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Upper Semicontinuity of Attractors for the Discretization of Strongly Damped Wave Equations

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Resumo

Neste artigo consideramos problemas hiperbólicos com amortecimento da forma

$$u_{tt} + \eta A^{\frac{1}{2}} u_t + a u_t + A u = f(u)$$

e sua discretização

$$\ddot{U} + 2\eta\, A_n^{1/2} \dot{U} + 2a\, \dot{U} = -A_n U + \mathrm{f}(U)$$

com $D(A) = \{u \in H^2(0,1) : u_x(0) = u_x(1) = 0\}$, $A : D(A) \subset X \to X$, $Au = -u_{xx} + \delta u$, $\delta > 0$, $a > 0$, e $\eta \geq 0$. Comparando os semigrupos gerados por ambas as equações, provamos que a dinâmica assintótica da discretização é semicontinua superiormente relativamente ao parâmetro n, quando $n \to \infty$.

UPPER SEMICONTINUITY OF ATTRACTORS FOR THE DISCRETIZATION OF STRONGLY DAMPED WAVE EQUATIONS

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ABSTRACT. We consider damped hyperbolic problems of the form

$$u_{tt} + \eta A^{\frac{1}{2}} u_t + a u_t + Au = f(u)$$

and its discretization

$$\ddot{U} + 2\eta\, A_n^{1/2}\dot{U} + 2a\,\dot{U} = -A_n U + \mathrm{f}(U)$$

with $D(A) = \{u \in H^2(0,1) : u_x(0) = u_x(1) = 0\}$, $A : D(A) \subset X \to X$, $Au = -u_{xx} + \delta u$, $\delta > 0$, $a > 0$, $\eta \geq 0$. By comparing the semigroups generated by both equations, we prove that asymptotic dynamics of this discretization is upper continuous for $n \to \infty$.

1. INTRODUCTION

In most problems, the ideal situation is having the asymptotic dynamics of one equation the same as the asymptotic dynamics of its discretization. Although, when we are studying the linear wave equation, we note that the spectrum of discretization and the spectrum of its continuous counterpart are far away from each other, no matter how fine the discretization is. This fact is restrictive in the hyperbolic equation (case $\eta = 0$). It also happens in the parabolic equation but the nonconvergent part is controlled by the fact that the real part of eigenvalues is negative and too large in absolute value (the corresponding modes do not interfere in the asymptotics).

In order to overcome this problem we propose to approach the linear wave equation by a "parabolic equation" (the strong damped wave equation) and then to make an approximation for the strongly damped wave equations by its discretization.

The equation that we consider is

$$\text{(1.1)} \qquad \begin{aligned} &u_{tt} + 2\eta\, A^{1/2}u_t + 2a\, u_t = -Au + f(u), \quad 0 < x < 1,\ t > 0 \\ &u_x(0) = u_x(1) = 0,\ t > 0, \end{aligned}$$

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and its discretization is given by

$$\ddot{U} + 2\eta\, A_n^{1/2}\dot{U} + 2a\,\dot{U} = -A_n U + \mathrm{f}(U) \tag{1.2}$$

where $\eta > 0$, $a > 0$, $Au = -u_{xx} + \frac{\delta}{2}u$, A_n is a $n \times n$, $A_n = \Delta_n + \frac{\delta}{2}I$ and Δ_n given by

$$\Delta_n = n^2 \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}, \tag{1.3}$$

$f : \mathbb{R} \to \mathbb{R}$ is a C^2 function satisfying a dissipative condition

$$\limsup_{|u| \to +\infty} \frac{f(u)}{u} \leq -\delta, \quad \delta > 0 \tag{1.4}$$

$\mathrm{f}(U) = (f(u_1), \cdots, f(u_n))^\top$ and $U = (u_1, \cdots, u_n)^\top$.
Note that δ may be taken equal zero for Dirichlet boundary conditions.

We denote by A, the operator $A : D(A) \subset X^0 \to X^0$ given by $Au = -u_{xx} + \frac{\delta}{2}u$, $X = L^2 = X^0$ and $D(A) = \{u \in H^2(0,1); u'(0) = u'(1) = 0\} = X^1$.
For convenience, we rewrite (1.1) in a abstract form

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = C_\eta \begin{bmatrix} u \\ v \end{bmatrix} + h(\begin{bmatrix} u \\ v \end{bmatrix}) \tag{1.5}$$

where $D(C_\eta) = X^1 \times X^{\frac{1}{2}} = Y^1$,

$$C_\eta = \begin{bmatrix} 0 & I \\ -A & -2(\eta A^{1/2} + a) \end{bmatrix} \text{ and } h(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} 0 \\ f^e(u) \end{bmatrix}.$$

For $\eta > 0$, $-C_\eta$ is a sectorial operator and generates an analytic semigroup of contractions (see [3, 4]). For $\eta \geq 0$, the equation (1.1) generates a C^1-semigroup T_η on $Y^0 = H^1 \times L^2$. $T_\eta(t)$, $t \geq 0$ is a gradient system asymptotically smooth. Furthermore, $T_\eta(t)$ admits a global attractor, $\mathcal{A}_\eta$. By using regularity results we have $\mathcal{A}_\eta \subset Y^1$.
In order to keep the similarity, we rewrite (1.2) in a matrix form

$$\frac{d}{dt}\begin{bmatrix} U \\ V \end{bmatrix} = C_{\eta n} \begin{bmatrix} U \\ V \end{bmatrix} + H(\begin{bmatrix} U \\ V \end{bmatrix}) \tag{1.6}$$

where

$$C_{\eta n} = \begin{bmatrix} 0 & I_n \\ -A_n & -2(\eta A_n^{1/2} + a) \end{bmatrix} \text{ and } \quad H(\begin{bmatrix} U \\ V \end{bmatrix}) = \begin{bmatrix} 0 \\ \mathrm{f}(U) \end{bmatrix}$$

For (1.6), we have a global attractor $\mathcal{A}_{\eta n}$.

Considering

$$\delta_Y(A,B) = \sup_{x\in A} \inf_{y\in B} d_Y(x,y) \tag{1.7}$$

we can define the continuity of a family of sets $B_\eta \subset Y$ in the following form: a family B_η is continuous in η_0 if it is upper semicontinuous at η_0; that is, $\delta_Y(B_\eta, B_{\eta_0}) \to 0$ as $\eta \to \eta_0$, and lower semicontinuous at η_0; that is, $\delta_Y(B_{\eta_0}, B_\eta) \to 0$ as $\eta \to \eta_0$.

It can be proved, [2], that the family of global attractors $\mathcal{A}_\eta$, $\eta \geq 0$ is continuous (lower and upper) in $\eta = 0$. Note that $\eta = 0$ in (1.5), give us the linear wave equation.

In this paper we study the upper semicontinuity of the family of global attractors $\mathcal{A}_{\eta n}$ for n large enough, where $\mathcal{A}_{\eta_\infty} \equiv \mathcal{A}_\eta$.

In order to compare the problems (1.5) and (1.2) it was necessary to consider the space of finite dimension embedded in the phase space of the continuous problem. We also consider two norms in $\mathbb{R}^n \times \mathbb{R}^n$ which are the discretization of norms in the continuous space (Y^0 and Y^1). Studying the problem, we realize it was not possible to reduce the phase space dimension using the invariant manifold. That happens because there is not a large gap in the spectrum of the linear operator, since $\lim_{k\to\infty} \mathrm{Re}(\lambda_{\pm(k+1)}) - \mathrm{Re}(\lambda_{\pm k}) = \eta$, where λ_k é o k^{th} eigenvalue of C_η.

We work with this problem for a fix $\eta > 0$ as follow. First, we analyse the closeness of the linear semigroups in the norm Y^0. In order to do that, we decompose the semigroups in two parts. In one of them, defined on an infinite space dimension, $\mathrm{Re}(\lambda_{\pm k}) \to -\infty$, where $k \to \infty$. It means that the semigroup norm can be set arbitrarily small. So our problem becomes to compare the semigroups in a finite dimension space. We did that using the convergence of the eigenvalues and eigenvectors of the discrete problem to the continuous problem. Then, we compare the nonlinear semigroups. Last, we prove the upper semicontinuity of the global attractors $\mathcal{A}_{\eta n}$. This procedure was used in [1]

The main result of this paper is

Theorem 1.1. *The family of global attractors $\mathcal{A}_{\eta n}$ is upper semicontinuous at $n = \infty$, for any $\eta > 0$.*

Theorem 1.1, and the fact the family $\mathcal{A}_\eta$ is upper continuous [2], leads to the following important result

Theorem 1.2. *Let $\mathcal{A}$ be the attractor of (1.1) for $\eta = 0$. Then, there exists a sequence (η, n_η) such that $\delta(\mathcal{A}_{\eta n_\eta}, \mathcal{A})$ converges to zero when $\eta \to 0$.*

These results can be summarized in the following diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "\eta\to 0"] & \mathcal{A}_\eta \arrow[l, shift left] \\ & \mathcal{A}_{\eta n} \arrow[u, "n\to\infty,\, \eta\ \text{fixo}"'] \arrow[ul, dashed] \end{tikzcd}$$

where the arrows denotes upper semicontinuity when it points to the limit problem and denotes lower semicontinuity when it points to the family problems.

This paper is organized as follows. In Section 1, we recall some properties of C_η and $C_{\eta n}$. We also define the norms and some relations between $\mathbb{R}^n \times \mathbb{R}^n$ and Y^0. In Section 3 we make the comparison of the linear semigroups. The comparison of the nonlinear semigroups is done in Section 4. Finally, the last section proves the upper semicontinuity of attractors $C_{\eta n}$.

2. Spectral properties of C_η and $C_{\eta n}$.

Let $\nu_k = (k\pi)^2 + \frac{\delta}{2}$ be the eigenvalues of A for $k = 0, 1, \cdots$. The eigenvalues, $\lambda_{\pm k}$, of C_η are the solutions of

$$\lambda^2 + (2\eta\nu_k^{\frac{1}{2}} + 2a\nu_k)\lambda + \nu_k = 0$$

and they are given by:

$$\lambda_{\pm k} \quad = \quad -(\eta\nu_k^{1/2} + a) \pm \sqrt{(\eta\nu_k^{1/2} + a)^2 - \nu_k}$$

For each $\eta > 0$, exist an $k_0 = k_0(\eta) \geq 0$ such that $\lambda_{\pm k}$ is a real number for $k < k_0$ and $\lambda_{\pm k}$ is a complex number for $k \geq k_0$.

The correspondents eigenfunctions are given by:

$$\phi_{\pm k} = \begin{bmatrix} e_k \\ \lambda_{\pm k} e_k \end{bmatrix} \tag{2.1}$$

where $e_k = \cos(k\pi x)$ is a eigenfunction of A with respect the eigenvalue ν_k. If $\lambda_{\pm k}$ is a double eigenvalue then $\psi_k = \begin{bmatrix} 0 \\ e_k \end{bmatrix}$ is a generalized eigenfunction associated with $\lambda_{\pm k}$. If $\lambda_{\pm k}$ is a complex eigenvalue then we consider the following vectors $\psi_{+k} = \mathrm{Re}(\phi_{\pm k})$ and $\psi_{-k} = \mathrm{Im}(\phi_{\pm k})$, in the real eigenspace associated with $\lambda_{\pm k}$.

We have the following properties:

1) the family $(\phi_{+k})_{k=0}^{k_0}, (\psi_{+k})_{k=k_0}^{\infty}$ is orthogonal in Y^0;
2) the family $(\phi_{-k})_{k=0}^{k_0}, (\psi_{-k})_{k=k_0}^{\infty}$ is orthogonal in Y^0;
3) $\langle \phi_{-i}, \phi_{+j} \rangle_{Y^0} = 0$, $\langle \psi_{-i}, \psi_{+j} \rangle_{Y^0} = 0$, $\langle \phi_{-i}, \psi_{+j} \rangle_{Y^0} = 0$, $\langle \psi_{-i}, \phi_{+j} \rangle_{Y^0} = 0$ if $i \neq j$.

Similarly, the eigenvalues of A_n are given by $\nu_k^n = 4n^2\sin^2\frac{k\pi}{2n} + \frac{\delta}{2}$ and the associated eigenvectors are $e_k^n = (\cos k\pi x_1, \cdots, \cos k\pi x_n)$ for $k = 0, \cdots, n-1$. The eigenvalues, $\lambda_{\pm k}^n$, of $C_{\eta n}$ are the solutions of the equation $\lambda^2 + (2\eta(\nu_k^n)^{\frac{1}{2}} + 2a\nu_k^n)\lambda + \nu_k^n = 0$ and are given by:

$$\lambda_{\pm k}^n = -(\eta(\nu_k^n)^{1/2} + a) \pm \sqrt{(\eta(\nu_k^n)^{1/2} + a)^2 - \nu_k^n}$$

The correspondents eigenvectors are given by:

$$\phi_{\pm k}^n = \begin{bmatrix} e_k^n \\ \lambda_{\pm k}^n e_k^n \end{bmatrix} \tag{2.2}$$

where e_k^n is the eigenvector of A_n associated with the eigenvalue ν_k^n. If $\lambda_{\pm k}^n$ is a double eigenvalue then $\psi_k^n = \begin{bmatrix} 0 \\ e_k^n \end{bmatrix}$ is a generalized eigenvector associated with $\lambda_{\pm k}^n$

If $\lambda_{\pm k}^n$ is a complex eigenvalue then we consider the following vectors $\psi_{+k}^n = \mathrm{Re}(\phi_{\pm k}^n)$ and $\psi_{-k}^n = \mathrm{Im}(\phi_{\pm k}^n)$, in the real eigenspace associated with $\lambda_{\pm k}$.

We also get that for each $\eta > 0$ and $n > 0$ exist a $k_0 = k_0(\eta, n) \geq 0$ such that $\lambda_{\pm k}^n$ is a real number for $k < k_0$ and $\lambda_{\pm k}^n$ is a complex number for $k_0 < k < n$.

In order to compare the problems (1.5) and (1.6), it is necessary to consider in $\mathbb{R}^n \times \mathbb{R}^n$ a compatible norm with the norm in Y^0. Therefore, we define in $\mathbb{R}^n \times \mathbb{R}^n$ the following inner product:

$$\langle \begin{bmatrix} U \\ V \end{bmatrix}, \begin{bmatrix} W \\ Z \end{bmatrix} \rangle_0 = \langle A_n U, W \rangle_{\mathbb{R}^n} + \langle V, Z \rangle_{\mathbb{R}^n} \tag{2.3}$$

where $\langle U, W \rangle_{\mathbb{R}^n} = \sum_{i=1}^n \frac{1}{n} u_i w_i$ is the inner product L^2 discretized. We denote for Y_n^0 the space $\mathbb{R}^n \times \mathbb{R}^n$ with the inner product given above.

About the eigenvectors of $C_{\eta n}$ in the space Y_n^0 we have:

1) the family $(\phi_{+k}^n)_{k=0}^{k_0}, (\psi_{+k}^n)_{k=k_0}^n$ is orthogonal in Y_n^0;
2) the family $(\phi_{-k}^n)_{k=0}^{k_0}, (\psi_{-k}^n)_{k=k_0}^n$ is orthogonal in Y_n^0;
3) $\langle \phi_{-i}^n, \phi_{+j}^n \rangle_{Y_n^0} = 0$, $\langle \psi_{-i}^n, \psi_{+j}^n \rangle_{Y_n^0} = 0$, $\langle \phi_{-i}^n, \psi_{+j}^n \rangle_{Y_n^0} = 0$, $\langle \psi_{-i}^n, \phi_{+j}^n \rangle_{Y_n^0} = 0$ if $i \neq j$.

We need to associate another inner product in $\mathbb{R}^n \times \mathbb{R}^n$ compatible with the inner product in Y^1, that means,

$$\langle \begin{bmatrix} U \\ V \end{bmatrix}, \begin{bmatrix} W \\ Z \end{bmatrix} \rangle_1 = \langle A_n U, A_n W \rangle_{\mathbb{R}^n} + \langle A_n V, Z \rangle_{\mathbb{R}^n} \tag{2.4}$$

We denote by Y_n^1 the space $\mathbb{R}^n \times \mathbb{R}^n$ with the inner product given above. We make the distinction between the inner products by the index 0 or 1.

With a simple evaluation we get that

$$\langle A_n U, W \rangle_{\mathbb{R}^n} = \sum_{i=1}^{n-1} n(u_{i+1} - u_i)(w_{i+1} - w_i) + \frac{\delta}{2} \sum_{i=1}^{n} \frac{1}{n} u_i w_i.$$

We use the index Y_n^0 or Y_n^1 to indicate the norm considered in $\mathbb{R}^n \times \mathbb{R}^n$. Furthermore, we use in $\mathbb{R}^n$, three diferente norms given by $\|U\|_{L_d^2} = \langle U, U \rangle_{\mathbb{R}^n}^{1/2}$ which we call L^2-discretized, $\|U\|_{H_d^1} = \langle A_n U, U \rangle_{\mathbb{R}^n}^{1/2}$ which we call H^1 discretized and $\|U\|_{H_d^2} = \langle A_n U, A_n U \rangle_{\mathbb{R}^n}^{1/2}$ which we call H^2 discretized. In order to avoid mistakes, we denote by $\|U'\|_{L^2_d} = \langle \Delta_n U, U\rangle^{\frac{1}{2}}$ the L^2 norm discretized of the discretized derivative.

We also use a decomposition of the spaces $\mathbb{R}^n \times \mathbb{R}^n$ and $H^1 \times L^2$.

We write $H^1 \times L^2 = \oplus E_k$ where E_k is the generalized real eigenspace 2-dimensional associated with the eigenvalues $\lambda_{\pm k}$. If $\lambda_{\pm k}$ are real then $E_k = [\phi_{+k}, \phi_{-k}]$. If $\lambda_{\pm k}$ are complex, we consider the vectors $\psi_{+k} = \mathrm{Re}\phi_{+k}$ and $\psi_{-k} = \mathrm{Im}\phi_{+k}$ the base of E_k.

We denote by $\sphericalangle_{+k}^{-k}$ the angle between ψ_{+k} and ψ_{-k}. We observe that $\cos(\sphericalangle_{+k}^{-k}) < 1 - \xi$, for some $\xi > 0$ and for any k. In fact, considering $\|e_k\|_{L^2} = 1$ we get

$$\cos(\sphericalangle_{+k}^{-k}) = \frac{\|e_k\|_{H^1}^2 + \mathrm{Re}(\lambda_{+k})\mathrm{Im}(\lambda_{+k})}{\sqrt{\|e_k\|_{H^1}^4 + (\mathrm{Re}^2(\lambda_{+k}) + \mathrm{Im}^2(\lambda_{+k}))\|e_k\|_{H^1}^2 + (\mathrm{Re}(\lambda_{+k})\mathrm{Im}(\lambda_{+k}))^2}},$$

remembering that $\mathrm{Re}(\lambda_{+k}) = O(-\eta\|e_k\|_{H^1})$ and $\mathrm{Im}(\lambda_{+k}) = O((1-\eta^2)^{\frac{1}{2}}\|e_k\|_{H^1})$ then,

$$\lim_{k\to\infty} \cos^2(\sphericalangle_{+k}^{-k}) \leq \frac{(1+\eta(1-\eta^2)^{\frac{1}{2}})^2}{2+\eta(1-\eta^2)^{\frac{1}{2}}} \leq 1 - \xi$$

for some $\xi > 0$. With this fact, we obtain the equivalence between the sum norm, the max norm and inner product norm in each E_k, with equivalence constants independent of k.

Therefore, for $(u, v) \in H^1 \times L^2$ we write

$$(u, v) = \sum_{k=1}^{\infty} \left((u, v)_k^+ \phi_{+k} + (u, v)_k^- \phi_{-k}\right).$$

Using the orthogonal properties of E_k, we have $\|(u, v)\|_{Y^0} = (\sum_{k=1}^{\infty} \|(u, v)_k\|^2)^{\frac{1}{2}}$ where $(u, v)_k$ is a projection of (u, v) in the space E_k.

Similarly, we write $\mathbb{R}^n \times \mathbb{R}^n = \oplus E_k^n$ where E_k^n is a two dimensional space associated with the eigenvalues $\lambda_{\pm k}^n$. If $\lambda_{\pm k}^n$ are real eigenvalues then $E_k^n = [\phi_{+k}^n, \phi_{-k}^n]$, where $\phi_{\pm k}^n$ is the normalized eigenvector associated with $\lambda_{\pm k}^n$. If $\lambda_{\pm k}^n$ is complex we consider the vectors $\psi_{+k}^n = \mathrm{Re}\phi_{+k}^n$ and $\psi_{-k}^n = \mathrm{Im}\phi_{+k}^n$ a base de E_k^n.

Thus, $(U, V) \in \mathbb{R}^n \times \mathbb{R}^n$ is

$$(U, V) = \sum_{k=1}^{n} \left((U, V)_k^+ \phi_{+k}^n + (U, V)_k^- \phi_{-k}^n\right)$$

and $\|(U, V)\|_{Y_n^0} = (\sum_{k=1}^{n} \|(U, V)_k\|^2)^{1/2}$, where $(U, V)_k$ is a projection of (U, V) in the E_k^n which are orthogonal.

In order to make the comparison proposed, we use a technique of Numerical Analysis which is denominated *Internal Approximation of a Normed Space*, see [10]. We define a family $\{\mathbb{R}^n \times \mathbb{R}^n, P_{2n}, i_{2n}\}$, $n \in \mathbb{N}$ where $P_{2n} : Y^0 \to \mathbb{R}^n \times \mathbb{R}^n$ and $i_{2n} : \mathbb{R}^n \times \mathbb{R}^n \to Y^0$ are denominated projection and inclusion respectively.

Let $(U, V) = (u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n) \in \mathbb{R}^n \times \mathbb{R}^n$, the inclusion application, i_{2n}, is defined by $i_{2n}(U, V) = (u(x), v(x))$ where $u(x)$ and $v(x)$ are given by

$$u(x) = u_1\chi_{[0,\frac{1}{2n})} + \sum_{i=1}^{n_1}(u_i + (u_{i+1} - u_i)n(x - x_i))\chi_{[\frac{2i-1}{2n},\frac{2i+1}{2n})} + u_n\chi_{[\frac{2n-1}{2n},1]} \tag{2.5}$$

and

$$v(x) = \sum_{i=1}^{n} v_i\chi_{I_i} \tag{2.6}$$

where I_i is the interval $[\frac{i-1}{n}, \frac{i}{n})$.

We also defined a projection of Y^0 in $\mathbb{R}^n \times \mathbb{R}^n$ in the following way. For each e_k we define $P_n(e_k) = U = (u_1, u_2, \cdots, u_n) \in \mathbb{R}^n$ where $u_i = e_k(x_i)$, hence, $P_n(e_k) = e_k^n$, thus we define $P_{2n}(\phi_{\pm k}) = (P_n(e_k), \lambda_{\pm k}P_n(e_k))$ and $P_{2n}(\psi_{+k}) = (P_n(e_k), \mathrm{Re}(\lambda_{\pm k})P_n(e_k))$ and $P_{2n}(\psi_{-k}) = (P_n(e_k), \mathrm{Im}(\lambda_{+k})P_n(e_k))$ and we extend by linearity on Y^0.

For the inclusion and projection applications we have

Theorem 2.1. *The inclusion application, $i_{2n} : \mathbb{R}^n \times \mathbb{R}^n \to H^1 \times L^2$ is continuous. Furthermore, the continuity is uniform in n.*

Proof: In fact, let $u(x)$ given by (2.5), then

$$\begin{aligned}\|u(x)\|_{L^2}^2 &= \frac{u_1^2}{2n} + \sum_{j=1}^{n-1}\int_{x_j}^{x_{j+1}} u^2(x)dx + \frac{u_n^2}{2n} \\ &\leq \frac{u_1^2}{2n} + \sum_{j=1}^{n-1}(\frac{u_{j+1}^2}{2n} + \frac{u_j^2}{2n}) + \frac{u_n^2}{2n} = \sum_{j=1}^{n}\frac{u_j^2}{n} = \|U\|_{L_d^2}^2\end{aligned}$$

and,

$$\|u(x)\|_{H^1}^2 = \sum_{j=1}^{n-1} n(u_{j+1} - u_j)^2 = \|U\|_{H_d^1}^2$$

and let $v(x)$ given by (2.6), then $\|v(x)\|_{L^2}^2 = \sum_{j=1}^{n}\frac{1}{n}v_i^2 = \|V\|_{L_d^2}^2$. Thus,

$$\|i(U,V)\|_{Y^0} = (\|u(x)\|_{H^1}^2 + \frac{\delta}{2}\|u(x)\|_{L^2}^2 + \|v(x)\|_{L^2}^2)^{\frac{1}{2}} \leq \|(U,V)\|_{Y_n^0}$$

Theorem 2.2. *The projection application, P_{2n}, is continuous. Furthermore, the continuity is uniform in n.*

Proof: In fact, let $U = P_n(\cos(k\pi x)) = (\cos(k\pi x_1), \ldots, \cos(k\pi x_1))$ then we have

$$\|U\|_{L_d^2}^2 = \sum_{1}^{n}\frac{1}{n}u_i^2 = \sum_{1}^{n}\frac{1}{n}\cos^2(k\pi x_i) \leq 1 = 2\|\cos(k\pi x)\|_{L^2}^2 \tag{2.7}$$

and

$$\begin{aligned}
\|U\|^2_{H^1_d} &= \sum_1^{n-1} n(u_{i+1}-u_i)^2 = \sum_1^{n-1} n(\cos(k\pi x_{i+1}) - \cos(k\pi x_i))^2 \\
&= \sum_1^{n-1} n(k\pi)^2 \operatorname{sen}^2(k\pi \bar{x}_i)\frac{1}{n^2} \leq (k\pi)^2 \leq 2\|k\pi \operatorname{sen}(k\pi x)\|^2_{L^2}
\end{aligned}$$

Thus,

$$\begin{aligned}
\|P_{2n}(\phi_{\pm k})\|_{Y^0_n} &= \|(P_n(e_k), \lambda_{\pm k}P_n(e_k))\|_{H^1_d \times L^2_d} \\
&= (\|P_n(e_k)\|^2_{H^1_d} + \frac{\delta}{2}\|P_n(e_k)\|^2_{L^2_d} + |\lambda_{\pm k}|\|P_n(e_k)\|^2_{L^2_d})^{\frac{1}{2}} \\
&\leq (2\|e_k\|_{H^1} + \frac{\delta}{2}2\|e_k\|_{L^2} + |\lambda_{\pm k}|2\|e_k\|^2_{L^2})^{\frac{1}{2}} \\
&= \sqrt{2}\|\phi_{\pm k}\|_{Y^0}
\end{aligned}$$

By using (2.1) and (2.2) we have that these applications are stable (see [10]).
Another result is

Theorem 2.3. *i) Let $\lambda_{\pm k}$ be the eigenvalues of C_η and $\lambda^n_{\pm k}$ the eigenvalues of $C_{\eta n}$, then for each k fixed we have that $\lambda^n_{\pm k} \to \lambda_{\pm k}$ when $n \to \infty$.*
ii) Let $\phi_{\pm k}$ be the eigenvectors of C_η and $\phi^n_{\pm k}$ the eigenvectors of $C_{\eta n}$, then for each k fixed we have that $i(\phi^n_{\pm k}) \to \phi_{\pm k}$ when $n \to \infty$, and $P_n(e_k) = e^n_n$.

For the inclusion application we use only i and the dimension of the space is omitted.

3. Comparison of Linear Semigroups

Let be $e^{C_\eta t}$ and $e^{C_{\eta n} t}$ the semigroups generated by C_η and $C_{\eta n}$ respectively. We have the following result comparing the semigroups

Theorem 3.1. *For each $\epsilon > 0$, there is a $n_o(\epsilon)$ such that $\forall n \geq n_0$*

$$\|e^{C_\eta t}(u_0, v_0) - i(e^{C_{\eta n} t}P_{2n}(u_0, v_0))\|_{Y^0} \leq \epsilon t^{-\alpha}\|(u_0, v_0)\|_{C^{1+\alpha}\times C^\alpha}, t > 0 \tag{3.1}$$

for all $(u_0, v_0) \in C^{1+\alpha} \times C^\alpha$ and

$$\|e^{C_\eta t} i(U_0, V_0) - i(e^{C_{\eta n} t}(U_0, V_0))\|_{Y^0} \leq \epsilon t^{-\alpha}\|(U_0, V_0)\|_{Y^0_n}, t > 0 \tag{3.2}$$

for all $(U_0, V_0) \in \bigcup_n \mathcal{A}_{\eta n}$.

Proof: We make the proof for the first inequality and when it is necessary we note the changes for the second one.

Let $\epsilon > 0$ be a real parameter. We consider two cases
i) for $0 < t \leq \epsilon$. In this case, when t is small, we use that $e^{-\delta t/2}$ is bounded by $K\epsilon^\nu t^{-\alpha}$ for $\alpha > \nu > 0$. Hence,

$$\begin{aligned}
\|e^{C_\eta t}(u_0, v_0) - i(e^{C_{\eta n} t}P_{2n}(u_0, v_0))\|_{Y^0} &\leq K' e^{-\delta/2t}\|(u_0, v_0)\|_{Y^0} \\
&\leq M\epsilon^\nu t^{-\alpha}\|(u_0, v_0)\|_{Y^0}.
\end{aligned}$$

ii) for $t > \epsilon$, we need estimate

$$\|e^{C_\eta t}(u_0, v_0) - i(e^{C_{\eta n} t} P_{2n}(u_0, v_0))\|_{Y^0}.$$

In this case, we decompose Y^0 in two subspaces. In the subspace of finite dimension, we have the uniform convergence of eigenvalues and inclusion of eigenvectors for the eigenvalues and eigenfunctions of the continuous problem. In the subspace of infinite dimension, we have that the real part of eigenvalues goes to infinity.

By using that $\lambda^n_{\pm k} \to \lambda_{\pm k}$ when $n \to \infty$ and $\mathrm{Re}(\lambda_{\pm k}) \to -\infty$, when $k \to \infty$ and considering $\beta \in (0, 1)$ a fixed number then there are $K(\epsilon)$ and $N(\epsilon)$ such that

$$e^{Re(\lambda^n_{\pm k})t} \leq \epsilon t^{-\beta}, \quad e^{Re(\lambda_{\pm k})t} \leq \epsilon t^{-\beta} \text{ for all } n \geq N(\epsilon) \text{ and } k \geq K(\epsilon). \tag{3.3}$$

Using $K = K(\epsilon)$ given in (3.3), we consider the following subspaces $\oplus E^n_k$, $1 \leq k \leq K$, $\oplus E^n_k$, $K + 1 \leq k \leq n$ of $\mathbb{R}^n \times \mathbb{R}^n$ and $\oplus E_k$, $1 \leq k \leq K$, $\oplus E_k$, $K + 1 \leq k < \infty$ of Y^0. Then,

$$\begin{aligned}
&\|e^{C_\eta t}(u_0, v_0) - i(e^{C_{\eta n} t} P_{2n}(u_0, v_0))\|_{Y^0} \\
&\qquad \leq \|e^{C_\eta t}\sum_{k=1}^{K}(u_0, v_0)_k - i(e^{C_{\eta n} t}\sum_{k=1}^{K}(P_{2n}(u_0, v_0))_k)\|_{Y^0} \\
&\qquad + \|e^{C_\eta t}\sum_{k=K+1}^{\infty}(u_0, v_0)_k\|_{Y^0} + \|i(e^{C_{\eta n} t}\sum_{k=K+1}^{n}(P_{2n}(u_0, v_0))_k)\|_{Y^0}
\end{aligned}$$

By the continuity, uniform in n, of the applications inclusion and projection, we have:

$$\begin{aligned}
&\|i(e^{C_{\eta n} t}\sum_{k=K+1}^{n}(P_{2n}(u_0, v_0))_k)\|_{Y^0} \leq M\|e^{C_{\eta n} t}\sum_{k=K+1}^{n}(P_{2n}(u_0, v_0))_k\|_{Y^0_n} \\
&= (\sum_{k=K+1}^{n}\|e^{C_{\eta n} t}(P_{2n}(u_0, v_0))_k\|^2_{Y^0_n})^{1/2} \leq (\sum_{k=K+1}^{n}(e^{Re\lambda^n_{\pm} t}\|(P_{2n}(u_0, v_0))_k\|_{Y^0_n})^2)^{1/2} \\
&\leq \epsilon t^{-\beta}(\sum_{k=K+1}^{n}\|(P_{2n}(u_0, v_0))_k\|^2_{Y^0_n})^{1/2} \leq \epsilon t^{-\beta}\|P_{2n}(u_0, v_0)\|_{Y^0_n} \\
&\leq M\epsilon t^{-\beta}\|(u_0, v_0)\|_{Y^0}
\end{aligned}$$

In the similar way, we have

$$\|e^{C_\eta t}\sum_{k=K+1}^{\infty}(u_0, v_0)_k\|_{Y^0} \leq M\epsilon t^{-\beta}\|(u_0, v_0)_k\|_{Y^0}$$

We consider another operator B_n, which possess the same eigenvalues of $C_{\eta n}$ but, associated with the eigenvectors of C_η. Thus, we have

$$\begin{aligned}
\|e^{C_\eta t}\sum_{k=1}^{K}(u_0,v_0)_k \;&-\; i(e^{C_{\eta n}t}\sum_{k=1}^{K}(P_{2n}(u_0,v_0))_k)\|_{Y^0} \\
&\leq \; \|e^{C_\eta t}\sum_{k=1}^{K}(u_0,v_0)_k - e^{B_n t}\sum_{k=1}^{K}(u_0,v_0)_k\|_{Y^0} \\
&+ \; \|e^{B_n t}\sum_{k=1}^{K}(u_0,v_0)_k - i(e^{C_{\eta n}t}\sum_{k=1}^{K}(P_{2n}(u_0,v_0))_k)\|_{Y^0}
\end{aligned}$$

If each $\lambda^n_{\pm k}$, for $1 \leq k \leq K$, is real then

$$\begin{aligned}
&\|\sum_{k=1}^{K} e^{C_\eta t}(u_0,v_0)_k - \sum_{k=1}^{K} e^{B_n t}(u_0,v_0)_k\|_{Y^0} \\
&\leq (\sum_{k=1}^{K}\|(e^{\lambda_{+k}t} - e^{\lambda^n_{+k}t})(u_0,v_0)_{+k} + (e^{\lambda_{-k}t} - e^{\lambda^n_{-k}t})(u_0,v_0)_{-k}\|^2_{Y^0})^{1/2} \\
&\leq 2M \max_{1\leq k\leq K}\{|e^{\lambda_{+k}t} - e^{\lambda^n_{+k}t}|, |e^{\lambda_{-k}t} - e^{\lambda^n_{-k}t}|\}.(\sum_{k=1}^{K}\|(u_0,v_0)_k\|^2)^{1/2} \\
&\leq 2Mt \max_{1\leq k\leq K}\{|e^{\tilde{\lambda}^n_{+k}t}||\lambda_{+k} - \lambda^n_{+k}|, |e^{\bar{\lambda}^n_{-k}t}||\lambda_{-k} - \lambda^n_{-k}|\}\|(u_0,v_0)\|_{Y^0} \\
&\leq \epsilon t^{-\beta}\|(u_0,v_0)\|_{Y^0}
\end{aligned}$$

for $n \geq n_1 \geq n_0$, where $\bar{\lambda}^n_{-k}$ is between λ_{-k} and λ^n_{-k}; and $\tilde{\lambda}^n_{+k}$ is between λ_{+k} and λ^n_{+k}.

In the case $\lambda_{\pm k}$ complex, we denote by $(u_0,v_0)_k$ the component of (u_0,v_0) in E_k and $(u_0,v_0)_k = 2a(cos\delta, -sen\delta)$ in the base ψ_{+k},ψ_{-k}. In this case, we have

$$e^{C_\eta t}(u_0,v_0)_k = 2ae^{\alpha_k t}(\cos(\beta_k t+\delta)\psi_{+k} - \mathrm{sen}(\beta_k t+\delta)\psi_{-k}),$$

$$e^{B_n t}(u_0,v_0)_k = 2ae^{\alpha^n_k t}(\cos(\beta^n_k t+\delta)\psi_{+k} - \mathrm{sen}(\beta^n_k t+\delta)\psi_{-k}),$$

where $\lambda_{\pm k} = \alpha_k \pm \beta_k$ and $\lambda^n_{\pm k} = \alpha^n_k \pm \beta^n_k$, then

$$\begin{aligned}
&\|e^{C_\eta t}(u_0,v_0)_k - e^{B_n t}(u_0,v_0)_k\|_{Y^0} \\
&\leq \|2ae^{\alpha_k t}[(\cos(\beta_k t+\delta) - \cos(\beta^n_k t+\delta))\psi_{+k} - (\mathrm{sen}(\beta_k t+\delta) - \mathrm{sen}(\beta^n_k t+\delta))\psi_{-k}]\| \\
&\quad +\|2a[\cos(\beta^n_k t+\delta)\psi_{+k} - \mathrm{sen}(\beta^n_k t+\delta)\psi_{-k}]\||e^{\alpha_{+k}t} - e^{\alpha^n_{+k}t}| \\
&\quad \leq e^{\alpha_{+k}t}t|\beta_k - \beta^n_k|\| - 2a\,\mathrm{sen}(\bar{\beta}^n_k + \delta)\psi_{+k} - 2a\cos(\bar{\beta}^n_k t+\delta)\psi_{-k}\| \\
&\quad +e^{\bar{\alpha}_{+k}t}t|\alpha_k - \alpha^n_k|\|2a\cos(\beta^n_k t+\delta)\psi_{+k} - 2a\,\mathrm{sen}(\beta^n_k t+\delta)\psi_{-k}\| \\
&\quad \leq L(e^{\alpha_{+k}t}t|\beta_k - \beta^n_k| + e^{\bar{\alpha}_{+k}t}t|\alpha_k - \alpha^n_k|)\|(u_0,v_0)_k\|
\end{aligned}$$

If $(P_{2n}(u_0,v_0))_k = a^n_{+k}\psi^n_{+k} + a^n_{-k}\psi^n_{-k}$ and $(u_0,v_0)_k = a_{+k}\psi_{+k} + a_{-k}\psi_{-k}$ then

$$\begin{aligned}
&\|e^{B_n t}\sum_{k=1}^{K}(u_0,v_0)_k - i(e^{C_{\eta n}t}\sum_{k=1}^{K}(P_{2n}(u_0,v_0))_k)\|_{Y^0}\\
&\leq \|\sum_{k=1}^{K} e^{B_n t}(u_0,v_0)_k - \sum_{k=1}^{K} e^{B_n t}(a^n_{+k}\psi_{+k} + a^n_{-k}\psi_{-k})\|_{Y^0}\\
&+\|\sum_{k=1}^{K} e^{B_n t}(a^n_{+k}\psi_{+k} + a^n_{-k}\psi_{-k}) - \sum_{k=1}^{K} i(e^{C_{\eta n}t}(P_{2n}(u_0,v_0))_k)\|_{Y^0}\\
&\leq e^{\alpha^n_1 t}\sum_{k=1}^{K}(|a_{+k}-a^n_{+k}|\|\psi_{+k}\| + |a_{-k}-a^n_{-k}|\|\psi_{-k}\|)\\
&+\|\sum_{k=1}^{K} e^{B_n t}(a^n_{+k}\psi_{+k} + a^n_{-k}\psi_{-k}) - \sum_{k=1}^{K} i(e^{C_{\eta n}t}a^n_{+k}\psi^n_{+k} + a^n_{-k}\psi^n_{-k})\|_{Y^0}
\end{aligned}$$

Hence, we need to estimate $|a_{+k}-a^n_{+k}|$ and $|a_{-k}-a^n_{-k}|$. In order to calculate this, we write $a^n_{+k} = \frac{b^n_{+k}}{c^n_{+k}}$ and $a_{+k} = \frac{b_{+k}}{c_{+k}}$ where

$$b^n_{+k} = \langle P_{2n}(u_0,v_0),\psi^n_{+k}\rangle\|\psi^n_{-k}\|^2 - \langle P_{2n}(u_0,v_0),\psi^n_{-k}\rangle\langle\psi^n_{-k},\psi^n_{+k}\rangle,$$

$$c^n_{+k} = \|\psi^n_{+k}\|^2\|\psi^n_{-k}\|^2 - \langle\psi^n_{+k},\psi^n_{-k}\rangle^2$$

and

$$b_{+k} = \langle(u_0,v_0),\psi_{+k}\rangle\|\psi_{-k}\|^2 - \langle(u_0,v_0),\psi_{-k}\rangle\langle\psi_{-k},\psi_{+k}\rangle,$$

$$c_{+k} = \|\psi_{+k}\|^2\|\psi_{-k}\|^2 - \langle\psi_{+k},\psi_{-k}\rangle^2$$

and

$$|a_{+k}-a^n_{+k}| \leq \frac{|c^n_{+k}||b_{+k}-b^n_{+k}| + |c_{+k}-c^n_{+k}||b^n_{+k}|}{|c_{+k}||c^n_{+k}|}.$$

Thus, it is sufficient estimate $|b_{+k}-b^n_{+k}|$ and $|c_{+k}-c^n_{+k}|$. We consider two cases:
I)$(u_0,v_0) = i(U_0,V_0)$ and $P_{2n}(u_0,v_0) = (U_0,V_0)$ for $(U_0,V_0)\in\mathcal{A}_{\eta n}$;
II) (u_0,v_0) in $C^{1+\alpha}\times C^{\alpha}$.
Since that
i) $\|\psi_{+k}\|_{Y^0} = \|\psi^n_{+k}\|_{Y^0_n} + O(\frac{1}{n})$,
ii) $\langle\psi_{+k},\psi_{-k}\rangle_{Y^0} = \langle\psi^n_{+k},\psi^n_{-k}\rangle_0 + O(\frac{1}{n})$
iii)$\langle i(P_n(u_0)),\cos(k\pi x)\rangle_{H^1} = \sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}} n(u_{i+1}-u_i)k\pi\,\mathrm{sen}(k\pi x)dx$,
iv)$\langle P_n(u_0),P_n(\cos(k\pi x))\rangle_{H^1_d} = \sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}} n(u_{i+1}-u_i)k\pi\,\mathrm{sen}(k\pi\bar{x}_i)dx$.
then

$$|\langle i(P_n(u_0)),\cos(k\pi x)\rangle_{H^1} - \langle P_n(u_0),P_n(\cos(k\pi x))\rangle_{H^1_d}| \leq k^2\pi^2\sum_{i=1}^{n-1}\frac{1}{n}(u_{i+1}-u_i).$$

If $(u_0, v_0) = i(U_0, V_0)$ for some $(U_0, V_0) \in \mathcal{A}_{\eta n}$ then, by using estimates similar to [2], we have $\bigcup_n \mathcal{A}_{\eta n}$ is bounded in $H_d^2 \times H_d^1$ and $n|u_{i+1} - u_i| \leq \|U\|_{H_d^1} + \|U\|_{H_d^2} \leq 2K$ for $1 \leq i \leq n-1$, thus

$$|\langle i(P_n(u_0)), \cos(k\pi x)\rangle_{H^1} - \langle P_n(u_0), P_n(\cos(k\pi x))\rangle_{H_d^1}| \leq \frac{k^2\pi^2}{n}(\|U_0\|_{H_d^1} + \|U_0\|_{H_d^2}).$$

We also have
v)$\langle i(P_n(v_0)), \cos(k\pi x)\rangle_{L^2} = \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} v_i \cos(k\pi x)dx$,
iv)$\langle P_n(v_0), P_n(\cos(k\pi x))\rangle_{L_d^2} = \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} v_i \cos(k\pi x_i)dx$.
Hence,

$$|\langle i(P_n(v_0)), \cos(k\pi x)\rangle_{L^2} - \langle P_n(v_0), P_n(\cos(k\pi x))\rangle_{L_d^2}| \leq k\pi \sum_{i=1}^n \frac{1}{n^2} v_i.$$

We are in the case of $(u_0, v_0) = i(U_0, V_0)$ for some $(U_0, V_0) \in \mathcal{A}_{\eta n}$ then, using that $\bigcup_n \mathcal{A}_{\eta n}$ is bounded in $H_d^2 \times H_d^1$ and $|v_i| \leq \|V\|_{L_d^2} + \|V\|_{H_d^1} \leq 2K$ for $1 \leq i \leq n$ then,

$$|\langle i(P_n(v_0)), \cos(k\pi x)\rangle_{L^2} - \langle P_n(v_0), P_n(\cos(k\pi x))\rangle_{L_d^2}| \leq \frac{k^2\pi^2}{n}(\|V_0\|_{L_d^2} + \|V_0\|_{H_d^1}).$$

Therefore,

$$|\langle i(P_{2n}(u_0, v_0)), \psi_{\pm k}\rangle_{Y^0} - \langle P_{2n}(u_0, v_0), \psi_{\pm k}^n\rangle_{Y_n^0}| \leq \frac{\tilde{K}}{n}\|(U_0, V_0)\|_{Y_n^1}.$$

If $(u_0, v_0) \neq i(U_0, V_0)$ and $(u_0, v_0) \in C^{1+\alpha} \times C^\alpha$ then we have $|u_{i+1} - u_i| \leq n^{-1}\|u_0\|_{C^{1+\alpha}}$, $|v_i| \leq \|v\|_{C^\alpha}$. Thus

$$\begin{aligned}
&|\langle u_0, cos(k\pi x)\rangle_{H^1} - \langle P_n(u_0), P_n(\cos(k\pi x))\rangle_{H_d^1}| \\
&\quad \leq |\langle u_0, \cos(k\pi x)\rangle_{H^1} - \langle i(P_n(u_0)), \cos(k\pi x)\rangle_{H_d^1}| \\
&\quad +|\langle i(P_n(u_0)), \cos(k\pi x)\rangle_{H^1} - \langle P_n(u_0), P_n(\cos(k\pi x))\rangle_{H_d^1}|.
\end{aligned}$$

However,

$$\|(u_0, v_0) - i(P_{2n}(u_0, v_0))\|_{Y^0} \leq \frac{1}{n^\alpha}\|(u_0, v_0)\|_{C^{1+\alpha}\times C^\alpha}.$$

Hence,

$$|\langle (u_0, v_0), \psi_{\pm k}\rangle_{Y^0} - \langle P_{2n}(u_0, v_0), \psi_{\pm k}^n\rangle_{Y_n^0}| \leq \frac{\tilde{K}}{n^\alpha}\|(u_0, v_0)\|_{C^{1+\alpha}\times C^\alpha}.$$

Therefore, for case I)

$$|b_{+k} - b_{+k}^n| \leq Mn^{-1}\|(U_0, V_0)\|_{Y_n^1},$$

for case II),

$$|b_{+k} - b_{+k}^n| \leq Mn^{-\alpha}\|(u_0, v_0)\|_{C^{1+\alpha}\times C^\alpha}$$

and in analogous form, for k, $1 \leq k \leq K$ we get

$$|c_{+k} - c_{+k}^n| \leq Mn^{-1}.$$

Analogously, we obtain $|a_{-k} - a_{-k}^n|$.

We came back to estimate $e^{\alpha_1^n t}\sum_{k=1}^{K}(|a_{+k}-a_{+k}^n|\|\psi_{+k}\|+|a_{-k}-a_{-k}^n|\|\psi_{-k}\|)$.
In the case I)

$$\begin{aligned}
e^{\alpha_1^n t}\sum_{k=1}^{K}(|a_{+k}-a_{+k}^n|\|\psi_{+k}\|+|a_{-k}-a_{-k}^n|\|\psi_{-k}\|)&\\
\leq e^{\alpha_1^n t}\tilde{M}n^{-1}\|(U_0,V_0)\|_{Y_n^1}\sum_{k=1}^{K}(\|\psi_{+k}\|+\|\psi_{-k}\|)&\\
\leq \epsilon t^{-\beta}\|(U_0,V_0)\|_{Y_n^1}&
\end{aligned}$$

and in the case II)

$$\begin{aligned}
e^{\alpha_1^n t}\sum_{k=1}^{K}(|a_{+k}-a_{+k}^n|\|\psi_{+k}\|+|a_{-k}-a_{-k}^n|\|\psi_{-k}\|)&\\
\leq e^{\alpha_1^n t}\tilde{M}n^{-\alpha}\|(u_0,v_0)\|_{C^{1+\alpha}\times C^{\alpha}}\sum_{k=1}^{K}(\|\psi_{+k}\|+\|\psi_{-k}\|)&\\
\leq \epsilon t^{-\beta}\|(u_0,v_0)\|_{C^{1+\alpha}\times C^{\alpha}}&
\end{aligned}$$

Now we go to estimate

$$\|\sum_{k=1}^{K}e^{B_n t}(a_{+k}^n\psi_{+k}+a_{-k}^n\psi_{-k})-\sum_{k=1}^{K}i(e^{C_{\eta n}t}a_{+k}^n\psi_{+k}^n+a_{-k}^n\psi_{-k}^n)\|_{Y^0}$$

In order to do this, we consider a complex inclusion, that means the inclusion of real part and the inclusion of imaginary part. In this case, we are considering complex solutions.

Since that $a_{+k}^n\psi_{+k}+a_{-k}^n\psi_{-k}=d_{+k}^n\phi_{+k}+d_{-k}^n\phi_{-k}$ and $a_{+k}^n\psi_{+k}^n+a_{-k}^n\psi_{-k}^n=d_{+k}^n\phi_{+k}^n+d_{-k}^n\phi_{-k}^n$ where $d_{+k}=1/2(a_{+k}^n-ia_{-k}^n)$ and $d_{-k}=1/2(a_{+k}^n+ia_{-k}^n)$ then

$$e^{B_n t}(a_{+k}^n\psi_{+k}+a_{-k}^n\psi_{-k})=e^{\lambda_{+k}^n t}d_{+k}^n\phi_{+k}+e^{\lambda_{-k}^n t}d_{-k}^n\phi_{-k}$$

and

$$\begin{aligned}
i(e^{C_{\eta n}t}(a_{+k}^n\psi_{+k}^n+a_{-k}^n\psi_{-k}^n)) &= i(e^{\lambda_{+k}^n t}d_{+k}^n\phi_{+k}^n+e^{\lambda_{-k}^n t}d_{-k}^n\phi_{-k}^n)\\
&= d_{+k}^n i(e^{\lambda_{+k}^n t}\phi_{+k}^n)+d_{-k}^n i(e^{\lambda_{-k}^n t}\phi_{-k}^n)
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|e^{B_n t}(a_{+k}^n\psi_{+k}+a_{-k}^n\psi_{-k})-i(e^{C_{\eta n}t}(a_{+k}^n\psi_{+k}^n+a_{-k}^n\psi_{-k}^n))\|\\
&\quad\leq |d_{+k}^n|\|e^{\lambda_{+k}^n t}\phi_{+k}-i(e^{\lambda_{+k}^n t}\phi_{+k}^n)\|+|d_{-k}^n|\|e^{\lambda_{-k}^n t}\phi_{-k}-i(e^{\lambda_{-k}^n t}\phi_{-k}^n)\|\\
&\quad= |d_{+k}^n|\|e^{\lambda_{+k}^n t}\phi_{+k}-e^{\lambda_{+k}^n t}i(\phi_{+k}^n)\|+|d_{-k}^n|\|e^{\lambda_{-k}^n t}\phi_{-k}-e^{\lambda_{-k}^n t}i(\phi_{-k}^n)\|\\
&\quad\leq |d_{+k}^n|e^{\alpha_{+k}^n t}\|\phi_{+k}-i(\phi_{+k}^n)\|+|d_{-k}^n|e^{\alpha_{-k}^n t}\|\phi_{-k}-i(\phi_{-k}^n)\|\\
&\quad\leq |d_{+k}^n|e^{\alpha_{+k}^n t}K/n\|\phi_{+k}\|+|d_{-k}^n|e^{\alpha_{-k}^n t}K/n\|\phi_{-k}\|\\
&\quad\leq e^{\alpha_{+k}^n t}\tilde{K}/n\|(u_0,v_0)_k\|\leq \epsilon t^{-\beta}\|(u_0,v_0)_k\|
\end{aligned}$$

Hence,

$$\|\sum_{k=1}^{K} e^{B_n t}(a^n_{+k}\psi_{+k} + a^n_{-k}\psi_{-k}) - \sum_{k=1}^{K} i(e^{C_{\eta n}t}a^n_{+k}\psi^n_{+k} + a^n_{-k}\psi^n_{-k})\|_{Y^0} \leq \epsilon t^{-\beta}\|(u_0, v_0)\|_{Y^0}$$

Finally, in the case I),

$$\|e^{B_n t}\sum_{k=1}^{K}(u_0, v_0)_k - i(e^{C_{\eta n}t}\sum_{k=1}^{K}(P_{2n}(u_0, v_0))_k)\|_{Y^0} \leq \epsilon t^{-\beta}\|(u_0, v_0)\|_{Y^0},$$

and in the case II)

$$\|e^{B_n t}\sum_{k=1}^{K}(u_0, v_0)_k - i(e^{C_{\eta n}t}\sum_{k=1}^{K}(P_{2n}(u_0, v_0))_k)\|_{Y^0} \leq \epsilon t^{-\beta}\|(u_0, v_0)\|_{C^{1+\alpha}\times C^{\alpha}}.$$

4. Comparison of nonlinear semigroups

About the nonlinear semigroups we have

Theorem 4.1. *Let $T_\eta(t)$ and $T_{\eta n}(t)$ be the nonlinear semigroups generated by (1.5) and (1.6), respectively, then*

$$\|T_\eta(t, i(U_0, V_0)) - i(T_{\eta n}(t, (U_0, V_0)))\|_{Y^0} \leq M\epsilon K_0 t^{-\beta}, \tag{4.1}$$

for $t \in (0, \tau)$, $(U_0, V_0) \in \mathcal{A}_{\eta n}$ and for $n \leq n(\epsilon)$

Proof: By the variation of constants formula and for $(U_0, V_0) \in \mathcal{A}_{\eta n}$ we have

$$T_{\eta n}(t, (U_0, V_0)) = e^{C_{\eta n}t}(U_0, V_0) + \int_0^t e^{C_{\eta n}(t-s)}H(T_{\eta n}(s, (U_0, V_0)))ds \tag{4.2}$$

$$T_\eta(t, i(U_0, V_0)) = e^{C_\eta t}i(U_0, V_0) + \int_0^t e^{C_\eta(t-s)}h(T_\eta(s, i(U_0, V_0)))ds \tag{4.3}$$

Then, for $t \in (0, \tau)$

$$\begin{aligned}
&\|T_\eta(t, i(U_0, V_0)) - i(T_{\eta n}(t, (U_0, V_0)))\|_{Y^0} \\
&\leq \|e^{C_\eta t}i(U_0, V_0) - i(e^{C_{\eta n}t}(U_0, V_0))\|_{Y^0} \\
&+\|\int_0^t e^{C_\eta(t-s)}h(T_\eta(s, i(U_0, V_0))) - i(e^{C_{\eta n}(t-s)}P_{2n}h(T_\eta(s, i(U_0, V_0))))ds\|_{Y^0} \\
&+\|\int_0^t i(e^{C_{\eta n}(t-s)}P_{2n}h(T_\eta(s, i(U_0, V_0)))) - i(e^{C_{\eta n}(t-s)}H(T_{\eta n}(s, (U_0, V_0))))ds\|_{Y^0} \\
&\leq \epsilon t^{-\beta}\|(U_0, V_0)\|_{Y^0_n} \\
&+ \int_0^t \|e^{C_\eta(t-s)}h(T_\eta(s, i(U_0, V_0))) - i(e^{C_{\eta n}(t-s)}P_{2n}h(T_\eta(s, i(U_0, V_0))))\|_{Y^0}ds \\
&+ \int_0^t \|i(e^{C_{\eta n}(t-s)}P_{2n}h(T_\eta(s, i(U_0, V_0)))) - i(e^{C_{\eta n}(t-s)}H(T_{\eta n}(s, (U_0, V_0))))\|_{Y^0}ds
\end{aligned}$$

Since that $i(U_0, V_0)$ is bounded in $H^1 \times L^2$ we have $(T_\eta(s, i(U_0, V_0)))_1$ is bounded in C^α, thus $\|h(T_\eta(s, i(U_0, V_0)))\|_{C^{1+\alpha}\times C^\alpha}$ is bounded for all $(U_0, V_0) \in \bigcup_n \mathcal{A}_{\eta n}$. We also have $H(T_{\eta n}(s,(U_0, V_0))) = P_{2n}(h(i(T_{\eta n}(s,(U_0, V_0)))))$ then

$$\begin{aligned}
&\|T_\eta(t, i(U_0, V_0)) - i(T_{\eta n}(t,(U_0, V_0)))\|_{Y^0} \\
&\leq \epsilon t^{-\beta}\|(U_0, V_0)\|_{Y_n^0} + \epsilon \int_0^t (t-s)^{-\beta}\|h(T_\eta(s, i(U_0, V_0)))\|_{C^{1+\alpha}\times C^\alpha} ds \\
&+ \int_0^t \|e^{C_{\eta n}(t-s)}\|\|P_{2n}(h(T_\eta(s, i(U_0, V_0)))) - P_{2n}(h(i(T_{\eta n}(s,(U_0, V_0)))))\|_{Y_n^0} ds \\
&\leq \epsilon t^{-\beta}K_0 + \epsilon\tau\frac{t^{-\beta}K_0}{1-\beta} + L\int_0^t \|(T_\eta(s, i(U_0, V_0))) - i(T_{\eta n}(s,(U_0, V_0)))\|_{Y^0} ds
\end{aligned}$$

Hence, by Gronwall Inequality, we have that exists a constant $M(\beta, \tau, L)$ such that

$$\|T_\eta(t, i(U_0, V_0)) - i(T_{\eta n}(t,(U_0, V_0)))\|_{H^1\times L^2} \leq M\epsilon K_0 t^{-\beta}, \tag{4.4}$$

for $t \in (0, \tau)$, $(U_0, V_0) \in \mathcal{A}_{\eta n}$ and for $n \leq n(\epsilon)$

5. Upper Semicontinuity of global attractors $\mathcal{A}_\eta$ and $i(\mathcal{A}_{\eta n})$ in $H^1 \times L^2$

Since that $\bigcup_n \mathcal{A}_{\eta n}$ is bounded in $\mathbb{R}^n \times \mathbb{R}^n$ and also $\|i(U,V)\|_{H^1\times L^2} \leq \|(U,V)\|_{\mathbb{R}^n\times\mathbb{R}^n}$ then $\|i(\bigcup_n \mathcal{A}_{\eta n})\|_{H^1\times L^2} \leq \|\bigcup_n \mathcal{A}_n\|_{\mathbb{R}^n\times\mathbb{R}^n} \leq K$.

The global attractor $\mathcal{A}_\eta$ attracts bounded of $H^1 \times L^2$ thus, $\forall \delta > 0$, exists $\tau = \tau(\delta)$ such that

$$\delta_{Y_0}(T_\eta(\tau, i(\phi_n)), \mathcal{A}_\eta) \leq \delta/2$$

for all $\phi_n \in \mathcal{A}_{\eta n}$ and for all n.
The attractors $\mathcal{A}_{\eta n}$ are invariant, thus if $\psi_n \in \mathcal{A}_{\eta n}$ then exists $\phi_n \in \mathcal{A}_{\eta n}$ such that $T_{\eta n}(\tau, \phi_n) = \psi_n$.
Hence, we choose $n_0(\delta) = n(\epsilon(\delta)) > 0$ such that

$$\|T_\eta(\tau, i(\phi_n)) - i(T_{\eta n}(\tau, \phi_n))\| \leq M\epsilon\tau^{-\beta}\|\phi_n\| \leq \delta/2$$

for $n \geq n_0(\delta)$.
Therefore,

$$\delta_{Y_0}(i(\psi_n), \mathcal{A}_\eta) \leq \delta_{Y_0}(i(\psi_n), T_\eta(\tau, i(\phi_n))) + \delta_{Y_0}(T_\eta(\tau, i(\phi_n)), \mathcal{A}_\eta) \leq \delta$$

for all $\psi_n \in \mathcal{A}_{\eta n}$ and for all $n \geq n_0(\delta)$.

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