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On the Location and Profile of Spike-Layer Nodal Solutions to Nonlinear Schrödinger Equations

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Resumo

Estudamos a existência e o fenômeno de concentração de soluções nodais para a equação

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \quad \text{em} \quad \Omega,$$

na qual Ω é um domínio em $\mathbb{R}^N$, não necessariamente limitado, V é uma função Hölder contínua e $f \in C^1(\mathbb{R})$ é uma função ímpar com crescimento subcrítico e superlinear. Sob condições locais em V, provamos que os extremos globais dessas soluções localizam-se em torno de um mínimo local de V.

On the Location and Profile of Spike-Layer Nodal Solutions to Nonlinear Schrödinger Equations *

CLAUDIANOR O. ALVES †– SÉRGIO H. M. SOARES‡

Abstract The existence and concentration behavior of a nodal solution are established for the equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \quad \text{in} \quad \Omega$$

where Ω is a domain in $\mathbb{R}^N$, not necessarily bounded, V is a positive Hölder continuous function and $f \in C^1(\mathbb{R})$ is a odd function having subcritical and superlinear growth. Under additional local condition on V, it is proved that such a solution has just one positive and one negative peaks, which are located around local minima of V.

1 Introduction

In this paper we are concerned with the problem

$$(P_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \Delta u + V(x)u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a domain in $\mathbb{R}^N$, not necessarily bounded, with empty or smooth boundary, V is a Hölder continuous function satisfying

(V_1) $V(x) \geq \alpha > 0$, for all $x \in \mathbb{R}^N$,

(V_2) There exists a bounded domain Λ compactly contained in Ω such that

$$V_0 = \inf_{x \in \Lambda} V(x) < \inf_{x \in \partial\Lambda} V(x),$$

and $f \in C^1(\mathbb{R})$ is a odd function satisfying

(f_1) $f(s) = o(|s|)$, as $s \to 0$.

(f_2) $\displaystyle\lim_{|s| \to \infty} \frac{f(s)}{|s|^p} = 0$ for some $1 < p < \frac{N+2}{N-2}$.

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Equações diferenciais Parciais

(f_3) For some $2 < \theta \leq p+1$ such that

$$0 < \theta F(s) \leq sf(s) \quad \text{for all } s \neq 0.$$

(f_4) The function $s \longmapsto \frac{f(s)}{s}$ is increasing in $(0, +\infty)$.

Equations of this kind arise in different models. For instance, they are object of special physical and mathematical interest to study the standing waves of the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\hbar^2 \Delta \Psi + (V(x) + E)\Psi - f(\Psi), \ \forall x \in I\!R^N, \qquad (NLS)$$

when $f(s) = |s|^{p-1}s$, $1 < p < \frac{N+2}{N-2}$. A standing wave of (NLS) is a solution of the form $\Psi(x,t) = \exp(-iEt/\hbar)u(x)$. In this case, u is a solution of (P_ε).

Existence and concentration of positive solutions for the problem (P_ε) have been extensively studied in recent years.

In [4], Floer and Weinstein studied (P_ε) in the case $N = 1, f(s) = s^3, \Omega = I\!R$ and potentials V with nondegenerate critical points. Based on Lyapunov - Schmidt reduction, they proved that, for small ε, this problem has a solution which concentrates around each nondegenerate critical point of V. Later, Oh in [8, 9] extended this result to higher dimensions, provided that $1 < p < \frac{N+2}{N-2}$, with potentials V belong to a Kato class.

In [10], Rabinowitz found a positive ground state solution (positive solution of minimal energy) for (P_ε) for small ε, when $f(s)$ behaves like s^p, $1 < p < \frac{N+2}{N-2}$ and V verifies the following global property

$$\liminf_{|x| \to \infty} V(x) > \inf_{x \in I\!R^N} V(x). \qquad (V_\infty)$$

This solution indeed concentrates around a global minimum of V, as $\varepsilon \to 0$, as shown later by Wang in [11]. Alves and Souto in [1] proved the same results obtained in [10] and [11] assuming that V satisfies (V_∞) and $f(s) = \lambda |s|^{q-1}s + |s|^{2^*-2}s$, where $2^* = 2N/(N-2)$.

In [3], del Pino and Felmer proved the local analogue of Wang's result. It is shown that if V satisfies (V_1) and (V_2), then (P_ε) has solution u_ε with just one local maximum, which concentrates around a minimum of V in Λ. This local minimum may exhibit arbitrary degeneracy. Alves, do Ó and Souto in [2] proved that the result obtained in [3] also holds when the nonlinearity has the critical form $f(s) = h(s) + |s|^{2^*-2}s$, with $h(s)$ behaves like s^p.

In [6, 7], Noussair and Wei showed that if $V \equiv 1$ and Ω is a bounded domain, problem (P_ε) has a nodal solution u_ε at the least energy level to (P_ε) and they proved that, for small ε, it possesses exactly one positive and one negative peaks, and that the peaks converge, as $\varepsilon \to 0$, to two points, whose locations depend on the geometry of Ω. In [7], it is considered the corresponding Neumann problem. They proved that nodal solutions always exist, and studied how the mean curvature function on $\partial\Omega$ affects the shape of such solutions.

In this paper, motivated by works [3] and [6], we study the existence and concentration behavior of nodal solutions for the problem (P_ε), when Ω is a domain in $I\!R^N$, not necessarily bounded, and V satisfies $(V_1)-(V_2)$. Our study may be seen as a complement for the above-mentioned results. More precisely, we prove the following theorem.

Theorem 1.1. *Suppose $f \in C^1(I\!R)$ is a odd function and satisfies $(f_1)-(f_4)$ and V satisfies $(V_1)-(V_2)$. Then, there exists $\varepsilon_0 > 0$ such that problem (P_ε) possesses a nodal solution $u_\varepsilon \in H_0^1(\Omega)$ for every $\varepsilon \in (0, \varepsilon_0)$. Moreover, u_ε possesses just one positive local maximum point P_ε^1 (hence global) and one negative local minimum point P_ε^2 (hence global), which are in Λ. We also have that P_ε^i concentrates around $x_0^i \in \Lambda$, where $V(x_0^i) = V_0$, i.e. $P_\varepsilon^i \to x_0^i$, as $\varepsilon \to 0$, for $i = 1, 2$ and*

$$|u_\varepsilon(x)| \leq M\left[\exp\left(-\beta\left|\frac{x-P_\varepsilon^1}{\varepsilon}\right|\right) + \exp\left(-\beta\left|\frac{x-P_\varepsilon^2}{\varepsilon}\right|\right)\right].$$

for certain positive constants β and M.

To prove Theorem 1.1, we first modify the nonlinearity into one more convenient in order to apply minimization arguments together implicit function theorem to obtain the existence. To show the phenomenon of concentration, we establish a upper estimate of the energy of the solution obtained and make a careful study of its profile obtaining a relation between its peaks points, which imply that these points are concentrated around local minima of V.

The paper is schemed as follows: In section 2, we introduce the penalized problem and by minimization we prove the existence of a nodal solution for it, while in sections 3 and 4, we establish a upper estimate for the energy of u_ε and preliminaries results, respectively. In section 5, we show the phenomenon of concentration and, in section 6, we complete the proof of Theorem 1.1.

2 Preliminaries

In this section, we fix some notations and prove the existence of nodal solution to (P_ε).

2.1 Notations

Throughout this paper, we denote by H the Hilbert space given by

$$H = \{u \in H_0^1(\Omega);\ \int_\Omega V(x)u^2\,dx < \infty\}$$

endowed with inner product

$$< u, v >_V = \int_\Omega (\nabla u \nabla v + V(x)uv)\,dx.$$

whose associated norm we denote by $||\cdot||$. We recall that the solutions of (P_ε) are critical points of functional $I_\varepsilon : H \longrightarrow I\!R$ given by

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega (\varepsilon^2|\nabla u|^2 + V(x)u^2)\,dx - \int_\Omega F(u)\,dx$$

where $F(u) = \int_0^u f(s)ds$. By (f_2), follows that I_ε is well defined and belongs to $C^1(H, I\!R)$.

To establish the existence of nodal solutions, we modify this functional adapting some arguments explored in [3]. Let θ be a number as given by (f_3), and let us choose $k > 0$ such that $k > \frac{\theta}{\theta - 2}$. Let $a > 0$ the value at which $\frac{f(a)}{a} = \frac{\alpha}{k}$, where α is as in (V_1). Let us set

$$\hat{f}(s) = \begin{cases} f(s) & \text{if} \quad |s| \leq a, \\ \frac{\alpha}{k} s & \text{if} \quad |s| > a. \end{cases}$$

Now, let $\delta > 0$ sufficiently small, and $\eta \in C^\infty([a-\delta, a+\delta])$ satisfying

1. $\eta(s) \leq \hat{f}(s)$ for all $s \in [a - \delta, a + \delta]$.
2. $\eta(a - \delta) = \hat{f}(a - \delta)$ and $\eta(a + \delta) = \frac{\alpha}{k}(a + \delta)$.
3. $\eta'(a - \delta) = f'(a - \delta)$ and $\eta'(a + \delta) = \frac{\alpha}{k}$.
4. $s \longmapsto \frac{\eta(s)}{s}$ is increasing in $[a - \delta, a + \delta]$.

Using the functions η and $\hat{f}$, we define the function

$$\tilde{f}(s) = \begin{cases} \hat{f}(s) & \text{if} \quad s \notin [a-\delta, a+\delta] \quad \text{and} \quad s \geq 0, \\ \eta(s) & \text{if} \quad s \in [a - \delta, a + \delta]. \end{cases}$$

Considering the odd extension of $\tilde{f}$, still denoted by $\tilde{f}$, let us define

$$g(x, s) = \chi_\Lambda(x) f(s) + (1 - \chi_\Lambda(x))\tilde{f}(s),$$

where Λ is a bounded domain as in the assumptions of Theorem 1.1 and χ_Λ denotes its characteristic function. It is easy to check that $(f_1) - (f_4)$ implies that g is a Carathéodory function, for fixed $x \in \Omega$, $s \longmapsto g(x, s)$ is of class C^1 in $I\!R$, and satisfies the following conditions:

(g_1) $g(x, s) = o(|s|)$ as $s \to 0$, uniformly in $x \in \Omega$.
(g_2) There exists $1 < p < \frac{N+2}{N-2}$ such that

$$\lim_{|s| \to +\infty} \frac{g(x, s)}{|s|^p} = 0.$$

(g_3) There exists a bounded subset Λ of Ω, $\text{int}(\Lambda) \neq \emptyset$, and $2 < \theta < p + 1$ such that

$$\begin{array}{lll} (i) & 0 < \theta G(x, s) \leq s g(x, s), & x \in \Lambda, s \neq 0 \\ (ii) & 0 < 2G(x, s) \leq s g(x, s) \leq \frac{1}{k} V(x) s^2, & x \in \Omega \setminus \Lambda, s \neq 0, \end{array}$$

where $G(x, s) = \int_0^s g(x, t)\, dt$.
(g_4) The function $s \longmapsto \frac{g(x,s)}{s}$ is nondecreasing for all $x \in \Omega$ a. e. and $s \neq 0$.

2.2 Existence of Ground State Nodal Solutions

Hereafter we use $(P_\varepsilon)_g$ to denote the following penalized problem

$$(P_\varepsilon)_g \qquad \begin{cases} -\varepsilon^2\Delta u + V(x)u = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Note that if u is a nodal solution of $(P_\varepsilon)_g$ with $|u(x)| \leq a-\delta$ for every $x \in \Omega\setminus\Lambda$, then u is also a nodal solution to (P_ε).

Associated with $(P_\varepsilon)_g$ is the energy functional $J_\varepsilon : H \longrightarrow I\!R$ defined by

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega (\varepsilon^2|\nabla u|^2 + V(x)u^2)\,dx - \int_\Omega G(x,u)\,dx.$$

Using the properties of function g and standard arguments, we have that $J_\varepsilon \in C^1(H, I\!R)$. A necessary condition for $u \in H$ to be a critical point of J_ε is that $J_\varepsilon'(u)u = 0$. This condition defines the *Nehari manifold*

$$\mathcal{N}_\varepsilon = \left\{u \in H; u \not\equiv 0, \int_\Omega (\varepsilon^2|\nabla u|^2 + V(x)u^2)dx = \int_\Omega g(x,u)u\,dx\right\}.$$

We note that there exists a positive constant $\nu = \nu(\varepsilon)$, which satisfies the following inequality

$$\int_\Lambda |u|^{p+1}\,dx > \nu, \ \forall\ u \in \mathcal{N}_\varepsilon. \tag{2.1}$$

In fact, since $u \in \mathcal{N}_\varepsilon$, from (g_3), we have the inequality

$$\int_\Omega (\varepsilon^2|\nabla u|^2 + V(x)u^2)\,dx \leq \int_\Lambda g(x,u)u\,dx + \int_\Omega \frac{1}{k}V(x)u^2\,dx.$$

Hence,

$$\int_\Omega (\varepsilon^2|\nabla u|^2 + (1-\frac{1}{k})V(x)u^2)\,dx \leq \int_\Lambda g(x,u)u\,dx.$$

Combining the definition of g with conditions (g_1) and (g_2), we get

$$\int_\Omega (\varepsilon|\nabla u|^2 + V(x)u^2)\,dx \leq \int_\Lambda (\delta u^2 + c_\delta|u|^{p+1})\,dx,$$

for some $0 < \delta < \frac{V_0}{2}$. Thus,

$$||u||^2 \leq \int_\Lambda 2c_\delta|u|^{p+1}\,dx.$$

By Sobolev embedding and above inequality, there exists $\nu_1 > 0$ such that

$$||u|| > \nu_1. \tag{2.2}$$

Consequently, from (2.2) there exists ν verifying (2.1).

Now, given $\mu \in (0, \nu/2)$, define the sets $\mathcal{X}_\varepsilon$ and $\mathcal{M}_\varepsilon$ by

$$\mathcal{X}_\varepsilon = \left\{ u \in H; \int_\Lambda (u^{\pm})^{p+1}\, dx \geq \mu \right\} \quad \text{and} \quad \mathcal{M}_\varepsilon = \mathcal{N}_\varepsilon \cap \mathcal{X}_\varepsilon.$$

We observe that if $u \in \mathcal{M}_\varepsilon$, then

$$J_\varepsilon(u) = \int_\Omega (\frac{1}{2} g(x,u)u - G(x,u))dx.$$

The last equality and (g_3) imply that

$$J_\varepsilon(u) \geq \frac{(\theta-2)}{2} \int_\Lambda G(x,u)\, dx \geq 0.$$

Thus, we can concluded that J_ε is bounded from below on $\mathcal{M}_\varepsilon$, which allows us to define

$$c_\varepsilon = \inf_{\mathcal{M}_\varepsilon} J_\varepsilon(u). \tag{2.3}$$

Theorem 2.1. *The level c_ε is achieved by some $u_\varepsilon \in \mathcal{M}_\varepsilon$. Moreover, u_ε is a weak solution and hence a classical nodal solution of $(P_\varepsilon)_g$.*

Proof. By Ekeland Variational Principal, there exists a minimizing sequence (u_n) for (2.3) satisfying

$$J_\varepsilon(u_n) \leq c_\varepsilon + \frac{1}{n} \tag{2.4}$$

and

$$J_\varepsilon(v) \geq J_\varepsilon(u_n) - \frac{1}{n}\|v-u_n\|, \ \forall v \in \mathcal{M}_\varepsilon. \tag{2.5}$$

Claim 2.1. The minimizing sequence (u_n) is bounded in H.
In fact, for each $u \in H$ we have,

$$J_\varepsilon(u) \geq \frac{1}{2}\int_\Omega (\varepsilon^2|\nabla u|^2 + V(x)u^2)\, dx - \int_\Omega \frac{1}{\theta} g(x,u)u\, dx - \int_\Omega \frac{1}{k} V(x)u^2\, dx.$$

Then, if $u \in \mathcal{M}_\varepsilon$, we obtain

$$J_\varepsilon(u) \geq \frac{(\theta-2)}{2\theta}\int_\Omega (\varepsilon^2|\nabla u|^2 + V(x)u^2)\, dx - \frac{1}{2k}\int_\Omega (\varepsilon^2|\nabla u|^2 + V(x)u^2)\, dx,$$

whence

$$J_\varepsilon(u) \geq \frac{1}{2}\left(\frac{\theta-2}{\theta} - \frac{1}{k}\right)\|u\|^2 \geq 0,$$

where the last inequality holds thanks to our choice of k and the proof of Claim 2.1 is over.

Here and in what follows, we denote by u_ε the weak limit of $\{u_n\}$ in H. To verify that c_ε is achieved, we will show that (u_n) is a Palais-Smale sequence. We start proving that the following limit holds

$$\int_\Omega (\varepsilon^2 \nabla u_n \nabla\varphi + V(x)u_n\varphi)\, dx - \int_\Omega g(x,u_n)\varphi\, dx \to 0, \text{ as } n \to \infty\ \forall \varphi \in H. \tag{2.6}$$

To prove (2.6), we use the techniques utilized in [6]. First, we note that there exists a subsequence of u_n, still denoted by u_n, satisfying one the four following possibilities:

i) $\int_\Lambda (u_n^{\pm})^{p+1}\,dx = \mu$;

ii) $\int_\Lambda (u_n^{\pm})^{p+1}\,dx > \mu$;

iii) $\int_\Lambda (u_n^{+})^{p+1}\,dx = \mu$ and $\int_\Lambda (u_n^{-})^{p+1}\,dx > \mu$;

iv) $\int_\Lambda (u_n^{+})^{p+1}\,dx > \mu$ and $\int_\Lambda (u_n^{-})^{p+1}\,dx = \mu$.

It is easy to see that (i) cannot occur because otherwise

$$\int_\Lambda |u|^{p+1}\,dx \leq 2\mu < \nu$$

and this contradicts (2.1).

Now, if (ii) occurs, for each n, we consider the function $h_n : I\!R^2 \longrightarrow I\!R$ given by

$$\begin{aligned} h_n(t,s) = {} & \int_\Omega (\varepsilon^2 |\nabla(u_n + t\varphi + su_n)|^2 + V(x)|u_n + t\varphi + su_n|^2)\,dx - \\ & - \int_\Omega g(x, u_n + t\varphi + su_n)(u_n + t\varphi + su_n)\,dx. \end{aligned}$$

From the definition of the function g, we have that h_n are of C^1 class, $h_n(0,0) = 0$ and

$$\frac{\partial h_n}{\partial s}(0,0) = \int_\Omega u_n(g(x,u_n) - g'(x,u_n)u_n)\,dx.$$

Denoting $H_n^{\pm}(x) = u_n^{\pm}(g(x,u_n^{\pm}) - g'(x,u_n^{\pm})u_n^{\pm})$, we have

$$\frac{\partial h_n}{\partial s}(0,0) = \int_\Omega (H_n^{+}(x) + H_n^{-}(x))\,dx.$$

From the definition of the function g, $H_n^{\pm}(x) \leq 0$ and

$$\frac{\partial h_n}{\partial s}(0,0) \leq \int_{\{x\in\Omega;\, u_n^{+}(x)\leq a-\delta\}} H_n^{+}(x)\,dx + \int_{\{x\in\Omega;\, u_n^{-}(x)\leq a-\delta\}} H_n^{-}(x)\,dx.$$

Now, since $u \in H_0^1(\Omega), u \not\equiv 0$ and $u \geq 0$, then $|\{x \in \Omega; u^{\pm}(x) \leq a - \delta\}| \neq 0$. Thus, by (f_4),

$$\frac{\partial h_n}{\partial s}(0,0) < 0.$$

Therefore, by the implicit function theorem, there is a function s_n defined and of C^1 class on some interval $(-\delta_n, \delta_n)$, $\delta_n > 0$, such that $s_n(0) = 0$ and $h_n(t, s_n(t)) = 0, t \in (-\delta_n, \delta_n)$. Thus, for every $t \in (-\delta_n, \delta_n)$, we have $u_n + t\varphi + s_n(t)u_n \in \mathcal{N}_\varepsilon$. Moreover, from the continuity of s_n, we can assume that $u_n + t\varphi + s_n(t)u_n \in \mathcal{M}_\varepsilon$.

Claim 2.2. The sequence $\{s_n'(0)\}$ is bounded.
In fact,

$$s_n'(0) = \frac{-\frac{\partial h_n}{\partial t}(0,0)}{\frac{\partial h_n}{\partial s_n}(0,0)} = \frac{-2\int_\Omega \left(\varepsilon^2 \nabla u_n \nabla \varphi + V(x) u_n^2 \varphi + (g'(x,u_n)u_n + g(u_n))\varphi\right) dx}{\int_\Omega u_n (g(x,u_n) - u_n g'(x,u_n))\, dx}$$

Since u_ε is the weak limit of (u_n) in H, using Sobolev embedding and the fact that $u_n \in \mathcal{X}_\varepsilon$, we have

$$\int_\Lambda (u_\varepsilon^{\pm})^{p+1}\, dx \geq \mu,$$

so $u_\varepsilon \not\equiv 0$ in Λ. Now, as

$$\left| \int_\Omega u_n (g(x,u_n) - u_n g'(x,u_n))\, dx \right| = \int_\Omega (g'(x,u_n) u_n^2 - u_n g(x,u_n))\, dx,$$

from Fatou's lemma, we have

$$\begin{aligned} \liminf_{n\to\infty} \int_\Omega (g'(x,u_n)u_n^2 - u_n g(x,u_n))\, dx \geq \\ \int_\Omega (g'(x,u_\varepsilon)u_\varepsilon^2 - u_\varepsilon g(x,u_\varepsilon))\, dx \equiv C_1 > 0. \end{aligned}$$

Therefore,

$$\frac{1}{\left|\int_\Omega u_n (g(x,u_n) - u_n g'(x,u_n))\, dx\right|} \leq C, \tag{2.7}$$

for some positive constant C. Thus, from (2.7),

$$|s_n'(0)| \leq \frac{1}{C} \left| 2 \int_\Omega (\varepsilon^2 \nabla u_n \nabla \varphi + (V(x) u_n - g'(x,u_n)u_n + g(x,u_n))\varphi\, dx \right|. \tag{2.8}$$

Using the fact that (u_n) is bounded, it follows that $|s_n'(0)| \leq C$, for some positive constant C, independent of n. Now, from (2.5), we have

$$J_\varepsilon(u_n + t\varphi + s_n(t)u_n) - J_\varepsilon(u_n) \geq \frac{-1}{n} \|t\varphi + s_n(t)u_n\|. \tag{2.9}$$

From (2.8) and (2.9), we have

$$\lim_{n\to\infty} \frac{d}{dt} J_\varepsilon(u_n + t\varphi + s_n(t)u_n)\Big|_{t=0} = 0,$$

hence

$$\begin{aligned} 0 &= \lim_{n\to\infty} \Big[\int_\Omega (\varepsilon^2 \nabla u_n \nabla \varphi + V(x) u_n \varphi)\, dx - \int_\Omega g(x,u_n)\varphi\, dx \\ &+ s'_n(0) \Big(\int_\Omega (\varepsilon^2 |\nabla u_n|^2 + V(x) u_n^2)\, dx - \int_\Omega g(x,u_n) u_n\, dx \Big) \Big] \\ &= \lim_{n\to\infty} \left[\int_\Omega (\varepsilon^2 \nabla u_n \nabla \varphi + V(x) u_n \varphi)\, dx - \int_\Omega g(x,u_n)\varphi\, dx \right], \end{aligned}$$

which verifies (2.6) and implies

$$\int_\Omega (\varepsilon^2 \nabla u_\varepsilon \nabla \varphi + V(x) u_\varepsilon \varphi)\, dx - \int_\Omega g(x, u_\varepsilon)\varphi\, dx = 0 \ \forall \varphi \in H.$$

Recalling that $u_\varepsilon^{\pm} \neq 0$, it follows that u_ε is a nodal solution of $(P_\varepsilon)_g$. To verify that c_ε is achieved, we will show that (u_n) is a Palais-Smale sequence. Effectively, by (2.5), it suffices to show that $J_\varepsilon^{'}(u_n) \to 0$. Given $\varphi \in H$, $||\varphi|| \leq 1$, from (2.5), we have

$$\frac{J_\varepsilon(u_n + t\varphi + s_n(t)u_n) - J_\varepsilon(u_n)}{t} \geq -\frac{1}{n}||\varphi|| - ||\frac{s_n(t)}{t} u_n||.$$

Thus, taking $t \to 0$ above, we obtain

$$J_\varepsilon'(u_n)\varphi \geq -\frac{1}{n} - \frac{1}{n}||s_n'(0)u_n||.$$

Replacing φ by $-\varphi$ in the last inequality, we find

$$|J_\varepsilon^{'}(u_n)\varphi| \leq \frac{1}{n} + \frac{1}{n}||s_n^{'}(0)u_n||. \tag{2.10}$$

Then taking the limit as $n \to 0$ in (2.10), it follows that $||J_\varepsilon^{'}(u_n)|| \to 0$. Using the information above, we have

$$\begin{cases} J_\varepsilon(u_n) \to c_\varepsilon & \text{in } I\!R, \\ J_\varepsilon^{'}(u_n) \to 0 & \text{in } H^{-1}. \end{cases}$$

In order to conclude our analysis, suppose that (iii) holds (case (iv) is analogous). In this case we consider the function $h_n : I\!R^3 \longrightarrow I\!R^2$ given by

$$h_n(t,s,w) = \left(J_\varepsilon^{'}(\psi_n(t,s,w))\psi_n(t,s,w), \int_\Lambda |\psi_n(t,s,w)|^{p+1}\, dx\right)$$

where $\psi_n(t,s,w) = u_n + t\varphi + su_n^- + wu_n^+$. From definition of g, it is clear that h_n is of C^1 class, $h(0,0,0) = (0,\mu)$, and the Jacobian of h_n at $(0,0,0)$ is

$$\begin{bmatrix} A_n & B_n & C_n \\ (p+1)\int_\Lambda (u_n^+)^p \varphi\, dx & 0 & (p+1)\mu \end{bmatrix}$$

where

$$A_n = 2\int_\Omega (\varepsilon^2 \nabla u_n \nabla \varphi + V(x) u_n \psi)\, dx - \int_\Omega (g'(x,u_n)u_n + g(x,u_n)\varphi)dx,$$

$$B_n = 2\int_\Omega (\varepsilon^2 |\nabla u_n^-|^2 + V(x)|u_n^-|^2)\, dx - \int_\Omega (g'(x,u_n^-){u_n^-}^2 + g(x,u_n^-)u_n^-)\, dx,$$

$$C_n = 2\int_\Omega (\varepsilon^2 |\nabla u_n^+|^2 + V(x)|u_n^+|^2)\, dx - \int_\Omega (g'(x,u_n^+){u_n^+}^2 + g'(x,u_n^+)u_n^+)\, dx.$$

Now, we prove B_n does not converge to zero. In fact, since $u_n \in \mathcal{N}_\varepsilon$

$$-J'_\varepsilon(u_n^+)u_n^+ = J'_\varepsilon(u_n^-)u_n^- \doteq T_n$$

we claim that the sequence T_n does not converge to zero. Supposing $T_n \to 0$ we can prove that there is $t_n^{\pm} \to 1$ such that

$$t_n^+ u_n^+,\ t_n^- u_n^- \in \mathcal{N}_\varepsilon.$$

Consequently,

$$(t_n^+)^{p+1} \int_\Lambda (u_n^+)^{p+1}\, dx > \nu.$$

On the other hand,

$$\int_\Lambda (u_n^+)^{p+1}\, dx = (t_n^+)^{p+1} \int_\Lambda (u_n^+)^{p+1}\, dx + [1-(t_n^+)^{p+1}] \int_\Lambda (u_n^+)^{p+1}\, dx.$$

Thus

$$\liminf_{n\to\infty} \int_\Lambda (u_n^+)^{p+1}\, dx \geq \nu,$$

which implies that, given $\delta > 0$ small, there exists $n_0 \in \mathbf{N}$ such that

$$\int_\Lambda (u_n^+)^{p+1}\, dx > \nu - \delta > \mu \quad \forall n \geq n_0$$

contradicting (iii). Hence, T_n does not converge to zero.

We observe that we must have $t_n^+ > 1$. Otherwise

$$\int_\Lambda (u_n^+)^{p+1}\, dx \geq (t_n^+)^{p+1} \int_\Lambda (u_n^+)^{p+1}\, dx \geq \nu > \mu$$

which is contrary to (iii). Using that $t_n^+ > 1$, from $(g_1) - (g_2)$, we have

$$0 < J'_\varepsilon(u_n^+)u_n^+ = -T_n.$$

Thus, we may assume that $T_n \to T < 0$. Therefore we can conclude that B_n does not converge to zero. If not,

$$\int_\Omega (g'(x,u_n^-){u_n^-}^2 + g(x,u_n^-)u_n^-)\, dx = T_n - B_n \to T < 0,$$

but, from (g_4)

$$g'(x,u_n^-){u_n^-}^2 + g(x,u_n^-)u_n^- \geq 0 \ \ \forall n$$

obtaining a contradiction. Thus, B_n does not converge to zero.

From implicit function theorem, there are functions s_n and w_n defined and of C^1 class on some interval $(-\delta_n, \delta_n)$, such that $s_n(0) = w_n(0) = 0$ and $h_n(t, s_n(t), w_n(t)) = (0, \mu)$, for every $t \in (-\delta_n, \delta_n)$, that is

$$u_n + t_n\varphi + s_n(t)u_n^- + w_n(t)u_n^+ \in \mathcal{N}_\varepsilon \ \ \forall\, t \in (-\delta_n, \delta_n),$$

$$\textstyle\int_\Lambda [(u_n + t_n\varphi + s_n(t)u_n^- + w_n(t)u_n^+)^+]^{p+1}\, dx = \mu \ \ \forall\, t \in (-\delta_n, \delta_n).$$

Moreover, from the continuity of s_n and w_n, we can assume that

$$\int_{\Lambda}[(u_n + t_n\varphi + s_n(t)u_n^- + w_n(t)u_n^+)^-]^{p+1}\,dx > \mu \quad \forall\, t \in (-\delta_n, \delta_n).$$

Therefore, $u_n + t_n\varphi + s_n(t)u_n^- + w_n(t)u_n^+ \in \mathcal{M}_\varepsilon \quad \forall\, t \in (-\delta_n, \delta_n)$.

Since

$$s_n'(0) = \tfrac{A_n}{B_n} - \frac{C_n \int_\Lambda (u_n^+)^{p+1}\varphi\,dx}{(p+1)\mu B_n}$$

$$w_n'(0) = \tfrac{1}{\mu}\int_\Lambda (u_n^+)^{p+1}\varphi\,dx$$

and using that B_n does not converge to zero, we conclude that the sequences $(s_n'(0))$ and $(w_n'(0))$ are bounded. As in the proof of case (ii), we have the same conclusion, namely

$$\begin{cases} J_\varepsilon(u_n) \to c_\varepsilon & \text{in } \mathbb{R}, \\ J_\varepsilon'(u_n) \to 0 & \text{in } H^{-1}. \end{cases}$$

Now, from Lemma 1.1 in [3], the functional J_ε satisfies the Palais-Smale condition on level c_ε $[(PS)_{c_\varepsilon}]$. Consequently, we may suppose that (u_n) has a subsequence, still denoted by (u_n), with $u_n \to u_\varepsilon$, strongly in H. As $J_\varepsilon(u_n) \to c_\varepsilon$, we conclude that $J_\varepsilon(u_\varepsilon) = c_\varepsilon$. Finally, by the elliptic regularity arguments, u_ε is a classical nodal solution of $(P_\varepsilon)_g$. ■

3 Estimates of the Energy

We devote this section to establish some properties about the energy of the minimizing function u_ε obtained in last section with relation to functional J_ε. Let $w \in H^1(\mathbb{R}^N)$ be a positive least energy solution to

$$-\Delta w + V_0 w = f(w), \tag{3.1}$$

with $w(0) = \max\limits_{x\in\mathbb{R}^N} w(x)$. That is, w satisfies

$$c_{V_0} = I_{V_0}(w) = \inf_{\substack{v \,\in\, H^1(\mathbb{R}^N) \\ v \,\neq\, 0}} \sup_{\tau \geq 0} I_{V_0}(\tau v), \tag{3.2}$$

where I_{V_0} is defined as

$$I_{V_0}(v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + V_0 v^2)\,dx - \int_{\mathbb{R}^N} F(v)\,dx. \tag{3.3}$$

Lemma 3.1. *Given $\varepsilon > 0$, the function u_ε satisfies*

$$\limsup_{\varepsilon\to 0}(\varepsilon^{-N} J_\varepsilon(u_\varepsilon)) \leq 2c_{V_0}.$$

Proof. Consider $z_0 \in \mathrm{int}\Lambda$ with $V(z_0) = V_0$. Let $r > 0$ be such that $B_r(z_0) \subset \mathrm{int}\Lambda$, and set $Q_1, Q_2 \in \partial B_r(z_0)$, with $|Q_1 - Q_2| = 2r$. If necessary, take r small enough such that $B_{\frac{r}{4}}(Q_i) \subset \Lambda$, for $i = 1, 2$. Taking $\psi \in C^\infty(I\!R^N)$, $|\nabla\psi| \leq C$, $\psi(x) = 1$ if $|x| \leq \frac{r}{4}$, and $\psi(x) = 0$ if $|x| \geq \frac{r}{2}$, define

$$\psi_i(x) = \psi(x - Q_i),\ i = 1,\ 2.$$

Thus, $\mathrm{supp}(\psi_1) \cap \mathrm{supp}(\psi_2) = \emptyset$ and $\mathrm{supp}(\psi_i) \subset B_{\frac{r}{2}}(Q_i)$, for $i = 1,\ 2$. Let w_i, $i = 1,\ 2$, be a positive least energy solution to

$$-\Delta w + V(Q_i)w = f(w), \tag{3.4}$$

with $w_i(0) = \max\limits_{x \in I\!R^N} w_i(x)$. That is, w_i satisfies

$$c_{V(Q_i)} = I_{V(Q_i)}(w_i) = \inf_{\substack{v \in H^1(I\!R^N) \\ v \neq 0}} \sup_{\tau \geq 0} I_{V(Q_i)}(\tau v), \tag{3.5}$$

where $I_{V(Q_i)}$ is defined as

$$I_{V(Q_i)}(v) = \frac{1}{2}\int_{I\!R^N} (|\nabla v|^2 + V(Q_i)v^2)\,dx - \int_{I\!R^N} F(v)\,dx. \tag{3.6}$$

Define

$$w_{\varepsilon,Q_i}(x) = \psi_i(x) w_i(\frac{x - Q_i}{\varepsilon}),\ i = 1,\ 2.$$

Now, from $(g_1) - (g_3)$, for any $\varepsilon > 0$ there exist $t_{i_\varepsilon} > 0$, $i = 1,\ 2$, such that

$$J_\varepsilon(t_{i_\varepsilon} w_{\varepsilon,Q_i}) = \max_{t>0} J_\varepsilon(t w_{\varepsilon,Q_i}),\ i = 1,\ 2.$$

Since $\frac{d}{dt} J_\varepsilon(t w_{\varepsilon Q_i}) \mid_{t = t_{i_\varepsilon}} = 0$, we have that

$$\int_\Omega (\varepsilon^2 \nabla(t_{i_\varepsilon} w_{\varepsilon,Q_i})\nabla w_{\varepsilon,Q_i} + V(x) t_{i_\varepsilon}(w_{\varepsilon,Q_i})^2)\,dx - \int_\Omega g(x, t_{i_\varepsilon} w_{\varepsilon,Q_i}) w_{\varepsilon,Q_i}\,dx = 0.$$

Thus, after to multiply the equality above by t_{i_ε}, we have

$$J_\varepsilon'(t_{i_\varepsilon} w_{\varepsilon,Q_i}) t_{i_\varepsilon} w_{\varepsilon,Q_i} = 0,\ i = 1,\ 2.$$

Now, considering the function

$$w_{\varepsilon,Q_1,Q_2} = t_{1_\varepsilon} w_{\varepsilon,Q_1} - t_{2_\varepsilon} w_{\varepsilon,Q_2},$$

as $\mathrm{supp}(w^+_{\varepsilon,Q_1,Q_2}) \cap \mathrm{supp}(w^-_{\varepsilon,Q_1,Q_2}) = \emptyset$, we have

$$w^+_{\varepsilon,Q_1,Q_2} = t_{1_\varepsilon} w_{\varepsilon,Q_1}\ ,\ w^-_{\varepsilon,Q_1,Q_2} = t_{2_\varepsilon} w_{\varepsilon,Q_2}$$

and

$$J_\varepsilon(w_{\varepsilon,Q_1,Q_2}) = J_\varepsilon(w^+_{\varepsilon,Q_1,Q_2}) + J_\varepsilon(w^-_{\varepsilon,Q_1,Q_2}).$$

Moreover

$$J_\varepsilon'(w^+_{\varepsilon,Q_1,Q_2}) w^+_{\varepsilon,Q_1,Q_2} = 0,\quad J_\varepsilon'(w^-_{\varepsilon,Q_1,Q_2}) w^-_{\varepsilon,Q_1,Q_2} = 0,$$

which implies $w_{\varepsilon,Q_1,Q_2} \in \mathcal{M}_\varepsilon$, thus

$$c_\varepsilon \le J_\varepsilon(w_{\varepsilon,Q_1,Q_2}) = J_\varepsilon(w^+_{\varepsilon,Q_1,Q_2}) + J_\varepsilon(w^-_{\varepsilon,Q_1,Q_2}).$$

Now, with a direct computation, we have

$$J_\varepsilon(w^+_{\varepsilon,Q_1,Q_2}) \le \sup_{t>0} J_\varepsilon(tu) = \varepsilon^N\{c_{V(Q_1)} + o(1)\}$$

$$J_\varepsilon(w^-_{\varepsilon,Q_1,Q_2}) \le \sup_{t>0} J_\varepsilon(tu) = \varepsilon^N\{c_{V(Q_2)} + o(1)\}$$

with $o(1) \to 0$ as $\varepsilon \to 0$. Hence,

$$\limsup_{\varepsilon\to 0} \varepsilon^{-N} J_\varepsilon(u_\varepsilon) \le c_{V(Q_1)} + c_{V(Q_2)}.$$

Finally, taking $r \to 0$ in the last inequality and using the continuity of the minimax function $\mu \longmapsto c_\mu$ (see Theorem 3.21 in [10] or [1]), we have

$$\limsup_{\varepsilon\to 0} \varepsilon^{-N} J_\varepsilon(u_\varepsilon) \le 2c_{V_0}. \quad \blacksquare$$

4 Preliminary Analysis of u_ε

In this section, we make a careful study of the profile of u_ε obtained in Section 2, with purpose to show the behavior of peaks points of u_ε.

Lemma 4.1. *The points where hold positive local maximum and negative local minimum of u_ε are in Λ.*

Proof. Let x_ε be a positive local maximum of u_ε. Suppose by contradiction that $x_\varepsilon \in \Omega \setminus \Lambda$. Since $\Delta u_\varepsilon(x_\varepsilon) \le 0$, using the definition of g, we have

$$\alpha u_\varepsilon(x_\varepsilon) \le -\varepsilon^2 \Delta u_\varepsilon(x_\varepsilon) + V(x_\varepsilon)u_\varepsilon(x_\varepsilon) = \tilde{f}(u_\varepsilon(x_\varepsilon)) \le \frac{\alpha}{k} u_\varepsilon(x_\varepsilon),$$

but is impossible thanks to our choice of k. A similar argument implies that every negative local minimum of u_ε belong to Λ. $\blacksquare$

The next lemma is a crucial step to get the concentration. A version of this result can be found in [6].

Lemma 4.2. *Let $a > 0$ the number chosen in the definition of g, P^1_ε a local maximum of u^+_ε and P^2_ε a local minimum of u^-_ε. Then,*

1. $u_\varepsilon(P^1_\varepsilon) \ge a$ *and* $u_\varepsilon(P^2_\varepsilon) \le -a$.
2. $\left|\dfrac{P^1_\varepsilon - P^2_\varepsilon}{\varepsilon}\right| \to +\infty$, *as* $\varepsilon \to 0$.

Proof. Since $P^1_\varepsilon, P^2_\varepsilon \in \Lambda$, from definition of g, we have

$$\varepsilon^2 \Delta u_\varepsilon(P^i_\varepsilon) = \Big(V(P^i_\varepsilon) - \frac{f(u_\varepsilon(P^i_\varepsilon))}{u_\varepsilon(P^i_\varepsilon)}\Big) u_\varepsilon(P^i_\varepsilon).$$

Thus, as $\varepsilon^2\Delta u_\varepsilon(P_\varepsilon^1) \leq 0$, $\varepsilon^2\Delta u_\varepsilon(P_\varepsilon^2) \geq 0$, $u_\varepsilon(P_\varepsilon^1) > 0$ and $u_\varepsilon(P_\varepsilon^2) < 0$, it follows that

$$V(P_\varepsilon^i) - \frac{f(u_\varepsilon(P_\varepsilon^i))}{u_\varepsilon(P_\varepsilon^i)} \leq 0, \text{ for } i = 1, 2$$

which together (V_1) imply

$$\frac{f(u_\varepsilon(P_\varepsilon^i))}{u_\varepsilon(P_\varepsilon^i)} \geq \alpha > 0 \text{ for } i = 1, 2.$$

Consequently, if $|u_\varepsilon(P_\varepsilon^i)| \leq a$, for $i = 1, 2$ we obtain

$$\frac{\alpha}{k} = \frac{\frac{\alpha}{k}u_\varepsilon(P_\varepsilon^i)}{u_\varepsilon(P_\varepsilon^i)} \geq \frac{f(u_\varepsilon(P_\varepsilon^i))}{u_\varepsilon(P_\varepsilon^i)} \geq \alpha > 0,$$

which is a contradiction. Therefore, $|u_\varepsilon(P_\varepsilon^i)| \geq a$, for $i = 1, 2$. Item 1 is proved.

To prove item 2, we argue by contradiction assuming that $\left|\frac{P_\varepsilon^1 - P_\varepsilon^2}{\varepsilon}\right| \to \beta < +\infty$. We start our arguments considering the function

$$v_\varepsilon(y) = u_\varepsilon(\varepsilon y + P_\varepsilon^1).$$

Note that v_ε satisfies the problem

$$(P_\varepsilon') \qquad \begin{cases} -\Delta v_\varepsilon + V(\varepsilon y + P_\varepsilon^1)v_\varepsilon = g(\varepsilon y + P_\varepsilon^1, v_\varepsilon) & \text{in } \Omega_\varepsilon \\ v_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\{\Omega - P_\varepsilon^1\}$. Using the properties of function g, it is easy to show that v_ε is a critical point of the functional

$$J(v) = \frac{1}{2}\int_{\Omega_\varepsilon} (|\nabla v|^2 + V(\varepsilon y + P_\varepsilon^1)v^2)\,dx - \int_{\Omega_\varepsilon} G(\varepsilon y + P_\varepsilon^1, v)\,dx$$

and, by a simple change variable, $J(v_\varepsilon) = \varepsilon^{-N}J_\varepsilon(u_\varepsilon)$. Consequently, by upper estimate of $J_\varepsilon(u_\varepsilon)$, we have for each $\delta > 0$, $J(v_\varepsilon) \leq 2c_{V_0} + \delta$ for every ε sufficiently small. Using the last estimate, the fact that $J_\varepsilon'(v_\varepsilon)(v_\varepsilon) = 0$ and (g_3), by straightforward computations we can proved that v_ε is bounded in $H^1(I\!R^N)$. Thus, from elliptic estimates, we conclude that v_ε converges uniformly in C^2 sense on compact subsets of $I\!R^N$ to a function $v \in H^1(I\!R^N)$. Now, the sequence of functions $\chi_\varepsilon(y) \equiv \chi_\Lambda(\varepsilon y + P_\varepsilon^1)$ can be assumed to converge weakly in any L^p on compacts to a function $0 \leq \chi \leq 1$. Therefore, v satisfies the limiting problem

$$-\Delta v + V(\bar{z})v = \bar{g}(z, v) \text{ in } I\!R^N, \qquad (PL)$$

where

$$\bar{g}(z,s) = \chi(z)f(s) + (1 - \chi(z))\tilde{f}(s) \quad \text{and} \quad \bar{z} = \lim_{\varepsilon\to 0} P_\varepsilon^1.$$

Suppose that $\beta = 0$. By item 1 and mean value inequality, we have

$$2a \leq |v_\varepsilon(0) - v_\varepsilon(Z_\varepsilon)| \leq |\nabla v_\varepsilon(Q_\varepsilon)||Z_\varepsilon|,$$

where $Z_\varepsilon = (P^1_\varepsilon - P^2_\varepsilon)/\varepsilon$ and $Q_\varepsilon = t_\varepsilon Z_\varepsilon$ for some $t_\varepsilon \in [0,1]$. Using the hypothesis that $\beta = 0$, from convergence on compact sets, it follows that $|\nabla v_\varepsilon(Q_\varepsilon)|$ is bounded in $B_1(0)$ for small ε, consequently the last inequality does not hold. Therefore $\beta > 0$ and $P = \lim\limits_{\varepsilon\to 0} Z_\varepsilon \neq 0$. To conclude the proof, for each sequence (ε_n), with $\varepsilon_n \to 0$, taking a subsequence if necessary, we have $P = \lim\limits_{n\to\infty} Z_{\varepsilon_n}$ and $v_n \equiv v_{\varepsilon_n}$ converges in the C^2 sense on compacts to a function $v \in H^1(I\!R^N)$. As $v_n(0) = u_{\varepsilon_n}(P^1_{\varepsilon_n}) \geq a$ and $v_n(Z_{\varepsilon_n}) = u_{\varepsilon_n}(P^2_{\varepsilon_n}) \leq -a < 0$, it follows that $v(0) > 0$ and $v(P) < 0$, i.e., v is a nodal function. On the other hand, since the function v_n satisfies $(P^{'}_\varepsilon)$, with $\varepsilon = \varepsilon_n$, we conclude that v satisfies the limiting problem (PL). Associated to the equation (PL) we have the functional $\bar{J} : H^1(I\!R^N) \longrightarrow I\!R$ defined by

$$\bar{J}(u) = \frac{1}{2}\int_{I\!R^N} (|\nabla u|^2 + V(\bar{z})u^2)\,dx - \int_{I\!R^N} \bar{G}(z,u)\,dx, \;\; u \in H^1(I\!R^N)$$

where $\bar{G}(z,s) = \int_0^s \bar{g}(z,t)\,dt$ and $\overline{z} = \lim\limits_{n\to\infty} P^1_{\varepsilon_n}$. Then, v is a critical point of $\bar{J}$. Thus, $\bar{J}^{'}(v)\varphi = 0$ for every $\varphi \in H^1(I\!R^N)$. In particular, $\bar{J}^{'}(v)v^{\overset{+}{-}} = 0$, which implies $v \in M_{V(\bar{z})}$, where

$$M_{V(\bar{z})} = \{u \in H; u^{\overset{+}{-}} \not\equiv 0, \int_\Omega (|\nabla u^{\overset{+}{-}}|^2 + V(\bar{z})|u^{\overset{+}{-}}|^2)\,dx = \int_\Omega \bar{g}(z,u^{\overset{+}{-}})u^{\overset{+}{-}}\,dx\}.$$

We also set $J_n(u) \equiv J_{\varepsilon_n}(u)$, then

$$J_n(v_n) = J_{\varepsilon_n}(v_n) = \varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}).$$

Thus, from upper estimate of $J_{\varepsilon_n}(u_{\varepsilon_n})$ and the fact that v_n is solution of $(P^{'}_\varepsilon)$, with $\varepsilon = \varepsilon_n$, we have

$$\begin{aligned}
2c_{V_0} &\geq \limsup_{n\to\infty} \varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) = \limsup_{n\to\infty} J_n(v_n) \\
&= \limsup_{n\to\infty}\{J_n(v_n) - \frac{1}{2}J^{'}_n(v_n)v_n\} \\
&= \limsup_{n\to\infty} \int_{\Omega_n} (\frac{1}{2}g(P^1_{\varepsilon_n} + \varepsilon_n y, v_n)v_n - G(P^1_{\varepsilon_n} + \varepsilon_n y, v_n))\,dx \\
&= \limsup_{n\to\infty}\Big[\int_{\Omega_n} (\frac{1}{2}g(P^1_{\varepsilon_n} + \varepsilon_n y, v_n^+)v_n^+ - G(P^1_{\varepsilon_n} + \varepsilon_n y, v_n^+))\,dx \\
&+ \int_{\Omega_n} (\frac{1}{2}g(P^1_{\varepsilon_n} + \varepsilon_n y, v_n^-)v_n^- - G(P^1_{\varepsilon_n} + \varepsilon_n y, v_n^-))\,dx\Big].
\end{aligned}$$

Thus, by Fatou's lemma, we have

$$\begin{aligned}
2c_{V_0} &\geq \textstyle\int_{I\!R^N} (\frac{1}{2}\bar{g}(z,v^+)v^+ - \bar{G}(z,v^+))\,dx + \int_{I\!R^N} (\frac{1}{2}\bar{g}(z,v^-)v^- - \bar{G}(z,v^-))\,dx \\
&= [\bar{J}(v^+) - \tfrac{1}{2}\bar{J}^{'}(v^+)v^+] + [\bar{J}(v^-) - \tfrac{1}{2}\bar{J}^{'}(v^-)v^-] = \bar{J}(v^+) + \bar{J}(v^-)
\end{aligned} \tag{4.1}$$

where the last equality holds because $v \in M_{V(\bar{z})}$. Then

$$\begin{aligned} 2c_{V_0} &\geq \bar{J}(v^+) + \bar{J}(v^-) = \max_{\tau\geq 0} \bar{J}(\tau v^+) + \max_{\tau \geq 0} \bar{J}(\tau v^-) \\ &\geq \inf_{u\neq 0} \max_{\tau\geq 0} \bar{J}(\tau u) + \inf_{u\neq 0} \max_{\tau \geq 0} \bar{J}(\tau u) = 2c_{V(\bar{z})}. \end{aligned}$$

Since $V(\bar{z}) \geq V_0$ and the function minimax $\mu \longmapsto c_\mu$ is increasing and continuous function (see [10] or [1]), then

$$2c_{V_0} \geq 2c_{V(\bar{z})} \geq 2c_{V_0}.$$

Thus, $c_{V_0} = c_{V(\bar{z})}$ and we can conclude that $V(\bar{z}) = V_0$.

Claim 4.1. The functions $v^{\overset{+}{-}}$ satisfy $\bar{J}(v^{\overset{+}{-}}) > c_{V_0}$.
In fact, we know $\bar{J}(v^+) \geq c_{V(\bar{z})} = c_{V_0}$. If $\bar{J}(v^+) = c_{V_0}$, by Theorem 4.3 in [12], we have v^+ is solution of the following equation

$$-\Delta u + V_0 u = \bar{g}(x, u^+) \text{ in } \mathbb{R}^N.$$

Then, by the maximum principle, v^+ is positive in $\mathbb{R}^N$. But this is in obvious contradiction, because $v^- \not\equiv 0$. A similar argument implies $\bar{J}(v^-) > c_{V_0}$. Hence, using (4.1) and Claim 4.1 , we have

$$2c_{V_0} \geq \bar{J}(v^+) + \bar{J}(v^-) > 2c_{V_0},$$

which is a contradiction. Therefore, $\left|\frac{P_\varepsilon^1 - P_\varepsilon^2}{\varepsilon}\right| \to \infty$ as $\varepsilon \to \infty$. ■

5 Concentration of u_ε

We begin this section establishing a important proposition to verify that u_ε actually solves (P_ε), when ε is sufficiently small, and satisfies the properties announced in Theorem 1.1.

Proposition 5.1. *If $\varepsilon_n \downarrow 0$ and $z_n^i \in \bar{\Lambda}$, $i = 1, 2$, are such that*

$$u_{\varepsilon_n}(z_n^1) \geq b > 0 \quad \textit{and} \quad u_{\varepsilon_n}(z_n^2) \leq -b < 0$$

then

$$\lim_{n\to\infty} V(z_n^i) = V_0, \; i = 1, 2.$$

Proof. In fact, we argue by contradiction. Assume, passing to a subsequence if necessary, that $z_n^i \to \bar{z}_i \in \bar{\Lambda}, i = 1, 2$ with

$$V(\bar{z}_1) > V_0 \quad \text{and} \quad V(\bar{z}_2) \geq V_0.$$

As in the last section, we consider the sequences

$$v_n^i(z) = u_{\varepsilon_n}(z_n^i + \varepsilon_n z), \; i = 1, 2$$

and study theirs behavior as n goes to infinite. For $i = 1, 2$, the function v_n^i satisfies the problem

$$(P_n)_i \qquad \begin{cases} -\Delta v_n^i + V(\varepsilon_n z + z_n^i)v_n^i = g(\varepsilon z + z_n^i, v_n^i) & \text{in } \Omega_n^i \\ v_n^i = 0 & \text{on } \partial\Omega_n^i \end{cases}$$

where $\Omega_n^i = \varepsilon^{-N}\{\Omega - z_n^i\}$. Again, the sequence v_n^i is bounded in $H^1(I\!R^N)$, and from elliptic estimates it can be assumed to converge uniformly on compacts subsets of $I\!R^N$ to a function $v^i \in H^1(I\!R^N)$. Now, the sequence of functions $\chi_n^i(z) \equiv \chi_\Lambda(\varepsilon_n z + z_n^i)$ can be assumed to converge weakly in any L^p on compacts to a function $0 \leq \chi^i \leq 1$. Therefore, v^i satisfies the limiting problem

$$-\Delta v^i + V(\bar{z}_i)v^i = \bar{g}(z, v^i) \text{ in } I\!R^N \qquad (PL)_i$$

where

$$\bar{g}_i(z,s) = \chi^i(z)f(s) + (1 - \chi^i(z))\tilde{f}(s) \quad \text{and} \quad \bar{z}_i = \lim_{n\to\infty} z_{\varepsilon_n}^i.$$

We claim that v^i does not change sign, for $i = 1, 2$. More precisely, $v^1 \geq 0$ and $v^2 \leq 0$. In fact, suppose by contradiction that v^1 changes sign. Let $r > 0$ be such that v^1 changes sign on the closed ball $B[0,r]$. Set $Q_{1,n}^+, Q_{1,n}^- \in B[0,r]$ such that

$$(v_n^1)^+(Q_{1,n}^+) = \max_{z\in B[0,r]} (v_n^1)^+(z),$$

$$(v_n^1)^-(Q_{1,n}^-) = \max_{z\in B[0,r]} (v_n^1)^-(z).$$

Now, as v_n^1 converges uniformly on compacts subsets of $I\!R^N$ to v^1, there exists n_0 such that

$$(v_n^1)^+(Q_{1,n}^+) \geq C > 0 \quad \text{and} \quad -(v_n^1)^-(Q_{1,n}^-) \leq -C < 0, \forall n \geq n_0$$

for some positive constant C. Thus, for $n \geq n_0$

$$(v_n^1)^+(Q_{1,n}^+) = v_n^1(Q_{1,n}^+) = u_{\varepsilon_n}(\varepsilon_n Q_{1,n}^+ + z_n^1),$$

and

$$-(v_n^1)^-(Q_{1,n}^-) = v_n^1(Q_{1,n}^-) = u_{\varepsilon_n}(\varepsilon_n Q_{1,n}^- + z_n^1).$$

Considering

$$\tilde{P}_n^+ = \varepsilon_n Q_{1,n}^+ + z_n^1 \text{ and } \tilde{P}_n^- = \varepsilon_n Q_{1,n}^- + z_n^1$$

it follows that $\tilde{P}_n^+$ and $\tilde{P}_n^-$ are maximum and minimum points of u_{ε_n} on $B[0,r]$ respectively, with

$$u_{\varepsilon_n}(\tilde{P}_n^+) \geq C$$

and

$$u_{\varepsilon_n}(\tilde{P}_n^-) \leq -C.$$

Applying the same arguments employed in Lemma 4.2, we have

$$\left|\frac{\tilde{P}_n^+ - \tilde{P}_n^-}{\varepsilon_n}\right| \to \infty.$$

But this is impossible because

$$\left|\frac{\tilde{P}_n^+ - \tilde{P}_n^-}{\varepsilon_n}\right| = |Q_{1,n}^+ - Q_{1,n}^-| \leq 2r.$$

Hence, as $v_n^1(0) > 0$ and $v_n^1 \to v^1$ uniformly on compacts subsets of $I\!R^N$, it follows that $v^1 \geq 0$. Moreover, using the weak maximum principle follows that $v^1 > 0$ in $I\!R^N$. A similar argument implies that $v^2 < 0$ in $I\!R^N$.

Associated to limiting problem $(PL)_i$, we have the functional $\bar{J}_i : H^1(I\!R^N) \longrightarrow I\!R$ defined by

$$\bar{J}_i(u) = \frac{1}{2}\int_{I\!R^N} (|\nabla u|^2 + V(\bar{z}_i)u^2)\,dx - \int_{I\!R^N} \bar{G}_i(z,u)\,dx, \; u \in H^1(I\!R^N)$$

where $\bar{G}_i(z,s) = \int_0^s \bar{g}_i(z,t)\,dt$. Then, v^i is clearly a critical point of $\bar{J}_i$. Also associated to problem $(P_n)_i$ we have the functional

$$J_n^i(u) = \frac{1}{2}\int_{\Omega_n^i} (|\nabla u|^2 + V(\varepsilon_n z + z_n^i)u^2)\,dx - \int_{\Omega_n^i} \bar{G}_i(\varepsilon_n z + z_n^i, u)\,dx, \; u \in H^1(\Omega_n^i),$$

which satisfies the following equality involving u_ε and v_ε

$$\varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) = \varepsilon_n^{-N}\Big(J_{\varepsilon_n}(u_{\varepsilon_n}^+) + J_{\varepsilon_n}(u_{\varepsilon_n}^-)\Big) = J_n^1((v_n^1)^+) + J_n^2((v_n^2)^-)$$

From Lemma 2.2 in [3], the function v_n^i satisfies the following lower estimate

$$\liminf_{n\to\infty} J_n^1((v_n^1)^+) \geq \bar{J}_1(v^1),$$

$$\liminf_{n\to\infty} J_n^2((v_n^2)^-) \geq \bar{J}_2(v^2).$$

Consequently,

$$2c_{V_0} \geq \limsup_{n\to 0} \varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) \geq \bar{J}_1(v^1) + \bar{J}_2(v^2).$$

Now, as v^i is a critical point of $\bar{J}_i$, and $\bar{g}_i$ satisfies hypothesis (g_4), we have

$$\bar{J}_i(v^i) = \max_{\tau>0} \bar{J}_i(\tau v^i).$$

Since $f(s) \geq \tilde{f}(s)$ for all $s \geq 0$, v^i does not change sign and f is a odd function, we get

$$\bar{J}_i(v^i) = \max_{\tau>0} \bar{J}_i(\tau v^i) \geq \inf_{\substack{u \in H^1(I\!R^N) \\ u \neq 0}} \sup_{\tau>0} I_{\bar{z}_i}(\tau u) = c_{V(\bar{z}_i)},$$

where $I_{\bar{z}_i}$ is given by

$$I_{\bar{z}_i}(u) = \frac{1}{2}\int_{I\!R^N} (|\nabla u|^2 + V(\bar{z}_i)u^2)\,dx - \int_{I\!R^N} F(u)\,dx, \quad u \in H^1(I\!R^N).$$

Recall that we are assuming $V(\bar{z}_1) > V_0$ and $V(\bar{z}_2) \geq V_0$, then

$$2c_{v_0} \geq \limsup_{n\to 0} \varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}) \geq \bar{J}^1(v^1) + \bar{J}^2(v^2) > 2c_{V_0},$$

which is a contradiction, and the proof of Proposition 5.1 is complete. ∎

Corollary 5.2. *If $\overset{+}{m_\varepsilon^-}$ is given by*

$$\overset{+}{m_\varepsilon^-} = \max_{z\in\partial\Lambda} \overset{+}{u_\varepsilon^-}(z)$$

then

$$\lim_{\varepsilon\to 0} \overset{+}{m_\varepsilon^-} = 0.$$

Moreover, for every ε sufficiently small, u_ε possesses at most one positive local maximum $P_\varepsilon^1 \in \Lambda$ and one negative local minimum $P_\varepsilon^2 \in \Lambda$ verifying

$$\lim_{\varepsilon\to 0} V(P_\varepsilon^i) = V_0 = \min_{z\in\bar{\Lambda}} V(z),\ \ i = 1, 2$$

Proof. From Proposition 5.1, we need to prove only that u_ε possesses at most one positive local maximum and one negative local minimum in Λ. Suppose by contradiction that there exist a sequence $\varepsilon_n \to 0$ such that, for instance, possesses two positive local maximum $z_n^1, z_n^2 \in \Lambda$. Then, from Lemma 4.2, $u_{\varepsilon_n}(z_n^i) \geq a > 0$, $i = 1, 2$. Again, from Proposition 5.1, we have that these sequences stay away from the boundary of Λ. Set $v_n(z) = u_{\varepsilon_n}(z_n^1 + \varepsilon_n z)$. Note 0 and $Q_n = (z_n^2 - z_n^1)/\varepsilon_n$ are two local maximum points of v_n. Then, up to subsequence v_n^i converges in the C^2 sense over compacts to $v \in H^1(I\!R^N)$ positive solution of the equation

$$-\Delta v + V_0 v = f(v) \ \text{ in } I\!R^N.$$

Invoking the results due to Gidas, Ni and Nirenberg [5], the function v has a unique local maximum at zero which is a nondegenerate global maximum, it is radially symmetric and decay exponentially. Then, using the fact that $\{v_n\}$ converge to v on $B_R(0)$ for all $R > 0$, we can conclude that $|Q_n| \to \infty$. Employing similar arguments used in [3], we obtain

$$\liminf_{n\to\infty}(\varepsilon^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}^+)) \geq 2c_{V_0}.$$

Recalling that

$$\liminf_{n\to\infty}(\varepsilon_n^{-N} J_{\varepsilon_n}(u_{\varepsilon_n}^-)) \geq c_{V_0}$$

we have the following inequality

$$\liminf_{n\to\infty}(\varepsilon_n^{-N}J_{\varepsilon_n}(u_{\varepsilon_n})) \geq \liminf_{n\to\infty}(\varepsilon_n^{-N}J_{\varepsilon_n}(u_{\varepsilon_n}^+)) + \liminf_{n\to\infty}(\varepsilon_n^{-N}J_{\varepsilon_n}(u_{\varepsilon_n}^-)) \geq 3c_{V_0},$$

which is a contradiction with the estimates proved in the Lemma 3.1. Thus, u_ε has a unique positive maximum. Employing similar arguments we can prove the uniqueness of the negative local minimum of u_ε. ■

6 Proof of Theorem 1.1

By Proposition 5.1, there exists ε_0 such that for all $0<\varepsilon<\varepsilon_0$,

$$|u_\varepsilon(z)| < a-\delta \quad \text{for all } z \in \partial\Lambda.$$

Recalling that u_ε satisfies (P_ε), we have

$$-\varepsilon^2\Delta u + V(x)u = g(x,u) \quad \text{in } \Omega\setminus\Lambda. \tag{6.1}$$

Firstly, we will show that $|u_\varepsilon| \leq a$ in $\Omega\backslash\Lambda$. We begin showing that $u_\varepsilon^+(x) \leq a-\delta$. Since $u_\varepsilon \in H_0^1(\Omega)$, we can choose $(u_\varepsilon - (a-\delta))^+ \equiv (u_\varepsilon^+ - (a-\delta))^+$ as a test function in (6.1) so that after integration by parts we obtain

$$\int_{\Omega\backslash\Lambda} \varepsilon^2|\nabla(u_\varepsilon^+ - (a-\delta))^+|^2 + c(z)(u_\varepsilon^+ - (a-\delta))^+ = 0, \tag{6.2}$$

where

$$c(z) = V(z)u_\varepsilon - g(z,u_\varepsilon)\cdot$$

The definition of g yields that

$$c(z) = \begin{cases} V(z)u_\varepsilon^+ + \psi(u_\varepsilon^+(z)) & \text{if } u_\varepsilon^+(z) \in [a-\delta, a+\delta], \\ V(z)u_\varepsilon^+(z) - \frac{\alpha}{k}u_\varepsilon^+(z) & \text{if } u_\varepsilon^+(z) > a+\delta. \end{cases}$$

Since $\psi(s) \leq \frac{\alpha}{k}s$ for all $s \in [a-\delta, a+\delta]$ we have

$$c(z) \geq \Big(V(z) - \frac{\alpha}{k}\Big)u_\varepsilon^+(z) \geq (\alpha - \frac{\alpha}{k})u_\varepsilon^+(z) \geq 0.$$

Hence all terms in (6.2) are zero. We conclude in particular

$$u_\varepsilon^+(z) \leq a-\delta \quad \text{for all } z \in \Omega\setminus\Lambda.$$

A similar argument shows that

$$-u_\varepsilon^-(z) \geq -(a-\delta) \quad \text{for all } z \in \Omega\setminus\Lambda.$$

Consequently,

$$|u_\varepsilon(z)| \leq (a-\delta) \quad \text{for all } z \in \Omega\setminus\Lambda,$$

and u_ε is a nodal solution to equation (P_ε). To prove exponential decay of u_ε, consider again the functions

$$\begin{cases} v_\varepsilon^1(y) = u_\varepsilon^+(\varepsilon y + P_\varepsilon^1) \\ v_\varepsilon^2(y) = u_\varepsilon^-(\varepsilon y + P_\varepsilon^2) \end{cases}$$

For $i = 1, 2$, the function v_ε^i solves the equation

$$\begin{cases} -\Delta v_\varepsilon^i + V(\varepsilon y + P_\varepsilon^i) v_\varepsilon^i = f(v_\varepsilon^i) & \text{in } \widehat{\Omega}_\varepsilon^i \\ v_\varepsilon^i = 0 & \partial\widehat{\Omega}_\varepsilon^i \end{cases}$$

where $\widehat{\Omega}_\varepsilon^1 = \{x \in \Omega_\varepsilon; u_\varepsilon(x) \geq 0\}$ and $\widehat{\Omega}_\varepsilon^2 = \{x \in \Omega_\varepsilon; u_\varepsilon(x) \leq 0\}$. By convergence C^2 on compact sets, there exists $R_0 > 0$ such that

$$\frac{f(v_\varepsilon^i)}{v_\varepsilon^i} \leq \frac{\alpha}{2}, \quad \text{for all } 0 < \varepsilon < \varepsilon_0 \text{ and } |y| = R_0,$$

thus

$$-\Delta v_\varepsilon^i + \alpha v_\varepsilon^i \leq -\Delta v_\varepsilon^i + V(\varepsilon y + P_\varepsilon^i) v_\varepsilon^i = f(v_\varepsilon^i) \leq \frac{\alpha}{2} v_\varepsilon^i$$

or equivalently

$$-\Delta v_\varepsilon^i + \frac{\alpha}{2} v_\varepsilon^i \leq 0 \quad \text{for all } 0 < \varepsilon < \varepsilon_0 \text{ and } |y| = R_0.$$

From the boundedness of (u_ε), we have that $\{v_\varepsilon^i\}$ is bounded in $H^1(I\!R)$. Repeating similar arguments explored by Zhu [13], we find $M, \beta > 0$ such that

$$v_\varepsilon^i(y) \leq M \exp(-\beta|y|), \text{ for all } x \in \widehat{\Omega}_\varepsilon^i.$$

Thus,

$$u_\varepsilon^+(\varepsilon y + P_\varepsilon^1) \leq M \exp(-\beta|y|)$$

and

$$u_\varepsilon^-(\varepsilon y + P_\varepsilon^2) \leq M \exp(-\beta|y|).$$

Therefore

$$|u_\varepsilon(x)| \leq M \left[\exp\left(-\beta\left|\frac{x - P_\varepsilon^1}{\varepsilon}\right|\right) + \exp\left(-\beta\left|\frac{x - P_\varepsilon^2}{\varepsilon}\right|\right)\right].$$

Using the Theorem 2.1 and Corollary 5.2, the proof of Theorem 1.1 is complete. ■

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References

[1] C. O. ALVES - M. A. S. SOUTO, *On the existence and concentration behaviour of ground state solutions for a class of problems with critical growth*, Comm. in Pure Appl. Analysis, **3** (2002), 417-431.

[2] C. O. ALVES - J. M. DO Ó - M. A. S. SOUTO, *Local mountain-pass for a class of elliptic problems involving critical growth*, Nonl. Anal. **46** (2001), 495-510.

[3] M. DEL PINO - P. L. FELMER, *Local mountain passes for semilinear elliptic problems in unbounded domains,* Cal. Var. **121** (1996), 121-137.

[4] A. FLOER - A. WEINSTEIN, *Nonspreading wave packets for the cubic Schrödinger equations with bounded potential,* J. Funct. Anal. **69** (1986), 397-408.

[5] B. GIDAS - W. M. NI - L. NIRENBERG, *Symmetry of positive solutions of nonlinear equations in* $I\!R^N$, in "Math. Anal. and Applications, Part A, Advances in Math. Suppl. Studies 7A", L. Nachbin (eds), Academic Press, 1981.

[6] E. S. NOUSSAIR - J. WEI, *On the effect of domain geometry on the existence of nodal solutions in singular pertubations problems,* Indiana University Mathematics Journal **46** (1997), 1255-1271.

[7] E. S. NOUSSAIR - J. WEI, *On the location of spikes and profile of nodal solutions for a singularly perturbed Neumann problem,* Comm. Partial Differ. Eq. **23** (1998), 793-816.

[8] Y. J. OH, *Existence of semi-classical bound states of nonlinear Schrödinger equations with potentials on the class* $(V)_a$, Comm. Partial Differ. Eq. **13** (1988), 1499-1519.

[9] Y. J. OH, *Corrections to existence semi-classical bound states of nonlinear Schrödinger equations under multiple well potential,* Comm. Partial Differ. Eq. **14** (1989), 833-834.

[10] P. H. RABINOWITZ, *On a class of nonlinear Schrödinger equations*, Z. Angew Math. Phys. **43** (1992), 270-291.

[11] X. WANG, *On concentration of positive bound states of nonlinear Schrödinger equations,* Comm. Math. Physical **53** (1993), 229-244.

[12] M. WILLEM, *Minimax Theorems*, in "Progress in Nonlinear Differential Equations and Theirs Applications" 24, Birkhäuser, 1996.

[13] X-Ping ZHU, *A pertubation result on positive entire solutions of a semilinear elliptic equation,* J. Diff. Equations **92** (1991), 163-178.

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