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Global Attractors for Parabolic Problems with Nonlinear Boundary Conditions in Fractional Power Spaces

by

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1. Introduction and Statement of the Results

Let Ω be a bounded smooth domain of $\mathbb{R}^2$. In this paper we consider reaction diffusion systems with dispersion of the form

$$\left.\begin{aligned} & u_t - \mathrm{Div}(a\nabla u) + \textstyle\sum_{j=1}^{2} B_j(x)\frac{\partial u}{\partial x_j} + \lambda u + f(u) = 0, \qquad \text{in} \quad \Omega, \\ & \frac{\partial u}{\partial n_a} + g(u) = 0, \qquad \text{in} \quad \partial\Omega. \end{aligned}\right\} \tag{1.1}$$

where $u = (u_1, \cdots, u_N)^\top$, $N \geq 1$, $a(x) = \mathrm{diag}(a_1(x), \cdots, a_N(x))$, $a_i \in C^1(\overline{\Omega})$, $a_i(x) > m_0 > 0$, $x \in \Omega$, $1 \leq i \leq N$, $\frac{\partial u}{\partial n_a} = \langle a\nabla u, \vec{n}\rangle$, $\vec{n}$ is the outward normal and $B_j = \mathrm{diag}(b_j^1, \cdots, b_j^N)$ is continuous in $\bar{\Omega}$, $j = 1,2$. Let $\bar{\xi}_{i0}, \xi_{i0} > 0$, $1 \leq i \leq N$ and $f = (f_1, \cdots, f_N)^\top : \mathbb{R}^N \to \mathbb{R}^N$, $g = (g_1, \cdots, g_N)^\top : \mathbb{R}^N \to \mathbb{R}^N$ be smooth functions satisfying

$$\begin{aligned} & u_i(\lambda u_i + f_i(u)) > 0 \\ & u_i\, g_i(u) > 0, \end{aligned} \tag{1.2}$$

$u_i \notin \Sigma_{i0} =: [-\bar{\xi}_{i0}, \xi_{i0}]$, $1 \leq i \leq N$. In other words, at the boundary of rectangles with faces parallel to $\Sigma_0 := \Pi_{i=1}^N \Sigma_{i0}$ and containing Σ_0, the vector fields $\lambda u + f(u)$ and $g(u)$ point outward.

We are interested on global well posedness and on existence of global attractors for such problems.

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It has been proved in Carvalho and Rodriguez-Bernal [1994], for the scalar case, N=1, and $\Omega \subset \mathbb{R}^n$, $n \leq 3$, that under some growth assumptions on the nonlinearity f, the problem (1.1), with $B_j \equiv 0$, $j = 1,2$, has a global attractor in $H^1(\Omega)$. More specifically, for $n = 2$, f and g are required to satisfy, instead of (1.2), $\liminf_{|s|\to\infty} \frac{\lambda s + f(s)}{s} \geq 0$, $\liminf_{|s|\to\infty} \frac{g(s)}{s} \geq 0$, one of the inequalities being strict, and

$$\lim_{|s|\to\infty} \frac{|f'(s)|}{e^{\eta |s|^2}} = \lim_{|s|\to\infty} \frac{|g''(s)|}{e^{\eta |s|^2}} = 0, \ \forall \eta > 0,$$

(see also, Hale [1988], Hale and Raugel [1992], Carvalho [1992] and Carvalho and Oliveira [1992], for the case $g = 0$ and more restrictive hypothesis on f).

These growth assumptions are used to obtain local existence of solutions for (1.1) and also play a role in obtaining energy estimates necessary to guarantee that the solution operator for (1.1) defines a global dynamical system which is bounded dissipative.

We will show below, that by restricting the space of initial data, no growth assumptions are needed for local existence of solutions for (1.1).

In particular, we will consider spaces which are embedded in $C(\overline{\Omega})$. Since all functions in this space are bounded, no growth conditions on f and g are required for local existence. Our aim is to show that in such spaces the problem (1.1) has a global attractor and to obtain some good estimates for the size of the attractor in the uniform norm. This problem has been studied in Carvalho and Ruas-Filho [1993], for the case $g = 0$.

Before we proceed let us introduce some notation. Let $X = L^2(\Omega, \mathbb{R}^N)$ and $A : D(A) \subset X \to X$ be the operator $A = \mathrm{diag}(A_1, \cdots A_N)$ defined by

$$D(A_i) = \{\phi \in H^2(\Omega) : \tfrac{\partial \phi}{\partial n_{a_i}} = 0\},$$

$$A_i \phi = -\mathrm{Div}(a_i(x)\nabla\phi) + \textstyle\sum_{j=1}^{2} b_j^i(x)\frac{\partial \phi}{\partial x_j} + \lambda\phi,$$

where $\dfrac{\partial u}{\partial n_{a_i}} = a_i \langle \nabla u, \vec{n}\rangle$, $i = 1, \ldots, N$.

We can define the fractional powers A^α of A (see, for example, Henry [1981]) and the fractional power spaces $X^\alpha := D(A^\alpha)$ endowed with the graph norm, $\alpha \in \mathbb{R}$, where $X^\alpha = (X^{-\alpha})'$, if $\alpha < 0$. In this case we can always view A as a sectorial operator with compact resolvent from $X^{\alpha+1}$ into X^α which is positive and self adjoint if $B_j \equiv 0$, for $j = 1,2$.

Hale [1986] proved the existence of a local attractor for (1.1), with $g = 0$, which coincides with the embedding of the attractor for $\dot{u} + \lambda u + f(u) = 0$ into the subspace of constant functions of X^α, $\alpha > \frac{3}{4}$, if the diffusion coefficient $a(x)$ is large (see also, Hale and Rocha [1987a,b] and Hale and Sakamoto [1989]). However, the techniques employed by Hale [1986] would only apply to global attractors if some a priori bound on the size of the absorbing set could be obtained and only if the diffusion coefficient is large (see Carvalho [1992] and Carvalho and Oliveira [1992]). This a priori bounds are obtained in Carvalho and Ruas-Filho [1993] for the case $g = 0$, also working in X^α, for $\alpha > \frac{3}{4}$ and $n = 3$.

Here we prove the existence of a global attractor for the problem (1.1) regardless of the size of the diffusion coefficient $a(x)$ and for $g \neq 0$. We also give uniform bounds on the size of the attractor which will make the results of Carvalho [1992] and Carvalho and Oliveira [1992] applicable to the case $\alpha \neq \frac{1}{2}$ as in Hale [1986], Hale and Rocha[1987a,b] and Hale and Sakamoto [1989] (see also Fusco [1987]).

To carry on this project we need to obtain that the solution operator associated to (1.1) is globally defined, that orbits of bounded subsets of X^α, under the flow defined by (1.1), are bounded subsets of X^α and that there is a bounded set that attracts points of X^α. Since the solution operator associated to (1.1) is compact, Theorem 3.4.6 in Hale [1989] would guarantee the existence of a global attractor.

Let us mention that Hale [1986], Hale and Rocha [1987a,b], Hale and Sakamoto [1989] and Carvalho and Ruas-Filho [1993], work in X^α for $\alpha > \frac{3}{4}$ and $n = 3$. The reason for this is the embedding $X^\alpha \subset C(\overline{\Omega})$. However, for $\alpha > \frac{3}{4}$, the space X^α incorporates the boundary condition $\frac{\partial u}{\partial \vec{n}_a} = 0$ and therefore, this cannot be the right space to work on with $g \neq 0$. However, for $n = 2$ and $\frac{3}{4} > \alpha > \frac{1}{2}$, we have that $X^\alpha \subset C(\overline{\Omega})$ and X^α does not incorporate any boundary condition. Therefore, we work in this range of α.

Observe that for $\Omega \subset \mathbb{R}$ we can take $\alpha = \frac{1}{2}$ and all the results hold true without any changes. Therefore, we restrict the presentation to the two dimensional case.

Next let us state our main results. In Section 3, we prove

Theorem 1.1 *Under the above hypotheses, the solutions of* (1.1) *with initial data in* X^α, $\frac{3}{4} > \alpha > \frac{1}{2}$, *are globally defined and orbits of bounded subsets of* X^α, *under the flow defined by* (1.1), *are also bounded subsets of* X^α.

Then, in Section 4, we prove

Theorem 1.2 *Suppose that* $f, g : \mathbb{R}^N \to \mathbb{R}^N$ *are* C^1 *and* C^2, *respectively, and satisfy (1.2). Then, the problem (1.1) has a global attractor* $\mathcal{A}$ *in* X^α. *Furthermore,*

$$u(x) \in \Sigma_0,\ \forall x \in \Omega,\ \forall u \in \mathcal{A}.$$

Once we prove this theorem, the next result follows from Hale [1989] and from the fact that a constant equilibrium u_0 for (1.1) may only happen if $\lambda u_0 + f(u_0) = g(u_0) = 0$.

Theorem 1.3 *The elliptic problem*

$$-Div(a\nabla u) + \textstyle\sum_{j=1}^{2} B_j(x)\frac{\partial u}{\partial x_j} + \lambda u + f(u) = 0$$

$$\frac{\partial u}{\partial n_a} + g(u) = 0$$

has at least one solution which is nontrivial if $\lambda I + f$ *and* g *have no common zeros.*

If we assume that (1.1) has gradient structure; that is, there are scalar potentials F and G such that $\nabla F = f$ and $\nabla G = g$ and $B_j \equiv 0$, $j = 1, 2$, we prove that

Theorem 1.4 *Under the above assumptions, the attractor $\mathcal{A}$ for (1.1) verifies*

$$\mathcal{A} = W^u(E),$$

where $W^u(E)$ is the unstable set, in X^α, of the set of equilibrium points E.

This is done in Section 5. Finally, in Section 6, we prove analogous results for alternative dissipativeness conditions.

The tools employed are the invariance theory as in Henry [1981], the theory of invariant regions of Chueh, Conley and Smoller [1977] and the results in Rodriguez-Bernal [1993] and Carvalho and Rodriguez-Bernal [1994].

2. Local Existence

It is well known that $A = \mathrm{diag}(A_1, \cdots A_N)$ defined by

$$D(A_i) = \{\phi \in H^2(\Omega) : \tfrac{\partial \phi}{\partial n_{a_i}} = 0\},$$

$$A_i\phi = -\mathrm{Div}(a_i \nabla \phi) + \textstyle\sum_{j=1}^{2} b_j^i(x)\frac{\partial \phi}{\partial x_j} + \lambda\phi,$$

where $\dfrac{\partial u}{\partial n_{a_i}} = a_i\langle \nabla u, \vec{n}\rangle$, $i = 1, \ldots, N$, generates an analytic semigroup on X^α and that, for suitable λ, it satisfies the following estimates

$$\begin{aligned} \|e^{-At}u_0\|_{X^\alpha} &\le Me^{-\epsilon t}\|u_0\|_{X^\alpha}, \quad t \ge 0, \\ \|e^{-At}u_0\|_{X^\alpha} &\le Me^{-\epsilon t}t^{-\alpha}\|u_0\|_X, \quad t > 0, \end{aligned} \tag{2.1}$$

for some $\epsilon > 0$, $M \ge 1$. In particular, if $B_j \equiv 0$, $j = 1, 2$, λ can be taken any positive number. On the other hand, if dispersion is present, λ has to be taken large enough for the semigroup to decay exponentially.

Assume that $h : X^\alpha \to X^\beta$, $\beta \in \mathbb{R}$, is a map which is bounded and locally Lipschitz continuous and consider the abstract parabolic problem

$$\begin{aligned} &\tfrac{du}{dt} + Au + h(u) = 0 \\ &u(0) = u_0 \in X^\alpha \end{aligned} \tag{2.2}$$

in X^β. Then, it has a unique solution in X^α, whenever $\alpha \in [\beta, \beta + 1)$ and for any $u_0 \in X^\alpha$. This solution $u(t, u_0)$ is defined in a maximal interval of existence $[0, t_{\max})$ and either $t_{\max} = +\infty$ or $t_{\max} < \infty$ and $\|u(t, u_0)\|_{X^\alpha} \to \infty$ as $t \to t_{\max}$ (see Henry [1981]).

Writing problem (1.1) in the form (2.2) and using the variation of constants formula, we can write the solution through $u_0 \in X^\alpha$ as

$$T(t)u_0 = e^{-At}u_0 - \int_0^t e^{-A(t-s)}h(T(s)u_0)ds.$$

In the particular case of the problems (1.1) and $\frac{3}{4} > \alpha > \frac{1}{2}$, following Rodriguez-Bernal [1993], we consider the functionals

$$h(u) := f_\Omega(u) + g_\Gamma(u)$$

which act on test functions in the following way

$$\langle h(u), \phi\rangle = \langle f_\Omega(u), \phi\rangle + \langle g_\Gamma(u), \gamma(\phi)\rangle$$

where γ denotes the trace operator and $f_\Omega : X^\alpha \to L^2(\Omega, \mathbb{R}^N)$ and $g_\Gamma : X^\alpha \to H^{\frac{1}{2}}(\Gamma, \mathbb{R}^N)$ are the maps defined by $f_\Omega(u)(x) = f(u(x))$ and $g_\Gamma(u) = \gamma(g_\Omega(u))$ with $g_\Omega(u)$ defined by $g_\Omega(u)(x) = g(u(x))$. Assume that f_Ω, g_Γ are maps which are Lipschitz continuous in bounded subsets of X^α; then, g_Γ, is Lipschitz continuous in bounded subsets of X^α.

Note that $\langle f_\Omega(u), \phi\rangle$ is well defined acting on test functions $\phi \in L^2(\Omega, \mathbb{R}^N)$, and that $\langle g_\Gamma(u), \gamma(\phi)\rangle$ is well defined acting on test functions $\phi \in H^r(\Omega, \mathbb{R}^N)$, for any $r > \frac{1}{2}$, since in this case $\gamma(\phi) \in L^2(\Gamma, \mathbb{R}^N)$. Therefore, with the hypothesis above $h(u)$ is a functional in $H^{-r}(\Omega, \mathbb{R}^N)$, $r > \frac{1}{2}$. Furthermore,

$$\langle f_\Omega(u) - f_\Omega(v), \phi\rangle \le \|f(u) - f(v)\|_{L^2(\Omega,\mathbb{R}^N)} \|\phi\|_{L^2(\Omega,\mathbb{R}^N)} \le L\|u - v\|_{X^\alpha} \|\phi\|_{L^2(\Omega,\mathbb{R}^N)}$$

and

$$\langle g_\Gamma(u) - g_\Gamma(u), \gamma(\phi)\rangle \le \|g(u) - g(v)\|_{L^2(\Gamma,\mathbb{R}^N)} \|\gamma(\phi)\|_{L^2(\Gamma,\mathbb{R}^N)} \le L\|u - v\|_{X^\alpha} \|\phi\|_{H^r(\Omega,\mathbb{R}^N)},$$

which proves that $h : X^\alpha \to H^{-r}(\Omega, \mathbb{R}^N)$, $\frac{1}{2} < r < 1$, is a map which is Lipschitz continuous in bounded subsets of $H^1(\Omega, \mathbb{R}^N)$. Therefore $\beta := -\frac{r}{2} \in (-\frac{1}{2}, -\frac{1}{4})$, and we have that problem (2.2) has a unique solution in X^α for any $u_0 \in X^\alpha$ and for any $\alpha \in [\beta, \beta + 1)$. In particular the problem is locally well posed in X^α for $\frac{3}{4} > \alpha > \frac{1}{2}$.

So, all we need is proving that f_Ω and g_Γ are Lipschitz continuous in bounded sets for (1.1) to be locally well posed in X^α.

Lemma 2.1 *Assume that $f : \mathbb{R}^N \to \mathbb{R}^N$ and $g : \mathbb{R}^N \to \mathbb{R}^N$ are C^1 and C^2 functions, respectively. Let $f_\Omega : X^\alpha \to L^2(\Omega, \mathbb{R}^N)$, $g_\Omega : X^\alpha \to H^1(\Omega, \mathbb{R}^N)$ and $g_\Gamma : X^\alpha \to H^{\frac{1}{2}}(\Gamma, \mathbb{R}^N)$ be the maps defined by*

$$f_\Omega(\phi)(x) = f(\phi(x)), \quad g_\Omega(\phi)(x) = g(\phi(x)), \quad \textit{and} \quad g_\Gamma = \gamma \circ g_\Omega.$$

Then, f_Ω and g_Γ are well defined maps which are Lipschitz continuous in bounded sets of X^α.

Proof: We prove only the assertions on g_Γ and g_Ω to illustrate the techniques. Let $u, v \in X^\alpha$ such that $\|u\|_{X^\alpha} \le R$ and $\|v\|_{X^\alpha} \le R$. Then, since $X^\alpha \subset C(\overline{\Omega})$ with continuous embedding, we have that $\|u\|_\infty \le K\,R$ and $\|v\|_\infty \le K\,R$ where K is the embedding constant for $X^\alpha \subset C(\overline{\Omega})$. Hence, for some $L_1(R) > 0$, we have that

$$\|g(u) - g(v)\|_X \le L_1(R)\|u - v\|_X \le L_1(R)\|u - v\|_{X^\alpha}$$

and for some $L_2(R) > 0$

$$\begin{aligned} \|g'(u)\nabla u - g'(v)\nabla v\|_X &\leq L_2(R)\|u-v\|_\infty\|\nabla u\|_X + L_1(R)\|\nabla u - \nabla v\|_X \\ &\leq L_3(R)\|u-v\|_{X^\alpha}. \end{aligned}$$

The result for g_Ω follows trivially from these estimates. The result for g_Γ follows from the fact that the trace operator $\gamma : H^1(\Omega, \mathbb{R}^N) \to H^{\frac{1}{2}}(\Gamma, \mathbb{R}^N)$ is bounded. $\square$

Note that, no growth assumptions are made on the nonlinearities f and g.

3. Boundedness of the Semigroup

In this section we prove that solutions of (1.1), with initial data in X^α, are globally defined and that orbits of bounded subsets of X^α, under the flow defined by (1.1), are also bounded subsets of X^α.

Lemma 3.1 *Let $\beta = -\frac{r}{2}$, $r > \frac{1}{2}$ and $\alpha + \frac{r}{2} < 1$. If $\|T(t)u_0\|_{C(\overline{\Omega})} \leq N_1$, for all $t \in [0, t_{\max})$, for some $N_1 > 0$; then, $t_{max} = +\infty$ and $\|T(t)u_0\|_{X^\alpha} \leq N_2$, for all $t \geq 0$.*

Proof: Using the variation of constants formula we obtain

$$\|T(t)u_0\|_{X^\alpha} \leq M\|u_0\|_\alpha e^{-\epsilon t} + MK\int_0^t e^{-\epsilon(t-s)}(t-s)^{-\alpha-\frac{r}{2}}ds. \tag{3.1}$$

with $K = \sup_{t\in[0,t_{\max})}\left\{\|f_\Omega(T(t)u_0)\|_{L^2(\Omega,\mathbb{R}^N)} + \|g_\Gamma(T(t)u_0)\|_{L^2(\Gamma,\mathbb{R}^N)}\right\} < \infty$, since,

$$\begin{aligned} |\langle h(T(s)u_0), \phi\rangle_{-r,r}| &\leq \left|\int_\Omega f(T(s)u_0)\phi\right| + \left|\int_\Gamma g_\Gamma(T(s)u_0)\gamma(\phi)\right| \\ &\leq \left(\|f_\Omega(T(s)u_0)\|_{L^2(\Omega,\mathbb{R}^N)} + \|g_\Gamma(T(s)u_0)\|_{L^2(\Gamma,\mathbb{R}^N)}\right)\|\phi\|_{H^r(\Omega,\mathbb{R}^N)}. \end{aligned}$$

since the right hand side of (3.1) is bounded for $t \in [0, t_{\max})$, the solutions must exist for $t \geq 0$ and the result is proved. $\square$

Therefore, if we are able to obtain estimates in $C(\overline{\Omega})$ for $T(t)u_0$, $0 \leq t < t_{\max}$, a similar estimate can be obtained in X^α, $\frac{1}{2} < \alpha < 1 - \frac{r}{2}$ and the solutions are globally defined. To obtain such estimates in $C(\overline{\Omega})$ we introduce the notion of invariant regions as in Smoller [1983].

Definition 3.1 *A set $\Sigma \subset \mathbb{R}^N$ is called a positively invariant region for the local solution of (1.1) if any solution $T(t)u_0$ satisfying $u_0(x) \in \Sigma$, $\forall x \in \Omega$ is such that $(T(t)u_0)(x) \in \Sigma$, $\forall x \in \Omega$ and for all t in the maximal interval of existence of the solution.*

Our next result characterizes some of the invariant regions of the problem (1.1) (see also Carvalho and Ruas-Filho [1993] for the case $g = 0$).

Theorem 3.1 *Let $\bar{\xi}_j, \xi_j > 0$, $1 \le j \le N$ be such that $u_j(\lambda u_j + f_j(u)) > 0$ and $u_j g_j(u) > 0$ for all $u \in \mathbb{R}^N$ with $u_j \notin [-\bar{\xi}_j, \xi_j] =: \Sigma_j$. Then, the rectangle $\Sigma = \Pi_{j=1}^N \Sigma_j$ is an invariant region for the local solution of (1.1).*

Proof: If there is a solution $v(x,t) = (v^1(x,t), v^2(x,t), \cdots, v^N(x,t))$ of (1.1) with initial data $v(x,0) = (v^1(x,0), v^2(x,0), \cdots, v^N(x,0)) \in \Sigma$ for all $x \in \bar{\Omega}$, that does not stay in $\Sigma_1 = (-\infty, \xi_1] \times \mathbb{R}^{N-1}$ for all $t \in [0, t_{\max})$, then there is a t_0 and $x_0 \in \overline{\Omega}$ such that

$$v^1(x,t) < \xi_1, \quad 0 \le t < t_0, \quad x \in \Omega, \quad \text{and} \quad v^1(x_0,t_0) = \xi_1.$$

We observe that $x_0 \notin \Gamma$ since in this case it would be a maximum point for v^1 with negative normal derivative, since $g_1(v) > 0$. This leads to a contradiction, hence $x_0 \in \Omega$.

Therefore, if $v^1(x_0,t) < \xi_1$, $\forall t \in [0,t_0)$ and $v^1(x_0,t_0) = \xi_1$ implies $v^1_t(x_0,t_0) < 0$; then, Σ_1 is invariant.

Observe that

$$v^1_t(x_0,t_0) - \mathrm{Div}(a_1 \nabla v^1(x_0,t_0)) + \lambda v^1(x_0,t_0) + \sum_{j=1}^{2} b^1_j(x_0) \frac{\partial v^1}{\partial x_j}(x_0,t_0) + f_1(v(x_0,t_0)) = 0. \quad (3.2)$$

We claim that $\nabla v^1 = 0$ at (x_0,t_0). In fact, if $\frac{\partial v^1}{\partial x_i} > 0$ at (x_0,t_0), for some $1 \le i \le n$; then, $v(x_0,t_0) = \xi_1$ and $v^1(x,t_0) > \xi_1$ for some x with $|x - x_0|$ small. This implies that $v^1(x,t) > \xi_1$ for x near x_0 and $t < t_0$ near t. This contradicts the definition of t_0 and then $\frac{\partial v^1}{\partial x_i} \le 0$. Using the same reasoning we obtain that $\frac{\partial v^1}{\partial x_i} \ge 0$ at (x_0,t_0), for some $1 \le i \le n$, leads to a contradiction and the claim is proved.

Similarly, $v^1_{x_i x_i} \le 0$, for all $1 \le i \le n$. Therefore, $\mathrm{Div}(a_1 \nabla v^1(x_0,t_0)) = a_1 \Delta v^1 + \nabla a_1 \cdot \nabla v^1 = a_1 \Delta v^1 \le 0$ and from expression (3.2), we have

$$v^1_t(x_0,t_0) \le -f_1(v(x_0,t_0)) - \lambda v^1(x_0,t_0) < 0.$$

Therefore, $(-\infty, \xi_1] \times \mathbb{R}^{N-1}$ is an invariant region for the local solution of (1.1). In the same way we obtain that $[-\bar{\xi}_1, \infty) \times \mathbb{R}^{N-1}$ is also an invariant region. From the fact that the intersection of invariant regions is still invariant, the proof is completed. □

Corollary 3.1 *For any $u_0 \in X^\alpha$ the local solution $T(t)u_0$ of (1.1) through u_0 satisfies*

$$\|T(t)u_0\|_{L^\infty(\Omega,\mathbb{R}^N)} \le N_1,$$

for some $N_1 > 0$, whenever defined. Furthermore, if B is a bounded subset of X^α, there are constants K_1, K_2, K_3, depending only on B, such that

$$\|T(t)u_0\|_{X^\alpha} \le K_1 + K_2 \int_0^t (t-s)^{-\alpha - \frac{r}{2}} e^{-c(t-s)} ds \le K_3$$

$\forall t \ge 0$ and $\forall u_0 \in B$.

Proof: Given $u_0 \in X^\alpha \subset C(\overline{\Omega})$ we can always chose Σ as in Theorem 3.1 such that $u_0(x) \in \Sigma$, $\forall x \in \Omega$. If $B \subset X^\alpha$ is bounded, Σ can be chosen the same for every $u_0 \in B$ and the rest follows from Theorem 3.1 and Lemma 3.1.□

4. Existence of Global Attractors

In this section we use the invariance theory, as in Henry [1981] , to prove that there is a bounded set in X^α which attracts points of X^α.

To prove that the solution operator for (1.1) is point dissipative we must first prove that the orbit of a point $\phi \in X^\alpha$ is a compact subset of X^α. This is a consequence of the following result (see Henry [1981], Theorem 3.3.6).

Theorem 4.1 *In the problem (2.2) assume that the nonlinearity h is Lipschitz continuous in bounded subsets of X^α and that A is a sectorial operator with compact resolvent. If $T(t)\phi$ is a solution of (2.2) on $[0,\infty)$ with $\|T(t)\phi\|_\alpha$ bounded as $t \to \infty$; then, $\{T(t)\phi,\ t \geq 0\}$ is a compact subset of X^α. Furthermore, if B is a bounded subset of X^α and $T(t)B$ remains in a bounded subset of X^α as $t \to \infty$; then, $\{T(t)B,\ t \geq 1\}$ is a compact subset of X^α.*

The next theorem is a classical result from invariance theory and will be the main tool in the proof of point dissipativeness (see, for example, Henry [1981] and Hale [1989]).

Theorem 4.2 *Suppose that $u_0 \in X^\alpha$ and $\{T(t)u_0,\ t \geq 0\}$ lies in a compact set in X^α; then, $\omega(u_0)$ is nonempty, compact, invariant, connected, and $dist(T(t)u_0, \omega(u_0)) \to 0$ as $t \to +\infty$.*

Similarly, if there is a negative orbit through u_0 and $\{T(t)u_0,\ t \leq 0\}$ lies in a compact set in X^α; then, $\alpha(u_0)$ is nonempty, compact, invariant, connected, and $dist(T(t)u_0, \alpha(u_0)) \to 0$ as $t \to -\infty$.

Next, we recall the definition of Liapunov function.

Definition 4.1 *Let $\{T(t),\ t \geq 0\}$ be a dynamical system on X^α. A Liapunov function for $T(t)$ is a continuous real valued function $V : X^\alpha \to \mathbb{R}$ such that*

$$\dot{V}(\phi) = \limsup_{t \to 0+} \frac{V(T(t)\phi) - V(\phi)}{t} \leq 0$$

for all $\phi \in X^\alpha$.

The following result has a classical and simple proof that we present for completeness.

Theorem 4.3 *Let V be a Liapunov function for $T(t)$ on X^α and define $E = \{\phi \in X^\alpha : \dot{V}(\phi) = 0\}$ and $\mathcal{M}$ the maximal invariant subset of E. If $\{T(t)u_0,\ t \geq 0\}$ lies in a compact set in X^α, then $T(t)u_0 \to \mathcal{M}$ as $t \to +\infty$.*

Proof: By hypothesis, $V(T(t)u_0)$ is nonincreasing for $t \geq 0$ and bounded below, since the orbit $\{T(t)u_0,\ t \geq 0\}$ is precompact, so $\ell = \lim_{t\to\infty} V(T(t)u_0)$ exists. If $y \in \omega(u_0)$, then $V(y) = \ell$, so also $V(T(t)y) = \ell$, $t \geq 0$, and so $\dot{V}(y) = 0$. Thus $\omega(u_0) \subset E$, so $\omega(u_0) \subset \mathcal{M}$ and from Theorem 4.2 it attracts $T(t)u_0$ and the result is proved.$\square$

We are interested in obtaining that the solution operator $\{T(t), t \geq 0\}$ for the parabolic system (1.1) is point dissipative. For this, we will construct a suitable Liapunov function.

Let $f : \mathbb{R}^N \to \mathbb{R}^N$ and $g : \mathbb{R}^N \to \mathbb{R}^N$ be C^1 and C^2 functions, respectively, satisfying (1.2), that is

$$u_i(\lambda u_i + f_i(u)) > 0, \quad u_i g_i(u) > 0, \; u_i \notin [-\bar{\xi}_{i0}, \xi_{i0}] =: \Sigma_{i0}, \; 1 \le i \le N. \tag{4.1}$$

We take the rectangle $\Sigma_0 = \Pi_{i=1}^N \Sigma_{i0} \subset \mathbb{R}^N$ and define $\Sigma_\tau = \Pi_{i=1}^N [-\bar{\xi}_{i0} - \tau, \xi_{i0} + \tau]$, $\tau \ge 0$, then $\{\Sigma_\tau, \tau \ge 0\}$ covers $\mathbb{R}^N$. To simplify the notation, when dealing with these rectangles, we denote $l_i := -\bar{\xi}_{i0} - \tau$ and $r_i := \xi_{i0} + \tau$. Let $p_{\Sigma_0} : \mathbb{R}^N \to \mathbb{R}^+$ defined by

$$p_{\Sigma_0}(u) = \inf\{\tau \ge 0 : u \in \Sigma_\tau\} \tag{4.2}$$

and define $F_{\Sigma_0} : X^\alpha \to \mathbb{R}^+$ by

$$F_{\Sigma_0}(w) = \sup\{p_{\Sigma_0}(w(x)) : \; x \in \Omega\}. \tag{4.3}$$

It is easy to check that F_{Σ_0} is a continuous function and $F_{\Sigma_0}(w) = 0$ if and only if $w(x) \in \Sigma_0$ for all $x \in \Omega$.

Theorem 4.4 *Let f, g and Σ_τ, $\tau > 0$, be as before. Suppose that $u(\cdot, t) \in X^\alpha$ is a solution of (1.1). Then, for any $T > 0$ for which $F_{\Sigma_0}(u(\cdot, T)) = \tau > 0$, there exists $\eta > 0$ such that*

$$\limsup_{h \to 0} \frac{F_{\Sigma_0}(u(\cdot, T+h)) - F_{\Sigma_0}(u(\cdot, T))}{h} \le -\eta. \tag{4.4}$$

Proof: If $F_{\Sigma_0}(u(\cdot, T)) = \tau > 0$ and $\Sigma_\tau = \Pi_{i=1}^N [l_i, r_i]$, we have that $l_i \le u_i(x, T) \le r_i$, for all $x \in \Omega$ and all $1 \le i \le$ N. If $\bar{x} \in \overline{\Omega}$ is such that $u(\bar{x}, T) \in \partial\Sigma_\tau$, then $u(\bar{x}, T)$ is in one of the faces of Σ_τ, say the right-hand j^{th} face $\partial(\Sigma_\tau)_j^r$ of Σ_τ, that is $u_j(\bar{x}, T) = r_j$ for some j. Therefore, there exists $\eta > 0$ such that

$$-\lambda u_j(\bar{x}, T) - f_j(u(\bar{x}, T)) < -\eta.$$

Note that, $\bar{x}$ cannot be on Γ since in this case $\bar{x}$ would be a maximum point for $u_j(\cdot, T)$ and the normal derivative at this point would be negative, since $g_j(u(\bar{x}, T)) > 0$. Thus, $\nabla u_j(\bar{x}, T) = 0$ and $\Delta u_j(\bar{x}, T) \le 0$. Consequently, at $(\bar{x}, T)$, $\operatorname{Div}(a_j \nabla u_j) = a_j \Delta u_j + \nabla a_j \nabla u_j \le 0$ and

$$\frac{\partial u_j}{\partial t} = \operatorname{Div}(a_j \nabla u_j) - \sum_{k=1}^{2} B_k \frac{\partial u_j}{\partial x_k} - \lambda u_j - f_j(u) < -\eta,$$

so that $u_j(\bar{x}, T+h) < r_j - \eta h$ for small h. By continuity, this holds for all x in a neighborhood of $\bar{x}$. If $K_T = \{x : u(x, T) \in \partial\Sigma_\tau\}$, then K_T is compact, and by what we have just shown, there is an open set $\Omega \supset U \supset K_T$ such that if $x \in U$, and h is small,

$$u(x, T+h) \in \Sigma_{(\tau - \eta h)}.$$

If $x \in \overline{\Omega} \backslash U$, then $u(x, T)$ is interior to Σ_τ, so there is a compact set Q contained in the interior of Σ_τ, and an $h_0 > 0$ such that $u(x, T+h) \in Q$ if $|h| < h_0$ for all $x \in \overline{\Omega} \backslash U$.

Thus, for sufficiently small h, $u(x, T+h)$ is in $\Sigma_{(\tau - \eta h)}$ for all $x \in \Omega$, and so

$$F_{\Sigma_0}(u(\cdot, T+h)) \le \tau - \eta h.$$

Therefore,
$$\frac{F_{\Sigma_0}(u(\cdot,T+h)) - F_{\Sigma_0}(u(\cdot,T))}{h} \leq -\eta.$$
and the result is proved.□

As an immediate consequence, we have

Corollary 4.1 *The map $F_{\Sigma_0} : X^\alpha \to \mathbb{R}^+$ is a Liapunov functional for (1.1) and*

$$\{\phi \in X^\alpha : \dot{F}_{\Sigma_0}(\phi) = 0\} \subset \{\phi \in X^\alpha : F_{\Sigma_0}(\phi) = 0\} = \{\phi \in X^\alpha : \phi(x) \in \Sigma_0, \forall x \in \Omega\} = \mathcal{M}_\infty$$

Proof: The proof is obvious, since from the theorem, if $F_{\Sigma_0} > 0$, then it decreases strictly.□

Note that $\mathcal{M}_\infty$ is not bounded in X^α and therefore point dissipativeness does not follow from the corollary.

Theorem 4.5 *Suppose that $f, g : \mathbb{R}^N \to \mathbb{R}^N$ are C^1 and C^2, respectively, and satisfy (4.1). Then, the problem (1.1) has a global attractor $\mathcal{A}$ in X^α. Furthermore,*

$$u(x) \in \Sigma_0, \ \forall x \in \Omega, \ \forall u \in \mathcal{A}. \tag{4.5}$$

Proof: It follows from La Salle's invariance principle and the previous corollary that for any $u_0 \in X^\alpha$, the ω–limit set $\omega(u_0)$ is contained in $\mathcal{M}_\infty$. Thus, given $\tau > 0$ there exists $t_\tau > 0$ such that $T(t)u_0(x) \in \Sigma_\tau$, for all $t \geq t_\tau$, since $\omega(u_0) \subset \mathcal{M}_\infty$ and $\omega(u_0)$ attracts u_0 in X^α under $T(t)$ and in particular, it attracts u_0 in $C(\overline{\Omega})$. From the variation of constants formula

$$T(t)u_0 = e^{-A(t-t_\tau)}T(t_\tau)u_0 + \int_{t_\tau}^t e^{-A(t-s)}h(T(s)u_0)ds, \quad t \geq t_\tau > 0 \tag{4.6}$$

we have that

$$\|T(t)u_0\|_{X^\alpha} \leq M\, e^{-\epsilon(t-t_\tau)}(t-t_\tau)^{-\alpha}|\Omega|^{\frac{1}{2}}L(\tau) + M\, P(\tau)\,|\Omega|^{\frac{1}{2}} \int_{t_\tau}^t e^{-\epsilon(t-s)}(t-s)^{-\alpha-\frac{r}{2}}ds, \tag{4.7}$$

where $\|T(s)u_0\|_\infty \leq L(\tau)$, $\forall s \geq t_\tau$ and $P(\tau) = \sup_{s\in\Sigma_\tau}\left\{|\Omega|^{\frac{1}{2}}|f(s)| + |\Gamma|^{\frac{1}{2}}|g(s)|\right\}$.

Letting $t \to \infty$, in the above expression, we obtain

$$\limsup_{t\to\infty} \|T(t)u_0\|_{X^\alpha} \leq M\, P(\tau)\,|\Omega|^{\frac{1}{2}} \int_0^\infty e^{-\epsilon s}s^{-\alpha-\frac{r}{2}}ds, \tag{4.8}$$

and since the right hand side of (4.8) does not depend on $u_0 \in X^\alpha$ we obtain that $T(t)$ is point dissipative. Even more, we can make $\tau \to 0$ and then

$$\limsup_{t\to\infty} \|T(t)u_0\|_{X^\alpha} \leq M\, P(0)\,|\Omega|^{\frac{1}{2}} \int_0^\infty e^{-\epsilon s}s^{-\alpha-\frac{r}{2}}ds, \tag{4.9}$$

Since we already have that orbits of bounded subsets of X^α under $T(t)$ are bounded subsets of X^α and that $T(t)$ is a compact semigroup we have the existence of a global attractor $\mathcal{A}$ for $\{T(t);\ t \geq 0\}$, (see, for example, Hale [1989], Theorem 3.4.6).

Now, if $u \in \mathcal{A}$, there is a complete precompact orbit through u and $\sup_{t\in\mathbb{R}} \|T(t)u\|_{X^\alpha} < \infty$, which implies

$$\sup_{t\in\mathbb{R}} F_{\Sigma_0}(T(t)u) < \infty.$$

By hypothesis, $F_{\Sigma_0}(T(-t)u)$ is increasing for $t \geq 0$ and is bounded above; thus, the limit $\ell = \lim_{t\to-\infty} F_{\Sigma_0}(T(t)u)$ exists. If $z \in \alpha(u)$, then $F_{\Sigma_0}(z) = \ell$, so also $F_{\Sigma_0}(T(t)z) = \ell$, $t \in \mathbb{R}$, and so $\dot{F}_{\Sigma_0}(z) = 0$. Thus $\alpha(y) \subset \mathcal{M}_\infty$ and from Theorem 4.2 attracts $T(t)u$. But then for any $\tau > 0$, and for sufficiently negative t_τ, $T(t_\tau)u(x) \in \Sigma_\tau$, for all $x \in \Omega$, which from Theorem 3.1 is an invariant region, and therefore $T(t)u(x) \in \Sigma_\tau$, for all $x \in \Omega$, for $t \geq t_\tau$. In particular, with $t = 0$, $u(x) \in \Sigma_\tau$, for all $x \in \Omega$. Since $\tau > 0$ is arbitrary, we get the result.□

Remark 4.1 *It is very important, for some applications, to be able to obtain some a priori estimates that do not depend on the size of the diffusion coefficient. See for example Carvalho* [1992] *and Carvalho and Oliveira* [1992] *Carvalho and Rodriguez-Bernal [1994]. This is an advantage that the technique of invariant regions has over those employed in Hale* [1989], *page 77, even in the case $\alpha = \frac{1}{2}$. The estimates obtained there, for the size of $\mathcal{A}$, strongly depend on the size of the diffusion coefficient.*

Corollary 4.2 *The elliptic problem*

$$-Div(a\nabla u) + \sum_{j=1}^{2} B_j(x)\frac{\partial u}{\partial x_j} + \lambda u + f(u) = 0, \quad in\ \Omega$$

$$\frac{\partial u}{\partial n_a} + g(u) = 0, \quad in\ \Gamma,$$

has at least one solution which is nontrivial whenever $\lambda I + f$ and g have no common zeros.

5. Gradient-Like Systems

In this section we study the structure of the global attractor in X^α, for systems of reaction diffusion equations without dispersion; that is, we consider the problem (1.1) with $B_j = 0$, $j = 1, 2$, that is

$$\left.\begin{aligned} &u_t - \mathrm{Div}(a\nabla u) + \lambda u + f(u) = 0, \quad \text{in } \Omega, \\ &\frac{\partial u}{\partial n_a} + g(u) = 0, \quad \text{in } \partial\Omega. \end{aligned}\right\} \tag{5.1}$$

where u and a are as in (1.1) and $\lambda > 0$. The nonlinearities $f = (f_1, \cdots, f_N)^\top : \mathbb{R}^N \to \mathbb{R}^N$ $g = (g_1, \cdots, g_N)^\top : \mathbb{R}^N \to \mathbb{R}^N$ are assumed to be a C^1 and C^2 functions, respectively, satisfying (1.2) and for which there exists scalar potentials $F : \mathbb{R}^N \to \mathbb{R}$ and $G : \mathbb{R}^N \to \mathbb{R}$ such that

$$\nabla F(u) = f(u), \quad \forall u \in \mathbb{R}^N$$

$$\nabla G(u) = g(u), \quad \forall u \in \mathbb{R}^N$$

Let $V : X^\alpha \to \mathbb{R}$ be the function defined by

$$V(\phi) = \frac{1}{2}\int_\Omega \langle a\nabla\phi, \nabla\phi\rangle + \frac{\lambda}{2}\int_\Omega |\phi|^2 + \int_\Omega F(\phi) + \int_\Gamma G(\gamma(\phi))$$

Then V is continuous and

$$\dot{V}(u) = -\|u_t\|^2 \leq 0,$$

along solutions of (5.1). Therefore, $E = \{\phi \in X^\alpha : \dot{V}(\phi) = 0\}$ is the set of equilibrium points for (5.1) and with the notations of Theorem 4.3, $\mathcal{M} = E$.

We note that (5.1) is not a gradient system in X^α as defined in Hale [1989] because the Liapunov functional V used before does not verify the conditions there in. However, we have that the attractor for (5.1) has the same structure that the attractors of gradient systems have.

Theorem 5.1 *If $\{T(t), t \geq 0\}$ is the semigroup defined by (1.1) in X^α, $\mathcal{A}$ denotes its attractor, and E the set of equilibrium points, then $\mathcal{A} = W^u(E) = \{y \in X^\alpha : T(-t)y$ is defined for $t \geq 0$ and $T(-t)y \to E$ as $t \to \infty\}$. If, in addition, every element of E is hyperbolic, then E is a finite set and $\mathcal{A} = \cup_{x \in E} W^u(x)$.*

Proof: Note that if $y \in W^u(E)$ then $\{T(t)y,\ t \in \mathbb{R}\}$ compact and invariant. Therefore it lies in $\mathcal{A}$ and in particular $y \in \mathcal{A}$. For the other inclusion observe that if $y \in \mathcal{A}$, then there exists a complete precompact orbit through y and $\sup_{t\in\mathbb{R}} \|T(t)y\|_{X^\alpha} < \infty$, which implies

$$\sup_{t\in\mathbb{R}} V(T(t)y) < \infty.$$

By hypothesis, $V(T(-t)y)$ is increasing for $t \geq 0$ and is bounded above; thus, the limit $\ell = \lim_{t\to-\infty} V(T(t)y)$ exists. If $z \in \alpha(y)$, then $V(z) = \ell$, so also $V(T(t)z) = \ell$, $t \in \mathbb{R}$, and so $\dot{V}(z) = 0$. Thus $\alpha(y) \subset E$ and from Theorem 4.2 attracts $T(t)y$ and the inclusion is proved. The rest follows as in Hale [1989].$\square$

6. Alternative Dissipativeness Condition

In this section we give alternative dissipativeness conditions to (1.2) and under these, we prove the existence of a global attractor for (1.1) and give $C(\overline{\Omega})$–diffusion independent estimates for the functions in the attractor.

Before we can state these conditions we need to introduce the notion of eventually convex functions.

Definition 6.1 *A smooth function $V : \mathbb{R}^N \to \mathbb{R}$ is said nondegenerate eventually convex if there exists $\tau_0 > 0$ such that for every $u \in \mathbb{R}^N$, with $V(u) > \tau_0$, $\nabla_u V(u) \neq 0$ and $d^2V(u)(\eta,\eta) \geq 0$ for every $\eta \in \mathbb{R}^N$.*

Without loss of generality we can always assume that $\tau_0 = 0$.

Let $V : \mathbb{R}^N \to \mathbb{R}$ be a nondegenerate eventually convex function such that $V_\tau = \{u \in \mathbb{R}^N : V(u) \leq \tau\}$, $\tau \geq 0$ verify

$$\begin{aligned} (\lambda u + f(u)) \cdot \vec{n}(u) > 0 \\ g(u) \cdot \vec{n}(u) > 0 \end{aligned} \tag{6.1}$$

for every $u \in \partial V_\tau = \{u \in \mathbb{R}^N : V(u) = \tau\}$, $\tau > 0$, where $\vec{n}(u)$ is the outward normal to ∂V_τ at u.

Note that $\{V_\tau, \tau \geq 0\}$ is an increasing family of sets that covers $\mathbb{R}^N$ and $\nabla_u V(u)$ is an outward normal vector to ∂V_τ.

First we prove the analogous to Theorem 3.1

Theorem 6.1 *The convex set V_τ, $\tau > 0$, is an invariant region for the local solution of (1.1).*

Proof: If there is a solution $v(x,t) = (v^1(x,t), \cdots, v^N(x,t))$ of (1.1) with initial data $v(x,0) = (v^1(x,0), \cdots, v^N(x,0)) \in V_\tau$ for all $x \in \bar{\Omega}$, that does not stay in V_τ for all $t \in [0, t_{\max})$, then there is a t_0 and $x_0 \in \overline{\Omega}$ such that

$$V(v(x,t)) < \tau, \ 0 \leq t < t_0, \ x \in \overline{\Omega}, \text{ and } V(v(x_0,t_0)) = \tau.$$

We observe that $x_0 \notin \Gamma$ since in this case, if $\vec{n}$ denotes the unit outward normal to $\partial\Omega$ at x_0

$$\begin{aligned} 0 \leq \nabla_x V(v(x_0,t_0)) \cdot \vec{n} &= \sum_{i=1}^{n}\sum_{j=1}^{N} \frac{\partial V}{\partial u_j}\frac{\partial u_j}{\partial x_i} n_i = \sum_{j=1}^{N} \frac{\partial V}{\partial u_j} \sum_{i=1}^{n} \frac{\partial u_j}{\partial x_i} n_i \\ &= \sum_{j=1}^{N} \frac{\partial V}{\partial u_j}(v(x_0,t_0)) \cdot (\nabla v_j \cdot \vec{n}) \\ &= -\sum_{j=1}^{N} \frac{\partial V}{\partial u_j}(v(x_0,t_0)) \cdot g_j(v(x_0,t_0)) \\ &= -\nabla_v V(v(x_0,t_0)) \cdot g(v(x_0,t_0)) < 0 \end{aligned}$$

where for the last inequality we used the fact that the gradient of V is orthogonal to the level surface ∂V_τ and (6.1).

Therefore, if $V(v(x_0,t)) < \tau$, $\forall t \in [0,t_0)$ and $V(v(x_0,t_0)) = \tau$ implies $\frac{d}{dt}V(v(x_0,t_0)) < 0$; then, V_τ is invariant.

From the fact that $x_0 \in \Omega$ is a point of maximum for $V(v(\cdot,t_0))$, we have $\nabla_x V(v(x_0,t_0)) = 0$ and $\Delta_x V(v(x_0,t_0)) \leq 0$ and since V is convex outside V_0 we have that, at (x_0,t_0)

$$\begin{aligned} 0 \geq \Delta V(v) &= d^2V(v)(\nabla v, \nabla v) + \nabla V_v(v) \cdot \Delta v \\ &\geq \nabla V_v(v) \cdot \Delta v. \end{aligned}$$

Thus, at (x_0, t_0)

$$\begin{aligned}
\frac{d}{dt}V(v) &= \nabla_v V(v)\frac{\partial u}{\partial t} \\
&= \nabla_v V(v)\mathrm{Div}(a\nabla v) - \sum_{j=1}^{2} B_j(x_0)\nabla_v V(v)\cdot\frac{\partial u}{\partial x_j} - \nabla_v V(v)(\lambda v + f(v)) \\
&\leq \nabla_v V(v)\mathrm{Div}(a\nabla v) - \nabla_v V(v)(\lambda v + f(v)) \\
&\leq -\nabla_v V(v)(\lambda v + f(v)) < 0
\end{aligned}$$

where for the last inequality we used the fact that $0 \neq \nabla_v V(v(x_0, t_0))$ is an outward normal to ∂V_τ at $v(x_0, t_0)$ and (6.1).

Therefore, V_τ is an invariant region for the local solution of (1.1) and the proof is completed.

□

We now show that, under the above conditions, the solution operator $\{T(t),\, t \geq 0\}$ for the parabolic system (1.1) is point dissipative. For this, as in the previous section, we will construct a suitable Liapunov function.

Let $f : \mathbb{R}^N \to \mathbb{R}^N$ and $g : \mathbb{R}^N \to \mathbb{R}^N$ be C^1 and C^2 functions, respectively, satisfying (6.1) and $F_{V_0} : X^\alpha \to \mathbb{R}^+$ be defined by

$$F_{V_0}(w) = \sup\{V(w(x)) :\ x \in \Omega\}. \tag{6.2}$$

It is easy to check that F_{V_0} is a continous function and $F_{V_0}(w) = 0$ if and only if $w(x) \in V_0$ for all $x \in \Omega$. Then, we prove

Theorem 6.2 *Let f, g and V_τ, $\tau > 0$, be as before. Suppose that $u(\cdot, t) \in X^\alpha$ is a solution of (1.1). Then, for any $T > 0$ for which $F_{V_0}(u(\cdot, T)) = \tau > 0$, there exists $\eta > 0$ such that*

$$\limsup_{h\to 0} \frac{F_{V_0}(u(\cdot, T+h)) - F_{V_0}(u(\cdot, T))}{h} \leq -\eta. \tag{6.3}$$

Proof: If $F_{V_0}(u(\cdot, T)) = \tau > 0$ and if $\bar{x} \in \overline{\Omega}$ is such that $u(\bar{x}, T) \in \partial V_\tau$, there exists $\eta > 0$ such that

$$-(\lambda u(\bar{x}, T) + f(u(\bar{x}, T)))\cdot\nabla_u V(u(\bar{x}, t)) < -\eta.$$

Note that, $\bar{x}$ cannot be on Γ since in this case, $\bar{x}$ is a maximum point for $V(u(\cdot, T))$ and if $\vec{n}$ is the unit outward normal to $\partial\Omega$ at $\bar{x}$, as before,

$$0 \leq \nabla_x V(v(\bar{x}, T))\cdot\vec{n} = -\nabla_v V(v(\bar{x}, T))\cdot g(v(\bar{x}, T)) < 0$$

which is a contradiction. Thus, $\bar{x} \in \Omega$ and $\nabla_x V(u(\bar{x}, T)) = 0$ and $\Delta_x V(u(\bar{x}, T)) \leq 0$. Consequently, $\nabla_u V(u)\cdot \mathrm{Div}\ (a\nabla u) \leq 0$ at $(\bar{x}, T)$ and, as before,

$$\frac{\partial V(u)}{\partial t} = \nabla_u V(u)\cdot\mathrm{Div}(a\nabla u) - \sum_{k=1}^{2} B_k\nabla_u V(u)\cdot\frac{\partial u}{\partial x_k} - \nabla_u V(u)\cdot(\lambda u - f(u)) < -\eta,$$

so that $V(u(\bar{x},T+h)) < \tau - \eta h$ for small h. By continuity, this holds for all x in a neighborhood of $\bar{x}$. If $K_T = \{x : u(x,T) \in \partial V_\tau\}$, then K_T is compact, and by what we have just shown, there is an open set $\Omega \supset U \supset K_T$ such that if $x \in U$, and h is small,

$$u(x,T+h) \in V_{(\tau-\eta h)}.$$

If $x \in \bar{\Omega}\backslash U$, then $u(x,T)$ is interior to V_τ, so there is a compact set Q contained in the interior of V_τ, and an $h_0 > 0$ such that $u(x,T+h) \in Q$ if $|h| < h_0$ for all x.

Thus, for sufficiently small h, $u(x,T+h)$ is in $V_{(\tau-\eta h)}$ for all x, and so

$$F_{V_0}(u(\cdot,T+h)) \leq \tau - \eta h.$$

Therefore,

$$\frac{F_{V_0}(u(\cdot,T+h)) - F_{V_0}(u(\cdot,T))}{h} \leq -\eta.$$

and the result is proved.□

As an immediate consequence, we have

Corollary 6.1 *The map $F_{V_0} : X^\alpha \to \mathbb{R}^+$ is a Liapunov functional for (1.1) and*

$$\{\phi \in X^\alpha : \dot{F}_{V_0}(\phi) = 0\} \subset \{\phi \in X^\alpha : F_{V_0}(\phi) = 0\} = \{\phi \in X^\alpha : \phi(x) \in V_0, \forall x \in \Omega\} = \mathcal{V}_\infty$$

Proof: The proof is obvious, since from the theorem, if $F_{V_0} > 0$, then it decreases strictly.□

Note that $\mathcal{V}_\infty$ is not bounded in X^α and therefore point dissipativeness does not follow from the corollary.

Theorem 6.3 *Suppose that $f,g : \mathbb{R}^N \to \mathbb{R}^N$ are C^1 and C^2, respectively, and satisfy (6.1). Then, the problem (1.1) has a global attractor $\mathcal{A}$ in X^α. Furthermore,*

$$u(x) \in V_0, \ \forall x \in \Omega, \ \forall u \in \mathcal{A}. \tag{6.4}$$

The proof is as in the previous section, with V_0, V_τ and $\mathcal{V}_\infty$ replacing Σ_0, Σ_τ and $\mathcal{M}_\infty$, respectively.

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