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Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2577

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Nº 172

NOTAS

Série Matemática

São Carlos – SP
Mai./2003

REAL MILNOR FIBRATIONS AND (C)-REGULARITY

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ABSTRACT. In this paper we study Milnor fibrations associated to real isolated singularities defined by map-germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^2, 0$. The main result relates the existence of the Milnor fibration with the (c)-regularity of the family of hypersurfaces with isolated singularity obtained by projecting f into the family $L_{-\theta}$ of all lines through the origin in the plane $\mathbb{R}^2$.

RESUMO. O objetivo deste artigo é estudar fibrações de Milnor associadas a singularidades isoladas reais, definidas por germes de aplicações $f : \mathbb{R}^n, 0 \to \mathbb{R}^2, 0$. O resultado principal relaciona a existência de fibração de Milnor com a (c)-regularidade da família de hipersuperfícies com singularidade isolada obtida através da projeção de f sobre a família $L_{-\theta}$ de todas as retas pela origem no plano $\mathbb{R}^2$.

1. Introduction

It is well known [Mi] that if

$$\psi : \left(\mathbb{C}^{n+1}, 0\right) \longrightarrow (\mathbb{C}, 0),$$

is the germ of a holomorphic function with a critical point at 0 then for every sufficiently small $\epsilon > 0$, the map $\phi := \frac{\psi}{||\psi||} : S_\epsilon^{2n-1} \setminus K_\epsilon \to S^1$, is the projection map of a locally trivial fiber bundle, where $K = \psi^{-1}(0) \cap S_\epsilon$ is the link of 0. This is the **Milnor fibration** of ψ.

Date: May 19, 2003.

Research partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico, (CNPq), Brazil, grants # 146427/1999-8 and # 300066/88-0, and by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Brazil, grant # 97/10735-3.

Milnor also proved in the last chapter of his book a fibration theorem for real singularities. He showed that if

$$\psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)\,, n \geq p \geq 2,$$

is the germ of a real analytic map whose derivative $D\psi$ has rank p on a punctured neighbourhood of $0 \in \mathbb{R}^n$, then, for every sufficiently small sphere $S_\epsilon \subset \mathbb{R}^n$ centered at 0, one has a locally trivial fiber bundle,

$\phi = \frac{\psi}{\|\psi\|} : S_\epsilon \setminus N_K \to S^{p-1}$, where N_K denotes a tubular neighborhood of the link K in S_ϵ. Moreover, ϕ can be extended to $S_\epsilon \setminus K$ as the projection map of a fiber bundle, but this extension may not be given by the obvious map $\frac{\psi}{\|\psi\|}$.

The problem of studying real isolated singularities at 0, for which the map $\frac{\psi}{\|\psi\|}$ extends to all of $S_\epsilon \setminus K \to S^{p-1}$ as the projection map of a fibre bundle (as in the case of holomorphic maps) was first studied by A. Jacquemard in [Ja]. When $p = 2$, given a polynomial $\psi = (P, Q)$: $(\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$, with isolated singularity at 0, Jacquemard gave two conditions that were sufficient to guarantee that $\frac{\psi}{\|\psi\|}$ extends to all of $S_\varepsilon \setminus K$ as the projection map of a fiber bundle. The first condition (A) is geometric: the angle between the gradient vector fields of P and Q be bounded below; the second condition (B) is algebraic: the ideals generated by the partial derivatives of P and Q have the same integral closure in the local ring of analytic map-germs at $0 \in \mathbb{R}^n$.

In [S1, S2], Seade gave the following construction that provides infinite families of real analytic isolated singularities $\psi : \mathbb{R}^{2n}, 0 \to \mathbb{R}^2, 0$ satisfying the strong Milnor condition at 0 (see Definition 3.3). Given germs at $0 \in U \subset \mathbb{C}^n$ of holomorphic vector fields F and X, the function $\psi_{F,X} : U \subset \mathbb{C}^n \to \mathbb{C} \cong \mathbb{R}^2$ defined by:

$$\psi_{F,X}(z) = < F(z), X(z) > = \sum_{i=1}^{n} F_i(z) \cdot \overline{X}_i(z)\,,$$

is real analytic, and one can find the conditions for this map to have isolated singularity and satisfy the above condition at 0.

A complete description of such examples when F and X are monomial vector fields is given in [RSV].

In this paper we use Seade's method to give a sufficient condition for a real analytic map-germ $\psi : \mathbb{R}^n, 0 \to \mathbb{R}^2, 0$, with isolated singularity at 0 to satisfy the strong Milnor condition. Let $\pi_\theta : \mathbb{C} \to L_{-\theta}$ be the orthogonal projection to the line $L_{-\theta}$ forming angle $-\theta$ with the horizontal axis in $\mathbb{C} = \mathbb{R}^2$. Our main result stablishes that if the family $\Psi_\theta = \pi_\theta \circ \psi$ satisfies K. Bekka's (c)-regularity condition, then ψ satisfies the strong Milnor condition. As a consequence of this result, we can also prove that Jacquemard's hypothesis are equivalent to the (w)-regularity of the family Ψ_θ.

2. Basic results on regular families

In this section we briefly recall the definitions and basic results on regular families needed in the paper; see [BK], [B1] for details.

Let M be a smooth manifold and X and Y be submanifolds of M such that $Y \subset \overline{X}$.

Definition 2.1. The pair (X, Y) is (a)-regular at $y_0 \in Y$ if, for any sequence of points $\{x_i\}$ in X, $\{x_i\} \to y_0$ such that the corresponding sequence of tangent spaces $\{T_{x_i}X\}$ of X at x_i converges to T in the Grassmann space of dimX-planes, then $T_{y_0}Y \subset T$. We say that (X, Y) is (a)-regular if it is (a)-regular at every y_0 in Y.

Definition 2.2. Let $\rho : M \to \mathbb{R}$ a smooth non-negative function such that $\rho^{-1}(0) = Y$. The pair (X, Y) is (c)-regular at $y_0 \in Y$ with respect to the control function ρ if, for any sequence of points in X, $\{x_i\} \to y_0$ such that the sequence of planes $\{kerd\rho(x_i) \cap T_{x_i}X\}$ converges to a plane T in the Grassmann space of (dimX-1)-planes , then $T_{y_0}Y \subset T$. The pair (X, Y) is (c)-regular with respect to the control function ρ if it is (c)-regular for every point $y_0 \in Y$ with respect to the function ρ.

It is easy to see that (c)-regularity $\Longrightarrow$ (a)- regularity.

We shall assume now that M has a Riemannian metric. Let (T_Y, π, ρ) be a tubular neighbourhood of Y together with a projection associated to a smooth non-negative control function such that $\rho^{-1}(0) = Y$ and $\nabla\rho(x) \in ker d\pi(x)$

Definition 2.3. [BK]:

The pair (X, Y) satisfies condition (m) if there exists a positive real number $\epsilon > 0$ such that

$$(\pi, \rho) \mid_{X\cap T_Y^\epsilon} : X \cap T_Y^\epsilon \to Y \times \mathbb{R}$$

$$x \longmapsto (\pi(x), \rho(x))$$

is a submersion, where $T_Y^\epsilon := \{x \in T_Y / \rho(x) < \epsilon\}$

The following characterization of (c)-regularity is also due to K. Bekka.

Theorem 2.4. [BK]: *The pair (X,Y) is (c)-regular at $y_0 \in Y$ with respect to the control function ρ if and only if the pair (X,Y) is (a)-regular at $y_0 \in Y$ and satisfies condition (m).*

Let $F : \mathbb{R}^n \times \mathbb{R}, 0 \times \mathbb{R} \to \mathbb{R}, 0$ be a one-parameter family of function-germs with isolated singularity, $F_t(x) = F(x,t)$, $X_t = F_t^{-1}(0) \setminus \{0\} \subset \mathbb{R}^n, 0$, $\forall t \in \mathbb{R}$, $X = F^{-1}(0) \setminus (0 \times \mathbb{R}) \subset (\mathbb{R}^n \times \mathbb{R}, 0 \times \mathbb{R})$, $Y = 0 \times \mathbb{R}$. Then, the following result holds:

Theorem 2.5. ([B2],[AV]) *If the pair (X,Y) as above is $(c)-$regular with respect to some control function ρ, then there exists a family of homemorphisms $h_t : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ such that $h_t(X_t) = X_0$ and, $\rho(h_t(x)) = \rho(x)$ on $F^{-1}(0)$.*

3. Strong Milnor Condition and (c)-regularity

Definition 3.1. ([RSV]) Let $\psi : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a real analytic map-germ. If there exists a neighbourhood V of 0 in $\mathbb{R}^n$, such that $rank(d\psi(x)) = p$, $\forall x \in V \setminus \{0\}$, we say that ψ satisfies a *Milnor condition* at 0.

Under this hypothesis, it follows from [Mi] that there exist a sufficiently small $\epsilon_0 > 0$ and a tubular neighbourhood N_{K_ϵ} of $K_\epsilon = \psi^{-1}(0) \cap S_\epsilon^{m-1}$ in S_ϵ^{n-1} , such that, $\phi = \frac{\psi}{\|\psi\|} : S_\epsilon^{m-1} \setminus N_{K\epsilon} \to S^{p-1}$, is the projection map of a locally trivial fiber bundle.

It is possible to extend ϕ as the projection map of a fiber bundle to $S_\epsilon^{m-1} \setminus K_\epsilon$, but this extension is not necessarily given by $\frac{\psi}{\|\psi\|}$, (see [Mi], p.99).

Definition 3.2. If there exists a sufficiently small $\epsilon > 0$ such that, $\phi = \frac{\psi}{\|\psi\|} : S_\epsilon^{m-1} \setminus K_\epsilon \to S^{p-1}$, $0 < \epsilon \leq \epsilon_0$, is the projection map of a locally trivial fiber bundle, we say that ψ satisfies a *strong Milnor condition* at 0.

If $\psi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is the germ of a holomorphic function with isolated singularity at zero, $\psi(z) = (P(z), Q(z))$, where $P(z)$ and $Q(z)$, are respectively, the real and imaginary part of ψ, it follows from the Milnor Fibration Theorem for holomorphic functions, that ψ satisfies the strong Milnor condition. Nontrivial examples of such singularities not arising from holomorphic singularities are obtained in [RSV].

The first approach to study real analytic map-germs $\psi = (P, Q) : (\mathbb{R}^m, 0) \to (\mathbb{R}^2, 0)$ satisfying the strong Milnor condition was described by A. Jacquemard ([Ja], [Ja1]). We shall discuss Jacquemard's results in the next section.

In this section, we introduce a new approach to this problem. Following J. Seade's method in [S1] (see also [S2] and [RSV]), we associate to ψ, the family $\Psi_\theta = \pi_\theta \circ \psi$, where $\pi_\theta : \mathbb{R}^2 \simeq \mathbb{C} \to L_{-\theta}$ is the orthogohal projection into the line $L_{-\theta}$, which passes through the origin in $\mathbb{R}^2$, and makes an angle $-\theta$ with the horizontal axis. Our purpose is to consider the (c)-regularity of the family Ψ_θ as a sufficient condition to guarantee that ψ satisfies the strong Milnor condition.

Let $\psi(x) = (P(x), Q(x)) \sim P(x)+iQ(x)$ be the natural identification and $\Psi_\theta(x) = \pi_\theta \circ \psi(x)$. After making a linear coordinate change in

$\mathbb{R}^2 \simeq \mathbb{C}$, we obtain $\Psi_\theta(x) = Re(e^{i\theta}\psi(x)) = \cos(\theta)P(x) - \sin(\theta)Q(x)$. Let $M = \psi^{-1}(0)$ and $M_\theta = \Psi_\theta^{-1}(0)$.

The following Lemma is an easy generalization of Theorem 1.4 in [S1].

Lemma 3.3. *Let $U \subseteq \mathbb{R}^m$ be a neighbourhood of 0 such that for every $x \in U \setminus \{0\}$, ψ has maximal rank at x. Then the following hold:*

(i) $U = \cup_\theta(M_\theta \cap U)$, $0 \leq \theta < \pi$.

(ii) $M = \cap_\theta M_\theta = M_{\theta_1} \cap M_{\theta_2}$, *where* $M = \psi^{-1}(0)$, $\theta_1 \neq \theta_2$ $\theta_1, \theta_2 \in [0, \pi)$.

(iii) *For each* $\theta \in [0,\pi)$, $M_\theta = E_\theta \cup M \cup E_{\theta+\pi}$, *where* $E_\alpha = \widetilde{\phi}^{-1}(e^{i\alpha})$ *with* $\widetilde{\phi} : U \setminus M \to \mathbb{S}^1$, $\widetilde{\phi}(x) = i\frac{\overline{\psi(x)}}{\|\psi(x)\|}$.

(iv) *For each* $\theta \in [0,\pi)$, $M_\theta^* = M_\theta \setminus \{0\}$ *is a real smooth submanifold of real codimension* 1 *of* $U \setminus \{0\}$, *given by the union of* E_θ, $E_{\theta+\frac{\pi}{2}}$ *and* $M \setminus \{0\}$.

Proof. We start by proving (iii). Let $x \in M = \{x \in \mathbb{R}^m : \psi(x) = 0\}$. Then $\Psi_\theta(x) = \pi_\theta(\psi(x)) = 0$, and $M \subset M_\theta$.
To show that $E_\theta \subset M_\theta$, let $x \in E_\theta = \widetilde{\phi}^{-1}(e^{i\theta})$. In this case $\psi(x) \neq 0$, then $\widetilde{\phi}(x) = i\frac{\overline{\psi(x)}}{\|\psi(x)\|} = e^{i\theta}$. That is, $i\overline{\psi(x)} = \|\psi(x)\|e^{i\theta}$, or, $\overline{\psi(x)} = -i\|\psi(x)\|e^{i\theta}$, and hence $\psi(x) = \|\psi(x)\|e^{-i(\theta+\frac{\pi}{2})}$.

Thus, by definition, $\psi(x) \in L_{-\theta+\frac{\pi}{2}}$ and $x \in M_\theta$.

To show that $E_{\theta+\pi} \subset M_\theta$, we first note that $L_{-\theta} = L_{-\theta+\pi}$ and $\pi_\theta = \pi_{\theta+\pi}$, and hence the inclusion follows from the previous argument. Then, $E_\theta \cup M \cup E_{\theta+\pi} \subseteq M_\theta$.

To prove the reverse inclusion, if $x \in M_\theta = \Psi_\theta^{-1}(0)$, that is $\Psi_\theta(x) = \pi_\theta \circ \psi(x) = 0$, then $\psi(x) \in L_{-\theta+\frac{\pi}{2}}$ and $\psi(x) = \|\psi(x)\|e^{i(-\theta+\frac{\pi}{2})} = \|\psi(x)\|ie^{-i\theta}$. If $\psi(x) = 0$, then $x \in M$.

If $\psi(x) \neq 0$, we have $e^{i\theta} = \frac{i\overline{\psi(x)}}{\|\psi(x)\|} = \widetilde{\phi}(x)$. Then $x \in \widetilde{\phi}^{-1}(e^{i\theta}) = E_\theta$ and $M_\theta \subseteq E_\theta \cup M \cup E_{\theta+\pi}$. Hence, $M_\theta = E_\theta \cup M \cup E_{\theta+\pi}$

(*ii*) From the above result, it follows that $M \subseteq \cap_\theta M_\theta$. To get the reverse inclusion, we show that for every $\theta_1, \theta_2 \in [0,\pi)$, $\theta_1 \neq \theta_2$, $M_{\theta_1} \cap M_{\theta_2} = M$.

The inclusion $M \subseteq M_{\theta_1} \cap M_{\theta_2}$ is obvious.

Suppose that $M \subsetneq M_{\theta_1} \cap M_{\theta_2}$ and let $x \in M_{\theta_1} \cap M_{\theta_2}$, $x \notin M$. Since $M_{\theta_1} = E_{\theta_1} \cup M \cup E_{\theta_1+\pi}$, $M_{\theta_2} = E_{\theta_2} \cup M \cup E_{\theta_2+\pi}$ and $E_{\theta_1} = \widetilde{\phi}^{-1}(e^{i\theta_1})$, $E_{\theta_2} = \widetilde{\phi}^{-1}(e^{i\theta_2})$, we can assume $x \in E_{\theta_1} \cap E_{\theta_2}$ so that $\theta_1 - \theta_2 = 2n\pi$; $n \in \mathbb{Z}$. Since $\theta_1, \theta_2 \in [0, \pi)$ we get $\theta_1 = \theta_2$, which is a contradiction. Hence, $M_{\theta_1} \cap M_{\theta_2} = M$.

(i) $\cup_\theta(M_\theta \cap U) = U$. The inclusion $\cup_\theta(M_\theta \cap U) \subseteq U$ is clear. Let $x \in U$. If $\psi(x) = 0$, then $x \in M \subseteq \cup(M_\theta \cap U)$. If $\psi(x) \neq 0$, we have $\widetilde{\phi}(x) = \frac{i\overline{\psi(x)}}{\|\psi(x)\|} = e^{i(-\theta(x)+\frac{\pi}{2})}$, therefore $x \in \widetilde{\phi}^{-1}(e^{i(-\theta(x)+\frac{\pi}{2})}) = E_{-\theta(x)+\frac{\pi}{2}} \subseteq \cup_\theta(M_\theta \cap U)$.

(iv) Let $\Psi_\theta^{-1}(0) = \psi^{-1}(\pi_\theta^{-1}(0)) = \psi^{-1}(L_{-\theta+\frac{\pi}{2}}) = \psi^{-1}(L^+_{-\theta+\frac{\pi}{2}}) \cup \psi^{-1}(0) \cup \psi^{-1}(L^-_{-\theta+\frac{\pi}{2}})$, where we are considering $L_{-\theta+\frac{\pi}{2}} = L^+_{-\theta+\frac{\pi}{2}} \cup \{0\} \cup L^-_{-\theta+\frac{\pi}{2}}$, with $L^+_{-\theta+\frac{\pi}{2}}$ to be the positive semi-axis Oy, and $L^-_{-\theta+\frac{\pi}{2}}$ the negative semi-axis Oy. We have $\psi^{-1}(L^+_{-\theta+\frac{\pi}{2}}) = E_{\theta(x)}$, $\psi^{-1}(L^-_{-\theta+\frac{\pi}{2}}) = E_{\theta(x)+\frac{\pi}{2}}$ and $\psi^{-1}(0) = M$. Since ψ has an isolated singularity at the origin, $\psi(U \setminus \{0\})$ is an open set whose closure contains the origin. Hence $E_{\theta(x)}$ and $E_{\theta(x)+\frac{\pi}{2}}$ are real codimension 1 submanifolds of U and $M \setminus \{0\}$ is a codimension 2 real submanifold of U. Moreover, $\Psi_\theta |_{U\setminus\{0\}}$ is a submersion therefore M_θ is a submanifold of U. □

Our main result in this section is the following:

Theorem 3.4. *Let $\psi : (\mathbb{R}^m, 0) \to (\mathbb{R}^2, 0)$ be a real analytic map-germ with isolated singularity at the origin, such that the family $\Psi_\theta : (\mathbb{R}^m, 0) \to (\mathbb{R}, 0)$ satisfies the (c)-regularity condition with respect to the function $\rho(x,\theta) = \sum_i x_i^2$. Then ψ satisfies the strong Milnor condition.*

Proof. Let Ψ_θ be (c)-regular. It follows from Theorem 2.5 that there exists a family of germs of homeomorphisms $h_\theta : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$, with $h_0 = Id$ and $\|h_\theta(x)\| = \|x\|$, $\forall x \in \Psi_\theta^{-1}(0)$ (preserving small spheres centered at the origin), such that $\Psi_\theta \circ h_\theta(x) = \psi_0(x)$. Hence, the hypersurfaces $M_{\theta's}$ are homeomorphic. Since the (c)-regularity condition is equivalent to (a)-regularity and the condition (m), it follows, in particular, that there exists a sufficiently small $\epsilon > 0$, such that

M_θ is transverse to S_ϵ^{m-1}, $\forall\ \theta$. Since E_θ is a submanifold of U, and $M_\theta \cap S_\epsilon^{m-1} = (E_\theta \cap S_\epsilon^{m-1}) \cup (M \cap S_\epsilon^{m-1}) \cup (E_{\theta+\frac{\pi}{2}} \cap S_\epsilon^{m-1})$, we can use Lemma 3.3 to obtain:

(i) $S_\epsilon^{m-1} = \cup_\theta (M_\theta \cap S_\epsilon^{m-1})$,

(ii) $K_\epsilon = S_\epsilon^{m-1} \cap M = \cap_\theta (M_\theta \cap S_\epsilon^{m-1})$

(iii) for each $\theta \in [0, \pi)$, $M_\theta \cap S_\epsilon^{m-1} = (E_\theta \cap S_\epsilon^{m-1}) \cup (M \cap S_\epsilon^{m-1}) \cup (E_{\theta+\pi} \cap S_\epsilon^{m-1})$ and

(iv) each $M_\theta \cap S_\epsilon^{m-1}$ is a real submanifold of the sphere S_ϵ^{m-1}.

Let $F_\theta = E_\theta \cap S_\epsilon^{m-1}$. It follows from (iii) that $M_\theta \cap S_\epsilon^{m-1} = F_\theta \cup K_\epsilon \cup F_{\theta+\pi}$.

For each θ_1, θ_2, $(\Psi_{\theta_1} \circ h_{\theta_1})^{-1}(0) = \psi_0^{-1}(0) = (\Psi_{\theta_2} \circ h_{\theta_2})^{-1}(0)$. Then, $h_{\theta_1}^{-1}(M_{\theta_1}) = h_{\theta_2}^{-1}(M_{\theta_2})$

Hence, $h_{\theta_2} \circ h_{\theta_1}^{-1}(M_{\theta_1} \cap S_\epsilon^{m-1}) = M_{\theta_2} \cap S_\epsilon^{m-1}$

Let

$$\begin{aligned} \Gamma_\alpha : & \mathbb{R} \times S_\epsilon^{m-1} \to S_\epsilon^{m-1} \\ & (\theta, x) \longmapsto h_\theta \circ h_\alpha^{-1}(x) \end{aligned}$$

Since $\|h_\xi(x)\| = \|x\|$, Γ_α is well defined and $\forall\ \theta \in \mathbb{R}$ and $x \in F_\alpha$, $h_\theta \circ h_\alpha^{-1}(F_\alpha) = h_\theta \circ h_\alpha^{-1}(E_\alpha \cap S_\epsilon^{m-1}) = E_\theta \cap S_\epsilon^{m-1} = F_\theta$

The map Γ_α is transversal to the fibres and satisfies $\Gamma_\alpha(K_\epsilon) = \Gamma_\alpha(M \cap S_\epsilon^{m-1}) = \Gamma_\alpha(M) \cap S_\epsilon^{m-1} = M \cap S_\epsilon^{m-1} = K_\epsilon$, hence it leaves the link invariant. We proceed now as in [Mi], Theorem 4.8, and show that $\frac{\psi}{\|\psi\|}$ is the projection map of a locally trivial fibre bundle. □

Example 3.5. The converse of the above theorem does not hold in general. Consider the following map-germ from the plane to the plane:

$$\begin{cases} P(x,y) = x \\ Q(x,y) = y(x^2+y^2) \end{cases} \tag{1}$$

The map-germ $\psi = (P, Q)$ satisfy the strong Milnor condition, but the corresponding family Ψ_θ is not (a)-regular. In fact, it is easy to verify that ψ has an isolated singularity and the link K_ϵ is empty. Calculations show that $\frac{\psi}{\|\psi\|}$ is always regular for all spheres S_ϵ, and is onto S^1. Also, one can show that the family $\Psi_\theta(x,y) = \Psi(x,y,\theta)$ associated to ψ satisfies condition (m).

We consider the following sequence of points $p_i = (x_i, y_i, \theta_i)$, where $x_i \to 0$, $y_i = 0$, $\theta_i = \frac{\pi}{2}$. Clearly, $\Psi_\theta(p_i) = 0$, that is, p_i belongs to $X = \Psi_\theta^{-1}(0)$. We have $T_{p_i}X = \nabla\Psi(x_i, y_i, \theta_i)^\perp$. Let $y_0 = \lim_i p_i = (0, 0, \frac{\pi}{2})$ and $Y = (0,0,\theta)$, $\theta \in \mathbb{R}$. Then $(T_{y_0}Y)^\perp = \mathbb{R}^2 \times \{0\}$ and $T_{p_i}X^\perp = \nabla\Psi(x_i, y_i, \theta_i) = (\cos\theta_i - 2x_iy_i\sin\theta_i, -\sin\theta_i(x_i^2+3y_i^2), -\sin\theta_i x_i - \cos\theta_i y_i(x_i^2+y_i^2)) = (0, -x_i^2, -x_i)$. Then, $(0 : x_i^2 : x_i) = (0 : x_i : 1)$ converges to $(0:0:1)$, but $(0,0,1) \not\subseteq (T_{y_0}Y)^\perp = \mathbb{R}^2 \times \{0\}$. So the family does not satisfies the (c)-regularity condition.

We can now obtain a generalization of Theorem 4.0.2 in [RSV]:

Corollary 3.6. *If $\psi : (\mathbb{R}^m, 0) \to (\mathbb{R}^2, 0)$ is the germ of a quasi-homogeneous polynomial map-germ of type $(w_1, ..., w_n : d)$ with an isolated singularity at the origin, then ψ satisfies the strong Milnor condition .*

Proof. Being ψ quasi-homogeneous, it is well known that the family Ψ_θ is (c)-regular,(see [B1]). □

4. Comparing our result with A. Jacquemard's result

We recall the notion of the integral closure of an ideal ([Te]).

Definition 4.1. Let I be an ideal in the ring A. An element $h \in A$ is in the integral closure of I, denoted by $\overline{I}$, if there exists a monic polynomial $P(z) = z^n + \sum_i a_a z^i$, $a_i \in I^{n-i}$, such that $P(h) = 0$.

A. Jacquemard obtained in his Ph.D. thesis sufficient conditions to guarantee that a real analytic map germ $\psi : \mathbb{R}^m, 0 \to \mathbb{R}^2, 0$ satisfies the Milnor condition. His hypothesis were inspired by the Milnor's proof of the fibration theorem for holomorphic functions. Jacquemard's main result is the following.

Theorem 4.2 (Theorem 2 in [Ja]). *Let $\psi = (P, Q) : (\mathbb{R}^m, 0) \to (\mathbb{R}^2, 0)$ be a real analytic map-germ with an isolated singularity at 0. Suppose that there exists a neighbourhood V of 0 in $\mathbb{R}^m$ such that:*

(A) $\frac{\|\langle \nabla P(x), \nabla Q(x)\rangle\|}{\|\nabla P(x)\| \cdot \|\nabla Q(x)\|} \leq 1 - \rho$, $\forall\, x \in V \setminus \{0\}$, $0 < \rho \leq 1$.

(B) *The integral closure of the ideals generated by $\nabla P(x)$ and $\nabla Q(x)$ in the ring $\mathcal{A}_m$ of real analytic function germs, coincide.*

Then there exists a sufficiently small $\epsilon_0 > 0$, such that for all $0 < \epsilon \leq \epsilon_0$, $\phi = \frac{\psi}{\|\psi\|} : S_\epsilon^{m-1} \setminus K_\epsilon \to S^1$ is the projection map of a locally trivial fiber bundle. Furthermore, this fiber bundle is equivalent to the fibration given by Milnor in [Mi].

In [RSV] the authors modified the condition (B) in Theorem 4.2 replacing the integral closure by the real integral closure, as defined by Gaffney in [Ga]. In the complex analytic category, both conditions are equivalent (see [Te], [Ga]).

Definition 4.3. Let I be an ideal in the ring $\mathcal{A}_m$. The real integral closure of I, denoted by $\overline{I_{\mathbb{R}}}$, is the set of elements $h \in \mathcal{A}_m$ such that for every analytic curve $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^m, 0)$, $h \circ \gamma \in (\gamma^*(I))\mathcal{A}(1)$.

It is easy to verify from Jacquemard's proof , that Theorem 4.2 still holds when condition (B) is replaced by the condition $(B_{\mathbb{R}})$: $\overline{\langle \nabla P(x)\rangle}_{\mathbb{R}} = \overline{\langle \nabla Q(x)\rangle}_{\mathbb{R}}$ in the ring $\mathcal{A}_m$.

We consider, as in the previous section, the family

$$\Psi_\theta : (\mathbb{R}^m, 0) \to (\mathbb{R}, 0)$$
$$x \longmapsto Re(e^{i\theta}\psi(x)) = \cos(\theta)P(x) - \sin(\theta)Q(x).$$

Theorem 4.4. *Let $\psi = (P, Q) : (\mathbb{R}^m, 0) \to (\mathbb{R}^2, 0)$, with an isolated singularity at the origin. The coordinate functions $P(x)$ and $Q(x)$ satisfy the conditions (A) and $(B_{\mathbb{R}})$ if, and only if, $\overline{\langle \nabla \Psi_\theta(x)\rangle}_{\mathbb{R}} = \overline{\langle \nabla \Psi_0\rangle}_{\mathbb{R}}$, $\forall\, \theta \in [0, \pi)$.*

Proof. Let $\gamma : (\mathbb{R},0) \longrightarrow (\mathbb{R}^m,0)$ be the germ of a non constant real analytic curve. Since $\nabla\Psi_\theta(x) = \cos(\theta)\nabla P(x) - \sin(\theta)\nabla Q(x)$, it follows that $\nabla\Psi_\theta(\gamma(t)) = \cos(\theta)\nabla P(\gamma(t)) - \sin(\theta)\nabla Q(\gamma(t))$.

We now consider the following Taylor expansions :
$\nabla P(\gamma(t)) = a_1 t^{n_1} + \ldots.. \, , \, a_1 \in \mathbb{R}^m \setminus \{0\} \, , \, n_1 \in \mathbb{N}^*$.
$\nabla Q(\gamma(t)) = b_1 t^{m_1} + \ldots.. \, , \, b_2 \in \mathbb{R}^m \setminus \{0\} \, , \, m_1 \in \mathbb{N}^*$.

Since $\overline{\langle \nabla P(x)\rangle}_{\mathbb{R}} = \overline{\langle \nabla Q(x)\rangle}_{\mathbb{R}}$, then $n_1 = m_1$ and $\nabla\Psi_\theta(\gamma(t)) = (\cos(\theta)a_1 - \sin(\theta)b_1)t^{n_1} + \ldots..$

The hypothesis $\frac{\|\langle \nabla P(x), \nabla Q(x)\rangle\|}{\|\nabla P(x)\| \cdot \|\nabla Q(x)\|} \leq 1-\rho$ for some $0<\rho\leq 1$ implies that for each real analytic $\gamma : (\mathbb{R},0) \longrightarrow (\mathbb{R}^m,0)$, the principal part of $\frac{\|\langle \nabla P(x), \nabla Q(x)\rangle\|}{\|\nabla P(x)\| \cdot \|\nabla Q(x)\|}$ satisfies $\frac{\|\langle a_1, b_1\rangle\|}{\|a_1\| \cdot \|b_1\|} < 1$, that is, a_1 and b_1 are linearly independents. Hence, $\cos(\theta)a_1 - \sin(\theta)b_1 \neq 0$, $\forall\, \theta$, and therefore $\nu(\nabla\Psi_\theta) = n_1 = \nu(\nabla\Psi_0)$, where $\nu(f)$ denotes the order of the function f, when restricted to the curve γ.

Since the curve γ was chosen arbitrary, we conclude that $\overline{\langle \nabla\Psi_\theta\rangle}_{\mathbb{R}} = \overline{\langle \nabla P\rangle}_{\mathbb{R}}$.

Reciprocally, if $\overline{\langle \nabla\Psi_\theta\rangle}_{\mathbb{R}} = \overline{\langle \nabla\Psi_0\rangle}_{\mathbb{R}} = \overline{\langle \nabla P\rangle}_{\mathbb{R}}$, making $\theta = \frac{\pi}{2}$ we get condition $(B_{\mathbb{R}})$.

Observe that $\forall\, \theta$, $\cos(\theta)a_1 - \sin(\theta)b_1 \neq 0$, otherwise there exists θ_0 such that $\cos(\theta_0)a_1 - \sin(\theta_0)b_1 = 0$, and $\nu(\nabla\Psi_{\theta_0}) > \nu(\nabla P)$, which is a contradiction.

If $\cos(\theta) \neq 0$, we have $a_1 - \frac{\sin(\theta)}{\cos(\theta)}b_1 = a_1 - \tan(\theta)b_1 \neq 0$, therefore a_1 and b_1 are linearly independent.

The curve γ is taken arbitrarily, so it follows that $\frac{|\langle \nabla P(x), \nabla Q(x)\rangle|}{|\nabla P(x)||\nabla Q(x)|} \leq 1-\rho$, for some $0<\rho\leq 1$, and $\forall\, x$ sufficiently close to the origin. $\square$

We will show now that our conditions are weaker than Jacquemard's hypothesis. More precisely, we will show that conditions (A) and $(B_{\mathbb{R}})$ imply the (c)-regularity of the family Ψ_θ. Furthermore, we will exibit an example of an isolated singularity for which the corresponding family Ψ_θ is (c)-regular, but Jacquemard's hypothesis do not hold.

First we recall the following definition ([FP]).

Definition 4.5. [FP] Let $F_z : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}, 0)$, $z \in \mathbb{R}$ be a family of analytic map-germs. We say that the family is (w)-regular if there exists a constant $K > 0$ such that, $|\frac{\partial F(x,z)}{\partial z}| \leq K\|x\|.\|\frac{\partial F_z}{\partial x}\|$, where $\frac{\partial F_z}{\partial x}$ denotes the gradient of the function F_z.

Lemma 4.6. *Let $\Psi(x,t) = \cos(t)P(x) - \sin(t)Q(x)$. If, for each $t \in \mathbb{R}$, $\overline{\langle \nabla_x \Psi(x,t) \rangle}_{\mathbb{R}} = \overline{\langle \nabla P(x) \rangle}_{\mathbb{R}}$ in the ring $\mathcal{A}_m$, then $\overline{\langle \nabla_x \Psi(x,t) \rangle}_{\mathbb{R}} = \overline{\langle \nabla P(x) \rangle}_{\mathbb{R}}$ in the ring $\mathcal{A}_{m+1}$.*

Proof. To simplify notation, we shall denote by $\overline{I}_{\mathbb{R}}\mathcal{A}_m$ the real integral closure of the ideal I, and by $\overline{I}_{\mathbb{R}}$ the real integral closure of the ideal I in the ring $\mathcal{A}_{m+1}$.

Let $\alpha(x,t) \in \overline{\langle \nabla_x \Psi(x,t) \rangle}_{\mathbb{R}}$. Then for every non constant real analytic curve $\gamma(s) = (x(s), t(s))$ in $\mathbb{R}^m \times \mathbb{R}, 0$ with $\gamma(0) = (0,0)$, it follows that,

$$\begin{aligned} \nu(\alpha \circ \gamma(s)) &\geq \nu(\cos(t(s))\frac{\partial P}{\partial x_j}(x(s)) - \sin(t(s))\frac{\partial Q}{\partial x_j}(x(s))) \\ &\geq \min\{\nu(\cos(t(s))\frac{\partial P}{\partial x_j}(x(s))), \nu(\sin(t(s))\frac{\partial Q}{\partial x_j}(x(s)))\}, \end{aligned}$$

for all $j = 1, \ldots, n$.

If $\min\{\nu(\cos(t(s))\frac{\partial P}{\partial x_j}(x(s))), \nu(\sin(t(s))\frac{\partial Q}{\partial x_j}(x(s)))\} = \nu(\cos(t(s))\frac{\partial P}{\partial x_j}(x(s))) \geq \nu(\frac{\partial P}{\partial x_j}(x(s)))$, then $\overline{\langle \nabla_x \Psi(x,t) \rangle}_{\mathbb{R}} \subseteq \overline{\langle \nabla P(x) \rangle}_{\mathbb{R}}$.

Otherwise, $\min\{\nu(\cos(t(s))\frac{\partial P}{\partial x_j}(x(s))), \nu(\sin(t(s))\frac{\partial Q}{\partial x_j}(x(s)))\} = \nu(\sin(t(s))\frac{\partial Q}{\partial x_j}(x(s))) \geq \nu(\frac{\partial Q}{\partial x_j}(x(s)))$, $\forall\ j$. Therefore $\overline{\langle \nabla_x \Psi(x,t) \rangle}_{\mathbb{R}} \subseteq \overline{\langle \nabla Q(x) \rangle}_{\mathbb{R}}$, Since $\overline{\langle \nabla P(x) \rangle}_{\mathbb{R}} = \overline{\langle \nabla Q(x) \rangle}_{\mathbb{R}}$, one inclusion follows.

To prove that $\overline{\langle \nabla P(x) \rangle}_{\mathbb{R}} \subseteq \overline{\langle \nabla_x \Psi(x,t) \rangle}_{\mathbb{R}}$, we observe that $\frac{\partial \Psi}{\partial x_i}(x,t) = \frac{\partial P}{\partial x_i}(x) + f_1(t)\frac{\partial P}{\partial x_i}(x) + f_2(t)\frac{\partial Q}{\partial x_i}(x)$, where $f_1(t) = \cos(t) - 1$ and $f_2(t) = \sin(t)$. Then, $f_1(0) = f_2(0) = 0$ and it is now easy to show that for each non constant real analytic curve $\gamma(s) = (x(s), t(s))$ in $\mathbb{R}^m \times \mathbb{R}, 0$ with $\gamma(0) = (0,0)$, we have $\nu(\frac{\partial P}{\partial x_i}(x(s))) \geq \nu(\frac{\partial P}{\partial x_i} + f_1(t)\frac{\partial P}{\partial x_i} + f_2(t)\frac{\partial Q}{\partial x_i})(x(s), t(s))$ $i = 1, \ldots, n$. □

Lemma 4.7. *Let $\Psi(x,t) = \cos(t)P(x) - \sin(t)Q(x)$ as above and suppose that $\overline{\langle \nabla_x \Psi(x,t) \rangle}_{\mathbb{R}} = \overline{\langle \nabla P(x) \rangle}_{\mathbb{R}}$ in the ring $\mathcal{A}_m$, for every t. Then the family Ψ_t is (w)-regular.*

Proof. Since $\overline{\langle \nabla_x \Psi(x,t)\rangle_{\mathbb{R}}\mathcal{A}_m} = \overline{\langle \nabla P(x)\rangle_{\mathbb{R}}\mathcal{A}_m}$ then, by Lemma 4.7, there exist positive constants c_1, c_2, c_3 and c_4 such that, $(1): c_1|\nabla P(x)| \leq |\nabla_x \Psi(x,t)| \leq c_2|\nabla P(x)|$, and $(2): c_3|\nabla Q(x)| \leq |\nabla_x \Psi(x,t)| \leq c_4|\nabla Q(x)|$.

Hence,

$$\begin{aligned}
|\frac{\partial \Psi(x,t)}{\partial t}| &= |-\sin(t)P(x) - \cos(t)Q(x)| \\
&\leq |\sin(t)P(x)| + |\cos(t)Q(x)| \leq |P(x)| + |Q(x)|
\end{aligned}$$

(from Bochnak-Lojasiewicz's [B1] result)

$$\begin{aligned}
&\leq k_1\|x\|.\|\nabla P(x)\| + k_2\|x\|.\|\nabla Q(x)\| \\
&\leq k_1\|x\|.\|\nabla P(x)\| + k_3\|x\|.\|\nabla P(x)\| \\
&\leq K_1\|x\|.\|\nabla P(x)\| \\
&\leq K\|x\|.\|\nabla_x \Psi(x,t)\|.
\end{aligned}$$

Hence the family $\Psi(x,t)$ is (w)-regular. □

Theorem 4.8. *: With the hypothesis of Theorem* 4.4, *the family* Ψ_θ *is* (c)*-regular.*

Proof. The following holds both in the real and complex cases: (w)-regularity $\Rightarrow$ (c)-regularity [B1]. □

The following example shows that the converse of the above theorem does not hold in general.

Example 4.9.
$$\begin{cases} P = xy \\ Q = x^2 - y^4 \end{cases} \tag{2}$$

It is easy to verify that $\psi(x,y) = (P(x,y), Q(x,y))$ has an isolated singularity at the origin.

Let $\Psi(x,y,t) = \cos(t)xy - \sin(t)(x^2 - y^4)$, $X = \Psi^{-1}(0) \setminus \{0\} \times \mathbb{R}$, and $Y = \{0\} \times \mathbb{R}$. The family Ψ has infinite Milnor radius. Moreover, for every $t \neq \frac{(2k+1)\pi}{2}$, the family has a Morse singularity at the origin, hence it is (a)-regular. To verify condition (a) for the pair (X,Y) at

$t_0 = \frac{(2k+1)\pi}{2}$ we will show that $\lim_{(x,y,t)\to(0,0,t_0)} \frac{|\partial_t \Psi(x,y,t)|}{||\nabla_{x,y}\Psi(x,y,t)||} = 0$, where $\partial_t \Psi$ is the derivative of Ψ with respect to the parameters, and $\nabla_{x,y}\Psi$ denotes the gradient of Ψ with respect to the variables x, y. To prove this, let $\alpha : [0,\epsilon) \to \Psi^{-1}(0)$ be a non constant real analytic curve with $\alpha(0) = 0$, $\alpha(s) = (x(s), y(s), t(s))$, where

$x(s) = x_1 s^{n_1} + \ldots ; x_1 \neq 0, n_1 \geq 1,$

$y(s) = y_1 s^{n_2} + \ldots ; y_1 \neq 0, n_2 \geq 1,$

$cost(s) = \mu_0 s^n + \ldots, \mu_0 \neq 0, n \geq 1.$

For points $(x, y, t) \in \Psi^{-1}(0)$, the following equations hold:

$cos(t)xy = sin(t)(x^2 - y^4),$

$|\partial_t \Psi(x,y,t)|^2 = \frac{(xy)^2}{sin^2(t)}$

$||\nabla_{x,y}\Psi_t(x,y,t)||^2 = (x^2 + y^2)cos^2(t) + 4y^2(2y^4 + 2x^2 + y^2)sin^2(t)$

The leading terms of the restriction to the curve $\alpha(s)$ of the Taylor expansions of the above equations are respectively

(a) $\mu_0 x_1 y_1 s^{n+n_1+n_2} = x_1^2 s^{2n_1} - y_1^4 s^{4n_2} + \ldots,$

(b) $|\partial_t \Psi(s)|^2 = \frac{(x_1 y_1)^2 s^{2(n_1+n_2)} + \ldots}{sin^2(t(s))}$

(c) $||\nabla_{x,y}\Psi(s)||^2 = ({x_1}^2 s^{2n_1} + {y_1}^2 s^{2n_2}){\mu_0}^2 s^{2n} + 4{y_1}^2(2{y_1}^4 s^{4n_2} + 2{x_1}^2 s^{2n_1} + {y_1}^2 s^{2n_2})sin^2(t(s)) + \ldots.$

We recall that for $t_0 = \frac{(2k+1)\pi}{2}$, $sin(t(s))$ is a unit for every s suficiently close to zero.

If $n_1 \leq n_2$, it follows from (a) that $2n_1 = n + n_1 + n_2$ hence $n_1 = n + n_2 > n_2$, which is a contradiction. Hence, $n_1 > n_2$.

We can have the following possibilities:

(i) $n_1 > 2n_2$.

(ii) $n_1 = 2n_2$.

(iii) $n_1 < 2n_2$.

We easily see from (a), (b) and (c) that in the cases (i) and (iii), it follows that $\lim_{s\to 0} \frac{|\partial_t \Psi(s)|}{||\nabla_{x,y}\Psi(s)||} = 0$.

Let $n_1 = 2n_2$. In this case, condition (a) implies that $n_1 \leq n + n_2$ and hence $n_2 \leq n$.

The equality $n_2 = n$ holds if and only if the equality $n_1 = n + n_2$ also holds . In this case, in (b) we obtain $|\partial_t \Psi(s)|^2 = (x_1 y_1)^2 s^{4n}.s^{2n} + \ldots,$

and in (c), $||\nabla_{x,y}\Psi(s)||^2 = (y_1^2\mu_0^2 + 4y_1^4)s^{4n} + \ldots$.

Hence, $\lim_{s\to 0} \frac{|\partial_t\Psi(s)|}{||\nabla_{x,y}\Psi(s)||} = 0$.

If $n_2 < n$ (this inequality holds if and only if $x_1^2 = y_1^4$), then (b) becomes $|\partial_t\Psi(s)|^2 = (x_1y_1)^2 s^{4n_2}.s^{2n_2} + \ldots$,

and (c) $||\nabla_{x,y}\Psi(s)||^2 = 4y_1^4 s^{4n_2} + \ldots$.

Hence in this case we also obtain $\lim \frac{|\partial_t\Psi(s)|}{||\nabla_{x,y}\Psi_t(s)||} \longrightarrow 0$, when $s \to 0$.

The other cases follow in a similar way.

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