beta} \| c_2 + \left(\| (-A)^{-\beta} \| c_1 + \tilde{M} \sum_{t_i \leq t} c_i^1 \right) v^{\lambda}(t) \\ &+ C_{1-\beta} c_1 \int_0^t \frac{v^{\lambda}(s)}{(t-s)^{1-\beta}} ds + \frac{C_{1-\beta} c_2 a^{\beta}}{\beta} + \tilde{M} \int_0^t m_f(s) W_f(v^{\lambda}(s)) ds \\ &+ \tilde{M} \sum_{t_i \leq t} c_i^2. \end{aligned}$$

and that

$$\begin{aligned} v^{\lambda}(t) \; &\leq \; \left(M_a + K_a \tilde{M}(H+1) \right) \| \varphi \| + K_a \left(c_2 (\tilde{M} + \| (-A)^{-\beta} \| + \frac{a^{\beta} C_{1-\beta}}{\beta}) + \tilde{M} \sum_{i=1}^{n} c_i^2 \right) \\ &+ \mu v^{\lambda}(t) + K_a \tilde{M} \int_0^t m_f(s) W_f(v^{\lambda}(s)) ds \tilde{M}. \end{aligned}$$

Consequently,

$$v^{\lambda}(t) \; \leq \; c + \frac{K_a \tilde{M}}{1-\mu} \int_0^t m_f(s) W_f(v^{\lambda}(s)) ds.$$

Designating by $\beta^{\lambda}(t)$ the right hand side of the above expression we get

$$\beta_{\lambda}'(t) \leq \frac{K_a \tilde{M}}{1-\mu} m_f(t) W_f(\beta_{\lambda}(t))$$

and so that

$$\int_c^{\beta_{\lambda}(t)} \frac{ds}{W_f(s)} \; \leq \; \frac{K_a \tilde{M}}{1-\mu} \int_0^t m_f(s) ds < \int_c^{\infty} \frac{1}{W_f(s)} ds,$$

which imply that the functions $\beta^{\lambda}(\cdot)$ are uniformly bounded on I. Thus, the functions $u^{\lambda}(\cdot)$ are uniformly bounded on I.

Next we prove that Γ is completely continuous. To this end, by using the relation $A\int_0^t T(s)xds = T(t)x - x$, we rewrite Γ in the form $\Gamma = \sum_{i=1}^{4} \Gamma_i$ where $\Gamma_i x = 0$ on $(-\infty, 0]$ for $i = 1, 2, ...4$ and

$$\begin{aligned}
\Gamma_1 x(t) &= T(t)(g(0,\varphi) - g(t, x_t + y_t)),\\
\Gamma_2 x(t) &= -\int_0^t AT(t-s)g(s, x_s + y_s)ds,\\
\Gamma_3 x(t) &= \int_0^t T(t-s)f(s, x_s + y_s)ds + \int_0^t AT(t-s)g(t, x_t + y_t)ds,\\
\Gamma_4 x(t) &= \sum_{t_i<t} T(t-t_i)I_i(x_{t_i} + y_{t_i}),
\end{aligned}$$

when $t \in I$. Next we prove that each Γ_i is completely continuous. In the sequel $B_r = B_r[0, S(a)]$. At first we observe that

$$r^* := K_a r + (M_a + K_a \tilde{M} H) \parallel \varphi \parallel_{\mathcal{B}} \geq \parallel x_t + y_t \parallel_{\mathcal{B}}, \qquad (t,x) \in I \times B_r.$$

Step 1. The map Γ_1 is completely continuous

Firstly we study the equicontinuity of $\Gamma_1 B_r$. Let $0 < \epsilon < t_0 < t \leq a$ and W^1_{ϵ,r^*} be the compact set in (a). Using $\mathbf{H_3}$ and the fact that set of functions $\{s \to T(s)x : x \in W^1_{\epsilon,r^*}\}$ is equicontinuous on $[0,a]$, we fix $0 < \delta < \epsilon$ such that

$$\begin{aligned}
\parallel T(s+h)x - T(s)x \parallel &< \epsilon, & s \in I,\ x \in W^1_{\epsilon,r^*},\\
\parallel g(t, x_t + y_t) - g(t_0, x_{t_0} + y_{t_0}) \parallel &< \epsilon, & x \in B_r,
\end{aligned}$$

when $0 < h < \delta$ and $\mid t - t_0 \mid < \delta$. Under these conditions; for $\mid t - t_0 \mid < \delta$ we see that

$$\begin{aligned}
\parallel \Gamma_1 x(t_0) - \Gamma_1 x(t) \parallel &\leq \parallel (T(t_0) - T(t))g(0,\varphi) \parallel\\
&\quad + \parallel T(t_0) \parallel \parallel g(t_0, x_{t_0} + y_{t_0}) - g(t, x_t + y_t) \parallel\\
&\quad + \parallel (-A)^{-\beta}(T(t_0 - \epsilon) - T(t-\epsilon))T(\epsilon)(-A)^{-\beta} g(t, x_t + y_t) \parallel\\
&\leq \parallel (T(t_0) - T(t))g(0,\varphi) \parallel + (\parallel T(t_0) \parallel + \parallel (-A)^{-\beta} \parallel)\epsilon
\end{aligned}$$

which imply that $\Gamma_1(B_r)$ is equicontinuous from the right hand side at t_0. The equicontinuity from the left t hand side at t_0 and is proved in similar form, we omit details. Consequently, $\Gamma_1(B_r)$ is equicontinuous on I.

It's clear that $\Gamma_1(B_r)(0)$ is compact. Moreover, if $0 < \epsilon < t$, we get

$$\begin{aligned}
\Gamma_1(B_r)(t) &= \{T(t)g(0,\varphi) + T(t-\epsilon)(-A)^{-\beta}T(\epsilon)(-A^{\beta})g(t, x_t + y_t) : x \in B_r\}\\
&\subset T(t)g(0,\varphi) + T(t-\epsilon)(-A)^{-\beta} W^1_{\epsilon,r^*}
\end{aligned}$$

which proves that $\Gamma_1(B_r)(t)$ is relatively compact and so that Γ_1 is completely continuos.

Step 2. The map Γ_2 is completely continuous

At first we prove that $\Gamma_2 B_r$ is equicontinuous. Let $0 < \xi < t_0 < a$. Since the semigroup is analytic, the function $s \to (-A)^{1-\beta}T(s)$ is uniformly continuous in the uniform convergence topology on $[\xi, a]$, which permit to chose $\delta > 0$ such that

$$\parallel (-A)^{1-\beta}T(s)x - (-A)^{1-\beta}T(s')x \parallel < \epsilon, \qquad x \in B_{r^*}(0, X),$$

when $s, s' \leq [\xi, a]$ and $\mid s - s' \mid < \delta$. Now, for $0 < t - t_0 < \delta < \epsilon$ and $u \in B_r$ we find that

$$\begin{aligned}
&\| \Gamma_2 x(t) - \Gamma_2 x(t_0) \| \\
&\quad \leq \| \int_0^{t_0-\xi} (-A)^{1-\beta}(T(t-s) - T(t_0-s))(-A)^\beta g(s, x_s + y_s)ds \| \\
&\quad\quad + \int_{t_0-\xi}^{t_0} \| T(t-t_0) - I \| \| (-A)^{1-\beta} T(t_0-s)(-A)^\beta g(s, x_s + y_s) \| ds \\
&\quad\quad + \int_{t_0}^{t} \| (-A)^{1-\beta} T(t-s)(-A)^\beta g(s, x_s + y_s) \| ds \\
&\quad \leq (t_0 - \xi)\epsilon + +2\tilde{M} C_{1-\beta}(c_1 r^\star + c_2)(\frac{\xi^\beta}{\beta} + \frac{(t-t_0)^\beta}{\beta})
\end{aligned}$$

which show that $\Gamma_2(B_r)$ is equicontinuous from the right hand side at t_0. Similarly we can to prove that $\Gamma_2(B_r)$ is equicontinuous at $t_0 \geq 0$. Thus $\Gamma_2(B_r)$ is equicontinuous.

Now we prove that $\Gamma_2(B_r)(t)$ is relatively compact for $t \in I$. The case $t = 0$ is trivial. Let $0 < \epsilon < t \leq a$ and $W^1_{\frac{\epsilon}{2}, r^\star}$ be the compact set in (a). Since $(-A)^\beta T(\cdot)$ is strongly continuous on $[\frac{\epsilon}{2}, a]$, follows that $W_\epsilon = \{(-A)^{1-\beta} T(s)x : s \in [\frac{\epsilon}{2}, a], x \in W^1_{\frac{\epsilon}{2}, r^\star}\}$ is precompact in X which from the mean value theorem for the Bochner integral, [21] Lemma 2.1.3, imply that

$$\begin{aligned}
\Gamma_2 x(t) &= \int_0^{t-\epsilon} (-A)^{1-\beta} T(t - s - \frac{\epsilon}{2}) T(\frac{\epsilon}{2})(-A)^\beta g(s, x_s + y_s)ds \\
&\quad + \int_{t-\epsilon}^{t} (-A)^{1-\beta} T(t-s)(-A)^\beta g(s, x_s + y_s)ds, \qquad x \in B_r, \\
&\in (t-\epsilon)\overline{co(W_\epsilon)} + C_\epsilon, \qquad x \in B_r,
\end{aligned}$$

where $co(W_\epsilon)$ is the convex hull of W_ϵ and $diam(C_\epsilon) \leq 2C_{1-\beta}(c_1 r^\star + c_2)\frac{\epsilon^\beta}{\beta}$. These remark permit to infer that $\Gamma_2(B_r)(t)$ is relatively compact in X. Thus, Γ_2 is completely continuous.

Arguing as in step 2, follows that Γ_3 is completely continuous, we omit details.

Step 3 The map Γ_4 is completely continuous

The proof follows from Proposition 1, the assumptions on the operators I_i and the expressions

$$\begin{aligned}
[\widetilde{\Gamma_4 x}]_i(t_i) &= \sum_{j=1}^{i-1} T(t_i - t_j) I_j(x_{t_j} + y_{t_j}) + I_i(x_{t_i} + y_{t_i}), \quad i \geq 1,\, x \in B_r, \\
[\widetilde{\Gamma_4 x}]_i(t_{i+1}) &= \sum_{j=1}^{i} T(t_{i+1} - t_j) I_j(x_{t_j} + y_{t_j}), \quad i \geq 0,\, x \in B_r,
\end{aligned}$$

where $[\widetilde{\Gamma_4 x}]_j(t_k)$ are defined in (8)-(10). We omit details of this part of the proof.

These remarks and the Leray Schauder Alternative asserts that Γ has a fixed point in $\mathcal{PC}$. This point is a mild solution of the problem (1). The proof is complete. ∎

Theorem 3 *Assume that conditions* ($\mathbf{H}_1$)-($\mathbf{H}_2$) *are verified. Suppose in addition that the followings conditions hold.*

(a) *For each* $r > 0$ *and all* $\epsilon > 0$ *there is a compact set* $W_{\epsilon,r} \subset X$ *such that* $T(\epsilon)f(s,\psi) \in W_{\epsilon,r}$ *for every* $s \in I$, $\psi \in B_r(0,\mathcal{B})$.

(b) *There exist constants* L_g, L_j, $j = 1,2,...n$ *such that*

$$\begin{aligned} \| I_j(\psi_1) - I_j(\psi_2) \| &\leq L_j \| \psi_1 - \psi_2 \|, & \psi_i \in \mathcal{B}, \\ \| (-A)^\beta g(t,\psi_1) - (-A)^\beta g(t,\psi_2) \| &\leq L_g \| \psi_1 - \psi_2 \|_{\mathcal{B}}, & (t,\psi_i) \in I \times \mathcal{B}. \end{aligned}$$

If $K_a \left(L_g(\| (-A)^{-\beta} \| + \frac{C_{1-\beta}a^\beta}{\beta}) + \tilde{M}\sum_{i=1}^n L_i \right) + \liminf_{r\to\infty} \frac{1}{r}\int_0^a m_f(s)W_f(K_a r + \| y_s \|)ds < 1$, *where* $y(\cdot)$ *is the function introduced in* $(\mathbf{H_3})$, *then there exists a mild solution of (1).*

¶ Let $\Gamma : S(a) \to S(a)$ the map defined in the proof of Theorem 4 and consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where $\Gamma_i x = 0$ on $(-\infty, 0]$, $i = 1,2$, and

$$\begin{aligned} \Gamma_1 x(t) &= \int_0^t T(t-s)f(s,x_s+y_s)ds, \quad t \in I, \\ \Gamma_2 x(t) &= T(t)g(0,\varphi) - g(t,x_t+y_t) - \int_0^t AT(t-s)g(s,x_s+y_s)ds \\ &\quad + \sum_{t_i<t} T(t-t_i)I_i(x_{t_i}+y_{t_i}), \quad t \in I. \end{aligned}$$

We affirm that there exist $r > 0$ such that $\Gamma(B_r(0,S(a))) \subset B_r(0,S(a))$. If the affirmation is false, for every $r > 0$ there exist $x^r \in B_r$ and $t^r \in I$ such that $r <\| \Gamma x^r(t^r) \|$ and then

$$\begin{aligned} r <\| \Gamma x^r(t^r) \| &\leq \tilde{M} \| g(0,\varphi) \| + \| (-A)^{-\beta} \| L_g \| x^r_{t^r} \|_{\mathcal{B}} + \| g(t^r, y_{t^r}) \| \\ &\quad + \int_0^{t^r} \frac{C_{1-\beta}}{(t^r-s)^{1-\beta}} (L_g \| x^r_s \|_{\mathcal{B}} + \| g(s,y_s) \|_{\mathcal{B}})ds \\ &\quad + \int_0^{t^r} m_f(s)W_f(\| x^r_s \|_{\mathcal{B}} + \| y_s \|)ds + \tilde{M}\sum_{t_i<t^r}(L_i \| x^r_{t_i} \| + \| I_i(y_{t_i}) \|) \\ &\leq \tilde{M} \| g(0,\varphi) \| + \| (-A)^{-\beta} \| L_g K_a r + \| g(t^r, y_{t^r}) \| + L_g K_a r \frac{C_{1-\beta}a^\beta}{\beta} \\ &\quad + \int_0^{t^r} \frac{C_{1-\beta} \| g(s,y_s) \|}{(t^r-s)^{1-\beta}}ds + \int_0^a m_f(s)W_f(K_a r + \| y_s \|)ds \\ &\quad + \tilde{M}\sum_{i=1}^n (L_i K_a r + \| I_i(y_{t_i}) \|) \end{aligned}$$

and hence

$$1 \leq K_a \left(L_g(\| (-A)^{-\beta} \| + \frac{C_{1-\beta}a^\beta}{\beta}) + \tilde{M}\sum_{i=1}^n L_i \right) + \liminf_{r\to\infty} \frac{1}{r}\int_0^a m_f(s)W_f(K_a r + \| y_s \|_{\mathcal{B}})ds$$

which is absurd.

Let $r_0 > 0$ such that $\Gamma(B_{r_0}) \subset B_{r_0}$. Follows from the proof of Theorem 4 that Γ_1 is completely continuous on B_{r_0} and from the estimate

$$\| \Gamma_2 u - \Gamma_2 v \|_a \leq K_a \left[L_g \left(\| (-A)^{-\beta} \| + \frac{C_{1-\beta}a^\beta}{\beta} \right) + \tilde{M}\sum_{i=1}^n L_i \right] \| u - v \|_a,$$

that Γ_2 is a contraction on B_{r_0}. Thus, Γ is a condensing operator on B_{r_0}. Now, the existence of a mild solution of (1) is consequence of the Sadovskii point fixed Theorem. ■

3 The Second Order Problem

In this section we discuss existence of mild solutions for the impulsive neutral problem (4)-(7). Along of this section, A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators, $(C(t))_{t\in \mathbb{R}}$, on X; $S(t)$ is the sine function associated with $(C(t))_{t\in \mathbb{R}}$, that is,

$$S(t)x = \int_0^t C(s)x ds, \qquad x \in X,\ t \in \mathbb{R},$$

and $N, \tilde{N}$ are constants such that $\|C(t)\| \leq N$ and $\|S(t)\| \leq \tilde{N}$ for every $t \in I$. We refer the reader to [11] for the necessary concepts about cosine functions. Next we only mention a few results and notations needed to establish our results. Along of this section, $[D(A)]$ is the space

$$D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable}\},$$

endowed with the norm $\| x \|_A = \| x \| + \| Ax \|$, $x \in D(A)$. Moreover, in this section the notation E stands for the space formed by the vectors $x \in X$ for which the function $C(\cdot)x$ is of class C^1. It was proved by Kisińsky [18] that E endowed with the norm

$$\|x\|_1 = \|x\| + \sup_{0\leq t\leq a} \|AS(t)x\|, \qquad x \in E,$$

is a Banach space. The operator valued function $G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is a strongly continuous group of linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. From this it follows that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0,\ t \to 0,$ for each $x \in E$. Furthermore, if $x : [0, \infty) \to X$ is locally integrable then $y(t) = \int_0^t S(t-s)x(s)ds$ defines an E-valued continuous function. This is an consequence of the fact that

$$\int_0^t g(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)\, ds \\ \int_0^t C(t-s)x(s)\, ds \end{bmatrix}$$

defines an $E \times X$-valued continuous function.

The existence of solutions of the second order abstract Cauchy problem

$$\begin{cases} x''(t) &= Ax(t) + h(t), \quad 0 \leq t \leq a, \\ x(0) &= x_0, \quad x'(0) = x_1, \end{cases} \tag{12}$$

where $h : [0, a] \to X$ is an integrable function, has been discussed in [26]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem it has been treated in [27]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s)\, ds, \quad 0 \leq t \leq a, \tag{13}$$

is called, mild solution of (12) and that when $x_0 \in E$, $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)x_0 + C(t)x_1 + \int_0^t C(t-s)h(s)\,ds.$$

Initially we study the Neutral system

$$\begin{aligned} \frac{d}{dt}[x'(t) - g(t,x_t)] &= Ax(t) + f(t,x_t), \qquad t \in I = [0,a], && (14) \\ x_0 &= \varphi \in \mathcal{B}, \;\; x'(0) = y_0 \in X, && (15) \\ \Delta x(t_i) &= I_i^1(x_{t_i}), && (16) \\ \Delta x'(t_i) &= I_i^2(x_{t_i}). && (17) \end{aligned}$$

If $u(\cdot)$ is a solution of (14)-(15), g is $D(A)$-valued and $Ag \in \mathrm{L}^1(\mathrm{I}:\mathrm{X})$ then

$$\begin{aligned} u(t) &= C(t)\varphi(0) + S(t)[y_0 - g(0,\varphi)] - \int_0^t g(s,u_s)ds + \int_0^t AS(t-s)\int_0^s g(\tau,u_\tau)d\tau ds \\ &\quad + \int_0^t S(t-s)f(s,u_s)ds. && (18) \end{aligned}$$

The expression (18) is the motivation of the following definition.

Definition 2 *A function $x : (-\infty, a] \to X$ is a mild solution of the problem (14)-(17) if $x_0 = \varphi$; $x(\cdot)_{|I} \in \mathcal{PC}$ and*

$$\begin{aligned} x(t) &= C(t)\varphi(0) + S(t)[y_0 - g(0,\varphi)] + \int_0^t C(t-s)g(s,x_s)ds + \int_0^t S(t-s)f(s,x_s)ds \\ &\quad + \sum_{t_i<t} C(t-t_i)I_i^1(x_{t_i}) + \sum_{t_i<t} S(t-t_i)I_i^2(x_{t_i}), \qquad t \in I. \end{aligned}$$

Theorem 4 *Assume that conditions $\mathbf{H_1}$ and $\mathbf{H_4}$ are verified and that $g(\cdot)$ is completely continuous. Suppose, furthermore, that the followings conditions hold.*

(a) *The set $U(t,t',r) = \{S(t')f(s,\psi) : s \in [0,t], \| \psi \|_{\mathcal{B}} \le r\}$ is relatively compact in X for every $0 < t' \le t$ and all $r > 0$.*

(b) *The functions I_i^1 are completely continuous and there exist positive constants c_i^1 such that $\| I_i^1(x) \| \le c_i^1 \| x \| + c_i^2$ for every $x \in X$.*

(c) *For every $s \in I$ and all $r > 0$, the set $V(s,r) = \{S(s)I_i^2(x) : \| x \| \le r,\, i = 1,2...n\}$ is relatively compact in X and there are constants d_i^2, such that $\| I_j^2(x) \| \le d_j^1 \| x \| + d_j^2$ for every $x \in X$.*

If $\mu = K_a \sum_{i=1}^{n}(Nc_i^1 + \tilde{N}d_i^1) < 1$, $\dfrac{K_a}{1-\mu}\displaystyle\int_0^t (Nm_g(s) + \tilde{N}m_f(s))ds < \int_c^\infty \frac{ds}{W_g(s) + W_f(s)}$ *where*

$$c = (1-\mu)^{-1}\left[(M_a + NH) \| \varphi \|_{\mathcal{B}} + K_a\left(\tilde{N}(\| y_0 \| + \| g(0,\varphi) \|) + \sum_{i=1}^{n}(Nc_i^2 + \tilde{N}d_i^2)\right)\right],$$

then there exists a mild solution of (14)-(17).

Proof. On the space $S(\varphi) = \{x \in \mathcal{PC} : x(0) = \varphi(0)\}$ endowed with the uniform convergence topology, we define the map $\Gamma : S(\varphi) \to S(\varphi)$ by

$$\begin{aligned}\Gamma x(t) &= C(t)\varphi(0) + S(t)[y_0 - g(0,\varphi)] + \int_0^t C(t-s)g(s,\bar{x}_s)ds + \int_0^t S(t-s)f(s,\bar{x}_s)ds \\ &\quad + \sum_{t_i<t} C(t-t_i)I_i^1(\bar{x}_{t_i}) + \sum_{t_i<t} S(t-t_i)I_i^2(\bar{x}_{t_i}), \qquad t \in I,\end{aligned}$$

where $\bar{x} : (-\infty, a] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on I. Now, we obtain an *a priori* bound for the solution of the integral equation $x = \lambda\Gamma(x)$, $\lambda \in (0,1)$. Let x^λ be a solution of $x = \lambda\Gamma(x)$, $\lambda \in (0,1)$. If $v(t) := M_a \| \varphi \|_{\mathcal{B}} + K_a \| x^\lambda \|_t$ we get

$$\begin{aligned}\| x^\lambda(t) \| &\leq N \| \varphi(0) \| + \tilde{N}(\| y_0 \| + \| g(0,\varphi) \|) + N\int_0^t m_g(s)W_g(v^\lambda(s))ds \\ &\quad + \tilde{N}\int_0^t m_f(s)W_f(v^\lambda(s))ds + \sum_{t_i \leq t}(Nc_i^1 + \tilde{N}d_i^1)v^\lambda(t) + \sum_{t_i \leq t}(Nc_i^2 + \tilde{N}d_i^2),\end{aligned}$$

which imply that

$$\begin{aligned}v^\lambda(t) &\leq M_a \| \varphi \|_{\mathcal{B}} + K_a\left(N \| \varphi(0) \| + \tilde{N}(\| y_0 \| + \| g(0,\varphi) \| + \sum_{i=1}^n (Nc_i^2 + \tilde{N}d_i^2)\right) \\ &\quad + K_a\left(N\int_0^t m_g(s)W_g(v^\lambda(s))ds + \tilde{N}\int_0^t m_f(s)W_f(v^\lambda(s))ds\right) + \mu v^\lambda(t)\end{aligned}$$

and so that

$$v^\lambda(t) \leq c + \frac{K_a}{1-\mu}\left(N\int_0^t m_g(s)W_g(v^\lambda(s))ds + \tilde{N}\int_0^t m_f(s)W_f(v^\lambda(s))ds\right).$$

Denoting by β_λ to the right hand side of last inequality, we see that

$$\beta_\lambda'(t) \leq \frac{K_a}{1-\mu}(Nm_g(t) + \tilde{N}m_f(t))(W_g(\beta_\lambda(t)) + W_f(\beta_\lambda(t))),$$

and hence

$$\int_c^{\beta_\lambda(t)} \frac{ds}{W_g(s) + W_f(s)} \leq \frac{K_a}{1-\mu}\int_0^t (Nm_g(s) + \tilde{N}m_f(s))ds < \int_c^\infty \frac{ds}{W_g(s) + W_f(s)}.$$

Theses estimates permit to conclude that $\{\beta_\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded in $\mathcal{PC}$ which in turn implies that $\{x^\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded in $\mathcal{PC}$.

To prove that Γ is completely continuous, we introduce the decomposition $\Gamma = \sum_{i=1}^3 \Gamma_i$ where $(\Gamma_1 x)_0 = \varphi$, $(\Gamma_i x)_0 = 0$, $i = 2, 3$, and

$$\begin{aligned}\Gamma_2 x(t) &= \int_0^t C(t-s)g(s,\tilde{x}_s)ds + \int_0^t S(t-s)f(s,\tilde{x}_s)ds, \qquad t \in I, \\ \Gamma_3 x(t) &= \sum_{t_i<t} C(t-t_i)I_i^1(\bar{x}_{t_i}) + \sum_{t_i<t} S(t-t_i)I_i^2(\bar{x}_{t_i}), \qquad t \in I.\end{aligned}$$

Using the assumptions (b), (c), Proposition 1 and the strongly continuity of $C(\cdot)$ follows that Γ_3 is completely continuous. Next we prove that Γ_2 is completely continuous. In what follow, for $r > 0$, $B_r = \{x \in S(\varphi) : \| x(\theta) - \varphi(0) \|_a \leq r\}$ and $r^* := (M_a + K_a H) \| \varphi \| + K_a r$. In order to use the Ascoli-Arzela Theorem we will divide this part of proof in two steps.

Step 1. The set $\Gamma_2 B_r = \{\Gamma_2 x : x \in B_r\}$ is equicontinuous on I.

Let $t \in I$ and $\epsilon > 0$. Since $C(\cdot)$ is strongly continuous and $g(\cdot)$ is completely continuous, there exists $\delta > 0$ such that

$$\| (C(t+h) - C(t))g(t,\psi) \| < \epsilon, \qquad \| \psi \|_{\mathcal{B}} \leq r^*,\ t \in I,$$

if $| h | \leq \delta$. Under the previous conditions, for $x \in B_r$ and $|h| \leq \delta$ with $t+h \in I$ we get

$$\begin{aligned}
&\| \Gamma_2 x(t+h) - \Gamma_2 x(t) \| \\
&\quad \leq \int_0^t \| (C(t+h-s) - C(t-s))g(s,\bar{x}_s) \| ds + NW_g(r^*) \int_t^{t+h} m_g(s) ds \\
&\qquad + \int_0^t \| S(t+h-s) - S(t-s) \| \| f(s,\bar{x}_s) \| ds + \tilde{N} W_f(r^*) \int_t^{t+h} m_f(s) ds \\
&\quad \leq \epsilon t + NW_g(r^*) \int_t^{t+h} m_g(s) ds + W_f(r^*) \left(Nh \int_0^t m_f(s) ds + \tilde{N} \int_t^{t+h} m_f(s) ds \right),
\end{aligned}$$

which prove the assertion.

Step 2. For every $t \in I$, the set $\Gamma_2 B_r(t) = \{\Gamma_2 x(t) : x \in B_r\}$ is relatively compact in X.

Let $t \in I$ and $\epsilon > 0$. If $x \in B_r$, from the estimate

$$\| f(\theta, x_\theta) \| \ \leq\ m_f(\theta) W_f(\| x_\theta \|_{\mathcal{B}}) \leq m_f(\theta) W_f(r^*), \qquad \theta \in I,$$

follows that the set $U = \{f(t-s, \bar{x}_{t-s}) : s \in [0,t], x \in B_r\}$ is bounded in X. Using that $S : I \to \mathcal{L}(\mathcal{X})$ is uniformly Lipschitz on I, we can chose $0 = s_1 < s_2 < \dots < s_k = t$ such that for all $i = 1, 2, \dots k-1$ and every $s, s' \in [s_i, s_{i+1}]$,

$$\| S(s')y \ - \ S(s)y \| < \epsilon \qquad y \in U.$$

Using the mean value theorem for the Bochner integral ([21] Lemma 2.1.3) and the fact that $V = \{C(s)g(\theta,\psi) : s, \theta \in I,\ \| \psi \| \leq r^*\}$ is relatively compact in X; for $x \in B_r$ we get

$$\begin{aligned}
\Gamma_2 x(t) &= \int_0^t C(t-s)g(s,\tilde{x}_s) ds + \sum_{i=1}^{k-1} \int_{s_i}^{s_{i+1}} (S(s) - S(s_i)) f(t-s, \tilde{x}_{t-s}) ds \\
&\quad + \sum_{i=1}^{k-1} \int_{s_i}^{s_{i+1}} S(s_i) f(t-s, \tilde{x}_{t-s}) ds \\
&\in t\overline{co(V)} + aB_\epsilon(0,X) + \sum_{i=1}^{k-1} (s_{i+1} - s_i) \overline{co(U(t, s_i, r^*))},
\end{aligned}$$

where $co(V)$ denote the convex hull of V. Thus, $\Gamma_2 B_r(t)$ is relatively compact in X.

From the steps 1, 2, we infer that $\Gamma_2 B_r$ is relatively compact in $C(I;X)$. The continuity of Γ_2 follows from the axioms of the phase space and the Lebesgue dominated convergence Theorem. Thus, Γ_2 is completely continuous.

These remarks proves that Γ is completely continuous. Finally, the Leray Schauder Alternative Theorem imply that there exist a mild solution of (14)-(17). ■

In the most of the situation of practical interest the sine function is compact, it's the motivation of the next result.

Corollary 1 *Assume that $S(t)$ is compact for every $t > 0$ and that the assumptions of Theorem 4, except condition* (a)*, are verified. If $f(\cdot)$ takes bounded into the bounded sets, then there exists a mild solution of (14)-(17).*

Theorem 5 *Let* ($\mathbf{H}_1$), ($\mathbf{H}_4$) *and condition* (a) *in Theorem 4 be satisfied. Assume, furthermore, that there exist positives constant L_g, L_j^i, $i = 1, 2, ..n, j = 1, 2$ such that*

$$\begin{aligned} \| I_i^j(x) - I_i^j(y) \| &\leq L_i^j \| x - y \|, && x, y \in X, \\ \| g(t, \psi_1) - g(t, \psi_2) \| &\leq L_g \| \psi_1 - \psi_2 \|_{\mathcal{B}}, && (t, \psi_i) \in I \times \mathcal{B}. \end{aligned}$$

If $K_a(NL_g a + \sum_{i=1}^{n}(NL_i^1 + \tilde{N}L_i^2)) + \tilde{N} \liminf_{r \to \infty} \int_0^a m_f(s) W_f(K_a r + (M_a + K_a H) \| \varphi \|) ds < 1,$ *then there exists a mild solution of (14)-(17).*

Proof. Let $\Gamma : S(\varphi) \to S(\varphi)$ the map defined in the proof of Theorem 4 and consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where

$$\begin{aligned} \Gamma_1 x(t) &= C(t)\varphi(0) + S(t)[y_0 - g(0, \varphi)] + \int_0^t C(t - s) g(s, \bar{x}_s) ds \\ &\quad + \sum_{t_i < t} C(t - t_i) I_i^1(\bar{x}_{t_i}) + \sum_{t_i < t} S(t - t_i) I_i^2(\bar{x}_{t_i}), \qquad t \in I, \\ \Gamma_2 x(t) &= \int_0^t S(t - s) f(s, \bar{x}_s) ds, \qquad t \in I. \end{aligned}$$

Let B_r be the set $B_r = \{x \in S(\varphi) : \sup_{\theta \in I} \| x(\theta) - \varphi(0) \| \leq r\}$ and $\mathcal{X}^{\varphi} : (-\infty, a] \to X$ be the extension of φ such that $\mathcal{X}^{\varphi}(t) = \varphi(0)$ on I. We affirm that there exist $r > 0$ such that $\Gamma(B_r) \subset B_r$. In fact, if the property is false, for each $r > 0$ there exist $x^r \in B_r$ and $t^r \in [0, a]$ such $\| \Gamma x^r(t^r) - \varphi(0) \|_a > r$ and then

$$\begin{aligned} r &\leq \| \Gamma x^r(t^r) - \varphi(0) \|_a \\ &\leq \| C(\cdot) - I \|_a \| \varphi(0) \| + N(\| y_0 \| + \| g(0, \varphi) \|) + N L_g K_a r a \\ &\quad + N \int_0^a \| \mathcal{X}_s^{\varphi} \|_{\mathcal{B}} ds + \tilde{N} \int_0^a m_f(s) W_f(K_a r + \| \mathcal{X}_s^{\varphi} \|_{\mathcal{B}}) ds \\ &\quad + N \sum_{i=1}^{n} \left[L_i^1 K_a r + \| I_i^1(\mathcal{X}_{t_i}^{\varphi}) \|_{\mathcal{B}} \right] + \tilde{N} \sum_{i=1}^{n} \left[L_i^2 K_a r + \| I_i^2(\mathcal{X}_{t_i}^{\varphi}) \|_{\mathcal{B}} \right] \end{aligned}$$

and hence

$$K_a(NL_g a + \sum_{i=1}^{n}(NL_i^1 + \tilde{N}L_i^2)) + \tilde{N} \liminf_{r \to \infty} \int_0^a m_f(s) W_f(K_a r + (M_a + K_a H) \| \varphi \|) ds < 1,$$

which is contradictory with the assumptions.

Let $r_0 > 0$ such that $\Gamma(B_{r_0}) \subset B_{r_0}$. We known from the proof of Theorem 4 that Γ_1 is completely continuous on B_{r_0} and since

$$\| \Gamma_2 x - \Gamma_2 y \|_a \leq K_a \left(NL_g a + \sum_{i=1}^{n} \left(NL_i^1 + \tilde{N}L_i^2 \right) \right) \| x - y \|_a,$$

we can to conclude Γ is condensing on B_{r_0}. The result is now consequence of the Sadovskii point fixed Theorem. ∎

To complete this paper, we study the neutral system (4)-(7). In general, the results are similar to those in the first part of this section and for this reason their proofs will be summarized or simply omitted. To begin we introduce the followings conditions.

H_5 The function $f : I \times \mathcal{B} \times X \to X$ satisfy the following conditions:

(i) $f(t, \cdot) : \mathcal{B} \times X \to X$ is continuous a.e. $t \in I$;

(ii) For each $(\psi, x) \in \mathcal{B} \times \mathcal{X}$, the function $f(\cdot, \psi, x) : I \to X$ is strongly measurable;

(iii) There exists a continuous function $m_f : [0, \infty) \to [0, \infty)$ and a continuous nondecreasing function $W_f : [0, \infty) \to (0, \infty)$ such that

$$\| f(t, \psi, x) \| \leq m_f(t) W_f(\| \psi \|_{\mathcal{B}} + \| x \|), \qquad (t, \psi, x) \in I \times \mathcal{B} \times \mathcal{X}.$$

H_6 The function g is E-valued and verifies the following conditions

(i) $g(t, \cdot) : \mathcal{B} \times X \to E$ is continuous a.e. $t \in I$;

(ii) the function $g(\cdot, \psi, x) : I \to E$ is strongly measurable for every $(\psi, x) \in \mathcal{B} \times \mathcal{X}$.

(iii) There exists a continuous function $\widetilde{m}_g : [0, \infty) \to [0, \infty)$ and a continuous nondecreasing function $\widetilde{W}_g : [0, \infty) \to (0, \infty)$ such that

$$\| g(t, \psi, x) \|_1 \leq \widetilde{m}_g(t) \widetilde{W}_g(\| \psi \|_{\mathcal{B}} + \| x \|), \qquad (t, \psi, x) \in I \times \mathcal{B}.$$

(iv) There exist positive constants c_1, c_2 such that

$$\| g(t, \psi, x) \| \leq c_1(\| \psi \|_{\mathcal{B}} + \| x \|) + c_2, \qquad (t, \psi, x) \in I \times \mathcal{B} \times \mathcal{X}.$$

H_7 If Λ is the space $\Lambda = \{x : (-\infty, a] \to X : x_0 = \varphi,\ x|_{[0,a]} \in \mathcal{PC}\}$ endowed with the uniform convergence topology, then, for every $r > 0$ the set of functions

$$\{t \to g(t, x_t, x(t)) : x \in \Lambda, \| x \|_a \leq r\}$$

is equicontinuos on $[0, a]$.

Definition 3 *A function $x : (-\infty, a] \to X$ is a mild solution of (4)-(7) if $x_0 = \varphi$; $x(\cdot)_{|I} \in \mathcal{PC}$, $x'(\cdot) \in \mathcal{PC}$, the impulsive conditions (6)-(7) are satisfied and*

$$\begin{aligned} x(t) &= C(t)\varphi(0) + S(t)[y_0 - g(0, \varphi, y_0)] + \int_0^t C(t - s) g(s, x_s, x'(s)) ds \\ &+ \int_0^t S(t - s) f(s, x_s, x'(s)) ds \\ &+ \sum_{t_i < t} C(t - t_i) I_i^1(x_{t_i}, x'(t_i)) + \sum_{t_i < t} S(t - t_i) I_i^2(x_{t_i}, x'(t_i)), \qquad t \in I. \end{aligned}$$

Theorem 6 *Assume that $(\varphi(0), y_0) \in X \times E$, that $\mathbf{H_5}$- $\mathbf{H_7}$ hold, that $g(\cdot)$ is completely continuous and that the followings conditions are verified.*

(a) *The functions I_i^j are completely continuous and there exist constants c_i^j, d_i^j such that*

$$\begin{aligned} \| I_i^1(\psi, x) \| &\leq c_i^1 \| (\psi, x) \| + c_i^2, \qquad (\psi, x) \in \mathcal{B} \times X, \\ \| I_i^2(\psi, x) \| &\leq d_i^1 \| (\psi, x) \| + d_i^2, \qquad (\psi, x) \in \mathcal{B} \times X. \end{aligned}$$

(b) *Any of the followings properties hold:*

(b.1) *The functions I_i^1 are E-valued completely continuous and there exist constants $\tilde{c}_i^1, \tilde{c}_i^2$ such that $\| I_i^1(\psi, x) \|_1 \leq \tilde{c}_i^1 \| (\psi, x) \| + \tilde{c}_i^2$, for all $(\psi, x) \in \mathcal{B} \times X$,*

(b.2) *$S(t)$ is compact for each $t > 0$, the functions I_i^1 are $[D(A)]$-valued continuous and there are constants $\tilde{c}_i^1, \tilde{c}_i^2$ such that $\| AS(t)I_i^1(\psi, x) \| \leq \tilde{c}_i^1 \| (\psi, x) \| + \tilde{c}_i^2$, for every $(t, \psi, x) \in I \times \mathcal{B} \times X$.*

(c) *For every $r > 0$ the set $f(I \times B_r(0, \mathcal{B} \times X))$ is relatively compact in X.*

If $\mu = K_a \sum_{i=1}^{n} \left[K_a(N\tilde{c}_i^1 + \tilde{N}d_i^1) + \tilde{c}_i^1 + \tilde{N}d_i^1\right] + c_1 < 1$ *and*

$$\frac{1}{1-\mu} \int_0^t \left((NK_a + 1)\widetilde{m}_g(s) + (K_a\tilde{N} + N)m_f(s)\right) ds < \int_c^\infty \frac{ds}{W_g(s) + W_f(s)}$$

where

$$c = (1-\mu)^{-1} \left[(M_a + K_aNH) \| \varphi \|_{\mathcal{B}} + (K_a\tilde{N} + N)(\| y_0 \| + \| g(0, \varphi, y_0) \|) + c_2 + \Theta_1\right]$$

and $\Theta_1 = K_a \sum_{i=1}^{n}(N\tilde{c}_i^2 + \tilde{N}d_i^2) + \sum_{i=1}^{n}(\tilde{c}_i^2 + Nd_i^2)$, *then there exists a mild solution of (4)-(7).*

Proof. Let $S(\varphi)$ be the space in the proof of Theorem 4. On the space $S = S(\varphi) \times \mathcal{PC}$ we define the map $\Gamma : S \to S$ where $\Gamma(x, y) = (\Gamma_1(x, y), \Gamma_2(x, y))$ and

$$\begin{aligned}
\Gamma_1(x,y)(t) &= C(t)\varphi(0) + S(t)[y_0 - g(0, \varphi, y_0)] + \int_0^t C(t-s)g(s, \bar{x}_s, y(s))ds \\
&\quad + \int_0^t S(t-s)f(s, \bar{x}_s, y(s))ds \\
&\quad + \sum_{t_i<t} C(t-t_i)I_i^1(\bar{x}_{t_i}, y(t_i)) + \sum_{t_i<t} S(t-t_i)I_i^2(\bar{x}_{t_i}, y(t_i)), \qquad t \in I, \\
\Gamma_2(x,y)(t) &= AS(t)\varphi(0) + C(t)[y_0 - g(0, \varphi, y_0)] + g(t, \bar{x}_t, y(t)) \\
&\quad + \int_0^t AS(t-s)g(s, \bar{x}_s, y(s))ds + \int_0^t C(t-s)f(s, \bar{x}_s, y(s))ds \\
&\quad \sum_{t_i<t} AS(t-t_i)I_i^1(\bar{x}_{t_i}, y(t_i)) + \sum_{t_i<t} C(t-t_i)I_i^2(\bar{x}_{t_i}, y(t_i)), \qquad t \in I,
\end{aligned}$$

and $\bar{x} : (-\infty, a] \to X$ is defined as in the proof of theorem 4. Follows from the assumptions that Γ is well defined and continuous. If $(x^\lambda, y^\lambda) = \lambda\Gamma(x^\lambda, y^\lambda)$, $\lambda \in (0, 1)$, we get

$$\begin{aligned}
\| x^\lambda(t) \| &\leq N \| \varphi(0) \|_{\mathcal{B}} + \tilde{N}[\| y_0 \| + \| g(0, \varphi, y_0) \|] \\
&\quad + N \int_0^t \widetilde{m}_g(s)\widetilde{W}_g(M_a \| \varphi \|_{\mathcal{B}} + K_a \| x^\lambda \|_s + \| y^\lambda(s) \|)ds \\
&\quad + \tilde{N} \int_0^t m_f(s)W_f(M_a \| \varphi \|_{\mathcal{B}} + K_a \| x^\lambda \|_s + \| y^\lambda(s) \|)ds \\
&\quad + N \sum_{t_i<t} \| I_i^1(\bar{x}_{t_i}^\lambda, y^\lambda(t_i)) \| + \tilde{N} \sum_{t_i<t} \| I_i^2(\bar{x}_{t_i}^\lambda, y^\lambda(t_i)) \|
\end{aligned}$$

and

$$\begin{aligned}\| y^\lambda(t) \| &\leq \| AS(\cdot)\varphi(0) \|_a + N[\| y_0 \| + \| g(0,\varphi,y_0) \|] + c_1 \| \bar{x}_t^\lambda \| \pm c_2 \\ &+ \int_0^t \widetilde{m}_g(s)\widetilde{W}_g(M_a \| \varphi \|_{\mathcal{B}} + K_a \| x^\lambda \|_s + \| y^\lambda(s) \|)ds \\ &+ N \int_0^t m_f(s)W_f(M_a \| \varphi \|_{\mathcal{B}} + K_a \| x^\lambda \|_s + \| y^\lambda(s) \|)ds \\ &\sum_{t_i<t} \| AS(t-t_i)I_i^1(\bar{x}_{t_i}^\lambda, y^\lambda(t_i)) \| + N \sum_{t_i<t} \| I_i^2(\bar{x}_{t_i}^\lambda, y^\lambda(t_i)) \| .\end{aligned}$$

If $\alpha_\lambda(t) = M_a \| \varphi \|_{\mathcal{B}} + K_a \| x^\lambda \|_t + \| y^\lambda(t) \|$, from the previous estimates we get

$$\begin{aligned}\alpha^\lambda(t) &\leq (M_a + K_a N H) \| \varphi \|_{\mathcal{B}} + (K_a \tilde{N} + N)[\| y_0 \| + \| g(0,\varphi,y_0) \|] + c_2 \\ &+ \| AS(\cdot)\varphi(0) \|_a + \left(K_a \sum_{i=1}^n \left[K_a(N\tilde{c}_i^1 + \tilde{N}d_i^1) + \tilde{c}_i^1 + \tilde{N}d_i^1 \right] + c_1 \right) \alpha^\lambda(t) \\ &+ (K_a N + 1) \int_0^t \widetilde{m}_g(s)\widetilde{W}_g(\alpha^\lambda(s))ds + (\tilde{N}K_a + N) \int_0^t m_f(s)W_f(\alpha^\lambda(s))ds\end{aligned}$$

which imply that

$$\alpha^\lambda(t) \leq c + \frac{1}{1-\mu} \left[(K_a N + 1) \int_0^t \widetilde{m}_g(s)\widetilde{W}_g(x_\lambda(s))ds + (\tilde{N}K_a + N) \int_0^t m_f(s)W_f(x^\lambda(s))ds \right]$$

If $\beta_\lambda(t)$ is the right hand side of the last inequality, we see that

$$\beta_\lambda'(t) \leq \frac{1}{1-\mu} \left((K_a N + 1)m_g(t) + (K_a\tilde{N} + N)m_f(t) \right) (W_g(\beta_\lambda(t)) + W_f(\beta_\lambda(t)))$$

and hence

$$\begin{aligned}\int_{\beta_\lambda(0)}^{\beta_\lambda(t)} \frac{ds}{W_g(s) + W_f(s)} &\leq \frac{1}{1-\mu} \int_0^t \left[(K_a N + 1)m_g(s) + (K_a\tilde{N} + N)m_f(s) \right] ds \\ &< \int_c^\infty \frac{ds}{W_g(s) + W_f(s)},\end{aligned}$$

which prove that the set $\{z^\lambda = (x^\lambda, y^\lambda) : z^\lambda = \lambda \Gamma z^\lambda,\ \lambda \in (0,1)\}$ are bounded in S.

Using the same arguments employed in the proof of Theorem 4, follows that Γ_1 is completely continuous. Moreover, from the compactness assumptions on g, f, I_i^j; $\mathbf{H_7}$ and the fact that the family $(AS(t))_{t\in\mathbb{R}}$ is a strongly continuous family of bounded linear operators from E into X, follows that Γ_2 is completely continuous map. Thus Γ is completely continuous.

The existence of a mild solution is now consequence of Theorem 1. ∎

The proof of the next result is similar to the proof of Theorem 5.

Theorem 7 *Assume that $(\varphi(0), y_0) \in E \times X$, that the condition* (c) *in Theorem 6 holds, that the maps $I_i^1, i = 1, 2, ..n$, are E-valued and that there are constants L_i^j such that*

$$\begin{aligned}\| I_i^1(x,y) - I_i^1(z,w) \|_1 &\leq L_i^1 \| (x,y) - (z,w) \|, \qquad (x,y),(z,w) \in X^2, \\ \| I_i^2(x,y) - I_i^2(z,w) \| &\leq L_i^2 \| (x,y) - (z,w) \|, \qquad (x,y),(z,w) \in X^2.\end{aligned}$$

If

$$(K_a+1)\sum_{i=1}^{n}\left[Nc_i^1+\tilde{c}_i^1+(N+\tilde{N})d_i^1\right]+\liminf_{r\to\infty}\left[\Omega(r)\int_0^a \tilde{m}_g(s)ds+\Lambda(r)\int_0^a m_f(s)ds\right]<1,$$

where $\Omega(r)=\tilde{W}_g(M_a\parallel\varphi\parallel_{\mathcal{B}}+(K_a+1)r)$ *and* $\Lambda(r)=\tilde{W}_f(M_a\parallel\varphi\parallel_{\mathcal{B}}+(K_a+1)r)$ *then there exists a mild solution of the neutral system (4)-(7).*

4 Examples

In this section we illustrate some of the results in this work. In the rest of this work $X=L^2([0,\pi])$ and A is the operator $Af=f''$ with domain

$$D(A):=\{f(\cdot)\in L^2([0,\pi]): f''(\cdot)\in L^2([0,\pi]),\ f(0)=f(\pi)=0\}.$$

It's well know that A is the infinitesimal generator of a C_0-semigroup and of a strongly continuous cosine function on X, which will be denoted by $(T(t))_{t\geq 0}$ and $(C(t))_{t\in\mathbb{R}}$ respectively. Moreover, A has discrete spectrum, the eigenvalues are $-n^2, n\in N$, with corresponding normalized eigenvectors $z_n(\xi):=\left(\frac{2}{\pi}\right)^{1/2}\sin(n\xi)$ and the following properties hold:

(a) $\{z_n: n\in N\}$ is an orthonormal basis of X.

(b) If $f\in D(A)$ then $A(f)=-\sum_{n=1}^{\infty}n^2<f,z_n>z_n$.

(c) For every $f\in X$, $(-A)^{-\frac{1}{2}}f=\sum_{n=1}^{\infty}\frac{1}{n}<f,z_n>z_n$. The operator $(-A)^{\frac{1}{2}}$ is given by $(-A)^{\frac{1}{2}}f=\sum_{n=1}^{\infty}n<f,z_n>z_n$ on $D((-A)^{\frac{1}{2}})=\{f\in X:\sum_{n=1}^{\infty}n<f,z_n>z_n\in X\}$, $\|(-A)^{-1/2}\|=1$ and $\|(-A)^{\frac{1}{2}}T(t)\|\leq\frac{e^{\frac{-t}{2}}t^{-\frac{1}{2}}}{\sqrt{2}}$ for $t>0$.

(d) For $f\in X$, $C(t)f=\sum_{n=1}^{\infty}\cos(nt)<f,z_n>z_n$ and $T(t)f=\sum_{n=1}^{\infty}e^{-n^2t}<f,z_n>z_n$. Moreover, it follow from these expressions that $S(t)\varphi=\sum_{n=1}^{\infty}\frac{\sin(nt)}{n}<\varphi,z_n>z_n$, that $S(t)$ and $T(t)$ are compact for $t>0$ and that $\|C(t)\|\leq 1$, $\|T(t)\|\leq e^{-t}$ and $\|S(t)\|\leq 1$ for every $t\in[0,\pi]$.

(e) If Φ denotes the group of translations on X defined by $\Phi(t)x(\xi)=\tilde{x}(\xi+t)$, where $\tilde{x}$ is the extension of x with period 2π, then $C(t)=\frac{1}{2}(\Phi(t)+\Phi(-t))$; $A=B^2$ where B is the infinitesimal generator of the group Φ and $E=\{x\in H^1(0,\pi): x(0)=x(\pi)=0\}$, see [11] for details.

On the other hand, to modelled our examples we will use the space $\mathcal{B}:=C_r\times L^2(g;X)$, $r=0$, see [17], which consists of all classes of functions $\varphi:(-\infty,0]\to X$ such that φ

is continuous on $[-r,0]$, Lebesgue-measurable and $g(\cdot) \mid \varphi(\cdot) \mid^p$ is Lebesgue integrable on $(-\infty,-r)$, where $g(\cdot):(-\infty,-r)\to \mathbb{R}$ is a positive Lebesgue integrable function. The seminorm in $\|\cdot\|_{\mathcal{B}}$ is defined by

$$\| \varphi \|_{\mathcal{B}} := \sup\{\| \varphi(\theta)\| : -r \leq \theta \leq 0\} + \Big(\int_{-\infty}^{-r} g(\theta) \| \varphi(\theta) \|^p \, d\theta\Big)^{1/p}.$$

We will assume that $g(\cdot)$ satisfies conditions (g-6) and (g-7) in the terminology of [17]. This means that $g(\cdot)$ is integrable on $(-\infty,-r)$ and that there exists a non-negative and locally bounded function γ on $(-\infty,0]$ such that $g(\xi+\theta) \leq \gamma(\xi)\, g(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty,-r)\setminus N_\xi$, where $N_\xi \subseteq (-\infty,-r)$ is a set with Lebesgue measure zero.

In this case, $\mathcal{B}$ is a phase space which verifies axioms **(A)**, **(A1)** and **(B)**, [17] Theorem 1.3.8. Moreover, $H=1$; $M(t)=\gamma(-t)^{\frac{1}{2}}$ and $K(t) = 1 + \Big(\int_{-t}^{0} g(\tau)d\tau\Big)^{\frac{1}{2}}$ for all $t \geq 0$.

4.1 A first order neutral equation

Next we study the first order neutral equation with unbounded delay

$$[u(t,\xi) + \int_{-\infty}^{t}\int_{0}^{\pi} b(s-t,\eta,\xi)u(s,\eta)d\eta ds]' = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + \int_{-\infty}^{t} F(t,s-t,\xi,u(s,\xi))ds, \tag{19}$$

$$u(t,0) = u(t,\pi) = 0, \;\; t \geq 0, \tag{20}$$

$$u(\tau,\xi) = \varphi(\tau,\xi), \;\; \tau \leq 0, \;\; 0 \leq \xi \leq \pi, \tag{21}$$

$$\Delta u(t_i)(\xi) = \int_{-\infty}^{t_i} a_i(s-t_i)u(s,\xi)ds, \tag{22}$$

where $\varphi \in \mathcal{B}$ and

(a) $b(\theta,\eta,\zeta)$, $\dfrac{\partial b(\theta,\eta,\zeta)}{\partial \zeta}$, $\dfrac{\partial^2 b(\theta,\eta,\zeta)}{\partial \zeta^2}$ are measurable; $b(\theta,\eta,\pi) = b(\theta,\eta,0) = 0$ and

$$L_g := \max\{(\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi} \frac{1}{g(s)}(\frac{\partial^i b(s,\eta,\zeta)}{\partial \zeta^i})^2 d\eta ds d\zeta)^{\frac{1}{2}} : i=0,1\} < \infty;$$

(b) $f:\mathbb{R}^4 \to \mathbb{R}$ is continuous and that there are continuous functions $\eta_i^f : \mathbb{R}^3 \to \mathbb{R}$ such that

$$| f(s,\xi,t,x) | \leq \eta_1^f(t,s,\xi) + \eta_2^f(t,s) \mid x \mid, \qquad (s,\xi,t,x) \in \mathbb{R}^4;$$

(c) the functions $a_i(\cdot)$ are continuous and $c_i^1 := \Big(\int_{-\infty}^{0} \dfrac{a_i^2(\theta)}{g(\theta)} d\theta\Big)^{\frac{1}{2}} < \infty$ for each $i = 1,2,..n$.

Assuming that the conditions (a), (b), (c) and the assumptions (i)-(iii) of Example 3.1 in [14] are verified, the problem (19)-(22) can be modelled as the abstract impulsive Cauchy problem (1)-(3) by defining the substituting operators

$$
\begin{aligned}
g(t,\psi)(\xi) &:= \int_{-\infty}^{0}\int_{0}^{\pi} b(s,\,\eta,\,\xi)\psi(s,\,\eta)d\eta ds, && (23)\\
f(t,\psi)(\xi) &:= \int_{-\infty}^{0} F(t,s,\xi,\psi(s,\xi))ds, && (24)\\
I_i(\psi) &= \int_{-\infty}^{0} a_i(s)\psi(s,\xi)ds. && (25)
\end{aligned}
$$

Moreover, $G(t,\cdot)$ and I_i are bounded linear operators, the range of G is contained in $X_{\frac{1}{2}}$, $\| (-A)^{1/2}G(t,\cdot) \| \leq L_g$, $\| I_i \| \leq c_i^1$ for $i = 1,2,..n$ and $\| f(t,\psi) \| \leq d_1(t) + d_2(t) \| \psi \|_{\mathcal{B}}$, where

$$
d^1(t) := \left(2\int_0^{\pi}\left(\int_{-\infty}^{0}\eta_1^f(t,s,\xi)ds\right)^2 d\xi\right)^{\frac{1}{2}} \text{ and } d^2(t) := \left(2\pi\int_{-\infty}^{0}\frac{\eta_2^f(t,s)^2}{g(s)}dsd\xi\right)^{\frac{1}{2}}.
$$

Proposition 1 *Assume that the previous conditions are verified. If*

$$
\left(1+(\int_{-\pi}^{0}g(\tau)d\tau)^{\frac{1}{2}}\right)\left(L_g(1+\sqrt{2\pi})+\sum_{i=1}^{n}L_i+\int_0^{\pi}(d_1(s)+d_1(s))ds\right)<1,
$$

then there exist a mild solution of (19)-(22).

4.2 A second order neutral equation

Now we consider the neutral differential equation

$$
[u'(t,\xi)+\int_{-\infty}^{t}\int_0^{\pi}b(s-t,\eta,\xi)u(s,\eta)d\eta ds]' = \frac{\partial^2 u(t,\xi)}{\partial \xi^2}+\int_{-\infty}^{t}F(t,s-t,\xi,u(s,\xi))ds, \quad (26)
$$

$$
\begin{aligned}
u(t,0) &= u(t,\pi)=0, \quad t\geq 0, && (27)\\
u(\tau,\xi) &= \varphi(\tau,\xi), \qquad \tau\leq 0,\ \ 0\leq\xi\leq\pi, && (28)\\
\Delta u(t_i)(\xi) &= \int_{-\infty}^{t_i} a_i(s-t_i)u(s,\xi)ds, && (29)\\
\Delta u'(t_i)(\xi) &= \int_{-\infty}^{t_i} \tilde{a}_i(s-t_i)u(s,\xi)ds. && (30)
\end{aligned}
$$

To study the problem (26)-(28), we always that the assumptions in the previous examples hold and additionally that

(d) the functions $\tilde{a}_i(\cdot)$ are continuous and $\tilde{c}_i^1 := \left(\int_{-\infty}^{0}\frac{\tilde{a}_i^2(\theta)}{g(\theta)}d\theta\right)^{\frac{1}{2}} < \infty$ for each $i = 1,2,..n$.

Defining f, g as in (23), (24) and I_i^j by

$$
I_i^1(\psi)=\int_{-\infty}^{0}a_i(s)\psi(s,\xi)ds \text{ and } I_i^2(\psi)=\int_{-\infty}^{0}\tilde{a}_i(s)\psi(s,\xi)ds, \quad (31)
$$

the problem (26)-(28) can be modelled as (14)-(17). The next results follows directly of Theorem (5).

Proposition 2 *Let the previous conditions be satisfied. If*

$$\left(1+(\int_{-\pi}^{0} g(\tau)d\tau)^{\frac{1}{2}}\right)\left(\pi L_g+\sum_{i=1}^{n}(L_i^1+L_i^2)+\int_0^{\pi}(d_1(s)+d_1(s))ds\right)<1,$$

then there exist a mild solution of (19)-(22).

References

[1] Balachandran; K. Dauer, J. P. Existence of solutions of nonlinear neutral integrodifferential equations in Banach spaces. *J. Math. Anal. Appl.* 251 (2000), no. 1, 93–105.

[2] Balachandran, K.; Sakthivel, R.; Dauer, J. P. Controllability of neutral functional integrodifferential systems in Banach spaces. *Comput. Math. Appl.* 39 (2000), no. 1-2,

[3] Balachandran, K.; Anthoni, S. Marshal Controllability of second-order semilinear neutral functional differential systems in Banach spaces. *Comput. Math. Appl.* 41 (2001), no. 10-11, 1223–1235.

[4] Balachandran, K.; Anthoni, S. Marshal Existence of solutions of second order neutral functional-differential equations. *Tamkang J. Math.* 30 (1999), no. 4, 299–309.

[5] Balachandran, K.; Sakthivel, R. Existence of solutions of neutral functional integrodifferential equation in Banach spaces. *Proc. Indian Acad. Sci. Math. Sci.* 109 (1999), no. 3, 325–332.

[6] Benchohra, M.; Ntouyas, S. K. Nonlocal Cauchy problems for neutral functional differential and integrodifferential inclusions in Banach spaces. *J. Math. Anal. Appl.* 258 (2001), no. 2, 573–590.

[7] Benchohra, M.; Henderson, J.; Ntouyas, S. K. Existence results for impulsive multivalued semilinear neutral functional differential inclusions in Banach spaces. *J. Math. Anal. Appl.* 263 (2001), no. 2, 763–780

[8] Benchohra, M.; Ntouyas, S. K. Neutral functional differential and integrodifferential inclusions in Banach spaces. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.* (8) 4 (2001), no. 3, 767–782.

[9] Benchohra, Mouffak; Henderson, Johnny; Ntouyas, S. K. Semilinear impulsive neutral functional differential inclusions in Banach spaces. Appl. Anal. 81 (2002), no. 4, 951-963 (confirmar)

[10] Benchohra, M.; Ntouyas, S. K. Nonlocal Cauchy problems on semi-infinite intervals for neutral functional differential and integrodifferential inclusions in Banach spaces. *Math. Slovaca* 51 (2001), no. 5, 529–545.

[11] H. O. Fattorini, Second Order Linear Differential Equations in Banach Spaces, North-Holland Mathematics Studies, Vol. 108, North-Holland, Amsterdam, 1985.

[12] Hale, Jack K.; Kato, Junji. Phase space for retarded equations with infinite delay. *Funkcial. Ekvac.* 21 (1978), no. 1, 11–41.

[13] Hale, Jack K.; Verduyn Lunel, Sjoerd M. Introduction to functional-differential equations. Applied Mathematical Sciences, 99. Springer-Verlag, New York, 1993.

[14] Hernández, Eduardo; Henríquez, Hernán R. Existence of periodic solutions of partial neutral functional differential equations with unbounded delay. *J. Math. Anal. Appl.* 221 (1998), no. 2, 499–522.

[15] Hernández, Eduardo; Henríquez, Hernán R. Existence results for partial neutral functional differential equations with unbounded delay. *J. Math. Anal. Appl.* 221 (1998), no. 2, 452–475.

[16] Hernández, Eduardo. A second-order impulsive Cauchy problem. *Int. J. Math. Math. Sci.* 31 (2002), no. 8, 451–461.

[17] Hino, Yoshiyuki; Murakami, Satoru; Naito, Toshiki Functional-differential equations with infinite delay. Lecture Notes in Mathematics, 1473. Springer-Verlag, Berlin, 1991.

[18] J. Kisyński, On second order Cauchy's problem in a Banach space,*Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronm. Phys.* **18** (1970) 371-374.

[19] Lee, Haeng Joo; Park, Jeongyo; Park, Jong Yeoul Existence results for second-order neutral functional differential and integrodifferential inclusions in Banach spaces. Electron. J. Differential Equations 2002, No. 96, 13 pp. (electronic).

[20] Liu, James H. Nonlinear impulsive evolution equations. *Dynam. Contin. Discrete Impuls. Systems* 6 (1999), no. 1, 77–85.

[21] R. H. Martin, Nonlinear Operators and Differential Equations in Banach Spaces, Robert E. Krieger Publ. Co., Florida, 1987.

[22] Pazy, A. Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences, 44. Springer-Verlag, New York-Berlin, 1983.

[23] Rogovchenko, Yuri V. Impulsive evolution systems: main results and new trends. *Dynam. Contin. Discrete Impuls. Systems* 3 (1997), no. 1, 57–88.

[24] Rogovchenko, Yuri V. Nonlinear impulse evolution systems and applications to population models. *J. Math. Anal. Appl.* 207 (1997), no. 2, 300–315.

[25] B. N. Sadovskii, On a fixed point principle. *Funct. Anal. Appl.* **1** (1967), 74-76.

[26] C. C. Travis, G. F. Webb, Compactness, regularity, and uniform continuity properties of strongly continuous cosine families. *Houston J. Math.* **3** (4) (1977) 555-567.

[27] C. C. Travis, G. F. Webb, Cosine families and abstract nonlinear second order differential equations. *Acta Math. Acad. Sci. Hungaricae,* **32** (1978) 76-96.

Eduardo Hernández M.
Departamento de Matemática
Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo - Campus de São Carlos
Caixa Postal 668
13560-970 São Carlos, SP. Brazil
E-mail : lalohm@icmc.sc.usp.br

NOTAS DO ICMC

SÉRIE MATEMÁTICA

162/2003 HERNÁNDEZ M., E. – Asymptotic almost periodic solutions for a second order differential equations with nonlocal conditions.

161/2003 HERNÁNDEZ M., E. – Existence results for a second order partial differential equation with nonlocal conditions.

160/2003 CARBINATTO, M.C.; RYBAKOWSKI, K.P. - Morse decompositions in the absence of uniqueness II.

159/2003 FERNANDES, A.; SOARES, H. – Bilipschitz triviality of family of maps.

158/2002 JORGE PÉREZ, V.H.; LEVCOVITZ, D.; SAIA, M. J. – Polar multiplicities, and Euler obstruction of map germs from C^n to C^n.

157/2002 JORGE PÉREZ, V.H.; SAIA, M. J. – Euler obstruction, polar multiplicities and equisingularity of map germs in O (n,p), $n < p$.

156/2002 BRUCE, J.W.; TARI, F. – On families of square matrices.

155/2002 FERNANDES, A.; GUTIERREZ, C. – On local diffeomorphisms of R^n that are injective.

154/2002 BIASI, C.; GODOY, S.M.S. – Generalized homogeneous functions and the two-body problem.

153/2002 LABOURIAU, I.S.; RUAS, M.A.S. – Invariants for bifurcations.