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Resumo

Nós consideramos a equação integral linear multidimensional do tipo Volterra

$$x(t) + (*) \int_{R_t} \alpha(s)\, x(s)\, ds = f(t), \quad t \in R, \tag{1}$$

como limite de sistemas discretos do tipo Stieltjes e provamos resultados de existência de soluções contínuas. As funções x, α e f assumem valores em espaços de Banach e são definidas num intervalo compacto R de $\mathbb{R}^n$, R_t é um subintervalo de R dependendo de $t \in R$ e $(*) \int$ denota umas das integrais de Bochner-Lebesgue ou de Henstock. Os resultados apresentados generalizam aqueles de [1] e seguem o espírito dos resultados de [3]. Como conseqüência do tratamento utilizado, é possível estudar as propriedades de (1) pela tranferência das propriedades dos sistemas discretos. A integral de Henstock nos permite considerar funções que oscilam bastante.

LINEAR VOLTERRA INTEGRAL EQUATIONS AS THE LIMIT OF DISCRETE SYSTEMS

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ABSTRACT. We consider the multidimensional abstract linear integral equation of Volterra type

$$x(t) + (*)\int_{R_t} \alpha(s)\, x(s)\, ds = f(t), \quad t \in R, \tag{1}$$

as the limit of discrete Stieltjes-type systems and we prove results on the existence of continuous solutions. The functions x, α and f are Banach space-valued defined on a compact interval R of $\mathbb{R}^n$, R_t is a subinterval of R depending on $t \in R$ and $(*)\int$ denotes either the Bochner-Lebesgue integral or the Henstock integral. The results presented here generalize those in [1] and are in the spirit of [3]. As a consequence of our approach, it is possible to study the properties of (1) by transferring the properties of the discrete systems. The Henstock integral setting enables us to consider highly oscillating functions.

1. Introduction

We establish conditions for the existence of continuous solutions to the linear Volterra integral equation

$$x(t) + (*)\int_{R_t} \alpha(s)\, x(s)\, ds = f(t), \quad t \in R, \tag{2}$$

as the limit of discrete systems, where R is a compact interval of $\mathbb{R}^n$, R_t is a subinterval of R depending on $t \in R$ and x, α and f are Banach space-valued functions defined on R. The integral is considered in each the Bochner-Lebesgue and the Henstock senses.

We start by proving that the auxiliary Stieltjes equation

$$x(t) + \int_{R_t} d\alpha(s)\, x(s) = f(t), \quad t \in R, \tag{3}$$

is the limit of a system of discrete equations. This fact is a generalization of [1] for the multidimensional case. Next, we state the equivalence between (3) and the equation

$$x(t) + (L)\int_{R_t} \alpha(s)\, x(s)\, ds = f(t), \quad t \in R, \tag{4}$$

where α is Bochner-Lebesgue integrable and $(L)\int$ denotes the Bochner-Lebesgue integral, and we give conditions for the existence of continuous solutions to (4). Finally, the result

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for the equation

$$x(t) + (H)\int_{R_t} \alpha(s)\, x(s)\, ds = f(t), \quad t \in R, \tag{5}$$

where α is Henstock integrable and $(H)\int$ denotes the Henstock integral, is obtained from a sequence of equations of type (4) and a fixed point theorem for sequences of operators with fixed points in the spirit of [6], [4] and [3].

2. Basic notations and definitions

For simplicity of notation and proofs we consider the bidimensional case, but one can easily generalize the notation and proofs to dimension n, for $n > 2$.

Let R be a compact interval of $\mathbb{R}^2$, that is, R is a rectangle with sides parallel to the coordinate axes. Given points $t = (t_1, t_2)$ and $s = (s_1, s_2)$ in $\mathbb{R}^2$, with $t \leq s$ (i.e., $t_i \leq s_i$, $i = 1, 2$), we denote by $[t, s]$ the rectangle $[t_1, s_1] \times [t_2, s_2]$ and we call any such rectangle a closed interval of $\mathbb{R}^2$. We set $]t, s] :=]t_1, s_1] \times]t_2, s_2]$ and $|[t, s]| := m([t, s])$, where m denotes the Lebesgue measure.

Any finite set of closed nonoverlapping intervals J_i, $i = 1, 2, \ldots, n$, contained in R and such that $\cup_i J_i = R$ is called a division of R and we denote this division by $d = (J_i)$ or simply (J_i). By D_R we denote the set of all divisions of R. We write $|d| := n$ and $\Delta d := \sup\{|J_i| : i = 1, 2, \ldots, |d|\}$.

Given $d = (J_i)$ and $\overline{d} = (\overline{J}_j)$ in D_R, we denote by $d \vee \overline{d}$ the division of R obtained by all subintervals $J_i \cap \overline{J}_j$, $i = 1, 2, \ldots, |d|$, $j = 1, 2, \ldots, |\overline{d}|$. We say that d is finer than $\overline{d}$, and we write $d \leq \overline{d}$, if every interval of d is an interval of $\overline{d}$. The relation $\leq$ is an order relation on D_R and makes it filtered on the right. For $\varepsilon > 0$, we set $D_\varepsilon := \{d \in D_R : \Delta d \leq \varepsilon\}$, and for $d \in D_R$, we set $D_d := \{\overline{d} \in D_R : d \leq \overline{d}\}$. Then the classes $\{D_\varepsilon : \varepsilon > 0\}$ and $\{D_d : d \in D_R\}$ are filter bases on D_R and the latter is finer than the former. Thus given a function h defined on D_R with values on a topological space, if the limit $\lim_{\Delta d \to 0} h(d)$ exists over the filter basis $\{D_\varepsilon : \varepsilon > 0\}$, then the limit $\lim_{d \in D_R} h(d)$ exists over the filter basis $\{D_d : d \in D_R\}$.

A tagged division d of R is any pair (ξ_i, J_i), where $(J_i) \in D_R$ and, for each i, ξ_i is a point of J_i called the "tag" of J_i. By TD_R we mean the set of all tagged divisions of R. A tagged partial division d of R is any subset of a tagged division of R. In this case we write $d \in TPD_R$.

A gauge of a set $E \subset R$ is any function $\delta : E \to]0, +\infty[$. Given a gauge δ of R, we say that $d = (\xi_i, J_i) \in TPD_R$ is δ-fine, whenever $J_i \in B_{\delta(\xi_i)}(\xi_i)$ for each i, where $B_{\delta(\xi_i)}(\xi_i) := \{t \in R : \|t - \xi_i\| < \delta(\xi_i)\}$. Then a natural question arises: given a gauge δ of R, is there a tagged division of R which is δ-fine? The answer to this question is "yes" and it is known as the Cousin Lemma.

Throughout this paper we consider X an Y as Banach spaces. By $L(X, Y)$ we mean the space of all linear continuous functions from X to Y endowed with the usual norm.

Let $\mathcal{I}_R$ be the set of all closed intervals contained in R. A function $F : \mathcal{I}_R \to X$ is called an additive function (of intervals), and we write $F \in A(\mathcal{I}_R, X)$, if given nonoverlapping intervals $J_1, J_2 \in \mathcal{I}_R$, with $J = J_1 \cup J_2 \in \mathcal{I}_R$, we have

$$F(J) = F(J_1) + F(J_2).$$

Given functions $\alpha \in A(\mathcal{I}_R, L(X,Y))$ and $f : R \to X$, we say that f is Henstock integrable with respect to α or simply α-Henstock integrable, if there exists a function $F^\alpha \in A(\mathcal{I}_R, Y)$ such that for every $\varepsilon > 0$, there is a gauge δ of R such that for every δ-fine $d = (\xi_i, J_i) \in TD_R$, we have

$$\sum_i \|F^\alpha(J_i) - \alpha(J_i)f(\xi_i)\| < \varepsilon.$$

In this case we write $f \in H^\alpha(R, X)$ and we call F^α the indefinite integral of f with respect to α. If moreover $X = Y$ and $\alpha(J)$ is the identity in $L(X)$ for every $J \in \mathcal{I}_R$, then f is Henstock integrable with indefinite integral F and we write $f \in H(R, X)$.

In the sequel we consider $R = [a, b]$, where $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$, that is, $R = [a, b] = [a_1, b_1] \times [a_2, b_2]$. Given $\alpha \in A(\mathcal{I}_R, L(X,Y))$, $f \in H^\alpha(R, X)$, $J \in \mathcal{I}_R$ and $t \in R = [a, b]$, we set

$$(H)\int_J d\alpha(s)f(s) := F^\alpha(J) \qquad \text{and} \qquad \widetilde{f}^\alpha(t) := (H)\int_{[a,t]} d\alpha(s)f(s) = F^\alpha([a,t]).$$

Remark 2.1. *Let $\alpha \in A(\mathcal{I}_R, L(X,Y))$. The following assertions hold.*

(i) *If $f \in H^\alpha(R, X)$ and $J \in \mathcal{I}_R$, then $f \in H^\alpha(J, X)$ (integrability on subintervals).*

(ii) *If $f \in H^\alpha(R, X)$ and $J, J_1, J_2 \in \mathcal{I}_R$ are such that $J = J_1 \cup J_2$, then $F^\alpha(J) = F^\alpha(J_1) + F^\alpha(J_2)$ and hence $F^\alpha \in A(\mathcal{I}_R, Y)$ (additivity of the indefinite integral).*

(iii) *If $f \in H^\alpha(J_1, X) \cap H^\alpha(J_2, X)$, then $f \in H^\alpha(R, X)$ (this last statement can be proved by applying the Cauchy Criteria).*

Remark 2.2. *Suppose $f \in H(R, X)$ and $\widetilde{f}(t) = (H)\int_{[a,t]} f$, for every $t \in [a, b]$. It is always possible to associate the function $\widetilde{f}$ with a function of intervals of R, which we still denote by $\widetilde{f}$, in the following manner*

$$\widetilde{f}([t,s]) = \widetilde{f}(s) - \widetilde{f}((t_1, s_2)) - \widetilde{f}((s_1, t_2)) + \widetilde{f}(t),$$

for every $t = (t_1, t_2), s = (s_1, s_2) \in R$, with $t \leq s$. Reciprocally, it is always possible to associate a function $F \in A(\mathcal{I}_R, X)$ with a functions from R to X which we still denote by F. We write

$$F(t) = F([a,t]) - F([a,(a_1,t_2)]) - F([a,(t_1,a_2)]) + F([a,a]),$$

for all $t = (t_1, t_2) \in R$. With the above notation we have

$$F((a_1, t_2)) = F((t_1, a_2))$$

and in this case we write $F \in A_0(\mathcal{I}_R, X)$. Thus a function $F \in A(\mathcal{I}_R, X)$ can be associated with a function $F_0 \in A_0(\mathcal{I}_R, X)$ with $F_0(J) = F(J)$, for every $J \in \mathcal{I}_R$.

Given $F \in A(\mathcal{I}_R, X)$, we say that F satisfies the Strong Lusin Condition if for every $\varepsilon > 0$ and every $A \subset R$, with $m(A) = 0$, there is a gauge δ of A such that for every δ-fine $d = (\xi_i, J_i) \in TPD_R$ with $\xi_i \in A$ for each i, we have

$$\sum_i \|F(J_i)\| < \varepsilon.$$

In this case we write $F \in SL(\mathcal{I}_R, X)$.

The next theorem says that the indefinite integral of a Henstock integrable function fulfills the Strong Lusin Condition. A proof of it can be found in [5].

Theorem 2.1. *If $\alpha \in SL(\mathcal{I}_R, L(X,Y))$ and $f \in H^\alpha(R, X)$, then $F^\alpha \in SL(\mathcal{I}_R, Y)$, where F^α is the indefinite integral of f with respect to α.*

Let $f : R \to X$ be a function and X be any Banach space. If f is a Bochner-Lebesgue integrable function with finite integral $(L) \int_R f < \infty$ (we write $f \in \mathcal{L}_1(R, X)$ in this case), then the inclusion

$$\mathcal{L}_1(R, X) \subset H(R, X)$$

holds. However, if f is Riemann integrable (we write $f \in R(R, X)$), then we may have $f \notin H(R, X)$ and hence $f \notin \mathcal{L}_1(R, X)$. The reader may want to consult [2] for an example of such a function. In general, if $\alpha \in A(\mathcal{I}_R, L(X,Y))$ and $f : R \to X$ is Riemann integrable with respect to α, that is, there exists the Riemann-Stieltjes integral $\int_R d\alpha(s) f(s)$ (we write $f \in R^\alpha(R, X)$ in this case), then we may have $f \notin H^\alpha(R, X)$. However if the dimension of X is finite, then

$$R^\alpha(R, X) \subset H^\alpha(R, X).$$

We define the variation, $V(F)$, of a function $F \in A(\mathcal{I}_R, X)$ as the smallest $p \in [0, \infty]$ such that for every $(J_i) \in D_R$, we have

$$\sum_i \|F(J_i)\| < p.$$

If $V(F) < \infty$, then we say that F is a function (of intervals) of bounded variation and we write $F \in BV(\mathcal{I}_R, X)$. In an analogous way we define the variation of F in a closed subinterval J of R which we denote by $V_J(F)$. In this manner, one can easily generalize the notion of variation of a function of intervals over finite unions of nonoverlapping elements of $\mathcal{I}_R$.

We also define the semivariation of a function of intervals. The semivariation, $SV(\alpha)$ of a function $\alpha \in SV(\mathcal{I}_R, L(X,Y))$ is the smallest $p \in [0, \infty]$ such that for every $(J_i) \in D_R$, we have

$$\left\| \sum_i \alpha(J_i)\, x_i \right\| \leq p \sup_i \{\|x_i\| : x_i \in X\}.$$

When $SV(\alpha) < \infty$, α is of bounded semivariation and we write $\alpha \in SV(\mathcal{I}_R, X)$. By $SV_J(\alpha)$ we mean the semivariation of α in $J \in \mathcal{I}_R$ and the semivariation of α over finite unions of nonoverlapping elements of $\mathcal{I}_R$ can also be defined.

Remark 2.3. *The following properties hold:*

(i) *If* $\alpha \in SV(\mathcal{I}_R, L(X,Y))$, *then* α *is bounded and* $\|\alpha(J)\| \leq SV_J(\alpha)$, *for every* $J \in \mathcal{I}_R$.
(ii) $BV(\mathcal{I}_R, L(X,Y)) \subset SV(\mathcal{I}_R, L(X,Y))$.

Theorem 2.2. *Let* $f \in H(R,X)$. *Then* $f \in \mathcal{L}_1(R,X)$ *if and only if* $F \in BV(\mathcal{I}_R, X)$, *where* F *denotes the indefinite integral of* f. *In this case,* $V(F) = \|f\|_1$, *where* $\|f\|_1 = (L)\int_R \|f\|$.

A proof of Theorem 2.2 can be carried out by simply adapting the steps of the one-dimensional case (see, for instance, [8], 9, or [2], Lemma 4.5).

Other fundamental properties of the Henstock integral are left to the Appendix.

3. Discrete Stieltjes equations

Let $N \in \mathbb{N}$ and X^N be the set of all n-uples in X, where X is a Banach space. For each $m \in \mathbb{N}$, $m \leq N$, let $Q_m \in L(X)$. From the operators Q_m we obtain mappings $K_{p,q} : X^N \to X$, $p, q \in \mathbb{N}$, given by

$$K_{p,p} = 0$$
$$K_{p,q}x = \sum_{s=p+1}^{q} Q_s x_s, \qquad p < q \text{ and } x = (x_1, \ldots, x_N).$$

We define a linear discrete Stieltjes equation in X^N by

$$(\pi_q + K_{p,q})x = \pi_q u, \qquad x, u \in X^N, \quad 0 \leq p \leq q \leq N,$$

where π_q, $0 \leq q \leq N$, denotes the usual linear projection over X^N, that is, $\pi_q(x_1, \ldots, x_N) = x_q$. In particular,

$$(\pi_r + K_{0,r})x = \pi_r u, \qquad x, u \in X^N, \quad 0 \leq r \leq N. \tag{6}$$

Given a bounded linear operator $T : X \to Y$ (we write $T \in B(X,Y)$), we denote respectively by $R(T)$ and $N(T)$ the range and kernel of T. $T \in B(X,Y)$ is said to be a Fredholm operator if $R(T)$ is closed in Y and the dimensions of $N(T)$ and $N(T^*)$, where T^* denotes the adjoint operator of T, are finite.

Next we present a result, whose proof can be found in [1], Theorem 2.2, on the existence of a solution to the Stieltjes equation (6).

Theorem 3.1. *Consider the linear discrete Stieltjes equation in* X^N

$$(\pi_r + K_{0,r})x = \pi_r u, \qquad x, u \in X^N, \quad 0 \leq r \leq N. \tag{7}$$

Suppose

$$N(\pi_r + K_{r-1,r})^* = \{0\}, \qquad 1 \leq r \leq N, \tag{8}$$

and $R(\pi_r + K_{r-1,r})$ *is closed for every* r. *Then given* $u \in X^N$, *there exists* $x \in X^N$ *satisfying* (7).

We call any bounded mapping $h : R \to X$ a step function, if there is a division $d = (J_i) \in D_R$, with $J_i = [s^i, t^i]$ and $s^i, t^i \in R$, such that the restriction of h to $]s^i, t^i]$, we write $h\big|_{]s^i,t^i]}$, is constant for every i. In this case, we write $h \in E(R, X)$. Given $d = (J_i) \in D_R$, we can identify a step function with respect to a division $d \in D_R$, say x_d, with an element $(x_1, \ldots, x_{|d|}) \in X^{|d|}$ in the following manner:

$$x_i := x_d\big|_{]s^i,t^i]}, \quad i = 1, 2, \ldots, |d|.$$

Let $A \in B(\mathcal{I}_R, L(X))$ and $d = (J_i) \in D_R$. If we consider $N = |d|$ and $Q_m = Q_m(d)$ given by

$$Q_m = A(J_m), \quad m = 1, 2, \ldots, |d|,$$

and $K_{p,q} = K_{p,q}(d)$, where $K_{p,q} : X^{|d|} \to X$ is defined as above, then we obtain the linear Stieltjes equation (7), which we now refer to as equation (9), by the identification $m \mapsto J_m$:

$$(\pi_r + K_{0,r})x = \pi_r u, \qquad x, u \in X^{|d|}, \quad 0 \leq r \leq |d|. \tag{9}$$

In this manner, we can replace the hypothesis (8) in Theorem 3.1 by

$$N(I + A(J))^* = \{0\}, \qquad J \in \mathcal{I}_R, \tag{10}$$

which is independent of the division $d \in D_R$ and where I denotes the identity in $L(X)$.

Given $x \in X$ and $\alpha \in SV(\mathcal{I}_R, L(X))$, we have

$$\int_J d\alpha(s)x = \alpha(J)x, \quad J \in \mathcal{I}_R.$$

In particular,

$$\int_{[a,t]} d\alpha(s)x = \alpha([a,t])x, \quad t \in R.$$

Thus, if we define

$$A(J_i)x := \alpha(J_i)x, \quad i = 1, 2, \ldots, |d|,$$

for any division $d = (J_i) \in D_R$, with $J_i = [s^i, t^i]$ for each i, then $A(J) = \alpha(J)$ for all $J \in \mathcal{I}_R$. and hence hypothesis (10) becomes

$$N(I + \alpha(J))^* = \{0\}, \qquad J \in \mathcal{I}_R,$$

and equation (9) can be written as

$$x_r + \sum_{i=1}^{r} \int_{]s^i,t^i]} d\alpha(s)x_i = u_r, \qquad 1 \leq r \leq |d|,$$

or

$$x_d(t^r) + \sum_{i=1}^{r} \int_{]s^i,t^i]} d\alpha(s)x_d = u_d(t^r), \qquad 1 \leq r \leq |d|, \tag{11}$$

where x_d, u_d are step functions with respect to d, with $x_d\big|_{]s^i,t^i]} = x_i$ and $u_d\big|_{]s^i,t^i]} = u_i$ for each $i = 1, 2, \ldots, |d|$ and $t^0 = a$. Notice that, when $t^0 = a$ and $r = 0$, (11) is degenerate.

Under the above notations and due to Theorem 3.1, we have the following result.

Theorem 3.2. *Given $\alpha \in SV(\mathcal{I}_R, L(X))$ and $d = (J_i) \in D_R$, with $J_i = [s^i, t^i]$ for each i, consider the linear Stieltjes equation in $E(R,X)$*

$$x_d(t^r) + \sum_{i=1}^{r} \int_{]s^i,t^i]} d\alpha(s) x_d = u_d(t^r), \qquad 1 \leq r \leq |\, d|, \tag{12}$$

where x_d, u_d are step functions with respect to d. Suppose

$$N(I + \alpha(J))^* = \{0\}, \qquad J \in \mathcal{I}_R, \tag{13}$$

and $R(\alpha(J))$ is closed for every $J \in \mathcal{I}_R$. Then given u_d, there exists x_d satisfying (12).

4. Continuous Stieltjes equations

If $A : \mathcal{I}_R \to L(X)$ is the indefinite integral of $\alpha \in \mathcal{L}_1(\mathcal{I}_R, L(X))$ and $x \in C(R,X)$, then the Riemann-Stieltjes integral $\int_R dA\, x$ exists (see Theorems 2.2 and 7.4). Then we can consider the linear continuous operator $F_A : C(R,X) \to C(R,X)$ defined by

$$(F_A x)(t) = \int_{[a,t]} dA(s) x(s), \qquad t \in R,$$

(see Theorem 7.7) and the linear Volterra-Stieltjes integral equation

$$x(t) + \int_{[a,t]} dA(s) x(s) = f(t), \qquad t \in R, \tag{14}$$

where $f \in C(R,X)$.

In order to make the transition from discrete Stieltjes equations to equations of type (14), we need a theorem which extends the solutions of equation (12) to the entire interval. In other words, we need a result that shows that the Volterra-Stieltjes equation can be approximated by discrete equations of type (12). This result is presented next.

Theorem 4.1. *Consider $\alpha \in SV(\mathcal{I}_R, L(X)) \cap SL(\mathcal{I}_R, L(X))$ and let $\varepsilon > 0$. Then there exists a division $d = (J_i) \in D_R$, with $J_i = [s^i, t^i]$ for each $i = 1, 2, \ldots, |\, d|$, such that if x_d, u_d are step functions with respect to d and*

$$\left\| x_d(t^r) + \sum_{i=1}^{r} \int_{[s^i,t^i]} d\alpha(s) x_d(s) - u_d(t^r) \right\| < \varepsilon, \qquad \text{for all } 1 \leq r \leq d,$$

then

$$\left\| x_d(t) + \int_J d\alpha(s) x_d(s) - u_d(t) \right\| < \varepsilon, \qquad \text{for all } J = [s,t] \in \mathcal{I}_R.$$

In particular,

$$\left\| x_d(t) + \int_{[a,t]} d\alpha(s) x_d(s) - u_d(t) \right\| < \varepsilon, \qquad \text{for all } t \in R.$$

Proof. Let $\varepsilon > 0$ and $F(J) = \int_J d\alpha(s)x(s)$, for every $J \in \mathcal{I}_R$. Since $\alpha \in SL(\mathcal{I}_R, L(X))$ and $x \in H^\alpha(R, X)$, it follows from Theorem 2.1 and Lemma 7.1 that $F \in \mathcal{C}^\Delta(\mathcal{I}_R, X)$. Thus, given $J \in \mathcal{I}_R$, there exists $0 < \eta < 1$ such that $\|F(J) - F(I)\| < \varepsilon$, whenever $I \in \mathcal{I}_R$ is such that $m(I \triangle J) < \eta$.

Consider $d = (J_i) \in D_R$, with $J_i = [s^i, t^i]$ for every i, such that if $A_j = \{i : J_i \cap J_j \neq \emptyset\}$, then $\operatorname{dist}(J_j, R \setminus \cup_{i \in A_j} J_i) < \lambda\eta$, where

$$\lambda < \min\left\{1, \frac{1}{4(\max\{b_i - a_i : i = 1, 2\} + 1)}\right\}.$$

Given $J = [s, t] \in \mathcal{I}_R$, let $A = \{i : J_i \cap J \neq \emptyset\}$. Then $J \subset \cup_{i \in A} J_i$ and we can assume, without loss of generality, that $\cup_{i \in A} J_i$ is of type $\cup_{i=1}^r J_i$ for some r with $1 \leq r \leq |\, d|$. Then, since F is an additive function of intervals, it follows from the continuity of F that

$$m\left(\cup_{i \in A} J_i \setminus J\right) < \eta \implies \|F\left(\cup_{i \in A} J_i \setminus J\right)\| \leq \sum_{i \in A} \|F(J_i) - F(J)\| < v\varepsilon, \tag{15}$$

where v is the number of elements of A (clearly $v \leq |\, d|$).

Let x_d, u_d be step functions with respect to d, with $x_i = x_d\big|_{]s^i,t^i]}$ and $u_i = u_d\big|_{]s^i,t^i]}$, $i = 1, 2, \ldots, |\, d|$, and such that

$$\left\| x_d(t^r) + \sum_{i=1}^{r} \int_{]s^i,t^i]} d\alpha(s)x_d(s) - u_d(t^r) \right\| < \varepsilon.$$

(such a division d exists by Theorem 3.2) and let $j \in A$ be such that $t \in]s^j, t^j]$. Then $x_d(t) = x_j = x_d(t^j)$ and $u_d(t) = u_j = u_d(t^j)$. Thus it is enough to show that

$$\left\| \sum_{i=1}^{r} \int_{]s^i,t^i]} d\alpha(s)x_d(s) - \int_J d\alpha(s)x_d(s) \right\|$$

is sufficiently small. However

$$\left\| \sum_{i=1}^{r} \int_{]s^i,t^i]} d\alpha(s)x_d(s) - \int_J d\alpha(s)x_d(s) \right\| = \left\| \int_{\cup_{i \in A} J_i} d\alpha(s)x_d(s) - \int_J d\alpha(s)x_d(s) \right\| =$$

$$= \left\| \int_{\cup_{i \in A} J_i \setminus J} d\alpha(s)x_d(s) \right\| = \|F\left(\cup_{i \in A} J_i \setminus J\right)\|.$$

Therefore, in view of (15), we only need to prove that $m\left(\cup_{i \in A} J_i \setminus J\right) < \eta$. But this fact follows immediately from the choice of d. Indeed. Let k, h be the length of the sides of J. Then

$$m\left(\cup_{i \in A} J_i \setminus J\right) < (k + 2\lambda\eta)(h + 2\lambda\eta) - |J| = 2\lambda\eta(k + h) + 4\lambda^2\eta^2 <$$
$$< 2\lambda\eta(k + h) + 4\lambda\eta < 4\lambda\eta\left(\max\{b_i - a_i : i = 1, 2\} + 1\right) < \eta$$

and the proof is finished.

The next lemma is the basis for the proof of the result following it. For a proof of it, the reader can consult [15], Theorem V.1.4.

Lemma 4.1. *If $T : X \to Y$ is a Fredholm operator, then there exists a closed subspace X_0 of X such that*

$$X = X_0 \oplus N(T)$$

and there exists a subspace Y_0 of Y of dimension equal to the dimension of $N(T^*)$ such that

$$Y = R(T) \oplus Y_0.$$

Moreover, there exists an operator $T_0 \in B(Y, X)$ such that

(i) $N(T_0) = Y_0$;
(ii) $R(T_0) = X_0$;
(iii) $T_0T = I$ *in* X_0;
(iv) $TT_0 = I$ *in* $R(T)$;

where I denotes the identity operator.

Now we give the main result of this section which states conditions for the existence of a solution to the Volterra-Stieltjes equation (14).

Theorem 4.2. *Let A be the indefinite integral of a function $\alpha \in \mathcal{L}_1(\mathcal{I}_R, L(X))$ and consider the multidimensional linear Volterra-Stieltjes integral equation*

$$x(t) + \int_{[a,t]} dA(s)x(s) = f(t), \qquad t \in R, \tag{16}$$

where $x, f \in C(R, X)$. Suppose

$$N(I + A(J))^* = \{0\}, \qquad \text{for all } J \in \mathcal{I}_R, \tag{17}$$

and that $T = I + F_A$ is a Fredholm operator, where $(F_Ax)(t) = \int_{[a,b]} dA(s)x(s)$ for every $t \in R$, and I is the identity operator in $C(R, X)$. Then given $f \in C(R, X)$, there exists $x \in C(R, X)$ satisfying (16).

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $E(R, X)$ converging to f in the usual supremum norm. Due to (17) and the fact that $A \in BV(\mathcal{I}_R, L(X)) \cap SL(\mathcal{I}_R, L(X))$ (see Theorems 2.2 and 7.1), then by Theorems 3.2 and 4.1 we can define equations

$$x_n(t) + \int_{[a,t]} d\alpha(s)x_n(s) = f_n(t), \qquad t \in R, \tag{18}$$

each one having a solution $x_n \in E(R, X)$. Furthermore $Tx_n = (I + F_\alpha)x_n = f_n$ and, since $f_n \to f$, we have

$$Tx_n \to f. \tag{19}$$

If $\dim(B)$ denotes the dimension of a set B, since $T \in L(C(R, X))$ is a Fredholm operator, then Lemma 4.1 implies the following

a) there exists a closed subspace X_0 of $C(R, X)$ such that

$$C(R, X) = X_0 \oplus N(T); \tag{20}$$

b) there exists a subspace Y_0 of $\mathcal{C}(R, X)$ such that

$$\dim(Y_0) = \dim(N(T^*)) \tag{21}$$

$$\mathcal{C}(R, X) = R(T) \oplus Y_0; \tag{22}$$

c) there exists an operator $T_0 \in B(\mathcal{C}(R, X), \mathcal{C}(R, X))$ such that

$$N(T_0) = Y_0; \tag{23}$$

$$R(T_0) = X_0; \tag{24}$$

$$T_0 T = I \text{ in } X_0; \tag{25}$$

$$T T_0 = I \text{ in } R(T); \tag{26}$$

where I denotes the identity and T^* is the adjoint of T.

By (21) and (17), $\dim(Y_0) = 0$. Then (22) implies $\dim(\mathcal{C}(R, X)) = \dim(R(T))$ and hence $\dim(N(T)) = 0$. Thus $\dim(\mathcal{C}(R, X)) = \dim(X_0)$ by (20) and, since $X_0 \subset \mathcal{C}(R, X)$ is closed, then $X_0 = \mathcal{C}(R, X)$.

Since $f_n \to f$ and T_0 is bounded, we have

$$T_0 f_n \to T_0 f. \tag{27}$$

For each n, we have by (18) that $f_n = T x_n$ and therefore $T_0 f_n = T_0 T x_n$. Then by (25),

$$T_0 f_n = x_n. \tag{28}$$

Hence (27) and (28) implies

$$x_n \to T_0 f$$

and taking $x = T_0 f$, we have $x \in \mathcal{C}(R, X)$ since $\mathcal{C}(R, X)$ is complete.

Is remais to prove that x is a solution of (16). But this fact follows from (26), since $T T_0 f = f$ and therefore $T x = f$.

5. The linear integral equation of Volterra-Bochner-Lebesgue

In this section, we are concerned with the linear integral equation of Volterra

$$x(t) + (L)\int_{[a,t]} \alpha(s)x(s) = f(t), \qquad t \in R, \tag{29}$$

where $\alpha \in \mathcal{L}_1(R, L(X))$ and $x, f \in \mathcal{C}(R, X)$. We call equation (29) the linear integral equation of Volterra-Bochner-Lebesgue. By means of Theorem 4.2, we show the following result on the existence of a solution to (29).

Theorem 5.1. *Given $\alpha \in \mathcal{L}_1(R, L(X))$, consider the linear integral equation of Volterra-Bochner-Lebesgue*

$$x(t) + (L)\int_{[a,t]} \alpha(s)x(s) = f(t), \qquad t \in R, \tag{30}$$

where $x, f \in \mathcal{C}(R, X)$. Suppose

$$N(I + A(J))^* = \{0\}, \quad \textit{for each } J \in \mathcal{I}_R,$$

where A is the indefinite integral of α and $(I + F_A) \in L(C(R, X))$ is a Fredholm operator with I the identity in $C(R, X)$ and $(F_A x)(t) = (H) \int_{[a,t]} dA(s)x(s)$, for every $t \in R$. Then given $f \in C(R, X)$, there exists $x \in C(R, X)$ satisfying (30).

Proof. By Theorem 7.5, equation (30) is equivalent to

$$x(t) + \int_{[a,t]} dA(s)x(s) = f(t), \qquad t \in R,$$

and the result follows by Theorem 4.2.

Remark 5.1. *Under the hypotheses of Theorem 5.1, equation* (30) *in equivalent to the continuous Stieltjes equation which is the limit of discrete equations* (12). *It follows that* (30) *is the limit of discrete equations* (12).

6. The linear integral equation of Volterra-Henstock

Our main result (Theorem 6.1) states conditions for the existence of a solution to the linear integral equation of Volterra in the sense of the Henstock integral

$$x(t) + (H) \int_{[a,t]} \alpha(s)x(s) = f(t), \qquad t \in R,$$

where $\alpha \in H(R, L(X))$ is such that its indefinite integral, A, belongs to $SV(\mathcal{I}_R, L(X))$ and $x, f \in C(R, X)$. The following lemma is needed. See [11], Theorem 7.1.2.

Lemma 6.1. *Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of mappings from X to X such that each mapping has a fixed point, say, $x_n = T_n x_n$. Let $T : X \to X$ be such that for some integer m, T^m is a contraction, where T^m is the composition of T m times, and suppose $T_n \to T$ uniformly. Then $x_n \to x_0 = T x_0$.*

Theorem 6.1. *Given $\alpha \in H(R, L(X))$ such that $A \in SV(\mathcal{I}_R, L(X))$, where A is the indefinite integral of α, consider the linear integral equation of Volterra-Henstock*

$$x(t) + (H) \int_{[a,t]} \alpha(s)x(s) = f(t), \qquad t \in R, \tag{31}$$

as well as the linear Volterra-Bochner-Lebesgue integral equations

$$x(t) + (L) \int_{[a,t]} \chi_{X_n} \alpha(s)x(s) = f(t), \qquad t \in R, \tag{32}$$

obtained through Theorem 7.3, and the operator $T : C(R, X) \to C(R, X)$ defined by

$$(Tx)(t) = f(t) - (H) \int_{[a,t]} \alpha(s)x(s), \qquad t \in R,$$

where $x, f \in C(R, X)$ and $n \in \mathbb{N}$. Suppose

$$N(I + A_n(J))^* = \{0\}, \qquad J \in \mathcal{I}_R, \ n \in \mathbb{N}, \tag{33}$$

where $A_n(J) = (L)\int_{[a,t]} \chi_{X_n}\alpha(s)x(s)$, *for every* $J \in \mathcal{I}_R$, *and* $(I + F_{A_n}) \in L(C(R,X))$ *is a Fredholm operator, where* I *is the identity in* $C(R,X)$ *and* $(F_{A_n}x)(t) = \int_{[a,t]} dA_n(s)x(s)$, *for each* $t \in R$. *Then given* $f \in C(R,X)$, *each equation* (32) *admits a solution* $x_n \in C(R,X)$. *Suppose in addition that one of the following conditions is fulfilled:*

(i) $(x_n)_{n\in\mathbb{N}}$ *admits a convergent subsequence* $x_{n_k} \to x_0 \in C(R,X)$;

(ii) α *is bounded and* T^m *is a contraction for some* $m > 1$, *where* T^m *is the composition of* T m *times.*

Then given $f \in C(R,X)$, *there exists a solution* $x \in C(R,X)$ *satisfying* (31).

Proof. Given $n \in \mathbb{N}$, consider the continuous operator $T_n : C(R,X) \to C(R,X)$ defined by

$$(T_n x)(t) = f(t) - (L)\int_{[a,t]} \chi_{X_n}\alpha(s)x(s), \qquad t \in [a,b] = R.$$

By Theorem 5.1, given $f \in C(R,X)$, each equation (32) admits a solution $x_n \in C(R,X)$. Hence x_n is a fixed point of T_n.

Suppose (i) holds. We will show that x_0 is a solution of (31), that is, x_0 is a fixed point of T. But given $n_k \in \mathbb{N}$, we have

$$\|x_0 - Tx_0\| \leq \|x_0 - x_{n_k}\| + \|x_{n_k} - T_{n_k}x_{n_k}\| + \|T_{n_k}x_{n_k} - T_{n_k}x_0\| + \|T_{n_k}x_0 - Tx_0\|,$$

where the first summand tends to zero as $k \to 0$ by the convergence of the subsequence, the second summand equals zero since x_{n_k} is a fixed point of T_{n_k}, the third summand is smaller than $\|T_{n_k}\|\,\|x_{n_k} - x_0\| \to 0$ as $k \to 0$ since T_{n_k} is bounded, and the fourth summand tends to zero as $k \to 0$ by Theorem 7.3.

Now suppose (ii) holds. Since α in bounded, then T is bounded and hence Theorem 7.3 implies

$$\|T_n - T\| = \sup_{\|x\|\leq 1} \|T_n - T\| = \sup_{\|x\|\leq 1} \left\|(L)\int_{[a,t]} \chi_{X_n}\alpha(s)x(s) - (H)\int_{[a,t]} \alpha(s)x(s)\right\| \longrightarrow 0$$

as $n \to \infty$. Then by Lemma 6.1, $x_n \to x_0 = Tx_0$ and we finished the proof.

Remark 6.1. *Given* $\alpha \in H(R, L(X))$ *such that* $A \in SV(\mathcal{I}_R, L(X))$, *where* A *is the indefinite integral of* α, *let* $(X_n)_{n\in\mathbb{N}}$ *be the sequence of closed subsets of* R *obtained by Theorem 7.3. Then given* $J_0 \in \mathcal{I}_R$, $n \in \mathbb{N}$ *and* $x \in C(R,X)$, *we have*

$$\left\|(L)\int_{X_n\cap J_0} \alpha(s)x(s)ds - (H)\int_{J_0} \alpha(s)x(s)ds\right\| \leq$$

$$\leq \sup_{J\in\mathcal{I}_R} \left\|(L)\int_{X_n\cap J} \alpha(s)x(s)ds - (H)\int_J \alpha(s)x(s)ds\right\| < \frac{1}{n}.$$

Let $I_n = X_n \cap J_0$ *and* $\alpha_n = \chi_{I_n}\alpha$, *for every* $n \in \mathbb{N}$. *Then*

$$\left\|(L)\int_{J_0} \alpha_n(s)x(s)ds - (H)\int_{J_0} \alpha(s)x(s)ds\right\| < \frac{1}{n}, \qquad \textit{for every } n \in \mathbb{N}.$$

Besides, $(L)\int_{J_0}\alpha_n(s)x(s)ds = \int_{J_0} dA_n(s)x(s)$ *by Theorem 7.5, and for each* n*, the indefinite integral of* a_n*, say* A_n*, belongs to* $BV(\mathcal{I}_R, L(X)) \cap SL(\mathcal{I}_R, L(X))$*. Then by Theorem 7.1 and its proof, it follows that, under the hypotheses of Theorem 6.1, the equation* (31) *can be approximated by a system of discrete equations* (12).

7. Appendix

As we mentioned before, we dedicate this Appendix to the fundamental theory of the Henstock integral. We start by introducing the notions of continuity, variation and semivariation for functions of intervals. Then we give some basic theorems and properties of the integral.

By $A\triangle B$ we mean the symmetric difference between two sets A and B of R. Let $F \in A(\mathcal{I}_R, X)$. We say that F is continuous on $J \in \mathcal{I}_R$ if and only if for every $\varepsilon > 0$, there is $\eta > 0$ such that for every $I \in \mathcal{I}_R$ with $m(J\triangle I) \leq \eta$, we have

$$\|F(J) - F(I)\| < \varepsilon.$$

If F is continuous on each $J \in \mathcal{I}_R$, then F is a continuous function (of intervals) and we write $F \in \mathcal{C}^{\triangle}(\mathcal{I}_R, X)$. In particular, if we denote the interior of R by $\operatorname{int}(R)$, then we write $F \in \mathcal{C}^{\triangle}(\mathcal{I}_{\operatorname{int}(R)}, X)$ whenever F is continuous on each $J \in \mathcal{I}_R$, with $J \subset \operatorname{int}(R)$. By $\mathcal{C}(R, X)$ we mean the set of all functions $f : R \to X$ which are continuous in the usual sense.

Lemma 7.1. $SL(\mathcal{I}_R, X) \subset \mathcal{C}^{\triangle}(\mathcal{I}_R, X)$.

Proof. Let $F \in SL(\mathcal{I}_R, X)$. Then for every $\varepsilon > 0$ and every $A \subset R$, with $m(A) = 0$, there is a gauge δ of A such that for every δ-fine $d = (\xi_i, J_i) \in TPD_R$ with $\xi_i \in A$ for each i, we have

$$\sum_i \|F(J_i)\| < \varepsilon. \tag{34}$$

Given $J \in \mathcal{I}_R$, let A be the boundary of J. Then $m(A) = 0$. Let $d = (\xi_i, J_i) \in TPD_R$ be such that for each i, $\xi_i \in A$, $(J_i \cap J) \subset A$ and $J_i \cap \operatorname{int}(J) = \emptyset$. This means that the intervals J_i's are attached to J from the outside and (34) holds.

Given $i \in \{1, 2 \ldots |d|\}$, let l_i and k_i be the measures of the sides of J_i and consider $\eta > 0$ such that $\eta < \min\{l_i, h_i; i = 1, 2 \ldots |d|\}$. Then if $K \in \mathcal{I}_R$ is such that $K \cap J = \emptyset$, $K \cap \operatorname{int}(J) = \emptyset$ and $m(K) \leq \eta$, there are $i, j \in \{1, 2 \ldots |d|\}$ such that $K \subset (J_i \cup J_j)$. Hence $K \subset \cup_i J_i$.

Now, consider $I \in \mathcal{I}_R$ such that $m(I\triangle J) < \eta$. By the choice of η, there exists a finite number of intervals K_i each of them of type K as in the previous paragraph and such that $(I\triangle J) \subset \cup_i K_i$. Thus $(I\triangle J) \subset \cup_i J_i$ which implies

$$\|F(I) - F(J)\| \leq \sum_i \|F(J_i)\| < \varepsilon$$

and the proof is finished.

The next result follows from Theorem 2.1 and Lemma 7.1.

Theorem 7.1. *Let* $f \in H(R, X)$ *and* F *be its indefinite integral. Then* $F \in \mathcal{C}^{\triangle}(\mathcal{I}_R, X)$.

A proof of the next result can be carried out by straight adaptation of the one-dimensional case (see, for instance, [12], Theorem 2.5) and apply Theorem 2.1.

Theorem 7.2. *Let $f \in H(R,X)$ and F be its indefinite integral. Then*

$$\lim_{r\downarrow 0} \frac{F(Q(\xi,r))}{|Q(\xi,r)|} = f(\xi),$$

for m-almost every $\xi \in R$, where $Q(\xi,r)$ denotes the square with sides of length $r>0$ and centered in ξ.

Corollary 7.1. *Every $f \in H(R,X)$ is m-measurable.*

Proof. Given $n \in \mathbb{N}$ and $\xi \in R$, let

$$h_n(\xi) = \frac{F(Q(\xi,1/n))}{|Q(\xi,1/n)|}.$$

From the continuity of F on R (Theorem 7.1), each function h_n is continuous (see Remark 2.2) and hence m-measurable. Besides by Theorem 7.2, $h_n \to f$ m-almost everywhere on R. Therefore f is the limit m-almost everywhere of m-measurable functions and hence f is m-measurable.

Theorem 7.3. *Given $f \in H(R,X)$, there exists a sequence $\{X_n\}_{n\in\mathbb{N}}$ of closed sets X_n such that $X_n \uparrow R$ (i.e., $X_n \subset X_{n+1}$ and $\cup_n X_n = R$) and $f \in \mathcal{L}_1(X_n,R)$, for every $n \in \mathbb{N}$. Furthermore,*

$$\sup_{J\in\mathcal{I}_R} \left\| (L)\int_{X_n\cap J} f - (H)\int_J f \right\| < \frac{1}{n}.$$

In order to prove the first part of Theorem 7.3, one can adapt the proof of [7], Theorem 2.10 for the Banach space-valued case when Corollary 7.1 must be applied. For the second part, it is enough to adapt the proof of [17], Theorem 6 for the Banach space-valued case.

Theorem 7.4. *If $\alpha \in SV(\mathcal{I}_R, L(X,Y))$ and $f \in C(R,X)$, then $f \in R^\alpha(R,X)$, that is the Riemann-Stieltjes integral*

$$\int_R d\alpha(s)f(s) = \lim_{\Delta d\to 0} \sum_i \alpha(J_i)f(\xi_i),$$

exists. Besides $\left\| \int_R d\alpha(s)f(s) \right\| \leq SV(\alpha)\|f\|$.

A proof of Theorem 7.4 can be carried out by adapting the steps of the one-dimensional case (see, for instance, [10], Theorem II.1.1). We give such a proof in the following lines.

Proof. By the definition of $SV(\alpha)$,

$$\left\| \sum_i \alpha(J_i)\, f(\xi_i) \right\| \leq SV(\alpha)\|f\|. \tag{35}$$

Therefore, if $\Delta d \to 0$ and the integral exists, we have

$$\left\| \int_R d\alpha(s) f(s) \right\| \leq SV(\alpha) \|f\|.$$

Now, for every $\delta > 0$, we set

$$\sum_{\delta} = \left\{ \sigma_{d,\xi} = \sum_i \alpha(J_i) f(\xi_i) \in Y : \Delta d \leq \delta,\, d = (\xi_i, J_i) \in TD_R \right\}.$$

Then (35) implies that every $\sum_\delta$ is a bounded subset of Y. Clearly $\delta' \leq \delta''$ implies $\sum_{\delta'} \supset \sum_{\delta''}$. Thus the sets $\sum_{1/n}$ are a countable filter basis of bounded sets. Since Y is sequentially complete, in order to prove that this filter basis is convergent, it is enough to show that it is a Cauchy filter basis, that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $\sigma_{d,\xi}, \sigma_{\overline{d},\overline{\xi}} \in \sum_\delta$, with $d = (\xi_i, J_i), \overline{d} = (\overline{\xi}_j, \overline{J}_j) \in TD_R$, we have $\|\sigma_{d,\xi} - \sigma_{\overline{d},\overline{\xi}}\| \leq \varepsilon$. Then taking $d \vee \overline{d}$, it is sufficient to consider the case when $d \geq \overline{d}$ and we have

$$\sigma_{d,\xi} - \sigma_{\overline{d},\overline{\xi}} = \sum_i \alpha(J_i) \left[f(\xi_i) - f(\overline{\xi}_{j(i)}) \right],$$

where $\overline{\xi}_{j(i)} = \overline{\xi}_j$ whenever $J_i \subset \overline{J_j}$. It follows that

$$\|\sigma_{d,\xi} - \sigma_{\overline{d},\overline{\xi}}\| \leq SV(\alpha) \sup \{ \|f(t) - f(s)\| : t, s \in R,\ \|t - s\| \leq \delta \}$$

which is arbitrarily small with δ, since f is uniformly continuous, and the proof is complete.

Theorem 7.5. *Let $\alpha \in \mathcal{L}_1(R, L(X,Y))$, A be the indefinite integral of α and $f \in \mathcal{C}(R, X)$. Then $\alpha f \in \mathcal{L}_1(R, Y)$ and*

$$(L) \int_R \alpha(s) f(s) ds = \int_R dA(s) f(s).$$

Proof. At first, since $\alpha \in \mathcal{L}_1(R, L(X,Y))$, then $M\alpha \in \mathcal{L}_1(R, L(X,Y))$, for every $M \in \mathbb{R}$. Therefore the mapping $M\|\alpha(\cdot)\| : R \to \mathbb{R}$ is Lebesgue integrable.

Since $f \in \mathcal{C}(R, X)$, there exists $M > 0$ such that $\|f\| \leq M$. Moreover, the mapping $\|\alpha(\cdot) f(\cdot)\| : R \to \mathbb{R}$ is m-measurable and $\|\alpha f\| \leq M\|\alpha\|$. Hence since $\mathcal{L}_1(R, \mathbb{R})$ is a lattice, we have $\|\alpha f\| \in \mathcal{L}_1(R, \mathbb{R})$. Thus $\alpha f \in \mathcal{L}_1(R, Y) \subset H(R, Y)$.

Now we prove the equality. Let $\varepsilon > 0$. Since $\alpha \in \mathcal{L}_1(R, L(X,Y)) \subset H(R, L(X,Y))$, there exists a gauge δ_1 of R such that for every δ_1-fine $d = (\zeta_j, I_j) \in TD_R$, we have

$$\sum_i \|\alpha(\zeta_j)|I_j| - A(I_j)\| < \varepsilon.$$

By Theorem 2.2, $A \in BV(\mathcal{I}_R, Y)$. Then by Theorem 7.4, the Riemann-Stieltjes integral, $\int_R dA\, f$, exists. Thus, there exists a constant gauge δ_2 of R such that for every δ_2-fine $d = (\rho_k, K_k) \in TD_R$, we have

$$\left\| \sum_k A(K_k) f(\rho_k) - \int_R dA(s) f(s) \right\| < \varepsilon.$$

Now taking a gauge δ of R defined by $\delta(\xi) \leq \delta_s(\xi)$, $s = 1, 2$, for every $\xi \in R$, and a δ-fine $d = (\xi_i, J_i) \in TD_R$, we have

$$\left\| \sum_i \alpha(\xi_i) f(\xi_i) |J_i| - \int_R dA(s) f(s) \right\| \leq$$

$$\leq \sum_i \|\alpha(\xi_i) f(\xi_i) |J_i| - A(J_i) f(\xi_i)\| + \left\| \sum_i A(J_i) f(\xi_i) - \int_R dA(s) f(s) \right\| < \varepsilon \|f\| + \varepsilon$$

and the proof is complete after applying the Cauchy Criteria.

The next result can be proved by simply applying the definitions.

Theorem 7.6. *If $\alpha \in H(R, L(X,Y))$ and $f \in H^\alpha(R, X)$, where A is the indefinite integral of α, then $\alpha f \in H(R, Y)$ and*

$$(H) \int_R \alpha(s) f(s) ds = (H) \int_R dA(s) f(s).$$

It follows from Theorems 7.4 and 7.6 that

Corollary 7.2. *If $\alpha \in H(R, L(X,Y))$, $A \in SV(\mathcal{I}_R, L(X,Y))$, where A is the indefinite integral of α, and $f \in C(R, X)$, then $\alpha f \in H(R, X)$ with*

$$(H) \int_R \alpha(s) f(s) ds = \int_R dA(s) f(s).$$

The next result is an analogue for the Riemann-Stieltjes integral of the well-known Saks-Henstock Lemma. A proof of it follows the standard steps. See, for instance, [16], Lemma 16.

Lemma 7.2 (Saks-Henstock). *Suppose $\alpha \in A(\mathcal{I}_R, L(X,Y))$ and $f \in R^\alpha(R, X)$. Given $\varepsilon > 0$, let δ be a gauge of R such that for every δ-fine $d = (\xi_i, J_i) \in TD_R$,*

$$\left\| \sum_{i=1}^{|d|} \alpha(J_i) f(\xi_i) - \int_R d\alpha(t) f(t) \right\| < \varepsilon.$$

Then for any δ-fine $d' = (\zeta_j, I_j) \in TPD_R$, we have

$$\left\| \sum_{j=1}^{|d'|} \alpha(\zeta_j) f(I_j) - \int_{I_j} d\alpha(t) f(t) \right\| \leq \varepsilon.$$

Theorem 7.7. *Let A be the indefinite integral of a function $\alpha \in \mathcal{L}_1(R, L(X))$. Then F_A defined by*

$$(F_A f)(t) = \int_{[a,t]} dA(s) f(s), \qquad f \in C(R, X), \quad t \in R,$$

belongs to $L(C(R, X))$.

Proof.

(1) Since $\alpha \in \mathcal{L}_1(R, L(X)) \subset H(R, L(X))$, it follows from Theorem 2.1 and Lemma 7.1 that $A \in SL(\mathcal{I}_R, L(X)) \subset \mathcal{C}^{\triangle}(\mathcal{I}_R, L(X))$. Also $A \in BV(\mathcal{I}_R, L(X))$ by Theorem 2.2.

(2) Let $f \in C(R, X)$. We assert that $f \in H^{\alpha}(R, X)$. We know that $f \in R^{\alpha}(R, X)$ by Theorem 7.4. Then by Corollary 7.2, $\alpha f \in H(R, X)$ and

$$(H) \int_R \alpha(s) f(s) ds = \int_R dA(s) f(s).$$

Thus, given $\varepsilon > 0$, let δ be the gauge of R such that for every δ-fine $d = (\xi_i, J_i) \in TD_R$, we have

$$\sum_i \|(L) \int_{J_i} \alpha(s) ds - \alpha(\xi_i)|J_i|\| < \varepsilon \quad \text{and} \quad \sum_i \|(H) \int_{J_i} \alpha(s) f(s) ds - \alpha(\xi_i) f(\xi_i)|J_i|\| \varepsilon.$$

Then

$$\sum_i \|(H) \int_{J_i} dA(s) f(s) - A(J_i) f(\xi_i)\| = \sum_i \|(H) \int_{J_i} \alpha(s) f(s) ds - A(J_i) f(\xi_i)\| \leq$$

$$\leq \sum_i \|(H) \int_{J_i} \alpha(s) f(s) ds - \alpha(\xi_i) f(\xi_i)|J_i|\| + \sum_i \|\alpha(\xi_i) f(\xi_i)|J_i| - A(J_i) f(\xi_i)\| <$$

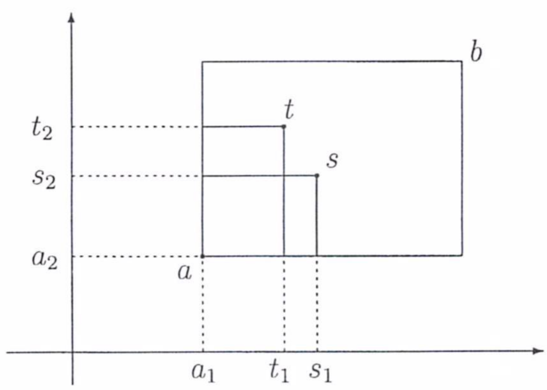
$$< \varepsilon + \varepsilon \|f\|$$

which proves the assertion.

(3) Let $\tilde{f}^A(t) = \int_{[a,t]} dA(s) f(s)$, for every $t \in R = [a, b]$. By Remark 2.2, we can consider $\tilde{f}^A$ as a function of intervals of R. Then by 1., Theorem 2.1 and Lemma 7.1, $\tilde{f}^A \in \mathcal{C}^{\triangle}(\mathcal{I}_R, X)$ and hence we have the continuity of $\tilde{f}^A$ as a function of points of R. Indeed. Given $t, s \in R$, we have

$$\tilde{f}^A(t) - \tilde{f}^A(s) = \int_{[a,t]} dA(s) f(s) - \int_{[a,s]} dA(s) f(s).$$

Consider the following figure:



Then

$$\int_{[a,t]} dA(s)f(s) - \int_{[a,s]} dA(s)f(s) = \int_{[(t_1,a_2),s]} dA(s)f(s) + \int_{[(a_1,s_2),t)]} dA(s)f(s).$$

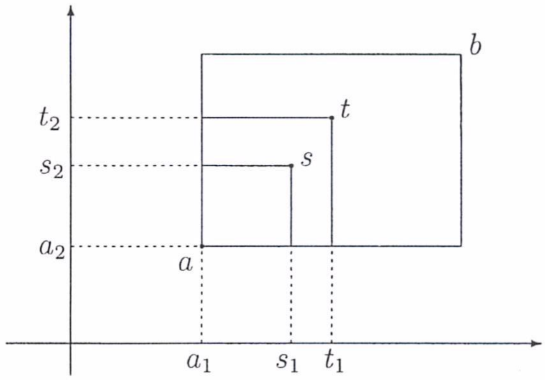
But $\|t-s\| \to 0$ implies $|[(t_1,a_2),s]| \to 0$ and $|[(a_1,s_2),t)]| \to 0$. Thus it follows from the continuity of the indefinite integral (i.e., the continuity of $\tilde{f}^A$ as a function of interval of R that

$$\|\tilde{f}^A([(t_1,a_2),s])\| \to 0 \qquad \text{and} \qquad \|\tilde{f}^A([(a_1,s_2),t)])\| \to 0.$$

Hence

$$\|\tilde{f}^A(t) - \tilde{f}^A(s)\| \to 0 \qquad \text{as} \qquad \|t-s\| \to 0.$$

Now consider the figure



The figures above show the possible positions of the points t and s. For the second figure we have

$$\int_{[a,t]} dA(s)f(s) - \int_{[a,s]} dA(s)f(s) =$$

$$= \int_{[(s_1,a_2),(t_1,s_2)]} dA(s)f(s) + \int_{[(a_1,s_2),(s_1,t_2)]} dA(s)f(s) + \int_{[s,t]} dA(s)f(s)$$

But $\|t-s\| \to 0$ implies

$$|[(s_1,a_2),(t_1,s_2)]| \to 0, \quad |[(a_1,s_2),(s_1,t_2)]| \to 0 \quad \text{and} \quad |[s,t]| \to 0.$$

Thus the continuity if the indefinite integral with respect to A, $\tilde{f}^A$, implies

$$\left\|\tilde{f}^A([(s_1,a_2),(t_1,s_2)])\right\| \to 0, \quad \left\|\tilde{f}^A([(a_1,s_2),(s_1,t_2)])\right\| \to 0 \quad \text{and} \quad \left\|\tilde{f}^A([s,t])\right\| \to 0,$$

and hence

$$\left\|\tilde{f}^A(t) - \tilde{f}^A(s)\right\| \to 0 \ \text{ as } \ |t-s| \to 0$$

and the continuity of $\tilde{f}^A$ as a function of points of R follows, that is, $F_A f = \tilde{f}^A \in C(R,X)$.

(4) The linearity of $\tilde{f}^A$ follows from the linearity of the integral.

(5) Now we prove the continuity of F_A. Given $f, g \in \mathcal{C}(R, X)$ and $t \in R$, consider $\varepsilon, \delta > 0$ from the definition of $(f-g) \in R^A(R, X)$ and let $d = (\xi_i, J_i) \in TD_{[a,t]}$ be δ-fine. Then by the Saks-Henstock Lemma for the Riemann-Stieltjes integral (Lemma 7.2) and by the properties of the variation of a function we have

$$\|F_A f(t) - F_A g(t)\| = \left\| \int_{[a,t]} dA(s)(f-g)(s) \right\| \leq$$

$$\leq \left\| \int_{[a,t]} dA(s)(f-g)(s) - \sum_i A(J_i)\left[f(\xi_i) - g(\xi_i)\right] \right\| + \left\| \sum_i A(J_i)\left[f(\xi_i) - g(\xi_i)\right] \right\| <$$

$$< \varepsilon + V_{[a,t]}(A)\|f-g\| \leq \varepsilon + V(A)\|f-g\|,$$

where $V_{[a,t]}(A)$ is the variation of A on $[a,t]$. Then

$$\|F_A f - F_A g\| = \sup_{t \in R} \|F_A f(t) - F_A g(t)\| \to 0 \quad \text{as} \quad \|f-g\| \to 0$$

and the proof is finished.

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