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Orthogonal Bases for Spaces of Complex Spherical Harmonics

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Bases ortogonais para os espaços dos harmônicos esféricos complexos

Resumo

O artigo propõe um método indutivo para construir bases para os espaços dos harmônicos esféricos sobre a esfera unitária Ω_{2q} de $\mathbb{C}^q$. As bases são mostradas ter muitas propriedades interessantes, entre elas ortogonalidade com respeito ao produto interno de $L^2(\Omega_{2q})$. Como ferramenta auxiliar, estudamos o produto interno $[f,g] = f(\overline{D})(\overline{g(z)})(0)$ sobre o espaço $\mathbb{P}(\mathbb{C}^q)$ dos polinômios nas variáveis $z, \overline{z} \in \mathbb{C}^q$, onde $f(\overline{D})$ é o operador diferencial com símbolo $f(\overline{z})$. Em relação aos espaços dos harmônicos esféricos, é mostrado que o produto interno $[\cdot,\cdot]$ reduz-se a um múltiplo do produto inteno de $L^2(\Omega_{2q})$. Bi-ortogonalidade em $(\mathbb{P}(\mathbb{C}^q), [\cdot,\cdot])$ é completamente investigado.

Orthogonal bases for spaces of complex spherical harmonics

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Abstract

This paper proposes an inductive method to construct bases for spaces of spherical harmonics over the unit sphere Ω_{2q} of $\mathbb{C}^q$. The bases are shown to have many interesting properties, among them orthogonality with respect to the inner product of $L^2(\Omega_{2q})$. As a bypass, we study the inner product $[f,g] = f(\overline{D})(\overline{g(z)})(0)$ over the space $\mathbb{P}(\mathbb{C}^q)$ of polynomials in the variables $z, \overline{z} \in \mathbb{C}^q$, in which $f(\overline{D})$ is the differential operator with symbol $f(\overline{z})$. On the spaces of spherical harmonics, it is shown that the inner product $[\cdot,\cdot]$ reduces to a multiple of the $L^2(\Omega_{2q})$ inner product. Biorthogonality in $(\mathbb{P}(\mathbb{C}^q), [\cdot,\cdot])$ is fully investigated. **2000 AMS Subj. Class.** 33C55,

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1. Introduction

This paper considers spaces of polynomials in the variables z and $\overline{z}$ of $\mathbb{C}^q$, $q \geq 1$. The unitary space $\mathbb{C}^q$ is assumed to be accompanied with its usual inner product

$$\langle z, w\rangle := z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_q\overline{w_q}, \quad z, w \in \mathbb{C}^q, \tag{1.1}$$

where we are writing $z = (z_1, z_2, \ldots, z_q)$ and $w = (w_1, w_2, \ldots, w_q)$. The major polynomial space considered here is $\mathbb{P}(\mathbb{C}^q)$, the unitary space of polynomials in the independent variables z and $\overline{z}$ of $\mathbb{C}^q$. Elements of this space can be written in the form

$$p(z) := p(z, \overline{z}) = \sum_{|\alpha|\leq m} \sum_{|\beta|\leq n} p_{\alpha,\beta} z^\alpha \overline{z}^\beta, \quad p_{\alpha,\beta} \in \mathbb{C}, \quad \alpha, \beta \in \mathbb{Z}_+^q, \tag{1.2}$$

for nonnegative integers m and n, where standard multi-index notation is in force. The subspace of $\mathbb{P}(\mathbb{C}^q)$ composed of polynomials that are homogeneous of degree m in z and of degree n in $\overline{z}$ will be denoted by $\mathbb{P}_{m,n}(\mathbb{C}^q)$. The dimension of $\mathbb{P}_{m,n}(\mathbb{C}^q)$ is given by ([2, p.17])

$$\delta(q, m, n) := \binom{m+q-1}{q-1}\binom{n+q-1}{q-1}. \tag{1.3}$$

The subspace of $\mathbb{P}_{m,n}(\mathbb{C}^q)$ composed of harmonic elements, that is, elements that are in the kernel of the complex Laplacian

$$\Delta_{2q} := 4\sum_{j=1}^{q} \frac{\partial^2}{\partial z_j \partial \overline{z_j}} \tag{1.4}$$

will be denoted by $\mathbb{H}_{m,n}(\mathbb{C}^q)$. Elements of this space play the role played by the solid harmonics in analysis on real spheres.

Next, we introduce spaces of polynomials restricted to the unit sphere

$$\Omega_{2q} := \{z \in \mathbb{C}^q : \langle z, z\rangle = 1\}. \tag{1.5}$$

The symbol $\mathcal{P}_{m,n}(\Omega_{2q})$ will stand for the space obtained from $\mathbb{P}_{m,n}(\mathbb{C}^q)$ by restricting its elements to Ω_{2q}. Finally, $\mathcal{H}_{m,n}(\Omega_{2q})$ will denote the space of *complex spherical harmonics* of degree m in z and degree n in $\overline{z}$, that is, the set of restrictions of elements of $\mathbb{H}_{m,n}(\mathbb{C}^q)$ to Ω_{2q}. The space $\mathcal{H}_{m,n}(\Omega_{2q})$ has dimension $d(q, m, n)$ given by ([2, p.17])

$$d(q, m, n) = \delta(q, m, n) - \delta(q, m-1, n-1), \quad m, n \neq 0, \tag{1.6}$$

$$d(q, m, 0) = \delta(q, m, 0), \quad \text{and} \quad \delta(q, 0, n) = \delta(q, 0, n). \tag{1.7}$$

This paper was motivated by the following three results: the orthogonal decomposition ([2])

$$\mathcal{P}_{m,n}(\Omega_{2q}) = \bigoplus_{j=0}^{m\wedge n} \mathcal{H}_{m-j,n-j}(\Omega_{2q}), \tag{1.8}$$

the dimension formula ([2,7])

$$d(q,m,n) = \sum_{k=0}^{m} \sum_{l=0}^{n} d(q-1,k,l), \quad q \geq 2, \tag{1.9}$$

and the fact that some elements of $\mathcal{H}_{m,n}(\Omega_{2q})$ can be constructed from given elements in $\mathcal{H}_{m-k,n-l}(\Omega_{2q})$, $k < m$, $l < n$, by multiplying them by special elements of $\mathbb{H}_{k,l}(\mathbb{C}^q)$ (see proof of Theorem 5.1 in [3]).

Looking at the real version of (1.8) in either [1, p.76] or [8, p.139] one observes that the proof there requires a special inner product on spaces of homogeneous polynomials. In the first half of the paper, we endow our polynomial spaces with the following similar inner product

$$[f,g] := [f,g]_q := f(\overline{D})\left(\overline{g(z)}\right)(0), \quad f,g \in \mathbb{P}(\mathbb{C}^q), \tag{1.10}$$

in which

$$\overline{D} := \left(\frac{\partial}{\partial \overline{z_1}}, \frac{\partial}{\partial \overline{z_2}}, \ldots, \frac{\partial}{\partial \overline{z_q}}\right), \tag{1.11}$$

and extract a number of interesting properties. Among them, we show that there is a positive constant C, depending on m, n and q, such that

$$[f,g] = C\langle f,g\rangle_2, \quad f,g \in \mathcal{H}_{m,n}(\Omega_{2q}). \tag{1.12}$$

The inner product in the right-hand side of (1.12) is the usual one in $L^2(\Omega_{2q})$, that is,

$$\langle f,g\rangle_2 := \int_{\Omega_{2q}} f(z)\overline{g(z)}d\sigma_q(z), \quad f,g \in L^2(\Omega_{2q}), \tag{1.13}$$

where σ_q is the unique positive Borel measure in Ω_{2q} such that

$$\sigma_q(\Omega_{2q}) = \frac{2\pi^q}{(q-1)!}. \tag{1.14}$$

The other properties we obtain are related to the Funk-Hecke formula ([5,6]) and with properties of bi-orthogonal systems in the polynomial spaces endowed with the inner product in (1.10). All the results mentioned above form the contents of Sections 2 and 3.

Formula (1.9) suggests that one should be able to construct a basis for $\mathcal{H}_{m,n}(\Omega_{2q})$ from given bases for the spaces $\mathcal{H}_{k,l}(\Omega_{2q-2})$, $k = 0,1,\ldots,m$, $l = 0,1,\ldots,n$. We prove this is the case using the special polynomials introduced in [3, p.3] as a generating function. In addition, we discuss orthogonality and representing properties that are implied by the result, completing the list of results forming Section 4.

2. The inner product $[\cdot,\cdot]$

To begin this section, we observe that the spaces $\mathcal{H}_{m,n}(\Omega_{2q})$ are pairwise orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_2$ ([3]). Throughout the paper, orthogonality will always refer to this inner product.

If $\mathcal{O}(2q)$ is the group of isometries of $\mathbb{C}^q$ that fix the origin then σ_q is $\mathcal{O}(2q)$-invariant in the following sense: $\sigma_q(\rho B) = \sigma_q(B)$ if $\rho \in \mathcal{O}(2q)$ and B is a Borel subset of Ω_{2q}. As a consequence, the following invariance property holds:

$$\langle f\circ\rho, g\circ\rho\rangle_2 = \langle f,g\rangle_2, \quad f,g\in L^2(\Omega_{2q}), \quad \rho\in\mathcal{O}(2q). \tag{2.1}$$

The following well-known result establishes the $\mathcal{O}(2q)$-invariance of complex spherical harmonics.

Lemma 2.1 *The space $\mathcal{H}_{m,n}(\Omega_{2q})$ is $\mathcal{O}(2q)$-invariant, that is, if $f \in \mathcal{H}_{m,n}(\Omega_{2q})$ and $\rho \in \mathcal{O}(2q)$ then $f\circ\rho \in \mathcal{H}_{m,n}(\Omega_{2q})$.*

Proof: It will be left to the reader. ■

Next, we return to formula (1.10).

Lemma 2.2 *Formula* (1.10) *defines an inner product in* $\mathbb{P}(\mathbb{C}^q)$.

Proof: It is very easy to see from the definitions that if $(i,j)\neq(k,l)$ then the spaces $\mathbb{P}_{i,j}(\mathbb{C}^q)$ and $\mathbb{P}_{k,l}(\mathbb{C}^q)$ are orthogonal with respect to $[\cdot,\cdot]$. In particular, we have

$$[z^\alpha\bar{z}^\beta, z^\gamma\bar{z}^\delta] = \begin{cases}\alpha!\beta!, & (\alpha,\beta)=(\gamma,\delta)\\ 0, & (\alpha,\beta)\neq(\gamma,\delta).\end{cases} \tag{2.2}$$

Now, let $f,g\in\mathbb{P}(\mathbb{C}^q)$. There are pairs of indices (k,l) and (m,n) in $\mathbb{Z}_+^2$ such that

$$f(z)=\sum_{i=0}^{k}\sum_{j=0}^{l} f_{i,j}(z), \quad g(z)=\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} g_{\mu,\nu}(z), \quad f_{i,j}\in\mathbb{P}_{i,j}(\mathbb{C}^q), \quad g_{\mu,\nu}\in\mathbb{P}_{\mu,\nu}(\mathbb{C}^q). \tag{2.3}$$

Hence,

$$[f,g]=\sum_{i=0}^{k}\sum_{j=0}^{l}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}[f_{i,j},g_{\mu,\nu}] = \sum_{\mu=0}^{k\wedge m}\sum_{\nu=0}^{l\wedge n}[f_{\mu,\nu},g_{\mu,\nu}]. \tag{2.4}$$

Expanding $f_{\mu,\nu}$ e $g_{\mu,\nu}$ in the form

$$f_{\mu,\nu}(z)=\sum_{|\alpha|=\mu}\sum_{|\beta|=\nu} a_{\alpha,\beta} z^\alpha\bar{z}^\beta, \quad g_{\mu,\nu}(z)=\sum_{|\gamma|=\mu}\sum_{|\delta|=\nu} b_{\gamma,\delta} z^\gamma\bar{z}^\delta, \quad a_{\alpha,\beta}, b_{\gamma,\delta}\in\mathbb{C}, \tag{2.5}$$

we finally deduce that

$$[f,g]=\sum_{\mu=0}^{k\wedge m}\sum_{\nu=0}^{l\wedge n}\sum_{|\alpha|=\mu}\sum_{|\beta|=\nu}\alpha!\beta!a_{\alpha,\beta}\overline{b_{\alpha,\beta}}. \tag{2.6}$$

Using this representation, it is now easy to verify that $[\cdot,\cdot]$ defines an inner product in the space $\mathbb{P}(\mathbb{C}^q)$. ■

As an example, we observe that the set

$$\bigcup_{m,n\in\mathbb{Z}_+}\left\{\frac{z^\alpha}{\sqrt{\alpha!}}\frac{\overline{z}^\beta}{\sqrt{\beta!}}:|\alpha|=m,\ |\beta|=n\right\} \tag{2.7}$$

is an orthonormal basis for $(\mathbb{P}(\mathbb{C}^q),[\cdot,\cdot])$. Another remark at this time is that Formula (1.10) reduces to

$$[f,g]=f(\overline{D})\left(\overline{g(z)}\right), \tag{2.8}$$

when the space $\mathbb{P}(\mathbb{C}^q)$ is replaced with its subspace $\mathbb{P}_{m,n}(\mathbb{C}^q)$.

Lemma 2.3 *The inner product $[\cdot,\cdot]$ possesses the following invariance property*

$$[f\circ\rho,f\circ\rho]=[f,f],\quad f\in\mathbb{P}_{m,n}(\mathbb{C}^q),\quad \rho\in\mathcal{O}(2q). \tag{2.9}$$

Proof: It suffices to prove the lemma in the case in which $f(z)=z^\alpha\overline{z}^\beta$, $|\alpha|=m$, $|\beta|=n$. If $\rho\in\mathcal{O}(2q)$ we can write

$$\rho(z)=\left(\sum_{j=1}^q a_{1j}z_j,\sum_{j=1}^q a_{2j}z_j,\ldots,\sum_{j=1}^q a_{qj}z_j\right),\quad a_{ij}\in\mathbb{C},\quad z\in\mathbb{C}^q. \tag{2.10}$$

Now, direct computation reveals that $(f\circ\rho)(\overline{D})(\overline{f(\rho(z))})=\alpha!\beta!=[f,f]$. ■

Next, we employ the vector space isomorphism

$$f\in\mathbb{H}_{m,n}(\mathbb{C}^q)\longmapsto f|_{\Omega_{2q}}\in\mathcal{H}_{m,n}(\Omega_{2q}) \tag{2.11}$$

to bring the inner product (1.10) into the space $\mathcal{H}_{m,n}(\Omega_{2q})$. If $f\in\mathcal{H}_{m,n}(\Omega_{2q})$ write $\widehat{f}$ to denote the unique element of $\mathbb{H}_{m,n}(\mathbb{C}^q)$ such that $\widehat{f}|_{\Omega_{2q}}=f$. Then the formula

$$[f,g]:=[\widehat{f},\widehat{g}],\quad f,g\in\mathcal{H}_{m,n}(\Omega_{2q}) \tag{2.12}$$

defines an inner product in $\mathcal{H}_{m,n}(\Omega_{2q})$.

Theorem 2.7 below will reveal that the spaces $(\mathcal{H}_{m,n}(\Omega_{2q}),[\cdot,\cdot])$ and $(\mathcal{H}_{m,n}(\Omega_{2q}),\langle\cdot,\cdot\rangle_2)$ are isomorphic. The following results will be helpful in proving that theorem. Details about them can be found in [2]. The symbol ε_q will stand for the vector $(0,0,\ldots,0,1)$ of $\mathbb{C}^q$.

Lemma 2.4 *If W is a nonzero finite-dimensional $\mathcal{O}(2q)$-invariant space of continuous functions on Ω_{2q} then there exists a unique f in $W\setminus\{0\}$ such that $f\circ\rho=f$, when $\rho\in\mathcal{O}(2q)$ and $\rho(\varepsilon_q)=\varepsilon_q$.*

Lemma 2.5 *Let f be in $\mathcal{H}_{m,n}(\Omega_{2q})$. The following assertions are equivalent:*
i) $f\circ\rho=f$ if $\rho\in\mathcal{O}(2q)$ and $\rho(\varepsilon_q)=\varepsilon_q$;
ii) There exists a complex number C such that

$$f(z)=Ce^{i(m-n)\theta}|\langle z,\varepsilon_q\rangle|^{|m-n|}P_{m\wedge n}^{(q-2,|m-n|)}(2|\langle z,\varepsilon_q\rangle|^2-1),\quad z\in\Omega_{2q},\tag{2.13}$$

in which θ is an argument of $\langle z,\varepsilon_q\rangle$ in $[0,2\pi)$.

Proposition 2.6 *Let $\mathcal{N}$ be a subspace of $\mathcal{H}_{m,n}(\Omega_{2q})$. If $\mathcal{N}$ is $\mathcal{O}(2q)$-invariant then either $\mathcal{N}=\{0\}$ or $\mathcal{N}=\mathcal{H}_{m,n}(\Omega_{2q})$.*

Proof: If $\mathcal{N}\neq\{0\}$ then $\mathcal{H}_{m,n}(\Omega_{2q})=\mathcal{N}\oplus\mathcal{N}^\perp$, in which $\mathcal{N}^\perp$ is the orthogonal complement of $\mathcal{N}$ in $\mathcal{H}_{m,n}(\Omega_{2q})$. Obviously, $\mathcal{N}^\perp$ is $\mathcal{O}(2q)$-invariant. The rest of the proof will show that $\mathcal{N}^\perp=\{0\}$. Indeed, if not, we may use Lemma 2.4 to choose $f\in\mathcal{N}\setminus\{0\}$ and $g\in\mathcal{N}^\perp\setminus\{0\}$ such that $f\circ\rho=f$ and $g\circ\rho=g$, when $\rho\in\mathcal{O}(2q)$ and $\rho(\varepsilon_q)=\varepsilon_q$. Lemma 2.5 furnishes a complex number C such that $f=C\,g$. It follows that $f=g=0$, a clear contradiction. ∎

Theorem 2.7 *There exists a positive constant C, depending on m, n and q, such that*

$$[f,g]=C\langle f,g\rangle_2,\quad f,g\in\mathcal{H}_{m,n}(\Omega_{2q}).\tag{2.14}$$

Proof: Since $F:=\{f\in\mathcal{H}_{m,n}(\Omega_{2q}):\langle f,f\rangle_2=1\}$ is a compact subset of $\mathcal{H}_{m,n}(\Omega_{2q})$, the continuous function

$$f\in F\longmapsto[f,f]\in\mathbb{R}\tag{2.15}$$

attains its maximum in a point f_0 of F. It follows that,

$$[f,f]\leq[f_0,f_0]\langle f,f\rangle_2,\quad f\in\mathcal{H}_{m,n}(\Omega_{2q}).\tag{2.16}$$

We will use this information to show that the bilinear form

$$\varphi:\mathcal{H}_{m,n}(\Omega_{2q})\times\mathcal{H}_{m,n}(\Omega_{2q})\longrightarrow\mathbb{C}\tag{2.17}$$

given by

$$\varphi(f,g) = [f_0, f_0]\langle f, g\rangle_2 - [f,g], \quad f,g \in \mathcal{H}_{m,n}(\Omega_{2q}) \tag{2.18}$$

is identically zero. Equivalently, we will show that

$$\mathcal{N} := \{f \in \mathcal{H}_{m,n}(\Omega_{2q}) : \varphi(f,g) = 0, \quad g \in \mathcal{H}_{m,n}(\Omega_{2q})\} \tag{2.19}$$

is the whole space $\mathcal{H}_{m,n}(\Omega_{2q})$. Since $\mathcal{N}$ is a subspace of $\mathcal{H}_{m,n}(\Omega_{2q})$, Proposition 2.6 tells us that it suffices to show that $\mathcal{N}$ is nonzero and $\mathcal{O}(2q)$-invariant. Let $\rho \in \mathcal{O}(2q)$ and $f \in \mathcal{N}$. Due to (2.16), φ is positive definite. Hence, we may apply Schwarz's inequality [4, p.375] to obtain

$$|\varphi(f \circ \rho, g)|^2 \leq \varphi(f \circ \rho, f \circ \rho)\varphi(g,g), \quad g \in \mathcal{H}_{m,n}(\Omega_{2q}). \tag{2.20}$$

However, Lemma 2.3 and property (2.1) imply that $\varphi(f\circ\rho, f\circ\rho) = \varphi(f,f) = 0$. It follows that $f \circ \rho \in \mathcal{N}$. Since a similar argument shows that $\varphi(f_0, g) = 0$, $g \in \mathcal{H}_{m,n}(\Omega_{2q})$, it is clear that $\mathcal{N}$ is nonzero. ■

Corollary 2.8 *There exists a positive constant C such that*

$$[f,g] = C\langle f|_{\Omega_{2q}}, g|_{\Omega_{2q}}\rangle_2, \quad f,g \in \mathbb{H}_{m,n}(\mathbb{C}^q). \tag{2.21}$$

Next, we compute the constant C in Theorem 2.7. The following lemma is taken from Rudin's book [7, p.16].

Lemma 2.9 *For multi-indices α and β we have*

$$\int_{\Omega_{2q}} z^\alpha \overline{z}^\beta \, d\sigma_q(z) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \dfrac{2\pi^q \alpha!}{(|\alpha| + q - 1)!} & \text{if } \alpha = \beta. \end{cases}$$

Take $f(z) = g(z) = z_1^m \overline{z_2}^n$ in the space $\mathbb{H}_{m,n}(\mathbb{C}^q)$. Formula (2.2) implies that $[f,g] = m!n!$ while Lemma 2.9 produces

$$\langle f, g\rangle_2 = \frac{2\pi^q m!n!}{(m+n+q-1)!}. \tag{2.22}$$

This proves the following theorem.

Theorem 2.10 *The constant C in Theorem 2.7 equals to $(m+n+q-1)!(2\pi^q)^{-1}$.*

We close the section by showing that Theorem 2.7 cannot hold in the bigger space $\mathcal{P}_{m,n}(\Omega_{2q})$. In fact, if $h(z) = z_1^m \overline{z_1}^n$ then $[h,h] = m!n!$ while Lemma 2.9 yields $\langle h,h\rangle_2 = 2\pi^q(m+n)!/(m+n+q-1)!$. Now, it is easily seen that the equality $[h,h] = C\langle h,h\rangle_2$

holds if and only if $C = m!n!(m+n+q-1)!(2\pi^q)^{-1}/(m+n)!$. This is not the value of C we have encountered in Theorem 2.10.

3. Bi-orthogonality in $(\mathbb{P}(\mathbb{C}^q), [\cdot,\cdot])$

In this section we investigate orthogonality in the space $(\mathbb{P}(\mathbb{C}^q), [\cdot,\cdot])$. We begin with a result related to basic elements of $(\mathbb{P}_{m,n}(\mathbb{C}^q), [\cdot,\cdot])$.

Theorem 3.1 *Let $\{f_\mu : \mu = 1,2,\ldots,\delta(q,m,n)\}$ and $\{g_\nu : \nu = 1,2,\ldots,\delta(q,m,n)\}$ be bases for $(\mathbb{P}_{m,n}(\mathbb{C}^q), [\cdot,\cdot])$. If $[f_\mu, g_\nu] = 0$, $\mu \neq \nu$ then*

$$\langle z,w\rangle^m \langle w,z\rangle^n = m!\,n! \sum_{\mu=1}^{\delta(q,m,n)} \frac{f_\mu(z)\overline{g_\mu(w)}}{[f_\mu, g_\mu]}, \quad z,w \in \mathbb{C}^q. \tag{3.1}$$

Proof: Since $\{f_\mu : \mu = 1,2,\ldots,\delta(q,m,n)\}$ is a basis for $\mathbb{P}_{m,n}(\mathbb{C}^q)$, there are polynomials p_μ, $\mu = 1,2,\ldots,\delta(q,m,n)$ such that

$$\langle z,w\rangle^m \langle w,z\rangle^n = \sum_{\mu=1}^{\delta(q,m,n)} p_\mu(w) f_\mu(z), \quad z,w \in \mathbb{C}^q. \tag{3.2}$$

Due to the hypothesis,

$$\begin{aligned} [\langle \cdot,w\rangle^m \langle w,\cdot\rangle^n, g_\nu] &= \sum_{\mu=1}^{\delta(q,m,n)} p_\mu(w)[f_\mu, g_\nu] \\ &= p_\nu(w)[f_\nu, g_\nu], \quad \nu = 1,2,\ldots,\delta(q,m,n), \quad w \in \mathbb{C}^q. \end{aligned}$$

On the other hand, writing g_ν in the form

$$g_\nu(z) = \sum_{|\alpha|=m} \sum_{|\beta|=n} c_{\alpha,\beta} z^\alpha \overline{z}^\beta \tag{3.3}$$

and computing, we obtain

$$\begin{aligned} [\langle \cdot,w\rangle^m \langle w,\cdot\rangle^n, g_\nu] &= m!\,n! \sum_{|\gamma|=m} \sum_{|\delta|=n} \sum_{|\alpha|=m} \sum_{|\beta|=n} \frac{\overline{w}^\gamma}{\gamma!} \frac{w^\delta}{\delta!} \overline{c_{\alpha,\beta}} [z^\gamma \overline{z}^\delta, z^\alpha \overline{z}^\beta] \\ &= m!\,n! \sum_{|\alpha|=m} \sum_{|\beta|=n} \overline{c_{\alpha,\beta}}\, \overline{w}^\alpha w^\beta \\ &= m!\,n!\,\overline{g_\nu(w)}, \quad \nu = 1,2,\ldots,\delta(m,n). \end{aligned}$$

Thus,

$$m!\,n!\,\overline{g_\nu(w)} = p_\nu(w)[f_\nu, g_\nu], \quad \nu = 1,2,\ldots,\delta(q,m,n), \quad w \in \mathbb{C}^q, \tag{3.4}$$

and, in particular, since each g_μ is not identically zero, $[f_\mu, g_\mu] \neq 0$, $\mu = 1,2,\ldots,\delta(m,n)$. Concluding,

$$p_\mu = m!n! \frac{\overline{g_\mu}}{[f_\mu, g_\mu]}, \quad \mu = 1,2,\ldots,\delta(m,n) \tag{3.5}$$

and the result follows. ■

If we let $z = w$ in the previous theorem we get the Pythagorian identity

$$\frac{\langle z, z\rangle^{m+n}}{m!\,n!} = \sum_{\mu=1}^{\delta(q,m,n)} \frac{f_\mu(z)\overline{g_\mu(z)}}{[f_\mu, g_\mu]}, \quad z \in \mathbb{C}^q. \tag{3.6}$$

When $z \in \Omega_{2q}$, it reduces to

$$\frac{1}{m!\,n!} = \sum_{\mu=1}^{\delta(q,m,n)} \frac{f_\mu(z)\overline{g_\mu(z)}}{[f_\mu, g_\mu]}. \tag{3.7}$$

If both bases in the previous theorem are equal and orthonormal with respect to $[\cdot,\cdot]$ then we deduce the addition formula

$$\langle z, w\rangle^m \langle w, z\rangle^n = m!\,n! \sum_{\mu=1}^{\delta(q,m,n)} f_\mu(z)\overline{f_\mu(w)}, \quad z, w \in \mathbb{C}^q. \tag{3.8}$$

This formula has a structure very similar to that of the addition formula for complex spherical harmonics ([2]). Finally, the following extension of (3.1) can be proved in a similar manner:

$$\langle z, u\rangle^m \langle v, z\rangle^n = m!\,n! \sum_{\mu=1}^{\delta(q,m,n)} \frac{f_\mu(z)\overline{g_\mu(u,\overline{v})}}{[f_\mu, g_\mu]}, \quad z, u, v \in \mathbb{C}^q. \tag{3.9}$$

In our next result, we establish a Funk-Hecke type theorem for elements in the space $(\mathbb{P}(\mathbb{C}^q), [\cdot,\cdot])$.

Theorem 3.2 *Let f be an element of $\mathbb{P}_{m,n}(\mathbb{C}^q)$ and g an element of $\mathbb{P}(\mathbb{C})$. Then, for each $w \in \mathbb{C}^q$, the map $z \in \mathbb{C}^q \mapsto g(\langle z, w\rangle)$ belong to $\mathbb{P}(\mathbb{C}^q)$. In addition, there exists a nonnegative constant λ, depending on m and n, such that*

$$[g(\langle\cdot, w\rangle), f] = \lambda\,\overline{f(w)}, \quad w \in \mathbb{C}^q. \tag{3.10}$$

Proof: For each pair (k,l), we will denote by $\{g^\mu_{k,l} : \mu = 1, 2, \ldots, \delta(q,k,l)\}$ an orthonormal basis for $(\mathbb{P}_{k,l}(\mathbb{C}^q), [\cdot,\cdot])$. Assume g has degree α in z and degree β in $\overline{z}$. Recalling Theorem 3.1, we can write

$$g(\langle z, w\rangle) = \sum_{k=0}^{\alpha}\sum_{l=0}^{\beta}\sum_{\mu=1}^{\delta(q,k,l)} k!\,l!\,g^\mu_{k,l}(z)\overline{g^\mu_{k,l}(w)}, \quad z, w \in \mathbb{C}^q. \tag{3.11}$$

We can find complex numbers a_j such that

$$f = \sum_{j=1}^{\delta(q,m,n)} a_j g^j_{m,n}. \tag{3.12}$$

It follows that

$$
\begin{aligned}
[g(\langle\cdot,w\rangle),f] &= \sum_{j=1}^{\delta(q,m,n)} \sum_{k=0}^{\alpha}\sum_{l=0}^{\beta} \sum_{\mu=1}^{\delta(q,k,l)} k!\,l!\,\overline{a_j}\,\overline{g^{\mu}_{k,l}(w)}[g^{\mu}_{k,l}, g^{j}_{m,n}] \\
&= \sum_{j=1}^{\delta(q,m,n)} \sum_{k=0}^{\alpha}\sum_{l=0}^{\beta} \sum_{\mu=1}^{\delta(q,k,l)} k!\,l!\,\overline{a_j}\,\overline{g^{\mu}_{k,l}(w)}\delta_{km}\delta_{ln}\delta_{\mu j}, \qquad w \in \mathbb{C}^q.
\end{aligned}
$$

Thus,

$$
[g(\langle\cdot,w\rangle),f] = \begin{cases} m!\,n!\,\overline{f(w)}, & \alpha \geq m \text{ and } \beta \geq n \\ 0, & \text{otherwise}, \end{cases} \tag{3.13}
$$

completing the proof of the theorem. ∎

Corollary 3.3 *The following formula holds*

$$
[\langle\cdot,w\rangle^m\langle w,\cdot\rangle^n, \langle\cdot,\zeta\rangle^m\langle\zeta,\cdot\rangle^n] = m!\,n!\langle\zeta,w\rangle^m\langle w,\zeta\rangle^n, \quad w,\zeta \in \mathbb{C}^q. \tag{3.14}
$$

The following theorem is a converse of Theorem 3.1.

Theorem 3.4 *Let $\{f_\mu : \mu = 1,2,\ldots,\delta(q,m,n)\}$ be a linearly independent subset of $\mathbb{P}_{m,n}(\mathbb{C}^q)$. Assume there is a subset $\{g_\mu : \mu = 1,2,\ldots,\delta(q,m,n)\}$ of $\mathbb{P}(\mathbb{C}^q)$ such that $[f_\mu, g_\mu] \neq 0$, $\mu = 1,2,\ldots,\delta(q,m,n)$ and*

$$
\langle z,w\rangle^m\langle w,z\rangle^n = m!\,n! \sum_{\mu=1}^{\delta(q,m,n)} \frac{f_\mu(z)\overline{g_\mu(w)}}{[f_\mu,g_\mu]} \qquad z,w \in \mathbb{C}^q. \tag{3.15}
$$

Then $\{f_\mu : \mu = 1,2,\ldots,\delta(q,m,n)\}$ and $\{g_\mu : \mu = 1,2,\ldots,\delta(q,m,n)\}$ are bases for $\mathbb{P}_{m,n}(\mathbb{C}^q)$ satisfying $[f_\mu, g_\nu] = 0$, $\mu \neq \nu$.

Proof: The use of (3.15) yields

$$
\begin{aligned}
m!\,n! \sum_{\mu=1}^{\delta(q,m,n)} \frac{f_\mu(z)\overline{g_\mu(\lambda w)}}{[f_\mu,g_\mu]} &= \langle z,\lambda w\rangle^m\langle\lambda w,z\rangle^n \\
&= \langle\overline{\lambda}z,w\rangle^m\langle w,\overline{\lambda}z\rangle^n \\
&= m!\,n! \sum_{\mu=1}^{\delta(q,m,n)} \frac{f_\mu(\overline{\lambda}z)\overline{g_\mu(w)}}{[f_\mu,g_\mu]} \\
&= m!\,n! \sum_{\mu=1}^{\delta(q,m,n)} \overline{\lambda}^m\lambda^n \frac{f_\mu(z)\overline{g_\mu(w)}}{[f_\mu,g_\mu]}, \qquad z,w \in \mathbb{C}^q, \quad \lambda \in \mathbb{C}.
\end{aligned}
$$

Hence

$$
\sum_{\mu=1}^{\delta(q,m,n)} \left(\overline{g_\mu(\lambda w)} - \overline{\lambda}^m\lambda^n\overline{g_\mu(w)}\right)\frac{f_\mu(z)}{[f_\mu,g_\mu]} = 0, \quad z,w \in \mathbb{C}^q, \quad \lambda \in \mathbb{C}. \tag{3.16}
$$

Since the set $\{f_\mu : \mu = 1,2,\ldots,\delta(q,m,n)\}$ is linearly independent, it follows that

$$g_\mu(\lambda w) - \lambda^m \overline{\lambda}^n g_\mu(w) = 0, \quad w \in \mathbb{C}^q, \quad \lambda \in \mathbb{C}, \tag{3.17}$$

that is, $g_\mu \in \mathbb{P}_{m,n}(\mathbb{C}^q)$, $\mu = 1,2,\ldots,\delta(q,m,n)$. To conclude the proof we apply Theorem 3.2 and Formula (3.15) appropriately to obtain

$$m!\,n!\,f_\nu(z) = [\langle\cdot,z\rangle^m\langle z,\cdot\rangle^n, \overline{f_\nu}] = m!\,n! \sum_{\mu=1}^{\delta(q,m,n)} \frac{f_\mu(z)[f_\nu,g_\mu]}{[f_\mu,g_\mu]}, \quad \nu = 1,2,\ldots,\delta(q,m,n).$$

The linear independence hypothesis allows us to conclude that $[f_\nu, g_\mu] = 0$, $\mu \neq \nu$. ■

Corollary 3.5 *If a linearly independent subset* $\{f_\mu : \mu = 1,2,\ldots,\delta(q,m,n)\}$ *of* $\mathbb{P}_{m,n}(\mathbb{C}^q)$ *satisfies*

$$\langle z,w\rangle^m\langle w,z\rangle^n = m!\,n! \sum_{\mu=1}^{\delta(q,m,n)} f_\mu(z)\overline{f_\mu(w)}, \quad z,w \in \mathbb{C}^q, \tag{3.18}$$

then it is orthonormal with respect to $[\cdot,\cdot]$.

Proof: It suffices to observe that, under the given hypotheses, the denominator in the sum on the right-hand side of the last equation in the proof of Theorem 3.4 disappears.■

4. Generating bases

This section presents a method to construct bases for the space $\mathcal{H}_{m,n}(\Omega_{2q})$. The method is inductive over the dimension of the sphere, that is, it presupposes the knowledge of a basis for $\mathcal{H}_{m,n}(\Omega_{2q-2})$. We begin with a technical lemma that exhibits a very special kernel in $\mathbb{H}_{m,n}(\mathbb{C}^q)$. As we said before, the idea behind the use of this kernel comes from the proof of Theorem 5.1 in [3].

For a fixed $q_1 \in \{1,2,\ldots,q\}$ we will employ the decomposition $\mathbb{C}^q = W^{q_1} \oplus V^{q-q_1}$, where $W^{q_1} = \{z \in \mathbb{C}^q : z_j = 0, j = q_1+1, q_1+2,\ldots,q\}$ and $V^{q-q_1} = \{z \in \mathbb{C}^q : z_j = 0, j = 1,2,\ldots,q_1\}$.

Lemma 4.1 *Let* $w \in W^{q_1} \cap \Omega_{2q}$ *and* $v \in V^{q-q_1} \cap \Omega_{2q}$. *Then*

$$G_{m,n}^{w,v}(z) := \langle z, v+w\rangle^m \langle v-w, z\rangle^n, \quad z \in \mathbb{C}^q \tag{4.1}$$

is an element of $\mathbb{H}_{m,n}(\mathbb{C}^q)$.

Proof: First observe that

$$\frac{\partial}{\partial \overline{z_j}} G^{w,v}_{m,n} = \begin{cases} -n\langle z, v+w\rangle^m \langle v-w, z\rangle^{n-1} w_j & j = 1,2,\ldots,q_1 \\ n\langle z, w+v\rangle^m \langle v-w, z\rangle^{n-1} v_j & j = q_1+1, q_1+2,\ldots,q. \end{cases}$$

Next, notice that

$$\sum_{j=q_1+1}^{q} \frac{\partial^2}{\partial z_j \partial \overline{z_j}} G^{w,v}_{m,n} = -\sum_{j=1}^{q_1} \frac{\partial^2}{\partial z_j \partial \overline{z_j}} G^{w,v}_{m,n}. \tag{4.2}$$

It follows that $\Delta_{2q}(G^{w,v}_{m,n}) = 0$. The homogeneity of $G^{w,v}_{m,n}$ with respect to z and $\overline{z}$ is clear. ■

If $q_1 = q-1$ in the previous lemma then $W^{q-1} \cap \Omega_{2q}$ is a copy of Ω_{2q-2}. In other words, elements of $W^{q-1} \cap \Omega_{2q}$ are of the form $\widehat{w} = (w, 0)$ with $w \in \Omega_{2q-2}$. Denoting the elements of $\mathbb{C}^q$ by $\widehat{z} = (z, z_q)$, $z \in \mathbb{C}^{q-1}$, and taking $v = \varepsilon_q = (0, 0, \ldots, 0, 1)$, the function in the previous lemma takes the form

$$G^{\widehat{w},v}_{m,n}(\widehat{z}) = (\langle z, w\rangle + z_q)^m (-\langle w, z\rangle + \overline{z_q})^n. \tag{4.3}$$

From now on, we will adopt the following simplified notation: $G^{w}_{m,n} := G^{\widehat{w},v}_{m,n}$. The main result of this section is as follows.

Theorem 4.2 *Let $\{g_j : j = 1, 2, \ldots, d(q,m,n)\}$ be a linearly independent subset of $\bigcup_{k=0}^{m} \bigcup_{l=0}^{n} \mathcal{H}_{k,l}(\Omega_{2q-2})$. Then $G^{w}_{m,n}$ has a decomposition in the form*

$$G^{w}_{m,n}(\widehat{z}) = \sum_{j=1}^{d(q,m,n)} f_j(\widehat{z}) g_j(w), \quad \widehat{z} = (z, z_q) \in \mathbb{C}^q, \quad w \in \Omega_{2q-2}, \tag{4.4}$$

in which $\{f_j : j = 1, 2, \ldots, d(q,m,n)\} \subset \mathbb{H}_{m,n}(\mathbb{C}^q)$.

Proof: Initially, we expand the right-hand side of (4.3) to write

$$G^{w}_{m,n}(\widehat{z}) = \sum_{|\alpha|+\mu=m} \frac{m!}{\mu!\,\alpha!} z^\alpha z_q^\mu \overline{w}^\alpha \sum_{|\beta|+\nu=n} \frac{n!}{\nu!\,\beta!} (-w)^\beta \overline{z}^\beta \overline{z}_q^\nu, \quad \widehat{z} \in \mathbb{C}^q. \tag{4.5}$$

Since $|\alpha| \leq m$ and $|\beta| \leq n$, a help of (1.8) allows us to find constants $a_j(\alpha, \beta)$ such that

$$\overline{w}^\alpha (-w)^\beta = \sum_{j=1}^{d(q,m,n)} a_j(\alpha,\beta) g_j(w), \quad w \in \Omega_{2q-2}. \tag{4.6}$$

Hence,

$$G^{w}_{m,n}(\widehat{z}) = \sum_{j=1}^{d(q,m,n)} \left(\sum_{|\alpha|+\mu=m} \sum_{|\beta|+\nu=n} a_j(\alpha,\beta) \frac{m!}{\mu!\,\alpha!} z^\alpha z_q^\mu \frac{n!}{\nu!\,\beta!} \overline{z}^\beta \overline{z}_q^\nu \right) g_j(w). \tag{4.7}$$

We now show that the expression

$$f_j(\widehat{z}) := \sum_{|\alpha|+\mu=m} \sum_{|\beta|+\nu=n} a_j(\alpha,\beta)\frac{m!}{\mu!\,\alpha!} z^\alpha z_q^\mu \frac{n!}{\nu!\,\beta!} \overline{z}^\beta \overline{z}_q^\nu, \tag{4.8}$$

defines an element of $\mathbb{H}_{m,n}(\mathbb{C}^q)$, for $j = 1,2,\ldots,d(q,m,n)$. The homogeneity of f_j of degree m with respect to $\widehat{z}$ and of degree n with respect to $\overline{\widehat{z}}$ is obvious. Applying the Laplacian in (4.7) we deduce

$$0 = \Delta_{2q}(G^w_{m,n})(\widehat{z}) = \sum_{j=1}^{d(q,m,n)} \Delta_{2q}(f_j)(\widehat{z}) g_j(w), \quad \widehat{z} \in \mathbb{C}^q, \quad w \in \Omega_{2q-2}. \tag{4.9}$$

The linear independence of the g_j implies that $\Delta_{2q}(f_j) = 0$. ■

The following lemma describes an integral operator that reproduces complex spherical harmonics. It is a complex version of the famous Funk-Hecke formula. A proof for this version can be found in [5,6]. In the statement of the lemma, $B[0,1]$ is the closed unit disk in $\mathbb{C}$, $d\nu_q(z)$ is the normalized Lebesgue measure given by

$$d\nu_q(z) := \frac{q-1}{\pi}\left(1 - x^2 - y^2\right)^{q-2} dx\, dy, \quad z = x + iy \in B[0,1], \tag{4.10}$$

$L^{p,q}(B[0,1])$ is the class of complex functions that are p-integrable in $B[0,1]$ with respect to ν_q and $P^{q-2}_{m,n}$ is the disk polynomial of degree $m+n$ associated with the integer $q-2$.

Lemma 4.3 *Let Y be an element of $\mathcal{H}_{m,n}(\Omega_{2q})$, and K an element of $L^{1,q}(B[0,1])$. Then for every w in Ω_{2q}, the mapping $z \in \Omega_{2q} \longmapsto K(\langle z,w\rangle)Y(z)$ is in $L^1(\Omega_{2q})$ and*

$$\int_{\Omega_{2q}} K(\langle z,w\rangle)Y(z)d\sigma_q(z) = \lambda^{q-2}_{n,m}(K)Y(w), \quad w \in \Omega_{2q}, \tag{4.11}$$

in which

$$\lambda^{q-2}_{n,m}(K) := \frac{2\pi^q}{(q-1)!}\int_{B[0,1]} K(z)\overline{P^{q-2}_{n,m}(z)}d\nu_q(z). \tag{4.12}$$

Theorem 4.4 *Let $\{g_j : j = 1,2,\ldots,d(q,m,n)\}$ be a linearly independent subset of $\bigcup_{k=0}^m \bigcup_{l=0}^n \mathcal{H}_{k,l}(\Omega_{2q-2})$ and let $\{f_j : j = 1,2,\ldots,d(q,m,n)\}$ be as in Theorem 4.2. If the set $\{g_j : j = 1,2,\ldots,d(q,m,n)\}$ is orthonormal then $\{f_j : j = 1,2,\ldots,d(q,m,n)\}$ is an orthogonal basis for $(\mathbb{H}_{m,n}(\mathbb{C}^q),[\cdot,\cdot])$.*

Proof: In the first step of the proof we show that $[G^w_{m,n}, G^\zeta_{m,n}] = K(\langle\zeta,w\rangle)$, for some function K. Indeed, recalling the hat notation introduced in the beginning of the section, we see that

$$[G^w_{m,n}, G^\zeta_{m,n}] = D_1^m\left[(\langle\zeta,z\rangle + \overline{z_q})^m\right] D_2^n\left[(-\langle z,\zeta\rangle + z_q)^n\right], \tag{4.13}$$

in which

$$D_1 := \overline{w_1}\frac{\partial}{\partial \overline{z_1}} + \overline{w_2}\frac{\partial}{\partial \overline{z_2}} + \cdots + \overline{w_{q-1}}\frac{\partial}{\partial \overline{z_{q-1}}} + \frac{\partial}{\partial \overline{z}_q} \tag{4.14}$$

and

$$D_2 := -w_1\frac{\partial}{\partial z_1} - w_2\frac{\partial}{\partial z_2} - \cdots - w_{q-1}\frac{\partial}{\partial z_{q-1}} + \frac{\partial}{\partial z_q}. \tag{4.15}$$

However,

$$D_1^m(\langle\zeta, z\rangle + \overline{z_q})^m = m!(\langle\zeta, w\rangle + 1)^m \tag{4.16}$$

and

$$D_2^n(-\langle z, \zeta\rangle + z_q)^n = n!(\langle w, \zeta\rangle + 1)^n \tag{4.17}$$

so that

$$[G^w_{m,n}, G^\zeta_{m,n}] = m!\, n!(\langle\zeta, w\rangle + 1)^m(\langle w, \zeta\rangle + 1)^n := K(\langle\zeta, w\rangle), \quad w, \zeta \in \Omega_{2q-2}. \tag{4.18}$$

Next, we use the previous theorem to deduce that

$$[G^w_{m,n}, G^\zeta_{m,n}] = \sum_{l=1}^{d(q,m,n)} p_l(w)\overline{g_l(\zeta)}, \quad w, \zeta \in \Omega_{2q-2}, \tag{4.19}$$

in which

$$p_l(w) = \sum_{j=1}^{d(q,m,n)} [f_j, f_l]g_j(w), \quad w \in \Omega_{2q-2}. \tag{4.20}$$

If the g_l form an orthonormal set we can apply the previous lemma to obtain

$$\lambda(j)g_j(w) = \int_{\Omega_{2q-2}} K(\langle\zeta, w\rangle)g_j(\zeta)d\sigma_{q-1}(\zeta) = p_j(w), \quad j = 1, 2, \ldots, d(q, m, n), \tag{4.21}$$

in which $\lambda(j)$ is a positive constant depending on g_j and K. Thus,

$$[G^w_{m,n}, G^\zeta_{m,n}] = \sum_{l=1}^{d(q,m,n)} \lambda(l)g_l(w)\overline{g_l(\zeta)}, \quad w, \zeta \in \Omega_{2q-2}. \tag{4.22}$$

A comparison with (4.19) yields the relation

$$\lambda(l)g_l(w) = \sum_{j=1}^{d(q,m,n)} [f_j, f_l]g_j(w), \quad w \in \Omega_{2q-2}, \quad l = 1, 2, \ldots, d(q, m, n). \tag{4.23}$$

It is now evident that $[f_j, f_l] = 0$, $j \neq l$ and that $[f_l, f_l] = \lambda(l)$, $l = 1, 2, \ldots, d(q, m, n)$. ■

Example 4.5 Let $m = n = 1$ and $q = 3$. Due to Lemma 2.9, the polynomials

$$g_1(w) = \frac{1}{\sqrt{2}\,\pi}, \quad g_2(w) = \frac{1}{\pi}w_1, \quad g_3(w) = \frac{1}{\pi}w_2, \quad g_4(w) = \frac{1}{\pi}\overline{w_1}, \quad g_5(w) = \frac{1}{\pi}\overline{w_2}, \tag{4.24}$$

$$g_6(w)=\frac{\sqrt{3}}{\pi}w_1\overline{w_2},\quad g_7(w)=\frac{\sqrt{3}}{\pi}\overline{w_1}w_2\quad\text{and}\quad g_8(w)=\frac{\sqrt{6}}{2\pi}(w_1\overline{w_1}-w_2\overline{w_2})\tag{4.25}$$

define an orthonormal subset of $\mathcal{H}_{0,0}(\Omega_4)\cup\mathcal{H}_{0,1}(\Omega_4)\cup\mathcal{H}_{1,0}(\Omega_4)\cup\mathcal{H}_{1,1}(\Omega_4)$. The kernel $G^w_{1,1}(\widehat{z})$ takes the form

$$z_3\overline{z_3}-\overline{z_1}z_3w_1-\overline{z_2}z_3w_2+z_1\overline{z_3}\,\overline{w_1}+z_2\overline{z_3}\,\overline{w_2}-\overline{z_1}z_2w_1\overline{w_2}-z_1\overline{z_2}w_2\overline{w_1}-z_1\overline{z_1}w_1\overline{w_1}-z_2\overline{z_2}w_2\overline{w_2}.$$

Computing the coefficients $a_j(\alpha,\beta)$ in (4.6), here written as $a_j(\alpha;\beta)$, we obtain

$$a_1(0,0;0,0)=\sqrt{2}\,\pi,\quad a_1(1,0;1,0)=-\frac{\sqrt{2}\,\pi}{2},\quad a_8(1,0;1,0)=-\frac{\pi}{\sqrt{6}}\tag{4.26}$$

$$a_1(0,1;0,1)=-\frac{\sqrt{2}\,\pi}{2},\quad a_8(0,1;0,1)=\frac{\pi}{\sqrt{6}},\quad a_2(0,0;1,0)=-\pi,\tag{4.27}$$

$$a_4(1,0;0,0)=\pi,\quad a_5(0,1;0,0)=\pi,\quad a_6(0,1;1,0)=-\frac{\pi}{\sqrt{3}}\tag{4.28}$$

$$a_7(1,0;0,1)=-\frac{\pi}{\sqrt{3}},\quad a_3(0,0;0,1)=-\pi,\tag{4.29}$$

while all the others equal zero. Looking at (4.8), we encounter

$$f_1(\widehat{z})=\frac{\sqrt{2}\,\pi}{2}\left(-z_1\overline{z_1}-z_2\overline{z_2}+2z_3\overline{z_3}\right),\quad f_2(\widehat{z})=-\pi z_3\overline{z_1},\quad f_3(\widehat{z})=-\pi z_3\overline{z_2},\tag{4.30}$$

$$f_4(\widehat{z})=\pi z_1\overline{z_3},\quad f_5(\widehat{z})=\pi z_2\overline{z_3},\quad f_6(\widehat{z})=-\frac{\pi}{\sqrt{3}}z_2\overline{z_1},\tag{4.31}$$

and

$$f_7(\widehat{z})=-\frac{\pi}{\sqrt{3}}z_1\overline{z_2},\quad f_8(\widehat{z})=\frac{\pi}{\sqrt{6}}\left(-z_1\overline{z_1}+z_2\overline{z_2}\right).\tag{4.32}$$

Theorem 4.4 implies that $\{f_j : j=1,2,\ldots,8\}$ is an orthogonal basis for $(\mathbb{H}_{1,1}(\mathbb{C}^3),[\cdot,\cdot])$. The isomorphism (2.11) provides us with an orthogonal basis for $\mathcal{H}_{1,1}(\Omega_6)$.

Corollary 4.6 *Assume the hypotheses in Theorem 4.4. If $\{g_j : j=1,2,\ldots,d(q,m,n)\}$ is orthonormal then*

$$f_j(\widehat{z})=p_j(z_q)\overline{g_j(z)},\quad z\in\Omega_{2q-2},\quad j=1,2,\ldots,d(q,m,n),\tag{4.33}$$

in which $\{p_j : j=1,2,\ldots,d(q,m,n)\}$ is a subset of $\mathbb{P}(\mathbb{C})$.

Proof: If $\{g_j : j=1,2,\ldots,d(q,m,n)\}$ is orthonormal, we can use (4.4) to deduce

$$f_j(\widehat{z})=\int_{\Omega_{2q-2}}G^w_{m,n}(\widehat{z})\,\overline{g_j(w)}\,d\sigma_{q-1}(w),\quad z\in\Omega_{2q-2}.\tag{4.34}$$

Expanding $G^w_{m,n}$ in the form

$$G^w_{m,n}(\widehat{z})=\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\frac{(-1)^\nu m!\,n!}{\mu!\nu!(m-\mu)!(n-\nu)!}z_q^{m-\mu}\overline{z_q}^{n-\nu}K_{\mu,\nu}(\langle w,z\rangle),\tag{4.35}$$

where $K_{\mu,\nu}(\langle z, w\rangle) = \langle z, w\rangle^\mu \langle w, z\rangle^\nu$, using Lemma 4.3 and arranging we obtain

$$f_j(\widehat{z}) = \left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \frac{(-1)^\nu m!\, n!}{\mu!\nu!(m-\mu)!(n-\nu)!} b_j(\mu,\nu) z_q^{m-\mu} \overline{z_q}^{n-\nu} \right) \overline{g_j(z)}, \quad z \in \Omega_{2q-2}, \tag{4.36}$$

where the $b_j(\mu,\nu)$ are constants produced by the Funk-Hecke formula. Defining

$$p_j(z) = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \frac{(-1)^\nu m!\, n!}{\mu!\nu!(m-\mu)!(n-\nu)!} b_j(\mu,\nu) z^{m-\mu} \overline{z}^{n-\nu}, \quad z \in \mathbb{C} \tag{4.37}$$

concludes the proof. ■

Corollary 4.7 *Assume the hypotheses in Theorem 4.4. If $\{g_j : j = 1, 2, \ldots, d(q,m,n)\}$ is orthonormal then*

$$(\langle \zeta, w\rangle + 1)^m (\langle w, \zeta\rangle + 1)^n = D \sum_{j=1}^{d(q,m,n)} \langle f_j, f_j\rangle_2 g_j(w) \overline{g_j(\zeta)}, \quad w, \zeta \in \Omega_{2q-2}, \tag{4.38}$$

in which $D = (m+n+q-1)!(2\pi^q\, m!n!)^{-1}$.

Proof: First we manipulate the sum in the right-hand side of (4.38) to obtain

$$\begin{aligned}
\sum_{\mu=1}^{d(q,m,n)} \langle f_\mu, f_\mu\rangle_2 g_\mu(w) \overline{g_\mu(\zeta)} &= \sum_{\mu=1}^{d(q,m,n)} \sum_{\nu=1}^{d(q,m,n)} \left(\int_{\Omega_{2q}} f_\mu(\widehat{z}) \overline{f_\nu(\widehat{z})} d\sigma_q(\widehat{z}) \right) g_\mu(w) \overline{g_\nu(\zeta)} \\
&= \int_{\Omega_{2q}} \sum_{\mu=1}^{d(q,m,n)} f_\mu(\widehat{z}) g_\mu(w) \overline{\sum_{\nu=1}^{d(q,m,n)} f_\nu(\widehat{z}) g_\nu(\zeta)}\, d\sigma_q(\widehat{z}) \\
&= \int_{\Omega_{2q}} G^w_{m,n}(\widehat{z}) \overline{G^\zeta_{m,n}(\widehat{z})}\, d\sigma_q(\widehat{z}), \quad w, \zeta \in \Omega_{2q-2}.
\end{aligned}$$

Recalling Lemma 4.1, Theorem 2.7 and Theorem 2.10, we conclude that

$$\sum_{\mu=1}^{d(q,m,n)} \langle f_\mu, f_\mu\rangle_2 g_\mu(w) \overline{g}_\mu(\zeta) = \frac{2\pi^q}{(m+n+q-1)!} [G^w_{m,n}, G^\zeta_{m,n}], \quad w, \zeta \in \Omega_{2q-2}. \tag{4.39}$$

Finally, (4.18) reduces (4.39) to

$$\sum_{\mu=1}^{d(q,m,n)} \langle f_\mu, f_\mu\rangle_2 g_\mu(w) \overline{g_\mu(\zeta)} = D^{-1} (\langle \zeta, w\rangle + 1)^m (\langle w, \zeta\rangle + 1)^n, \quad w, \zeta \in \Omega_{2q-2}, \tag{4.40}$$

with D as described in the statement of the corollary. ■

By letting $w = \zeta$ in Corollary 4.7 we deduce the following identity

$$\sum_{\mu=1}^{d(q,m,n)} \langle f_\mu, f_\mu\rangle_2 |g_\mu(w)|^2 = \frac{2^{m+n+1}\pi^q\, m!n!}{(m+n+q-1)!}, \quad w \in \Omega_{2q-2}. \tag{4.41}$$

We close this section presenting two independent results, one giving an estimate for the sum $\sum_{\mu=1}^{d(q,m,n)}\langle f_\mu, f_\mu\rangle_2$ and the other explaining why the construction in Theorem 4.2 preserves bi-orthogonality.

Corollary 4.8 *Assume the hypotheses in Theorem 4.4. If $\{g_j : j = 1, 2, \ldots, d(q,m,n)\}$ is orthonormal then*

$$\sum_{j=1}^{d(q,m,n)} \langle f_j, f_j\rangle_2 \leq \frac{2^{m+n+2}\pi^{2q-1}}{(q-1)!(q-2)!}. \tag{4.42}$$

Proof: First apply the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} G^w_{m,n}(\widehat{z})\overline{G^w_{m,n}(\widehat{z})} &\leq \langle \widehat{z}, \widehat{z}\rangle^m \langle (w,1),(w,1)\rangle^m \langle \widehat{z}, \widehat{z}\rangle^n \langle (-w,1),(-w,1)\rangle^n \\ &\leq 2^{m+n}\langle \widehat{z}, \widehat{z}\rangle^{m+n}, \quad \widehat{z} \in \mathbb{C}^q, \quad w \in \Omega_{2q-2}. \end{aligned}$$

Integration yields

$$\int_{\Omega_{2q}} G^w_{m,n}(\widehat{z})\overline{G^w_{m,n}(\widehat{z})} d\sigma_q(\widehat{z}) \leq 2^{m+n}\int_{\Omega_{2q}} d\sigma_q(\widehat{z}) = \frac{2^{m+n+1}\pi^q}{(q-1)!}, \quad w \in \Omega_{2q-2}. \tag{4.43}$$

On the other hand, if $\{g_j : j = 1, 2, \ldots, d(q,m,n)\}$ is orthonormal, the arguments at beginning of the proof of Corollary 4.7 imply that

$$\sum_{j=1}^{d(q,m,n)} |g_j(w)|^2 \langle f_j, f_j\rangle_2 \leq \frac{2^{m+n+1}\pi^q}{(q-1)!}, \quad w \in \Omega_{2q-2}. \tag{4.44}$$

Finally,

$$\begin{aligned} \sum_{j=1}^{d(q,m,n)} \langle f_j, f_j\rangle_2 &= \sum_{j=1}^{d(q,m,n)} \langle f_j, f_j\rangle_2 \int_{\Omega_{2q-2}} |g_j(w)|^2 d\sigma_{q-1}(w) \\ &\leq \frac{2^{m+n+1}\pi^q}{(q-1)!}\int_{\Omega_{2q-2}} d\sigma_{q-1}(w) = \frac{2^{m+n+2}\pi^{2q-1}}{(q-1)!(q-2)!}, \end{aligned}$$

completing the proof. ■

Corollary 4.9 *Let $\{g_j : j = 1, 2, \ldots, d(q,m,n)\}$ and $\{g_j' : j = 1, 2, \ldots, d(q,m,n)\}$ be orthonormal subsets of $\bigcup_{k=0}^m \bigcup_{l=0}^n \mathcal{H}_{k,l}(\Omega_{2q-2})$ and let $\{f_j : j = 1, 2, \ldots, d(q,m,n)\}$ and $\{f_j' : j = 1, 2, \ldots, d(q,m,n)\}$ be the corresponding sets resulting from the use of Theorem 4.2. If $\langle g_j, g_k'\rangle_2 = 0,\ j \neq k$, then $[f_j, f_k'] = 0,\ j \neq k$.*

Proof: We use Corollary 4.6 to write

$$f_j(\widehat{z}) = p_j(z_q)\overline{g_j(z)}, \quad z \in \Omega_{2q-2}, \quad j = 1, 2, \ldots, d(q,m,n), \tag{4.45}$$

and

$$f_j'(\widehat{z}) = p_j'(z_q)\overline{g_j'(z)}, \quad z \in \Omega_{2q-2}, \quad j = 1, 2, \ldots, d(q,m,n), \tag{4.46}$$

in which $\{p_j : j = 1, 2, \ldots, d(q, m, n)\}$ and $\{p_j' : j = 1, 2, \ldots, d(q, m, n)\}$ are subsets of $\mathbb{P}(\mathbb{C})$. It follows, with a help of Theorem 2.7, that

$$\begin{aligned}
[f_j, f_k']_q &= p_j\left(\frac{\partial}{\partial \overline{z_q}}\right)\left(\overline{p_k'(z_q)}\right)[g_j, g_k']_{q-1} \\
&= \frac{m+n+q-1)!}{2\pi^q} p_j\left(\frac{\partial}{\partial \overline{z_q}}\right)\left(\overline{p_k'(z_q)}\right)\langle g_j, g_k'\rangle_2.
\end{aligned}$$

The conclusion in the statement of the Corollary follows. ∎

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