

UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação
ISSN 0103-2577

On the Orbit Structure of $\mathbb{R}^n$ – Actions on
 n -Manifolds

J. L. Arraut
C. A. Maquera

Nº 174

NOTAS

Série Matemática



São Carlos – SP
Jul./2003

SYSNO	1319448
DATA	/ /
ICMC - SBAB	

ON THE ORBIT STRUCTURE OF $\mathbb{R}^n$ -ACTIONS ON n -MANIFOLDS

J. L. ARRAUT AND C. A. MAQUERA

RESUMO. Começamos provando que uma C^2 ação localmente livre de $\mathbb{R}^{n-1}$ sobre $T^{n-1} \times [0, 1]$ tangente ao bordo e sem órbitas compactas em seu interior tem todas suas órbitas não compactas do mesmo tipo topológico. Depois, consideramos o conjunto $A^2(\mathbb{R}^n, N)$ das C^2 ações de $\mathbb{R}^n$ sobre uma n -variedade analítica real fechada e orientável N . Definimos um subconjunto $\mathcal{A}_n \subset A^r(\mathbb{R}^n, N)$ e provamos que se $\varphi \in \mathcal{A}_n$ tem uma $T^{n-1} \times \mathbb{R}$ -órbita, então cada órbita de dimensão n é também uma $T^{n-1} \times \mathbb{R}$ -órbita. O subconjunto $\mathcal{A}_n$, é suficientemente grande para conter todas as ações analíticas reais que possuem pelo menos uma órbita de dimensão n . Também obtemos informação sobre a topologia de N .

ABSTRACT. We begin by proving that a locally free C^2 -action of $\mathbb{R}^{n-1}$ on $T^{n-1} \times [0, 1]$ tangent to the boundary and without compact orbits in the interior has all non-compact orbits of the same topological type. Then, we consider the set $A^2(\mathbb{R}^n, N)$ of C^2 -actions of $\mathbb{R}^n$ on a closed connected orientable real analytic n -manifold N . We define a subset $\mathcal{A}_n \subset A^r(\mathbb{R}^n, N)$ and prove that if $\varphi \in \mathcal{A}_n$ has a $T^{n-1} \times \mathbb{R}$ -orbit, then every n -dimensional orbit is also a $T^{n-1} \times \mathbb{R}$ -orbit. The subset $\mathcal{A}_n$, is big enough to contain all real analytic actions that have at least one n -dimensional orbit. We also obtain information on the topology of N .

1. INTRODUCTION

In this paper we begin by proving the following result that is a generalization of Corollary 2.6 in [3].

Theorem A. *Let ψ be a locally free C^2 -action of $\mathbb{R}^{n-1}$ on $T^{n-1} \times [0, 1]$, $n \geq 2$, tangent to the boundary. If there are no compact orbits in the interior, then all non-compact orbits have the same topological type.*

Due to Theorem 2.11, there is no restriction in assuming that the manifold with boundary, in Theorem A, is $T^{n-1} \times [0, 1]$.

N will denote a closed connected orientable real analytic n -manifold with $n \geq 2$. Let $\mathcal{H}_n$ be the family of orientable n -manifolds obtained by glueing two copies of $T^{n-2} \times D^2$. $\mathcal{H}_2$ contains only S^2 and $\mathcal{H}_3$ consists of 3-manifolds that admit a Heegaard splitting of genus one. Denote by $A^r(\mathbb{R}^n, N)$ the set of C^r -actions of $\mathbb{R}^n$ on N , $2 \leq r \leq \omega$, with C^r infinitesimal generators. It was proved in [1] that if $\varphi \in A^\omega(\mathbb{R}^n, N)$, then all n -dimensional orbits of φ have the same topological type, i.e., are $T^k \times \mathbb{R}^{n-k}$ -orbits for some fixed k , $0 \leq k \leq n$. Moreover, if the type is $T^{n-1} \times \mathbb{R}$, then N is either homeomorphic to T^n or $N \in \mathcal{H}_n$. It is not

Date: June 30, 2003.

1991 Mathematics Subject Classification. Primary: 57R30.

Key words and phrases. Action of $\mathbb{R}^n$, orbit structure.

The first author was partially supported by FAPESP of Brazil Grant 00/05385-8.

The second author was supported by FAPESP of Brazil Grant 99/11311-8 and 02/09425-0.

difficult to construct counterexamples of this results when $r = \infty$. In this paper we define a subset $\mathcal{A}_n \subset A^2(\mathbb{R}^n, N)$, see Definition 2.2, which contains all actions $\varphi \in A^\omega(\mathbb{R}^n, N)$ that have at least one n -dimensional orbit. Then, we prove:

Theorem B. *If $\varphi \in \mathcal{A}_n$ has one $T^{n-1} \times \mathbb{R}$ -orbit, then every n -dimensional orbit is also a $T^{n-1} \times \mathbb{R}$ -orbit. Moreover,*

- (1) *if $\text{Sing}_{n-2}^c(\varphi) = \emptyset$, then N is a T^{n-1} bundle over S^1 ;*
- (2) *if $\text{Sing}_{n-2}^c(\varphi) \neq \emptyset$, then $\text{Sing}_{n-2}^c(\varphi)$ is the union of two T^{n-2} -orbits and $N \in \mathcal{H}_n$.*

The connection between the two results is that the first is used in the proof of second. It would be interesting to obtain analogous results for actions in $A^2(\mathbb{R}^n, N)$ that have one $T^k \times \mathbb{R}^{n-k}$ -orbit with $0 \leq k < n - 1$.

2. PRELIMINARIES AND PROOF OF RESULTS

M will denote a closed connected and orientable real analytic m -manifold. A C^r -action of Lie group G on M is a C^r -map $\varphi : G \times M \rightarrow M$, $1 \leq r \leq \omega$, such that $\varphi(e, p) = p$ and $\varphi(gh, p) = \varphi(g, \varphi(h, p))$, for each $g, h \in G$ and $p \in M$, where e is the identity in G . $\mathcal{O}_p = \{\varphi(g, p); g \in G\}$ is called the φ -orbit of p . $G_p = \{g \in G; \varphi(g, p) = p\}$ is called the *isotropy group* of p . For each $p \in M$ the map $g \mapsto \varphi(g, p)$ induces an injective immersion of the homogeneous space G/G_p in M with image $\mathcal{O}_p$. When $G = \mathbb{R}^n$, the possible φ -orbits are injective immersions of $T^k \times \mathbb{R}^\ell$, $0 \leq k + \ell \leq n$, where $T^k = S^1 \times \cdots \times S^1$, k times.

For each $0 \leq i \leq n - 1$ let $\text{Sing}_i(\varphi) = \{p \in M; \dim \mathcal{O}_p = i\}$ and $\text{Sing}(\varphi) = \bigcup_{i=0}^{n-1} \text{Sing}_i(\varphi)$. If $p \in \text{Sing}(\varphi)$, $\mathcal{O}_p$ is called a *singular orbit* and when $p \in \text{Sing}_0(\varphi)$, $\mathcal{O}_p$ is also called a *point orbit* and p a *fixed point* by φ . We also write $p \in \text{Sing}_i^c(\varphi)$, $i = 1, \dots, n - 1$, when $\mathcal{O}_p$ is a T^i -orbit. If $\text{Sing}(\varphi) = M$, we call φ a *singular action*.

For each $w \in \mathbb{R}^n \setminus \{0\}$ φ induces a C^r -flow $(\varphi_w^t)_{t \in \mathbb{R}}$ given by $\varphi_w^t(p) = \varphi(tw, p)$ and its corresponding C^{r-1} -vector field X_w defined by $X_w(p) = D_1\varphi(0, p) \cdot w$. If $\{w_1, \dots, w_n\}$ is a base of $\mathbb{R}^n$ the associated vector fields $X_{w_1}, \dots, X_{w_n}$ determine completely the action φ and are called a set of *infinitesimal generators* of φ . Note that $[X_{w_i}, X_{w_j}] = 0$ for any two of them.

Definition 2.1. *Let $\varphi \in A^r(\mathbb{R}^n, N)$ and $p \in N$.*

a) φ is of type j at p , $0 \leq j \leq n$, if there exists a neighborhood V of p such that the union of the j -dimensional orbits of $\varphi|_V$ form an open and dense subset of V .

b) φ is j -finite at p , if there exists a neighborhood V of p that intersects only a finite number of j -dimensional orbits.

Definition 2.2. *We say that $\varphi \in \mathcal{A}_n \subset A^2(\mathbb{R}^n, N)$ if φ is of type n and n -finite at each $p \in \text{Sing}_i(\varphi)$ with $0 \leq i \leq n - 3$, $\text{Sing}_{n-2}(\varphi) = \text{Sing}_{n-2}^c(\varphi)$ and for each $p \in \text{Sing}_{n-2}^c(\varphi)$ there exists a neighborhood V_p of $\mathcal{O}_p$ in N that satisfies one of the following two properties:*

- (1) V_p is φ -invariant, homeomorphic to $T^{n-2} \times D^2$, where D^2 is an open disk, $V_p \cap (\bigcup_{i=1}^{n-2} \text{Sing}_i(\varphi)) = \mathcal{O}_p$ and $\text{Front}(V_p)$ is either a T^{n-1} -orbit or a T^{n-2} -orbit.
- (2) V_p contains at most a finite number of i -dimensional orbits with $i = n - 1, n$.

Infinitesimal generators adapted to a T^{n-1} -orbit. Assume that $\mathcal{O}_p$ is a T^{n-1} -orbit of $\varphi \in A^r(\mathbb{R}^n, N)$ and let G_p be its isotropy group. Call G_p^0 the connected component of G_p that contains the origin and let H be a $(n - 1)$ -dimensional subspace of $\mathbb{R}^n$ such that $\mathbb{R}^n = H \oplus G_p^0$. Note that $G_p \cap H$ is

isomorphic to $\mathbb{Z}^{n-1}$. Let $\{w_1, \dots, w_n\}$ be a base of $\mathbb{R}^n$ such that $\{w_1, \dots, w_{n-1}\}$ is a set of generators of $G_p \cap H$, $w_n \in G_p^0$ and write $X_i = X_{w_i}$; $i = 1, \dots, n$. Note that if $q \in \mathcal{O}_p$, then for every $k \in \{1, \dots, n-1\}$ the orbit of X_k by p is periodic of period one and also $X_n(q) = 0$. We shall say that $X_1, \dots, X_n$ is a set of *infinitesimal generators adapted to $\mathcal{O}_p$* . The action $\psi_\varphi \in A^r(\mathbb{R}^{n-1}, N)$, $r \geq 2$, with infinitesimal generators $X_1, \dots, X_{n-1}$ will be called the *action induced by φ and $\mathcal{O}_p$* . The understanding of the holonomy of $\mathcal{O}_p$ as an orbit of ψ_φ will bring light on the orbit structure of φ in the neighborhood of $\mathcal{O}_p$.

Let $\mathcal{O}_p$ be a T^{n-1} -orbit of $\psi \in A^r(\mathbb{R}^{n-1}, N)$, $\{w_1, \dots, w_{n-1}\}$ be a set of generators of its isotropy group G_p and $X_1 = X_{w_1}, \dots, X_{n-1} = X_{w_{n-1}}$. For each $k \in \{1, \dots, n-1\}$, let $\psi_k \in A^r(\mathbb{R}^{n-2}, N)$ be the action defined by $X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_{n-1}$. Put a Riemannian metric on N and let ξ be the norm one vector field defined in a neighborhood of $\mathcal{O}_p$ that is orthogonal to the orbits of ψ . Let S_k be the circle orbit of X_k through p , $k = 1, \dots, n-1$, and consider the ring $A = S^1 \times (-1, 1)$ with coordinates (θ, x) . Define $f_k : A \rightarrow N$ by $f_k(\theta, x) = \xi^x \circ X_k^\theta(p)$ and note that $f_k(S^1 \times \{0\}) = S_k$ and $f_k(0, 0) = p$. Fix $k \in \{1, \dots, n-1\}$. Since S_k , as a submanifold of $\mathcal{O}_p$, is transversal to the orbits of ψ_k , there exists $\varepsilon > 0$ such that f_k restricted to $A_\varepsilon = S^1 \times (-\varepsilon, \varepsilon)$ is an embedding transversal to the orbits of ψ_k . Let $D_k^{n-2}(\delta) = \{t = (t_1, \dots, t_{k-1}, t_{k+1}, \dots, t_{n-1}); t_j \in (-\delta, \delta)\}$ and consider the C^r -map $h_k : A_\varepsilon \times D_k^{n-2}(1) \rightarrow N$ defined by $h_k(\theta, x, t) = \psi_k(t, f_k(\theta, x))$. There exists $\delta > 0$ such that h_k restricted to $A_\varepsilon \times D_k^{n-2}(\delta)$ is a diffeomorphism onto its image V_k . Moreover, in these coordinates the infinitesimal generators of φ take the form:

$$(2.1) \quad \begin{aligned} X_i(\theta, x, t) &= \frac{\partial}{\partial t_i}, \quad i = 1, \dots, k-1, k+1, \dots, n-1 \\ X_k(\theta, x, t) &= \sum_{j=1}^{n-1} a_{jk}(\theta, x) \frac{\partial}{\partial t_j} + b_k(\theta, x) \frac{\partial}{\partial \theta} + c_k(\theta, x) \frac{\partial}{\partial x}. \end{aligned}$$

A map like h_k will be called a *cylindrical coordinate system adapted to $\mathcal{O}_p$ at S_k* . The vector field

$$\hat{X}_k = b_k(\theta, x) \frac{\partial}{\partial \theta} + c_k(\theta, x) \frac{\partial}{\partial x}$$

defines a local flow on A_ε having $S^1 \times \{0\} \subset A_\varepsilon$ as an orbit. When $\psi = \psi_\varphi$ for some $\varphi \in A^r(\mathbb{R}^n, N)$, then we also have

$$X_n(\theta, x, t) = \sum_{k=1}^{n-1} a_{nk}(\theta, x) \frac{\partial}{\partial t_k} + d_k(\theta, x) \frac{\partial}{\partial \theta} + e_k(\theta, x) \frac{\partial}{\partial x}.$$

The vector fields $\hat{X}_k$ and $\hat{X}_n = d_k \partial / \partial \theta + e_k \partial / \partial x$ define a local C^r -action $\hat{\varphi}_k$ of $\mathbb{R}^2$ on A having $S^1 \times \{0\}$ as a singular orbit.

The ring $\Sigma_k = f_k(A_\varepsilon)$ is transversal to the orbits of ψ and so is $J = \bigcap_{k=1}^{n-1} \Sigma_k$. Note that $p \in J$. The vector fields $\hat{Y}_k = (f_k)_* \hat{X}_k$ and $\hat{Y}_n = (f_n)_* \hat{X}_n$ are tangent to Σ_k and define a local C^r -action of $\mathbb{R}^2$ on Σ_k . The map $\alpha_k : [0, 1] \rightarrow \Sigma_k$ given by $\alpha_k(\tau) = \hat{Y}_k^\tau(p)$ is a parametrization of S_k . Let $\omega_k : (J, p) \rightarrow (J, p)$ be the Poincaré map of α_k and

$$(2.2) \quad \text{Hol} : \pi_1(\mathcal{O}_p, p) \cong \mathbb{Z}^k \rightarrow \text{Diff}^r(J, p)$$

the holonomy of $\mathcal{O}_p$ as a leaf of the foliation defined by the orbits of ψ . Then, $\omega_k = \text{Hol}([\alpha_k])$. Write J as the union of two intervals $J^+ \cup J^-$ with $J^+ \cap J^- = \{p\}$. Since $\mathcal{O}_p$ is two-sided in N , each ω_i leaves J^+ (J^-) invariant.

Remark 2.3. Note that $\{X_1, \dots, X_{k-1}, \widehat{X}_k, X_{k+1}, \dots, X_{n-1}, \widehat{X}_n\}$ define a local $\mathbb{R}^n$ -action $\widehat{\varphi}$ on $A \times D_k^{n-2}(\varepsilon)$ and that $\mathcal{O}_{(\theta, x, t)}(\widehat{\varphi}) = \mathcal{O}_{(\theta, x, t)}(h_k \circ \varphi \circ h_k^{-1})$ for each $(\theta, x, t) \in A \times D_k^{n-2}(\varepsilon)$.

The following lemma is essential in the proof of Theorem A.

Lemma 2.4. Let $\mathcal{O}_p$ be a T^{n-1} -orbit of $\psi \in A^2(\mathbb{R}^{n-1}, N)$ and assume that ψ has no T^{n-1} -orbits, aside $\mathcal{O}_p$, in a neighborhood V of $\mathcal{O}_p$. Then there exists a neighborhood I^+ of p in J^+ such that for each $k \in \{1, \dots, n-1\}$ one of the following statements is verified:

- (1) $\omega_k|_{I^+} = \text{id}$; i.e., every $\widehat{Y}_k$ -orbit near S_k is periodic.
- (2) Either $\omega_k|_{I^+}$ or $(\omega_k|_{I^+})^{-1}$ is a topological contraction, i.e., every $\widehat{Y}_k$ -orbit near S_k spirals towards S_k .

Proof. We give the proof for $k = n-1$; the other cases are similar. Assume that ω_{n-1} does not satisfy neither (1) nor (2). Then, there is a sequence $\{q_l \in J^+; l \in \mathbb{N}\}$ such that $\omega_{n-1}(q_l) = q_l$ and $\lim_{l \rightarrow \infty} q_l = p$. We claim that p is an isolated fixed point of ω_j for at least one $j \in \{1, \dots, n-2\}$. Otherwise, for each $1 \leq j \leq n-2$ there exists a sequence $\{q_{jk} \in J^+; k \in \mathbb{N}\}$ such that $\omega_j(q_{jk}) = q_{jk}$ and $\lim_{k \rightarrow \infty} q_{jk} = p$. If $q_{jk} \in V$ and $\omega_i(q_{jk}) = q_{jk}$ for each $i \in \{1, \dots, n-1\}$, then the ψ -orbit of q_{jk} is a T^{n-1} -orbit. Therefore, for each $q_{jk} \in V$ there exists $i \neq j$ such that $\omega_i(q_{jk}) \neq q_{jk}$. Let $q_k = \lim_{m \rightarrow \infty} \omega_i^m(q_{jk}) \neq p$. It follows from the commutativity of ω_i and ω_j that $q_k \in \text{Fix}(\omega_i) \cap \text{Fix}(\omega_j)$. If $\omega_\ell(q_k) \neq q_k$, then $p \neq \lim_{m \rightarrow \infty} \omega_\ell^m(q_k) \in \text{Fix}(\omega_\ell) \cap \text{Fix}(\omega_i) \cap \text{Fix}(\omega_j)$. Repeating this process, if necessary, we obtain a point $q \in \bigcap_{i=1}^{n-1} \text{Fix}(\omega_i)$ with $q \neq p$. But, this would imply that $\mathcal{O}_q$ is a T^{n-1} -orbit, contradicting one of the hypothesis. Thus, it exists $j \in \{1, \dots, n-2\}$ such that p is an isolated fixed point of ω_j , i.e., there exists a neighborhood I^+ of p in J^+ such that $I^+ \cap \text{Fix}(\omega_j) = \{p\}$. By N. Kopell Lemma [4], $\omega_{n-1}|_{I^+} = \text{id}$. This new contradiction proves that ω_{n-1} satisfies (1) or (2). $\square$

Proof of Theorem A. It is a classical result in foliation theory that the leave structure of a foliation in the neighborhood of a compact leaf is determined by the holonomy of such leaf. The holonomy of the orbit $T^{n-1} \times \{0\}$, given in Lemma 2.4, guarantees that there exist $d \in (0, 1)$ and $s \in \{0, \dots, n-2\}$ such that every ψ -orbit by points in $T^{n-1} \times [0, d]$ is homeomorphic to $T^s \times \mathbb{R}^{n-s-1}$. We claim that the saturated V of $T^{n-1} \times [0, d]$ by ψ is equal to $T^{n-1} \times [0, 1)$ and this would conclude the proof. In fact, if $V \neq T^{n-1} \times [0, 1)$, then $\text{Front}(V) \cap T^{n-1} \times \{1\} = \emptyset$. Let $C \neq T^{n-1} \times \{0\}$ be a connected component of $\text{Front}(V)$. C is a compact ψ -invariant subset and contains a minimal subset μ . By a theorem of Sacksteder [5, Theorem 7], μ can not be an exceptional minimal set. Thus μ is a compact orbit. This contradiction proves that $V = T^{n-1} \times [0, 1)$. $\square$

Lemma 2.5. If $\varphi \in A^1(\mathbb{R}^n, N)$ has an orbit $\mathcal{O}$ diffeomorphic to $T^{n-1} \times \mathbb{R}$, then $\text{Front}(\mathcal{O})$ is the union of at most two T^k -orbits with $k \in \{n-2, n-1\}$.

The proof of this lemma is given in [1].

Corollary 2.6. Let $\mathcal{O}$ be an T^{n-1} -orbit of $\varphi \in A^1(\mathbb{R}^n, N)$ and V be a neighborhood of $\mathcal{O}$ such that $V \setminus \mathcal{O}$ has exactly two connected components V_1 and V_2 . Assume that there exist points $p, q \in V_1$ such that $\dim \mathcal{O}_p = n$, $\mathcal{O}_p \neq \mathcal{O}_q$ and $\text{cl}(\mathcal{O}_p) \supset \mathcal{O} \subset \text{cl}(\mathcal{O}_q)$. Then, $\mathcal{O}_p$ is not a $T^{n-1} \times \mathbb{R}$ -orbit.

Proposition 2.7. Assume that $\varphi \in A^2(\mathbb{R}^n, N)$ with $n \geq 2$. If $\mathcal{O}$ is a $T^k \times \mathbb{R}^{n-k}$ -orbit with $k \leq n-2$, then $\text{cl}(\mathcal{O})$ can not contain a T^{n-1} -orbit.

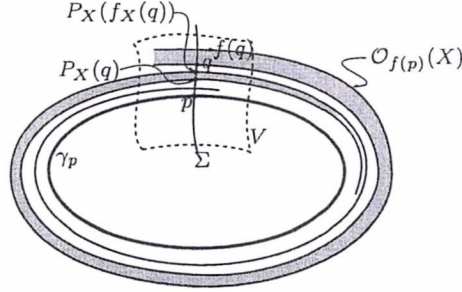


FIGURE 1.

If $X \in \mathfrak{X}^r(M^2)$, let $\mathcal{C}(X)$ be the set of diffeomorphisms $f \in \text{Diff}^r(M^2)$ that preserve orbits of X . Assume that the orbit γ_p of X by p is periodic of period τ . Let Σ be a cross section to X at p and $P_X : (\Sigma, p) \rightarrow (\Sigma, p)$ be Poincaré map.

Lemma 2.8. *There exists a cross section $\Sigma_p \subset \Sigma$ and a neighborhood $\mathcal{V} \subset \text{Diff}^r(M^2)$ of the identity map in the C^0 topology such that every $f \in \mathcal{C}(X) \cap \mathcal{V}$ induces a local diffeomorphism $f_X : (\Sigma_p, p) \rightarrow (\Sigma_p, p)$ of class C^r , with $f_X \circ P_X = P_X \circ f_X$.*

Proof. Let $\tau > 0$ be the period of γ_p . We can assume that $\Sigma = h^{-1}(\{0\} \times (-1, 1))$, where $h : V \rightarrow (-1, 1)^2$ is a flow box for X at p . Let $\pi : V \rightarrow \Sigma$ be the projection along of the orbits of X and recall that $P_X = \pi \circ X^\tau$. Put $U = h^{-1}((-1/2, 1/2)^2)$ and $\Sigma_\varepsilon = h^{-1}(\{0\} \times (-\varepsilon, \varepsilon))$. There exists a neighborhood $\mathcal{V}$ of $\text{id} \in \text{Diff}^r(M^2)$, in the C^0 topology, and $\varepsilon > 0$ such that $f(U) \subset V$ and $f(\Sigma_\varepsilon) \subset U$ for each $f \in \mathcal{V}$ and also $X^\tau(\Sigma_\varepsilon) \subset U$. Choose $\Sigma_p = \Sigma_\varepsilon$. For each $f \in \mathcal{V}$ there is defined a map $f_X : \Sigma_p \rightarrow \Sigma$ by $f_X(q) = \pi(f(q))$. We are going to show that $f_X \circ P_X = P_X \circ f_X$. Let $[x, y] \subset \Sigma$ be the arc with extremes x and y and define $x \leq y$ if $[p, x] \subseteq [p, y]$. Assume that $P_X(q) \leq q$ for every $q \in \Sigma_p$. For a fixed $f \in \mathcal{V}$ there are two possibilities $f_X(q) \geq q$ or $f_X(q) \leq q$. Let us consider the case $P_X(q) \leq q$ and $f_X(q) \geq q$. If $P_X(q) = q$, i.e., the orbit of q by X is periodic, then the orbit by $f_X(q)$ is also closed, therefore $f_X \circ P_X(q) = f_X(q) = P_X \circ f_X(q)$. If $P_X(q) < q$, then $f_X(P_X(q))$ belongs to the orbit of X by $f_X(q)$ and $P_X(q) \leq f_X(P_X(q)) < f_X(q)$. Thus, $f_X \circ P_X(q) = P_X \circ f_X(q)$. The other cases are analogous. $\square$

Proof of Proposition 2.7. We begin proving this proposition for $n = 2$. Let $\varphi \in A^2(\mathbb{R}^2, N)$ and assume that there exist an $\mathbb{R}^2$ -orbit P such that its closure, $\text{cl}(P)$, contains an S^1 -orbit $\mathcal{O}_0$, see Figure 2. Then, there exists at least one $\mathbb{R}$ -orbit $\mathcal{O}$ contained in $\text{Front}(P)$ such that $\text{cl}(\mathcal{O}) \supset \mathcal{O}_0$. If G_0 and G are the isotropy groups of $\mathcal{O}_0$ and $\mathcal{O}$, respectively, then $G = G_0^0$. Let $X_i = X_{w_i}$, $i = 1, 2$, where $\{w_1, w_2\}$ is a set of generators of G_0 such that $w_2 \in G_0^0$. Note that $X_2|_{\mathcal{O} \cup \mathcal{O}_0} \equiv 0$ and $\mathcal{O}_0$ is a periodic orbit of X_1 of period 1. Let Σ be a transversal section to X_1 at $p \in \mathcal{O}_0$ and $P_{X_1} : (\Sigma, p) \rightarrow (\Sigma, p)$ be the Poincaré map. Without loss of generality we can assume that P_{X_1} is a topological contraction, otherwise we consider the Poincaré map of $-X_1$. Let $\mathcal{V}$ and Σ_p be as in the Lemma 2.8 and $\delta > 0$ such that $X_2^t \in \mathcal{V}$ for all $t \in (-\delta, \delta)$. Fix a $t \neq 0$ in the interval $(-\delta, \delta)$ and put $f = X_2^t$. It follows from the commutativity of X_1 with X_2 that $f \in \mathcal{C}(X_1)$. Note that $f_{X_1}(q) = q$ if $q \in \mathcal{O} \cap \Sigma_p$ and that $f_{X_1}(q) \neq q$ if $q \in P$. By Lemma 2.8, $f_{X_1} \circ P_{X_1} = P_{X_1} \circ f_{X_1}$ and by N. Kopell Lemma $f_{X_1} = \text{id}$. This contradiction completes the proof in the case $n = 2$.

Assume now that $n \geq 3$ and that there exist a $T^k \times \mathbb{R}^{n-k}$ -orbit P , with $k < n - 1$, such that $\text{cl}(P)$ contains a T^{n-1} -orbit $\mathcal{O}_0$. If G_0 and G_P are the

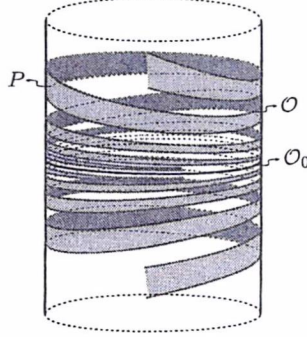


FIGURE 2.

isotropy groups of $\mathcal{O}_0$ and P , respectively, then $G_P \subset G_0$. Let $X_1, \dots, X_n$ the infinitesimal generators adapted to $\mathcal{O}_0$ such that the linear $(n-2)$ -subspace H of $\mathbb{R}^n$ generated by $\{w_1, \dots, w_{n-2}\}$ contains G_P . There exists a neighborhood V of $\mathcal{O}_0$ such that the action $\psi_\varphi \in A^2(\mathbb{R}^{n-1}, N)$ induced by φ and $\mathcal{O}_0$, i.e., generated by $X_1, \dots, X_{n-1}$, has not T^{n-1} -orbits inside V . Otherwise, by Corollary 2.6, either P is an $T^{n-1} \times \mathbb{R}$ -orbit or $\mathcal{O}_0 \not\subset \text{cl}(P)$. If $\mathcal{O}$ is a $T^s \times \mathbb{R}^{n-s-1}$ -orbit of φ such that $\mathcal{O} \subset \text{cl}(P)$, then $\mathcal{O}_0 \subset \text{cl}(\mathcal{O})$ and $s < n-1$. Consequently, if $G_{\mathcal{O}}$ is the isotropy group of $\mathcal{O}$, then $G_{\mathcal{O}} \subset G_0$ and $G_{\mathcal{O}}^0 = G_0^0$. Since $k, s < n-1$, we can assume that $H \cap G_{\mathcal{O}}$ is isomorphic to $\mathbb{Z}^s$ and $w_{n-1}, w_n \notin H$. Let $p \in \mathcal{O}_0$ and consider $\omega_1, \dots, \omega_{n-1}$ as in the proof of Lemma 2.4. By Lemma 2.4 ω_{n-1} (or ω_{n-1}^{-1}) is a topological contraction. Therefore, if $q \in \Sigma_{n-1} \cap P$, then $\mathcal{O}_{h_{n-1}^{-1}(q)}(\widehat{\varphi}_{n-1})$ is a $\mathbb{R}^2$ -orbit in A that contains the $\widehat{\varphi}_{n-1}$ -orbit $S^1 \times \{0\}$ in its closure. By the first part of the proof this is a contradiction. $\square$

Remark 2.9. Let $\varphi \in \mathcal{A}_n$, $p \in \text{Sing}_{n-2}^c(\varphi)$ and V_p a neighborhood of $\mathcal{O}_p$.

(a) Assume that V_p satisfies (1) and $\text{Front}(V_p)$ is a T^{n-1} -orbit. Since $r \geq 2$, it follows by Proposition 2.7, that there is no $T^s \times \mathbb{R}^{n-s}$ -orbit with $s \neq n-1$ inside V_p . Thus, we can say that one of the following possibilities is satisfied:

- (a1) $V_p \setminus \mathcal{O}_p$ is a $T^{n-1} \times \mathbb{R}$ -orbit;
- (a2) V_p contains infinitely many $T^{n-1} \times \mathbb{R}$ -orbits;
- (a3) $V_p \setminus \mathcal{O}_p$ contains only $(n-1)$ -dimensional orbits.

(b) Assume now that V_p satisfies (2), then:

- (b1) if there is one $T^s \times \mathbb{R}^{n-s-1}$ -orbit, $s \neq n-1$, such that its closure contains $\mathcal{O}_p$, then every n -orbit in $V_p \setminus \mathcal{O}_p$ is not homeomorphic to $T^{n-1} \times \mathbb{R}$;
- (b2) if there are no $T^s \times \mathbb{R}^{n-s-1}$ -orbits, $s \neq n-1$, which contain $\mathcal{O}_p$ in its closure, then there is only one n -orbit and it is homeomorphic to $T^{n-1} \times \mathbb{R}$.

Proposition 2.10. Let $\mathcal{O}_0$ be a T^{n-1} -orbit of $\varphi \in A^2(\mathbb{R}^n, N)$. Then, there exists a φ -invariant neighborhood V_0 of $\mathcal{O}_0$, such that every connected component U of $V_0 \setminus \mathcal{O}_0$ satisfies one of the following properties:

- (1) U is an $T^{n-1} \times \mathbb{R}$ -orbit;
- (2) U contains infinitely many T^{n-1} -orbits approaching $\mathcal{O}_0$ and every n -dimensional orbit inside U is a $T^{n-1} \times \mathbb{R}$ -orbit;
- (3) There exist $s \in \{0, 1, \dots, n-2\}$ such that U is the union of $T^s \times \mathbb{R}^{n-s-1}$ -orbits.

Proof. By the continuity of the infinitesimal generators of φ there is a neighborhood V of $\mathcal{O}_0$ such that every orbit by points in V_0 has dimension at least $n-1$. Since

$r \geq 2$, it follows from Proposition 2.7 that there are no $T^s \times \mathbb{R}^{n-s}$ -orbits, $s \neq n-1$ approaching $\mathcal{O}_0$. Therefore, if U one connected component of $V \setminus \mathcal{O}_0$, then there are two possibilities:

(i) *There are infinitely many n -orbits in U approaching F .* In this case we will show that (2) is verified. The other possibility is that there exists a sequence $\{\mathcal{O}_i\}_{i \in \mathbb{N}}$ of n -dimensional orbits inside U , which are not homeomorphic to $T^{n-1} \times \mathbb{R}$ and that approach $\mathcal{O}_0$. Since $\text{cl}(\mathcal{O}_i) \cap \text{Sing}_j(\varphi) = \emptyset$ for each $j \in \{0, \dots, n-2\}$, then the compact set $\text{cl}(\mathcal{O}_i) \setminus \mathcal{O}_i$ is the union of $(n-1)$ -dimensional orbits, of which at least one is compact, this contradicts the Proposition 2.7.

(ii) *There are only a finite number of n -orbits in U .* If it happens to exist a $T^{n-1} \times \mathbb{R}$ -orbit $\mathcal{O}$ such that $\text{Front}(\mathcal{O}) \supset \mathcal{O}_0$, then we can assume that $U = \mathcal{O}$, therefore (1) is verified. If there is no such $T^{n-1} \times \mathbb{R}$ -orbit, then we can assume that the orbit of each $p \in U$ is $(n-1)$ -dimensional. We also can assume that φ has no T^{n-1} -orbits inside U , otherwise (2) is verified. Reducing the size of V it follows, from Lemma 2.4 that there exist $s \in \{0, \dots, n-2\}$ such that $\mathcal{O}_p$ is a $T^s \times \mathbb{R}^{n-s-1}$ -orbit for each $p \in U$. Thus, (3) is verified and this completes the proof. $\square$

Theorem 2.11 (D. Tischler, [6]). *Let N be a compact orientable n -manifold with non-empty boundary. Suppose that ψ is a C^2 locally free action of $\mathbb{R}^{n-1}$ on N , then N is diffeomorphic to $T^{n-1} \times [0, 1]$.*

Figure 3 illustrates Theorem B for some $\varphi \in \mathcal{A}_2$.

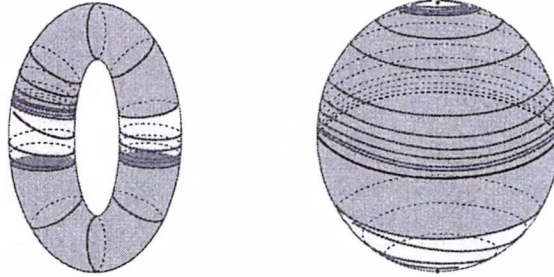


FIGURE 3.

Proof of Theorem B. Let $\mathcal{O}$ be a $T^{n-1} \times \mathbb{R}$ -orbit and $\mathcal{U}$ the family of all φ -invariant neighborhoods $U \supset \mathcal{O}$ homeomorphic to $T^{n-1} \times \mathbb{R}$ that do not contain a $T^s \times \mathbb{R}^{n-s}$ -orbit with $s < n-1$. The inclusion relation defines a partial order in $\mathcal{U}$ and by Zorn's Lemma there exists a maximal element U_M in $\mathcal{U}$. We are going to show that $N \setminus U_M$ is either an T^{n-1} -orbit or the union of two T^{n-2} -orbits. Assume that $N \setminus U_M$ has non-empty interior, then $\text{cl}(U_M) \setminus U_M$ has two connected components. By Definition 2.2 at each $p \in \text{Sing}_i(\varphi)$, $i = 1, \dots, n-3$, φ is of type n and n -finite. This fact and Lemma 2.5 implies that $(\text{cl}(U_M) \setminus U_M) \cap \text{Sing}_i(\varphi) = \emptyset$, $i = 1, \dots, n-3$. Moreover, there exists one connected component F of $\text{cl}(U_M) \setminus U_M$ that is not a T^{n-2} -orbit. We know that F is φ -invariant and will show that $F \cap \text{Sing}_{n-2}^c(\varphi) = \emptyset$. In fact, if there exists $p \in F$ such that $\mathcal{O}_p$ is a T^{n-2} -orbit and V_p is a neighborhood of $\mathcal{O}_p$ that satisfies Definition 2.2 (1), then $U_M \cup V_p$ would be a member of $\mathcal{U}$ containing U_M properly. If V_p satisfies condition (2), then, since $F \neq \mathcal{O}_p$, there are $T^s \times \mathbb{R}^{n-s-1}$ -orbits, $s \neq n-1$, arriving at $\mathcal{O}_p$ and by Remark 2.9 we would have $T^l \times \mathbb{R}^{n-l}$ -orbits, $l \neq n-1$, inside U_M . Therefore F is an T^{n-1} -orbit. If (1) or (2) of Proposition 2.10 is verified, then there exists an open φ -invariant set V homeomorphic to $T^{n-1} \times \mathbb{R}$, which does

not contain $T^s \times \mathbb{R}^{n-s}$ -orbits with $s \neq n-1$ and such that $\text{Front}(V) \subset F$. If (3) of Proposition 2.10 is verified, then by Theorem 2.11 there exists an open φ -invariant set V homeomorphic to $T^{n-1} \times \mathbb{R}$ and such that $\text{Front}(V) \subset F$. The open set $U_M \cup V \in \mathcal{U}$ and contains properly U_M , but this contradicts the fact that U_M is maximal. Thus, $\text{Front}(U_M) = N \setminus U_M$.

Assume that $\text{Sing}_{n-2}^c(\varphi) = \emptyset$, then $\text{Front}(U_M)$ is homeomorphic to T^{n-1} . Therefore, N is T^{n-1} bundle over S^1 . If $\text{Sing}_{n-2}^c(\varphi) \neq \emptyset$, then $\text{Front}(U_M)$ is the union of two T^{n-2} -orbits consequently, $N \in \mathcal{H}_n$. $\square$

REFERENCES

- [1] J. L. ARRAUT AND C. A. MAQUERA, *Structural stability of singular actions of $\mathbb{R}^n$ having a first integral*. Submitted for publication, (2003).
- [2] C. CAMACHO, *Morse-Smale $\mathbb{R}^2$ -actions on two manifolds*. Dynamical Systems, Editor M. M. Peixoto, Academic Press, (1973), 71–75.
- [3] G. CHATELET, H. ROSENBERG AND D. WEIL, *A classification of the topological types of $\mathbb{R}^2$ -actions on closed orientable 3-manifolds*. Publ. Math. IHES, 43, (1974), 261–272.
- [4] N. KOPELL, *Commuting Diffeomorphisms*, Proc. Amer. Math. Soc. Symp., 14, (1968), 165–184.
- [5] R. SACKSTEDER, *Foliations and pseudogroups*, Amer. J. Math. 87, (1965), 79–102.
- [6] D. TISCHLER, *Manifolds M^n of rank $n-1$* . Proc. Amer. Math. Soc., 94, 1, (1985), 158–160.

J. L. ARRAUT AND C. A. MAQUERA, UNIVERSIDADE DE SÃO PAULO - SÃO CARLOS, INSTITUTO DE CIÊNCIAS MATEMÁTICAS E DE COMPUTAÇÃO, AV. DO TRABALHADOR SÃO-CARLENSE 400, 13560-970 SÃO CARLOS, SP, BRAZIL

E-mail address: arraut@icmc.usp.br

E-mail address: cmaquera@icmc.usp.br

NOTAS DO ICMC

SÉRIE MATEMÁTICA

- 173/2003 FEDERSON, M.; TÁBOAS, P.Z. – Topological dynamics of retarded functional differential equations.
- 172/2003 SANTOS, R.N.A.; RUAS, M.A.S. – Real Milnor fibrations and (C) – regularity.
- 171/2003 ALVES, C.O.; SOARES, S.H.M. – On the location and profile of spike-layer nodal solutions to nonlinear schrödinger equations.
- 170/2003 OLIVEIRA, C.P.; MENEGATTO, V.A. – Orthogonal bases for spaces of complex spherical harmonics.
- 169/2003 OLIVEIRA, C.P.; MENEGATTO, V.A. – Annihilating properties of complex spherical convolution operators.
- 168/2003 CARVALHO, A. N.; PRIMO, M.R.T. – Spatial homogeneity in parabolic problems with nonlinear boundary conditions.
- 167/2003 CARVALHO, A. N.; BRUSCHI, S. M. – Upper semicontinuity of attractors for the discretization of strongly damped wave equations.
- 166/2003 GUÍÑEZ, V.; GUTIERREZ, C. – Rank – 1 codimension one singularities of positive quadratic differential forms.
- 165/2003 GUTIERREZ, C.; OLIVEIRA, R.D.S.; TEIXEIRA, M.A. – Positive quadratic differential forms: topological equivalence through Newton polydera.
- 164/2003 BIASI, C.; GUTIERREZ, C. – Finite branched coverings in a generalized inverse mapping theorem.