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The Markus-Yamabe Conjecture for differentiable vector fields of $\mathbb{R}^2$. *

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Dedicated to César Camacho on his 60th Birthday

ABSTRACT

(a) Let $X: \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable map (not necessarily C^1) and let Spec(X) be the set of (complex) eigenvalues of the derivative DX_p when p varies in $\mathbb{R}^2$. If, for some $\epsilon > 0$, Spec$(X) \cap [0, \epsilon) = \emptyset$ then X is injective.

(b) Let $X: \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable vector field such that $X(0) = 0$ and Spec$(X) \subset \{z \in \mathbb{C} : \Re(z) < 0\}$. Then, for all $p \in \mathbb{R}^2$, there is a unique positive trajectory starting at p; moreover the ω−limit set of p is equal to $\{0\}$. , 2003 ICMC-USP

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RESUMO

Seja $X: \mathbb{R}^2 \to \mathbb{R}^2$ uma aplicação diferenciável (não necessariamente C^1) e $\text{Spec}(X)$ o conjunto dos autovalores (complexos) da derivada DX_p quando p percorre todo o $\mathbb{R}^2$. Se, para algum $\epsilon > 0$, $\text{Spec}(X) \cap [0, \epsilon) = \emptyset$ então X é injetivo.

(b) Seja $X: \mathbb{R}^2 \to \mathbb{R}^2$ um campo de vetores diferenciável tal que $X(0) = 0$ e $\text{Spec}(X) \subset \{z \in \mathbb{C} : \Re(z) < 0\}$. Então, para todo $p \in \mathbb{R}^2$, existe uma única trajetória positiva partindo de p; além disso o conjunto ω−limite de p é igual a $\{0\}$.

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INTRODUCTION

The main results of this article are the following theorems:

Theorem 2.2.1. *Let* $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ *be a differentiable map (not necessarily of class* C^1*) If, for some* $\epsilon > 0$, $\operatorname{Spec}(X) \cap [0, \epsilon) = \emptyset$, *then* X *is injective.*

Theorem 3.3.1. *Let* $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ *be a differentiable vector field such that* $X(0) = 0$ *and* $\operatorname{Spec}(X) \subset \{z \in \mathbb{C} : \Re(z) < 0\}$. *Then, for all* $p \in \mathbb{R}^2$, *there is a unique positive trajectory starting at* p*; moreover, the* $\omega-$*limit set of* p *is equal to* $\{0\}$.

These results are relevant with respect to the following conjectures as it will be explained in the comments below. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map. We denote by $\operatorname{Spec}(F)$ the set of (complex) eigenvalues of the derivative DF_p, as p varies in $\mathbb{R}^n$. One of the several equivalent formulations of the famous KELLER JACOBIAN CONJECTURE states that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map having constant non-zero Jacobian, then F is injective. The MARKUS-YAMABE CONJECTURE states that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 Vector field such that $X(0) = 0$ and $\operatorname{Spec}(F) \subset \{z \in \mathbb{C} : \Re(z) < 0\}$, then the origin 0 is a global attractor. The CHAMBERLAND CONJECTURE [7] states that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a map of class C^1 such that, for some $\epsilon > 0$, $\operatorname{Spec}(F) \cap \{z \in \mathbb{C} : |z| < \epsilon\} = \emptyset$, then F is injective. It has already been proved that the Chamberland Conjecture implies the Weak Markus-Yamabe one [12]. In this respect, V. A. Alexandrov stated in [1] a conjecture which is close to that of Chamberlain.

Comments:

(1) Theorem 2.2.1. is optimal in the following sense. If the assumptions are relaxed to $0 \notin \operatorname{Spec}(X)$, then the conclusion, even for polynomials maps X, need no longer be true, as shown by Pinchuck's counterexample [25]. Also B. Smith and F. Xavier ([30], Theorem 4) proved that there exist integers $n > 2$ and non-injective polynomial maps $P : \mathbb{R}^n \to \mathbb{R}^n$ with $\operatorname{Spec}(P) \cap [0, \infty) = \emptyset$.

(2) Theorem 2.2.1. extends the A. Fernandes and C. Gutierrez injectivity result, which requires X be of class C^1. In the same way, Theorem 3.1 improves the solution of the bidimensional Markus-Yamabe Conjecture given by R. Fessler and C. Gutierrez for C^1 vector fields [13], [14] (see also [20] and [23]). It has already been proved that the Markus-Yamabe Conjecture is false for dimensions greater than two ([9]).

(3) Theorem 2.2.1. confirms, in a stronger way, the Chamberland and Alexandrov conjectures in dimension 2.

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(4) Theorem 2.2.1. does not imply the bidimensional real Keller Jacobian Conjecture, since given an even natural n, the polynomial map $X(x,y) = (-y, x + y^n)$ has constant Jacobian equal to one and satisfies $\mathrm{Spec}(X) = \mathbb{S}^1 \cup (\mathbb{R} \setminus \{0\})$

(5) Campbell [4] classified the two-dimensional C^1 maps whose eigenvalues are both 1. All such maps have an explicit inverse. The class of functions considered in Theorem 2.2.1. is much broader, but no explicit inverse is given. Also, the surjectivity of the maps studied by Campbell remains as an open problem for the case of differentiable maps. The articles [8] and [10] are related to [4].

This paper is organized as follows. The first section is devoted to prove Theorem 2.2.1. under a stronger assumption. The proof of Theorem 2.2.1. is completed in section 2. Section 3 is devoted to the proof of Theorem 3.3.1.

1. A PARTIAL INJECTIVITY RESULT

In this section we prove the following result

THEOREM 1.1.1. *Let $X = (f,g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable map. If, for some $\epsilon > 0$, $\mathrm{Spec}(X) \cap (-\epsilon, \epsilon) = \emptyset$, then X is injective.*

To this end we shall use the following A. V. Černavskii's Theorem [5], [6] (see also [31] and [27]).

THEOREM 1.1.2. *Let U be an open subset of $\mathbb{R}^2$ and $X = (f,g) : U \to \mathbb{R}^2$ be a differentiable map such that, for all $p \in U$, $DX(p)$ is non-singular. Then, for all $p \in U$, there exist a neighborhood $V = V(p)$ and $\varepsilon = \varepsilon(p) > 0$ such that $X|_V \colon V \to (f(p) - \varepsilon, f(p) + \varepsilon) \times (g(p) - \varepsilon, g(p) + \varepsilon)$ is a homeomorphism.*

As a direct consequence of this result we obtain.

COROLLARY 1.1.3. *Let $X = (f,g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable map such that, for all $p \in \mathbb{R}^2$, $DX(p)$ is non-singular. Then the level curves $\{f = \text{constant}\}$ (resp. $\{g = \text{constant}\}$) make up a C^0-foliation $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) on $\mathbb{R}^2$, without singularities, such that if L is a leaf of $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) then $g|_L$ (resp. $f|_L$) is strictly monotone; in particular $\mathcal{F}(f)$ and $\mathcal{F}(g)$ are transversal to each other.*

Orient $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) so that if L is an oriented leaf of $\mathcal{F}(f)$ (resp.$\mathcal{F}(g)$) then $g|_L$ (resp. $f|_L$) is increasing in conformity with the orientation of L.

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Let $a > 0$ and $\sigma, \gamma : [-a, a) \to \mathbb{R}^2$ be injective C^0-curves such that $\sigma(0) = \gamma(0) = 0$. We say that γ is *transversal* (resp. *tangent*) to σ at $\gamma(0) = \sigma(0)$, if there exist $\varepsilon > 0$. neighborhoods V of $\gamma(0)$ and U of $(0,0)$, in $\mathbb{R}^2$ and a homeomorphism $H : V \to U$ such that for all $|t| < \varepsilon$, $H \circ \sigma(t) = (t, 0)$ and $H \circ \gamma(t) = (t, t)$ (resp $H \circ \gamma(t) = (t, \phi(t))$, where $\phi(t) \geq 0$ and $\phi(0) = 0$). If γ is tangent to σ at $\gamma(0) = \sigma(0)$, we say that the tangency is *generic* if H and ϕ (as right above) can be taken so that $\phi(t) = |t|$.

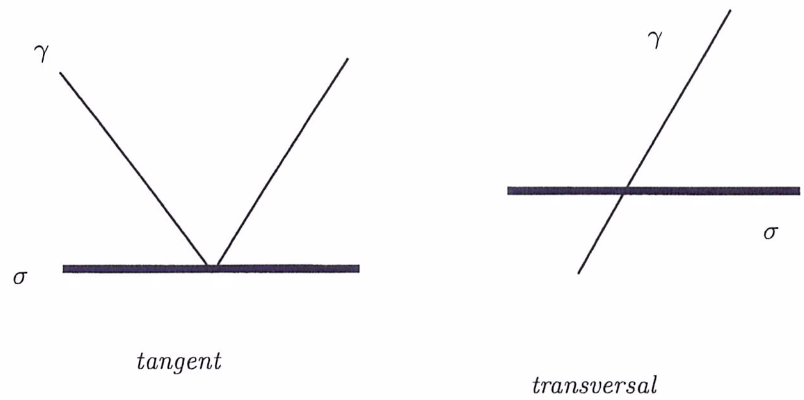

FIG. 1.

Let $h_0(x, y) = xy$ and consider the set

$$B = \{(x, y) \in [0, 2] \times [0, 2] : 0 < x + y \leq 2\} \ .$$

Definition 1. Let $X = (f, g) \colon \mathbb{R}^2 \to \mathbb{R}^2$ be as in Theorem 1.1.1. Given $h \in \{f, g\}$, we will say that $\mathcal{A} \subset \mathbb{R}^2$ is a *half-Reeb component* for $\mathcal{F}(h)$ (or simply a *hRc* for $\mathcal{F}(h)$) if there is a homeomorphism $H : B \to \mathcal{A}$ which is a topological equivalence between $\mathcal{F}(h)|_{\mathcal{A}}$ and $\mathcal{F}(h_0)|_B$ and such that

(1) The segment $\{(x, y) \in B : x + y = 2\}$ is sent by H onto a transversal section for the foliation $\mathcal{F}(h)$ in the complement of $H(1, 1)$; this section is called the *compact edge* of $\mathcal{A}$.

(2) Both segments $\{(x, y) \in B : x = 0\}$ and $\{(x, y) \in B : y = 0\}$ are sent by H onto full half-trajectories of $\mathcal{F}(h)$. These two semi-trajectories of $\mathcal{F}(h)$ are called the *non–compact edges* of $\mathcal{A}$.

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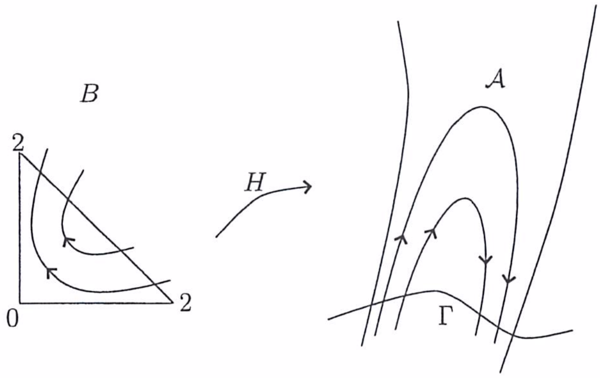

FIG. 2. A half-Reeb component.

The connection between half-Reeb components and injectivity is given by the following result.

PROPOSITION 1.1.4. *Suppose that* $X = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ *is a differentiable map such that* $0 \notin \mathrm{Spec}(X)$*. If* X *is not injective, then both* $\mathcal{F}(f)$ *and* $\mathcal{F}(g)$ *have hRc's.*

Proof: Suppose by contradiction that $\mathcal{F}(f)$ has no half-Reeb components. As the foliation $\mathcal{F}(f)$ has no singularities and is topologically equivalent to a smooth foliation [15], by Kaplan's classification of planar foliations [18], [19],(see [2] for a more recent exposition) we obtain that $\mathcal{F}(f)$ is topologically equivalent to the horizontal foliation of $\mathbb{R}^2$. Since $\mathcal{F}(f)$ is made up of level curves of f and f is strictly monotone along leaves of $\mathcal{F}(g)$, we obtain that each level curve of f must be connected. As g restricted to each level curve of f is strictly monotone, we arrive to the contradiction that X is injective.

As we want to give a self-contained exposition, we shall provide a proof of Proposition 1.1.4 independent on Kaplan's result ([18], [19]).

We again start supposing, by contradiction, that $\mathcal{F}(f)$ has no half-Reeb components. Without lost of generality we may assume that there are $p_1, p_2 \in \mathbb{R}^2$ such that $X(p_1) = X(p_2) = (0,0)$. For $i = 1, 2$, let α_i be the trajectory of $\mathcal{F}(f)$ passing through p_i. As $g|_{\alpha_i}$ is strictly monotone and $g(p_1) = g(p_2) = 0$, we obtain $\alpha_1 \cap \alpha_2 = \emptyset$. Let $\Omega(p_1, p_2)$ be the set of compact arcs of $\mathbb{R}^2$ whose endpoints are p_1 and p_2 and which meet transversally $\mathcal{F}(f)$ at $\{p_1, p_2\}$.

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(a) Among all elements of $\Omega(p_1, p_2)$ take $\Gamma \in \Omega(p_1, p_2)$ which minimizes the number of tangencies with $\mathcal{F}(f)$.

We claim that:

(b) $\alpha_i \cap \Gamma = \{p_i\}$, for $i = 1, 2$.

If we assume, by contradiction, that $\alpha_1 \cap \Gamma$ contains properly $\{p_1\}$, we may find $q \in \Gamma \setminus \{p_1, p_2\}$ and a closed subinterval α of α_1, with endpoint p_1, q, such that $\alpha \cap \Gamma = \{p_1, q\}$. We may assume that Γ is transversal to α at q. Let γ denote the connected component of $\Gamma \setminus \{q\}$ containing $\{p_2\}$. We can see that $\alpha \cup \gamma$ is an arc connecting p_1 and p_2 and also that $\mathcal{F}(f)$ is tangent to Γ at some point of $\Gamma \setminus (\gamma \cup \{p_1\} \cup \{q\})$. Under these conditions, we may approximate $\alpha \cup \gamma$ by an arc of $\Omega(p_1, p_2)$ which has less number of tangencies with $\mathcal{F}(f)$ than Γ. This contradiction with (a) proves (b).

As $f(p_1) = f(p_2) = 0$, $\mathcal{F}(f)$ is tangent to Γ at some point $q \notin \{p_1, p_2\}$. By using (a) that all tangencies of $\mathcal{F}(f)$ with Γ are generic. Therefore, by looking at the trajectories of $\mathcal{F}(f)$ around q, we may see that there exist closed subintervals $[p, q]$, $[q, Tp]$ of Γ with $[p, q] \cap [q, Tp] = \{q\}$, and a homeomorphism $T : [p, q] \to [q, Tp]$ such that,

(c1) $Tq = q$ and, for every $x \in (p, q]$, there is an arc of trajectory $[x, Tx]_f$ of $\mathcal{F}(f)$, starting at x, ending at Tx and meeting Γ exactly and transversally at $\{x, Tx\}$,

(c2) the family $\{[x, Tx]_f : x \in (p, q]\}$ depends continuously on x and tends to $\{q\}$ as $x \to q$.

From now on, suppose that

(d) $[p, q]$ is maximal with respect to properties (c1)-(c2) above

Then, using (b) and the fact that $\mathcal{F}(f)$ has no Reeb components, we obtain $\{p, Tp\} \cap \{p_1, p_2\} = \emptyset$. We claim that

(e) there is no arc of trajectory $[p, Tp]_f$ of $\mathcal{F}(f)$ connecting p and Tp such that the family $\{[x, Tx]_f : x \in (p, q]\}$ approaches continuously to $[p, Tp]_f$ as x goes to p.

In fact, suppose that (e) is false. Then, by using (d) and the fact that $\mathcal{F}(f)$ has no Reeb components, we conclude $[p, Tp]_f$ is tangent to Γ at least at one of the points of $\{p, Tp\}$. Under these circumstances, it is not difficult to approximate the curve, which is the union of $[p, Tp]_f$ with $\Gamma \setminus ((p, q] \cup [q, Tp))$, by a curve $\Gamma_1 \in \Omega(p_1, p_2)$ which has less tangencies with $\mathcal{F}(f)$ than Γ. This contradiction with (a) proves (e). Therefore, the sub-interval $[p, q] \cup [q, Tp]$ is the compact edge of a half-Reeb component of $\mathcal{F}(f)$ made up of two half trajectories of $\mathcal{F}(f)$ starting at p and Tp, respectively, together with the union of the arcs $[x, Tx]_f$, with $x \in (p, q]$. This finishes the proof. ∎

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For each $\theta \in \mathbb{R}$ let R_θ denote the linear rotation

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The following proposition will be useful. Its proof is contained in [14, Lemma 2.5].

PROPOSITION 1.1.5. *Let $X = (f,g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a non-injective, differentiable map such that $0 \notin \mathrm{Spec}(X)$. Let $\mathcal{A}$ be a hRc of $\mathcal{F}(f)$ and let $(f_\theta, g_\theta) = R_\theta \circ X \circ R_{-\theta}$, $\theta \in \mathbb{R}$. If $\Pi(\mathcal{A})$ is bounded, where $\Pi : \mathbb{R}^2 \to \mathbb{R}$ is given by $\Pi(x,y) = x$, then there is an $\epsilon > 0$ such that, for all $\theta \in (-\epsilon, 0) \cup (0, \epsilon)$, $\mathcal{F}(f_\theta)$ has a hRc $\mathcal{A}_\theta$ such that $\Pi(\mathcal{A}_\theta)$ is an interval of infinite length.*

Proof:

(a) Let $\theta \in \mathbb{R}$ be such that, for all $m \in \mathbb{Z}$, $\theta \neq \frac{m\pi}{2}$. Then $\mathcal{F}(f_\theta)$ and $\mathcal{F}(g_\theta)$ are transversal to both $R_\theta(\mathcal{F}(f))$ and $R_\theta(\mathcal{F}(g))$

In fact, we shall only prove that $\mathcal{F}(f_\theta)$ is transversal to $R_\theta(\mathcal{F}(f))$. If $\alpha : (a,b) \to \mathbb{R}^2$ is an injective curve contained in a leaf of $\mathcal{F}(f)$, then $f_\theta \circ R_\theta \circ \alpha(t) = (\cos\theta) f(\alpha(t)) - (\sin\theta) g(\alpha(t))$ which is strictly monotone, because $f(\alpha(t)) \equiv$ constant, $g(\alpha(t))$ is strictly monotone and $\sin\theta \neq 0$. Without lost of generality, we may assume that *nearby* its endpoints, the compact edge of $\mathcal{A}$ is made up of arcs of $\mathcal{F}(g)$. In this way there exist $a > 0$ and an injective, continuous curve $\gamma : (-a, 1+a)$ such that

(b1) $\gamma[0,1]$ is a compact edge of $\mathcal{A}$.

(b2) $\gamma|_{(-a,a)}$ and $\gamma|_{(1-a,1+a)}$ are contained in leaves of $\mathcal{F}(g)$.

(b3) for some $0 < \delta < a$ there exists an orientation reversing, injective function $\varphi_0 : [-\delta, \delta] \to (1-a, 1+a)$, such that $f(\gamma(s)) = f(\gamma(\varphi_0(s)))$, also, if $s \in (0,\delta)$, $\varphi_0(s) \in (1-a, 1)$ and there exist an arc of trajectory $T_0(s) \subset \mathcal{A}$ connecting $\gamma(s)$ with $\gamma(\varphi_0(s))$

If $\varepsilon > 0$ is small enough, than for each $|\theta| < \varepsilon$, there exist an orientation reversing continuous injective function $\varphi_\theta : [-\delta, \delta] \to (1-\varepsilon, 1+\varepsilon)$ such that

(c1) $f_\theta(\gamma(s)) = f_\theta(\gamma(\varphi_\theta(s))))$.

(c2) for all $s \in [\frac{\delta}{4}, \frac{\delta}{2}]$ there is an arc of trajectory $T_\theta(s) \subset R_\theta(\mathcal{A})$ of $\mathcal{F}(f_\theta)$ containing $\gamma(s)$ with $\gamma(\varphi_\theta(s))$,

(c3) $T_\theta(s)$ meets $\gamma[-\delta, 1+\varepsilon)$ exactly at $\{\gamma(s), \gamma(\varphi_\theta(s))\}$

(c4) $\varphi_\theta(-\delta) > 1$

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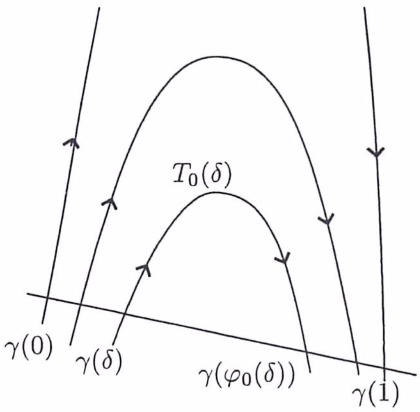

FIG. 3.

The continuous family $\{T_\theta(s) : \frac{\delta}{4} \leq s \leq \frac{\delta}{2}\}$ extends to a maximal continuous family $\{T_\theta(s) : \sigma \leq s \leq \frac{\delta}{2}\}$, with $\sigma \geq -\delta$ and satisfying (c2) and (c3) above.

The family $\{T_\theta(s)\}$ cannot be extended to $s = -\delta$ because $\varphi_\theta(-\delta) > 1$ and, as we can easily see,

such an arc $T_\theta(-\delta)$ would have a tangency with $R_\theta(\mathcal{F}(f))$ at a point of $R_\theta(\mathcal{A})$.

Therefore the set $\overline{\bigcup_{\sigma<s\leq\frac{\delta}{2}} T_\theta(s)}$ contains a half-Reeb Component B of $\mathcal{F}(f_\theta)$. Also we may easily se that one of the compact edges of B must be contained in $R_\theta(\mathcal{A})$. This implies that $\Pi(B)$ is an infinite length interval. ∎

The proof of the following lemma can be found in [28] (see also [3] and [16]). The proof below, due to C. G. T. A. Moreira (Gugu), is included for completeness.

LEMMA 1.1.6. *Let I be a bounded interval of $\mathbb{R}$ and $H\colon I \to \mathbb{R}$ be a bounded measurable function. If A denote the set of $x \in I$ such that*

$$\lim_{h\to 0} \frac{|H(x+h) - H(h)|}{|h|} = \infty$$

Then A is a (Lebesgue) measure set zero.

Proof: Suppose, by contradiction, that A has positive measure. Then there exist a positive measure compact subset $K \subset A$ such that $f|_K : K \to \mathbb{R}$ is continuous.

Then, for all $y \in f(K)$, $(f|_K)^{-1}(y)$ is a discrete set and so $(f|_K)^{-1}(y)$ is a finite set. Given positive integers n, r, let $C(n,r)$ be the set of $x \in K$ such that for some points

$x_1 < x_2 < \cdots < x_n$ in K with $x_{j+1} - x_j \geq \frac{1}{r}$ for $1 \leq j < n$ and $f(x_1) = f(x_2) = \cdots = f(x_n) = f(x)$. By the continuity of f and by the compactness of K, $C(n,r)$, is closed. It follows that the set

$$A(n) = \{x \in K : \#(f|_K)^{-1}(x)) \geq n\} = \bigcup_{r\geq 1} C(n,r)$$

is borelian. Hence,

$$B(n) = A(n) \setminus A(n+1) = \{x \in K : \#(f|_K)^{-1}(x)) = n\}$$

is also a borelian.

As $K = \sum_{n\geq 1} B(n)$ has positive measure, some $B(m)$ has positive measure. As

$$B(m) = \bigcup_{r\geq 1} B(m) \cap C(m,r)$$

there is some positive integer $\overline{r}$ such that $B(m) \cap C(m, \overline{r})$ has positive measure. By writings this last set as a finite union of sets having diameter less than $\frac{1}{\overline{r}}$, we have that one of these sets has positive measure and the restriction of f to this set is injective (by the definitions of $B(m)$ and $C(m,\overline{r})$). This set contains a positive measure compact set L such that $f|_L \colon L \to f(L)$ is continuous injective and so a homeomorphism. We obtain a contradiction by applying the Sard's Theorem [21] to $(f|_L)^{-1} \colon f(L) \to L$ which has zero derivative everywhere and so L should be a measure zero set. ∎

Proof of Theorem 1.1.1

Suppose by contradiction that $X = (f,g)$ is not injective. By Proposition 1.1.4, $\mathcal{F}(f)$ has a half-Reeb component $\mathcal{A}$. Let $\Pi \colon \mathbb{R}^2 \to \mathbb{R}$ be the orthogonal projection onto the first coordinate. By composing with a rotation if necessary (see Proposition 1.1.5) we may assume that $\Pi(\mathcal{A})$ is an unbounded interval. To simplify matters, let us suppose that $[b,\infty) \subset \Pi(\mathcal{A})$. Then, if $a > b$ is enough large,

(a) For any $x \geq a$, the vertical line $\Pi^{-1}(x)$ intersects exactly one trajectory $\alpha_x \subset \mathcal{A}$ of $\mathcal{F}(f)|_{\mathcal{A}}$ such that $\Pi(\alpha_x) \cap (x,\infty) = \emptyset$. In other words, x is the maximum for the restriction $\Pi|_{\alpha_x}$.

As α_x is a continuous curve, it follows that

(b) If $x \geq a$, $\alpha_x \cap \Pi^{-1}(x)$ is a compact subset of $\mathcal{A}$.

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Let $H : (a, \infty) \to \mathbb{R}$ be defined by

$$H(x) = \sup\{y : (x, y) \in \alpha_x \cap \Pi^{-1}(x)\}$$

As $\mathcal{F}(f)$ is a foliation, we may obtain that the function

(c) $\varphi(x) = f(x, H(x))$ is a strictly monotone continuous function which, when restricted to any interval $(a, b]$, is bounded; in particular, φ is differentiable a.e.

We claim that

(d) H is upper semicontinuous; thus, H is a measurable function.

In Fact, suppose by contradiction that H is not upper semicontinuous at $x_0 > a$. As H restricted to $(a, x_0 + 1)$ is bounded there exist $c \in \mathbb{R}$ and a sequence $x_n \to x_0$ such that $H(x_0) < c$ and $H(x_n) \to c$. However, φ is continuous. Hence,

$$(x_0, c) = \lim_{n\to\infty} (x_n, H(x_n) = \lim_{n\to\infty} \varphi(x_n) = \varphi(x_0) = (x_0, H(x_0)).$$

This contradiction proves (d).

By (d) above and lemma 1.1.6 if $a > 0$ is large enough, there exists a full measure subset M of (a, ∞) such that

(e) If $x \in M$, then φ is differentiable at x and

$$\liminf_{h\to 0} \frac{|H(x + h) - H(h)|}{|h|} < \infty$$

To proceed we shall only consider the case in which φ is strictly increasing. We claim that

(f) If $x \in M$, then $\varphi'(x) = f_x(x, H(x)) \geq \varepsilon$

In fact, if $x \in M$, there exists a sequence $h_n \to 0$ such that $\lim_{n\to\infty} \dfrac{k_n}{h_n} = \sigma \in \mathbb{R}$ where $k_n = H(x+h_n) - H(x)$. Also, by the structure of the level curves of $f|_{\mathcal{A}}$ and the assumptions that φ is increasing,

$$f(x, H(x)) = \inf\left\{f(x, y) : y \in \Pi^{-1}(x) \cap \mathcal{A}\right\}$$

This implies that $f_y(x, H(x)) = 0$. Hence, as f is differentiable at $(x, H(x))$, there are real valued functions ε_1, ε_2 defined in a neighborhood of $(0, 0)$ such that

$$f(x + h_n, y + k_n) = f(x, y) + f_x(x, H(x))h_n + \varepsilon_1(h_n, k_n)h_n + \varepsilon_2(h_n, k_n)k_n$$

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and $\lim_{n\to\infty} \varepsilon_1(h_n, k_n) = \lim_{n\to\infty} \varepsilon_2(h_n, k_n) = 0$. Therefore, for n large enough,

$$\frac{\varphi(x+h_n)-\varphi(x)}{h_n} = f_x(x,H(x)) + \varepsilon_1(h_n,k_n) + \varepsilon_2(h_n,k_n)\frac{k_n}{h_n}$$

which implies that

$$\varphi'(x) = \lim_{n\to\infty} \frac{\varphi(x+h_n)-\varphi(x)}{h_n} = f_x(x,H(x))$$

Therefore

$$DX(x,H(x)) = \begin{pmatrix} \varphi'(x) & 0 \\ g_x(x,H(x)) & g_y(x,H(x)) \end{pmatrix}$$

i. e. $\varphi'(x)$ is an eigenvalue of $DX(x,H(x))$. By the assumption of the theorem and the assumptions that φ is strictly increasing, (f) is proved.

As $f|_A$ is bounded, φ is bounded. Hence, there is a constant $K > 0$ such that for all $x > a$, $\varphi(a) \leq \varphi(x) \leq K$. Take $c > a$ so that $(c-a)\varepsilon > K$. Then we have that

$$K > \varphi(c) - \varphi(a) \geq \int_a^c \varphi'(x)dx \geq \int_a^c \varepsilon dx = (c-a)\varepsilon > K$$

This contradiction proves the theorem. ∎

2. INJECTIVITY RESULT

This section is devoted to the proof of the following

THEOREM 2.2.1. *Let $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a a differentiable map. If, for some $\epsilon > 0$,* $\text{Spec}(X) \cap [0,\epsilon) = \emptyset$, *then X is injective.*

We shall need the following

LEMMA 2.2.2. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map such that* $\det(F'(x)) \neq 0$ *for all x in $\mathbb{R}^n$. Given $t \in \mathbb{R}$, let $F_t\colon \mathbb{R}^n \to \mathbb{R}^n$ denote the map $F_t(x) = F(x) - tx$. If there exists a sequence $\{t_m\}$ of real numbers converging to 0 such that every map $F_{t_m}\colon \mathbb{R}^n \to \mathbb{R}^n$ is injective, then F is injective.*

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Proof: Choose $x_1, x_2 \in \mathbb{R}^n$ such that $F(x_1) = y = F(x_2)$. We will prove $x_1 = x_2$. By the Inverse Mapping Theorem 1.1.2, we may find neighborhoods U_1, U_2, V of x_1, x_2, y, respectively, such that, for $i = 1, 2$, $F|_{U_i} : U_i \to V$ is a homeomorphism and $U_1 \cap U_2 = \emptyset$. If m is large enough, then $F_{t_m}(U_1) \cap F_{t_m}(U_2)$ will contain a neighborhood W of y. In this way, for all $w \in W$, $\#(F_{t_m}^{-1}(w)) \geq 2$. This contradiction with the assumptions, proves the lemma. ∎

REMARK 2.2.3. *Even if $n = 1$ and the maps F_{t_m} in Lemma 2.2.2 are smooth diffeomorphisms, we cannot conclude that F is a diffeomorphism. For instance, if $F : \mathbb{R} \to (0,1)$ is an orientation reversing diffeomorphism, then for every $t > 0$, the map $F_t : \mathbb{R} \to \mathbb{R}$ (defined by $F_t(x) = F(x) - tx$) will be an orientation reversing global diffeomorphism.*

Proof of Theorem 2.2.1

We claim that for each $0 < t < \epsilon$, the map $F_t \colon \mathbb{R}^2 \to \mathbb{R}^2$, given by $F_t(x) = F(x) - tx$, is injective.

In fact, As $D(F_t)(x) = DF(x) - tI$, (where I is the Identity map), we obtain that if $0 < a < \min\{t, \epsilon - t\}$, then $\text{Spec}(F_t) \cap (-a, a) = \emptyset$. This theorem follows immediately from Lemma 2.2.2 and Theorem 1.1.1. ∎

3. GLOBAL ASYMPTOTIC STABILITY

This section is devoted to the proof of the following

THEOREM 3.3.1. *Let $X = (f, g) \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable vector field such that* $\text{Spec}(X) \subset \{z \in \mathbb{C} : \Re(z) < 0\}$. *Then, for all $p \in \mathbb{R}^2$, there is a unique positive trajectory starting at p; moreover, the $\omega-$limit set of p is equal to* $\{0\}$.

Let $X^* = (-g, f) : \mathbb{R}^2 \to \mathbb{R}^2$. Observe that X^* is orthogonal to $X = (f, g)$. In the following, the same notation as that for intervals of $\mathbb{R}$ will be used for oriented arcs of trajectory $[p, q], [p, q), \cdots$ (resp. $[p, q]^*, [p, q)^*, \cdots$), connecting the points p and q, of X (resp. of X^*.) The orientacion of these arcs is that induced by X (resp. by X^*). For any arc of trajectory $[p, q]^*$ we have the function $L(p, q)$, given by

$$L(p, q) = \left| \int_{[p,q]^*} \|X\| ds \right|$$

where ds denotes the arc length element.

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LEMMA 3.3.2. *Let A be a compact rectangle the boundary of which is made up of the following (oriented) arcs of trajectory: $[p_1, q_1]$,$[p_2, q_2]$ of X and $[p_1, p_2]^*$,$[q_1, q_2]^*$ of X^*. Then*

$$L(q_1, q_2) - L(p_1, p_2) < 0 \tag{1}$$

Proof: It follows from the Green's Formula, as presented in [24] (corollary 5.7) and the assumptions that Trace(DX) < 0 everywhere in A, that Trace(DX) is Lebesgue integrable in A and that

$$L(q_1, q_2) - L(p_1, p_2) = \int_A \text{Trace}(DX)dx \wedge dy \ < \ 0$$

This finishes the proof. ∎

LEMMA 3.3.3. *For every $p \in \mathbb{R}^2$, there is only one positive half-trajectory of X starting at p.*

Proof: Suppose, by contradiction, that there are two positive half-trajectories γ_p^+ and σ_p^+ starting at p. As $X(p) \neq 0$, we may take a triangle (i.e. a degenerate rectangle) the boundary of which is made up of two arcs of trajectory $[p, q_1] \subset \gamma_p^+$, $[p, q_2] \subset \sigma_p^+$ of X and one arc of trajectory $[q_1, q_2]^*$ of X^*. By applying (1) of Lemma 3.3.2 we will obtain

$$L(q_1, q_2) < 0.$$

This contradiction proves the lemma. ∎

In next lemma we shall need to what extent flow behavior persist when uniqueness fails. This has been studied, for instance, in [17], [26] and [29]. Given the arcs $(p, q), [p, q), \cdots$ (resp. $(p, q)^*, [p, q)^*, \cdots$) of X (resp. of X^*), their arc length will be denoted by let $\ell(p, q)$ (resp. $\ell(p, q)^*$).

LEMMA 3.3.4. *Let W be a relatively compact open neighborhood of $0 \in \mathbb{R}^2$ and $p_1 \in \mathbb{R}^2 \setminus W$. If $\epsilon > 0$ is small enough, there exists $\delta > 0$ such that if $[p_1, q_1]$ is an arc of trajectory of X, with $[p_1, q_1] \cap W = \emptyset$ and $[p_1, p_2]^*$ (resp. $[p_2, p_1]$) is an arc of trajectory of X^*, with $\ell(p_1, p_2)^* < \delta$, then there are arcs of trajectory $[p_2, q_2]$ of X and $[q_1, q_2]^*$ of X^* (resp. $[q_2, q_1]^*$ of X^*), such that $\ell(q_1, q_2)^* < \epsilon$.*

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Proof: Let $U \subset \mathbb{R}^2 \setminus W$ be an open disc centered at p_1. Let V be an open neighborhood of 0 such that $V \subset \overline{V} \subset W$. As $X(0) = 0$ and X is injective (see Theorem 2.2.1.), there exists $\rho > 0$ such that for all $p \in \mathbb{R}^2 \setminus V$, $||X(p)|| > \rho$. Let $\Delta = \sup\{||X(p)|| : p \in U\}$. Take $\epsilon > 0$ smaller than the distance between V and $\mathbb{R}^2 \setminus W$ and take $\delta > 0$ smaller than both $(\epsilon\rho)/\Delta$ and the radius of U.

Observe first that if such a rectangle $R(p_1, q_1; q_2)$ (made up of $[p_1, q_1], [p_1, p_2]^*, [p_2, q_2], [q_1, q_2]^*$) exists, then

$$\rho\ell(q_1, q_2)^* \leq L(q_1, q_2) < L(p_1, p_2) \leq \Delta\ell(p_1, p_2)^*$$

and so

$$\ell(q_1, q_2)^* \leq \frac{\Delta}{\rho}\,\ell(p_1, p_2)^*.$$

which, by the assumptions above, imply that $\ell(q_1, q_2)^* < \epsilon$.

Let m be the supremum of all $x \in [p_1, q_1]$ such that , for all $y \in [p_1, x)$, $R(p_1, y; q_2)$ exists. By using Proposition 2.1 and Corollary 2.2 of [26], it follows from the remarks above and that $m = q_1$. This proves the lemma. ∎

Let W^s denote the set of points in $\mathbb{R}^2$ whose ω-limit set is the origin:

$$W^s = \{p \in \mathbb{R}^2 : \omega(p) = \{0\}\}$$

From our assumption the origin is a local sink, so it follows from lemma right above that

COROLLARY 3.3.5. *W^s is a non-empty open set.*

LEMMA 3.3.6. *The vector field X has no closed trajectories; moreover, given $p \in \mathbb{R}^2$ the $\omega-$limit set of p, denoted by $\omega(p)$, is either $\{0\}$ or the empty set.*

Proof: It follows directly from the Greenś Formula that X has no closed orbits. Then by using arguments of the classical Poincaré Bendixson Theorem, we may easily obtain the conclusions of the lemma. ∎

Proof of Theorem 3.3.1

By using Corollary 3.3.5, W^s is a non-empty open set. By Lemma 3.3.4 it can be obtained that $R^2 \setminus W^s$ is an open set. By the connectedness of $\mathbb{R}^2$ we conclude that $W^s = \mathbb{R}^2$ ∎

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